Equidistribution speed towards the Green current for endomorphisms of $\mathbb{P}^k$

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Abstract

Let f be a non-invertible holomorphic endomorphism of $\mathbb{P}^k$. For a hypersurface H of $\mathbb{P}^k$, generic in the Zariski sense, we give an explicit speed of convergence of $f^{-n}(H)$ towards the dynamical Green $(1,1)$-current of f.

Key words: Green current, equidistribution speed.
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1 Introduction

Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ on the complex projective space $\mathbb{P}^k$. The iterates $f^n = f \circ \cdots \circ f$ define a dynamical system on $\mathbb{P}^k$. It is well-know that, if ω denotes the normalized Fubini-Study form on $\mathbb{P}^k$ then, the sequence $d^{-n}(f^n)^*(\omega)$ converges to a positive closed current T of bidegree $(1,1)$ called the Green current of f (see e.g. [8]). It is a totally invariant current, whose support is the Julia set of f and that exhibits interesting dynamical properties. In particular, for a generic hypersurface H of degree s, the sequence $d^{-n}(f^n)^*[H]$ converges to sT [6]. Here, $[H]$ denotes the current of integration on H and the convergence is in the sense of currents. In fact, if we denote by T^p the self-intersection $T \wedge \cdots \wedge T$, Dinh and Sibony proposed the following conjecture on equidistribution.

Conjecture 1.1. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and T its Green current. If H is an analytic set of pure codimension p and of degree s which is generic in the Zariski sense, then the sequence $d^{-pn}(f^n)^*[H]$ converges to sT^p exponentially fast.*

The aim of the paper is to prove the conjecture for $p = 1$. It is a direct consequence of the following more precise result on currents. Indeed, we only have to apply the theorem to $S := s^{-1}[H]$ for hypersurfaces H which does not contain any element of $\mathscr{A}_\lambda$.

Theorem 1.2. *Let f, T be as above and let $1 < \lambda < d$. There exists a finite family $\mathscr{A}_\lambda$ of periodic irreducible analytic sets such that if S is a positive closed $(1,1)$-current of mass 1, whose dynamical potential u verifies $\|u\|_{L^1(X)} \leq C$ for all X in $\mathscr{A}_\lambda$, then the sequence $S_n := d^{-n}(f^n)^*(S)$ converge exponentially fast to T. More precisely, for every $0 < \beta \leq 2$ and $\phi \in \mathscr{C}^\beta(\mathbb{P}^k)$ we get*

$$|\langle S_n - T, \phi\rangle| \leq A\|\phi\|_{\mathscr{C}^\beta}\left(\frac{\lambda}{d}\right)^{n\beta/2}, \tag{1.1}$$

where $A > 0$ depends on the constants C and β but is independent of S, ϕ and n.

Here, the space $L^1(X)$ is with respect to the volume form $\omega^{\dim(X)}$ on X and $\mathscr{C}^\beta(\mathbb{P}^k)$ denotes the space of $(k-1,k-1)$-forms whose coefficients are of class $\mathscr{C}^\beta$, equipped with the norm induced by a fixed atlas. *The dynamical potential* of S is the unique quasi-plurisubharmonic function u such that $S = dd^c u + T$ and $\max_{\mathbb{P}^k} u = 0$. Note that $\mathscr{A}_\lambda$ will be explicitly constructed. Theorem 1.2 still holds if we replace $\mathscr{A}_\lambda$ by an analytic subset, e.g. a finite set, which intersects all components of $\mathscr{A}_\lambda$.

Equidistribution problem without speed was considered in dimension 1 by Brolin [3] for polynomials and by Lyubich [17] and Freire-Lopes-Mañé [14] for rational maps. They proved that for every point a in $\mathbb{P}^1$, with maybe two exceptions, the preimages of a by f^n converge towards the equilibrium measure, which is the counterpart of the Green current in dimension 1.

In higher dimension, for $p = k$, simple convergence in Conjecture 1.1 was established by Fornæss-Sibony [12], Briend-Duval [2]. Recently in [9], Dinh and Sibony give exponential speed of convergence, which completes Conjecture 1.1 for $p = k$. The equidistribution of hypersurfaces was proved by Fornæss and Sibony for generic maps [13] and by Favre and Jonsson in dimension 2 [11]. The convergence for general endomorphisms and Zariski generic hypersurfaces was obtained by Dinh and Sibony in [6]. These papers state convergence but without speed. In other codimensions, the problem is much more delicate. However, the conjecture was solved for generic maps in [7], using the theory of super-potentials.

We partially follow the strategy developed in [13], [11] and [6], which is based on pluripontential theory together with volume estimates, i.e. a lower bound to the contraction of volume by f. These estimates are available outside some exceptional sets which are treated using hypothesis on the map f or on the current S.

The exceptional set $\mathscr{A}_\lambda$ will be defined in Section 5. It is in general a union of periodic analytic sets possibly singular. In our proof of Theorem 1.2, it is necessary to obtain the convergence of the trace of S_n to these analytic sets. So, we have to prove an analog of Theorem 1.2 where $\mathbb{P}^k$ is replaced with an invariant analytic set. The geometry of the analytic set near singularities is the source of important technical difficulties. We will collect in Section 2 and Section 3 several versions of Lojasiewicz's inequality which will allow us to work with singular analytic sets and also to obtain good estimates on the size of a ball under the action of f^n. Such estimates are crucial in order to obtain the convergence outside exceptional sets.

Theorem 1.2 can be reformulated as an L^1 estimate of the dynamical potential u_n of S_n (see Theorem 6.1). The problem is equivalent to a size control of the sublevel set $K_n = \{u_n \leq -(\lambda/d)^n\}$. Since T is totally invariant, we get that $u_n = d^{-n} u \circ f^n$ and $f^n(K_n) = \{u \leq -\lambda^n\}$. The above estimate on the size of ball can be applied provide that K_n is not concentrated near the exceptional sets. The last property will be obtained using several generalizations of exponential Hörmander's estimate for plurisubharmonic functions that will be stated in Section 4. A key point in our approach is that, by reducing the domain of integration, we obtain uniform exponential estimates for non-compact families of quasi-plurisubharmonic functions.

We close this introduction by setting some notations and conventions. The symbols $\lesssim$ and $\gtrsim$ mean inequalities up to constants which only depend on f or on the ambient space. To desingularize an analytic subset of $\mathbb{P}^k$, we always use a finite sequence of blow-ups of $\mathbb{P}^k$. Unless otherwise specified, the distances that we consider are naturally induced by embedding or smooth metrics for compact manifolds. For $K > 0$ and $0 < \alpha \leq 1$, we say that a function $u : X \to \mathbb{C}$ is (K, α)-Hölder continuous if for all x and y in X, we have $|u(x) - u(y)| \leq K \text{dist}(x, y)^\alpha$. We denote by $\mathbb{B}$ the unit ball of $\mathbb{C}^k$ and for $r > 0$, by $\mathbb{B}_r$ the ball centered at the origin with radius r. In $\mathbb{P}^k$, we denote by $B(x, r)$ the ball of center x and of radius r. And, for $X \subset \mathbb{P}^k$ an analytic subset, we denote by $B_X(x, r)$ the connected component of $B(x, r) \cap X$ which contains x. We call it the ball of center x and of radius r in X. It may have more than one irreducible component. Finally, for a subset $Z \subset X$, we denote by $Z_{X,t}$ or simply Z_t, the tubular t-neighborhood of Z in X, i.e. the union of $B_X(z, t)$ for all z in Z. A function on X is call (strongly) holomorphic if it has locally a holomorphic extension to a neighborhood of the ambient space.

2 Lojasiewicz's inequality and consequences

One of the main technical difficulties of our approach is related to singularities of analytic sets that we will handle using blow-ups along smooth varieties. In this section, we study the behavior of metric properties under blow-ups. It will allow us to establish volume and exponential estimates onto singular analytic sets.

We will frequently use the following Lojasiewicz inequalities. We refer to [1] for further details.

Theorem 2.1. *Let U be an open subset of $\mathbb{C}^k$ and let h, g be subanalytic functions in U. If $h^{-1}(0) \subset g^{-1}(0)$, then for any compact subset K of U, there exists a constant $N \geq 1$ such that, for all z in K, we have*

$$|h(z)| \gtrsim |g(z)|^N.$$

In this paper, we only use the notion of subanalytic function in the following case. Let U be an open subset of $\mathbb{C}^k$ and $A \subset U$ be an analytic set. Every compact of U has a neighborhood $V \subset U$ such that the function $x \mapsto \mathrm{dist}(x, A)$ and analytic functions on U are subanalytic on V. Moreover, the composition or the sum of two such functions is still subanalytic on V. In particular, we have the following property.

Corollary 2.2. *Let U be an open subset of $\mathbb{C}^k$ and let A, B be analytic subsets of U. Then for any compact subset K of U, there exists a constant $N \geq 1$ such that, for all z in K, we have*

$$\mathrm{dist}(z, A) + \mathrm{dist}(z, B) \gtrsim \mathrm{dist}(z, A \cap B)^N.$$

We briefly recall the construction of blow-up that we will use later. If U is an open subset of $\mathbb{C}^k$ which contains 0, the blow-up $\widehat{U}$ of U at 0 is the submanifold of $U \times \mathbb{P}^{k-1}$ defined by the equations $z_i w_j = z_j w_i$ for $1 \leq i, j \leq k$ where $(z_1, \ldots, z_k)$ are the coordinates of $\mathbb{C}^k$ and $[w_1 : \cdots : w_k]$ are the homogeneous coordinates of $\mathbb{P}^{k-1}$. The sets $w_i \neq 0$ define local charts on $\widehat{U}$ where the canonical projection $\pi : \widehat{U} \to U$, if we set for simplicity $i = 1$ and $w_1 = 1$, is given by

$$\pi(z_1, w_2, \ldots, w_k) = (z_1, z_1 w_2, \ldots, z_1 w_k).$$

If $V \subset \mathbb{C}^p$ is an open subset, *the blow-up* of $U \times V$ along $\{0\} \times V$ is defined by $\widehat{U} \times V$. This is the local model of a blow-up.

Finally, if X is a complex manifold, the blow-up $\widehat{X}$ of X along a submanifold Y is obtained by sticking copies of the above model and by using suitable atlas of X. The natural projection $\pi : \widehat{X} \to X$ defines a biholomorphism between $\widehat{X} \setminus \widehat{Y}$ and $X \setminus Y$ where the set $\widehat{Y} := \pi^{-1}(Y)$ is called the exceptional hypersurface. If A is an analytic subset of X not contained in Y, the strict transform of A is defined as the closure of $\pi^{-1}(A \setminus Y)$.

We have the following elementary lemma.

Lemma 2.3. *Let $\pi : \widehat{U} \times V \to U \times V$ be as above and $\widehat{Y}$ denote the exceptional hypersurface in $\widehat{U} \times V$. Assume that $U \times V$ is bounded in $\mathbb{C}^k \times \mathbb{C}^p$. Then for all $\widehat{z}, \widehat{z}' \in \widehat{U} \times V$, we have*

$$\mathrm{dist}(z, z') \gtrsim \mathrm{dist}(\widehat{z}, \widehat{z}')(\mathrm{dist}(\widehat{z}, \widehat{Y}) + \mathrm{dist}(\widehat{z}', \widehat{Y})),$$

where $z = \pi(\widehat{z})$ and $z' = \pi(\widehat{z}')$.

Proof. Since π leaves invariant the second coordinate, considering the maximum norm on $U \times V$, the general setting is reduced to the case of blow-up of a point. Hence we can take $V = \{0\}$. The lemma is obvious if z or z' is equal to 0. Since π is a biholomorphism outside $\widehat{Y}$, we can assume that $0 < \|z\|, \|z'\| < 1$. Moreover, up to an isometry of $\mathbb{C}^k$, we can assume that $\max |z_i| = |z_1|$ and $\max |z_i'| = |z_1'|$. Indeed, we can send z and z' into the plane generated by the first two coordinates and then use the rotation group of this plane. Therefore, in the chart $w_1 = 1$, we have $\widehat{z} = (z_1, w_2, \ldots, w_k)$, $\widehat{z}' = (z_1', w_2', \ldots, w_k')$ with $|w_i|, |w_i'| \leq 1$.

By triangle inequality, we have

$$|z_1 w_i - z_1' w_i'| \geq |z_1 w_i - z_1 w_i'| - |z_1 w_i' - z_1' w_i'| \geq |z_1||w_i - w_i'| - |z_1 - z_1'|.$$

Hence, by symmetry in z_1 and z_1' we get

$$2|z_1 w_i - z_1' w_i'| \geq (|z_1| + |z_1'|)|w_i - w_i'| - 2|z_1 - z_1'|.$$

Therefore, there is a constant $a > 0$ independent of z and z' such that

$$|z_1 - z_1'| + \sum_{i=2}^{k} |z_1 w_i - z_1' w_i'| \geq a(|z_1| + |z_1'|)(|z_1 - z_1'| + \sum_{i=2}^{k} |w_i - w_i'|).$$

The left-hand side corresponds to $\mathrm{dist}(z, z')$. We also have that $\mathrm{dist}(\widehat{z}, \widehat{z}') \simeq |z_1 - z_1'| + \sum_{i=2}^k |w_i - w_i'|$ and $\mathrm{dist}(\widehat{z}, \widehat{Y}) + \mathrm{dist}(\widehat{z}', \widehat{Y}) \simeq |z_1| + |z_1'|$. The result follows. □

A similar result holds for analytic sets. More precisely, consider an irreducible analytic subset X of $\mathbb{P}^k$ of dimension l and a smooth variety Y contained in X. Let $\overline{\pi} : \widehat{\mathbb{P}^k} \to \mathbb{P}^k$ be the blow-up along Y and π the restriction of $\overline{\pi}$ to the strict transform $\widehat{X}$ of X. Denote by $\overline{Y}$ and $\widehat{Y}$ the exceptional hypersurfaces in $\widehat{\mathbb{P}^k}$ and in $\widehat{X}$ respectively.

Lemma 2.4. *There exists $N \geq 1$ such that for all $\widehat{z}$ and $\widehat{z}'$ in $\widehat{X}$*

$$\operatorname{dist}(z, z') \gtrsim \operatorname{dist}(\widehat{z}, \widehat{z}')(\operatorname{dist}(\widehat{z}, \widehat{Y}) + \operatorname{dist}(\widehat{z}', \widehat{Y}))^N,$$

Proof. The previous lemma gives the inequality with $N = 1$ if we substitute $\widehat{Y}$ by $\overline{Y}$. By Corollary 2.2 applied to $A = \widehat{X}$ and $B = \overline{Y}$, there exists $N \geq 1$ such that $\operatorname{dist}(\widehat{x}, \overline{Y}) \gtrsim \operatorname{dist}(\widehat{x}, \widehat{Y})^N$ for all $\widehat{x}$ in $\widehat{X}$. The result follows. □

Here is the first estimate on contraction for blow-ups.

Lemma 2.5. *There exists a constant $N \geq 1$ such that for all $0 < t \leq 1/2$, if $\operatorname{dist}(\widehat{x}, \widehat{Y}) > t$ and $r < t/2$, then $\pi(B_{\widehat{X}}(\widehat{x}, r))$ contains $B_X(\pi(\widehat{x}), t^N r)$. Moreover, if N is large enough then the image by π of a ball of radius $0 < r \leq 1/2$ contains a ball of radius r^N in X.*

Proof. Let $\widehat{y}$ be a point in $\widehat{X}$ such that $\operatorname{dist}(\widehat{x}, \widehat{y}) = r$ and set $x = \pi(\widehat{x})$, $y = \pi(\widehat{y})$. The assumption on r gives that $\operatorname{dist}(\widehat{y}, \widehat{Y}) > t/2$. Therefore, we deduce from Lemma 2.4 that

$$\operatorname{dist}(x, y) \gtrsim rt^N.$$

The first assertion follows since $t \leq 1/2$ and π is a biholomorphism outside $\widehat{Y}$.

For a general ball B of radius r in $\widehat{X}$, we can reduce the ball in order to avoid $\widehat{Y}$ and then apply the first statement. More precisely, as $\dim(\widehat{Y}) \leq l-1$ there is a constant $c > 0$ such that for all $\rho > 0$, $\widehat{Y}$ is cover by $c\rho^{-2(l-1)}$ balls of radius ρ. On the other hand, by a theorem of Lelong [16, 4], the volume of a ball of radius ρ in $\widehat{X}$ varies between $c'^{-1}\rho^{2l}$ and $c'\rho^{2l}$ for some $c' > 0$. Hence, the volume of $\widehat{Y}_\rho$ is of order ρ^2. Take $\rho = c''r^l$ with $c'' > 0$ small enough. By counting the volume, we see that B is not contained in $\widehat{Y}_\rho$. Therefore, $B \setminus \widehat{Y}$ contains a ball of radius $\rho/3$. We obtain the result using the first assertion. □

In the same spirit, we have the following lemma.

Lemma 2.6. *Let $\widehat{Z}$ be a compact manifold, Z be a irreducible analytic subset of $\mathbb{P}^k$ and $\pi : \widehat{Z} \to Z$ be a surjective holomorphic map. Let A be an irreducible analytic subset of Z and define $\widehat{A} := \pi^{-1}(A)$. There exists $N \geq 1$ such that A_{t^N} is included in $\pi(\widehat{A}_t)$ for all $t > 0$ small enough. Moreover, if $\widehat{A}$ is the union on two analytic sets $\widehat{A}_1$, $\widehat{A}_2$ such that $A_2 := \pi(\widehat{A}_2)$ is strictly contained in A then $\pi(\widehat{A}_{1,t})$ contains $A_{t^N} \setminus A_{2,t^{1/2}}$.*

Proof. Since $\operatorname{dist}(\widehat{z}, \widehat{A}) = 0$ if and only if $\operatorname{dist}(\pi(\widehat{z}), A) = 0$, we can apply Theorem 2.1 to these functions which implies the existence of $N \geq 1$ such that

$$\operatorname{dist}(\widehat{z}, \widehat{A})^N \lesssim \operatorname{dist}(\pi(\widehat{z}), A).$$

Therefore, since π is surjective, $\pi(\widehat{A}_t)$ contains A_{t^N} for $t > 0$ small enough.

Now, let $\widehat{A} = \widehat{A}_1 \cup \widehat{A}_2$ be as above. Since π is holomorphic, there exists $c > 0$ such that $\pi(\widehat{A}_{2,t}) \subset A_{2,ct}$. Therefore, $\pi(\widehat{A}_{1,t})$ contains $A_{t^N} \setminus A_{2,t^{1/2}}$ for $t > 0$ sufficiently small. □

In the sequel, we will constantly use desingularization of analytic sets. The following lemma allows us to conserve integral estimates.

Lemma 2.7. *Let Z and $\widehat{Z}$ be irreducible analytic subsets of Kähler manifolds. Let $\pi : \widehat{Z} \to Z$ be a surjective proper holomorphic map. Then, for every compact $\widehat{L}$ of $\widehat{Z}$ there exists $q \geq 1$ such that if v is in $L^q_{loc}(Z)$ then $\widehat{v} := v \circ \pi$ is in $L^1(\widehat{L})$. Moreover, there exists $c > 0$, depending on $\widehat{L}$, such that*

$$\|\widehat{v}\|_{L^1(\widehat{L})} \leq c\|v\|_{L^q(\pi(\widehat{L}))}.$$

Proof. Using a desingularization, we can assume that $\widehat{Z}$ is a smooth Kähler manifold with a Kähler form $\widehat{\omega}$. Denote by ω, n, m a Kähler form on Z and the dimensions of Z and $\widehat{Z}$ respectively. Generic fibers of π are compact of dimension $m - n$ and form a continuous family. It follows that the integral of $\widehat{\omega}^{m-n}$ on that fibers is a constant.

Consider $\widehat{\lambda} = \pi^*(\omega^n) \wedge \widehat{\omega}^{m-n}$ on $\widehat{Z}$. The last observation implies that $\pi_*(\widehat{\lambda}) = \omega^n$ up to a constant. Therefore, if v is in $L^q_{loc}(Z)$ then $\widehat{v}$ is in $L^q_{loc}(\widehat{Z}, \widehat{\lambda})$. Moreover, we can write $\widehat{\lambda} = h\widehat{\omega}^m$ where h is a positive function. If there exists $\tau > 0$ such that $h^{-\tau}$ is integrable on $\widehat{L}$ with respect to $\widehat{\omega}^m$, we obtain for $p = 1 + \tau$ and q its conjugate that

$$\begin{aligned} \int_{\widehat{L}} |\widehat{v}|\widehat{\omega}^m = \int_{\widehat{L}} |\widehat{v}| h^{-1}\widehat{\lambda} &\leq \left(\int_{\widehat{L}} |\widehat{v}|^q \widehat{\lambda}\right)^{1/q} \left(\int_{\widehat{L}} h^{-p}\widehat{\lambda}\right)^{1/p} \\ &\lesssim \left(\int_{\pi(\widehat{L})} |v|^q \omega^n\right)^{1/q} \left(\int_{\widehat{L}} h^{-\tau}\widehat{\omega}^m\right)^{1/p} \\ &\lesssim \|v\|_{L^q(\pi(\widehat{L}))}. \end{aligned}$$

It remains to show the existence of τ. The set $\{h = 0\}$ is contained in the complex analytic set A where the rank of π is not maximal. More precisely, if π has maximal rank at z, then we can linearize π in a neighborhood of z. Therefore, $\widehat{\lambda}$ and $\widehat{\omega}^m$ are comparable in that neighborhood.

Since $\widehat{L}$ is compact, the problem is local. Let z_0 in $\widehat{L}$. We can find a small chart U at z_0 and a holomorphic function ϕ on U such $A \cap U$ is contained in $\{\phi = 0\}$. We can also replace $\widehat{\omega}$ and ω by standard Euclidean forms on U and on a neighborhood of $\pi(U)$. So, we can assume that h is analytic. By Lojasiewicz inequality, for every compact K of U, there exists $N \geq 1$ such that $h(z) \gtrsim |\phi(z)|^N$ for all z in K. On the other hand, exponential estimate (cf. [15] and Section 4) applied to the plurisubharmonic function $\log|\phi|$ says that $\phi^{-\alpha}$ is in $L^1(K, \widehat{\omega}^m)$ for some $\alpha > 0$. Therefore, $h^{-\alpha/N}$ belongs to $L^1(K, \widehat{\omega}^m)$. We obtain the desired property near z_0 by taking $\tau \leq \alpha/N$. □

Finally, the following results establish a relation between the regularity of functions on an analytic set and that of their lifts to a desingularization.

Proposition 2.8. *Let Z, $\widehat{Z}$, π be as in Lemma 2.7 and v be a function on Z. Assume that the lift of v to $\widehat{Z}$ is (K, α)-Hölder continuous. Then, for every compact L of Z, there exist constants $0 < \alpha' \leq 1$ and $a > 0$, independent of v, such that v is (aK, α')-Hölder continuous on L.*

Proof. Let Δ be the diagonal of $Z \times Z$. We still denote by π the map induced on the product $\widehat{Z} \times \widehat{Z}$ and we set $\widehat{\Delta} = \pi^{-1}(\Delta)$. As in Lemma 2.6, by Lojasiewicz inequality, we have

$$\operatorname{dist}(\widehat{a}, \widehat{\Delta})^M \lesssim \operatorname{dist}(\pi(\widehat{a}), \Delta),$$

if $\pi(\widehat{a}) \in L \times L$. Therefore, if we set $\widehat{a} = (\widehat{x}, \widehat{x}')$ and $\pi(\widehat{a}) = (x, x')$, then we can rewrite this inequality as

$$(\operatorname{dist}(\widehat{x}, \widehat{z}) + \operatorname{dist}(\widehat{x}', \widehat{z}'))^M \lesssim \operatorname{dist}(x, x'), \tag{2.1}$$

for some $\widehat{z}, \widehat{z}' \in \widehat{Z}$ such that $\pi(\widehat{z}) = \pi(\widehat{z}')$.

Now, let v be as in the proposition and $\widehat{v} = v \circ \pi$ denote its lift. Taking the same notation as above we have

$$|v(x) - v(x')| = |\widehat{v}(\widehat{x}) - \widehat{v}(\widehat{x}')| \leq |\widehat{v}(\widehat{x}) - \widehat{v}(\widehat{z})| + |\widehat{v}(\widehat{z}') - \widehat{v}(\widehat{x}')|,$$

since $\pi(\widehat{z}) = \pi(\widehat{z}')$ which implies that $\widehat{v}(\widehat{z}) = \widehat{v}(\widehat{z}')$. Therefore, the assumption on $\widehat{v}$ implies that

$$|v(x) - v(x')| \leq K(\operatorname{dist}(\widehat{x}, \widehat{z})^\alpha + \operatorname{dist}(\widehat{z}', \widehat{x}')^\alpha),$$

and finally (2.1) gives

$$|v(x) - v(x')| \leq aK \operatorname{dist}(x, x')^{\alpha/M},$$

where $a > 0$ depends only on α, L and π. □

Corollary 2.9. *For every compact L of Z, there exists $0 < \alpha \leq 1$ such that every continuous weakly holomorphic function on Z is α-Hölder continuous on L. Moreover, every uniformly bounded family of such functions is uniformly α-Hölder continuous on L.*

Proof. Recall that a continuous function on Z is weakly holomorphic if it is holomorphic on the regular part of Z. The result is known if Z is smooth with $\alpha = 1$. Therefore, in general, it is enough to apply Proposition 2.8 to a desingularization of Z. □

In Proposition 3.4, we will need a similar result in a local setting but with a uniform control of the constants. It is the aim of the two following results.

Proposition 2.10. *Let Z, $\widehat{Z}$, π and L be as in Proposition 2.8. Let v be a function defined on a ball $B_Z(y,r) \subset L$ with $0 < r \leq 1/2$. Assume that $\widehat{v} = v \circ \pi$ is (K,α)-Hölder continuous. Then, there exist constants $0 < \alpha' \leq 1$, $a > 0$ and $N \geq 1$, independent of v, y and r such that v is (aK, α')-Hölder continuous on $B_Z(y, r^N)$.*

Proof. The proof is the same as that of Proposition 2.8 except that we have to check that if x and x' are in $B_Z(y, r^N)$ with $N \geq 1$ large enough, then $\widehat{v}$ is well-defined at the points $\widehat{z}$ and $\widehat{z}'$ defined in (2.1).

Since π is holomorphic, we have $\operatorname{dist}(x, \pi(\widehat{z})) \lesssim \operatorname{dist}(\widehat{x}, \widehat{z})$. Therefore, by (2.1) we have

$$\operatorname{dist}(x, \pi(\widehat{z})) \lesssim \operatorname{dist}(\widehat{x}, \widehat{z}) + \operatorname{dist}(\widehat{x}', \widehat{z}') \lesssim \operatorname{dist}(x, x')^{1/M}.$$

Hence, if N is large enough, then $\widehat{z}$ and $\widehat{z}'$ belong to $\pi^{-1}(B_Z(y,r))$. Then, the result follows as in Proposition 2.8. □

Corollary 2.11. *For every compact L of Z, there are constants $0 < \alpha \leq 1$, $K > 0$ and $N \geq 1$ such that if v is a continuous weakly holomorphic function on $B_Z(y,r) \subset L$ with $|v| \leq 1$ then v is $(K/r, \alpha)$-Hölder continuous on $B_Z(y, r^N)$.*

Proof. Let $\pi : \widehat{Z} \to Z$ be a desingularization. Let $\widehat{z}$ be in $\pi^{-1}(B_Z(y, r/2))$. Since π is holomorphic, there is $a > 0$ such that $B_{\widehat{Z}}(\widehat{z}, r/a)$ is contained in $\pi^{-1}(B_Z(y,r))$. Therefore, by Cauchy's inequality, $\widehat{v}$ is a/r-Lipschitz on $\pi^{-1}(B_Z(y, r/2))$. Hence, the result follow from Proposition 2.10. □

3 Volume estimate for endomorphisms

The multiplicities of an endomorphism f are strongly related to volume estimates which were used successfully to solve equidistribution problems. In what follows, we generalize Lojasiewicz type inequalities obtained in [13], [6] and [9] to analytic sets, possibly singular. The aim is to control the size of a ball under iterations of f in an invariant analytic set. Singularities, in particular the points where the analytic sets are not locally irreducible, lead to technical difficulties.

In this section, X always denotes an irreducible analytic set of a smooth manifold. In order to avoid some problems related to the local connectedness of analytic sets, instead of the distance induced by an embedding of X, we consider the distance ρ defined by paths in X. Namely, if $x, y \in X$ then $\rho(x, y)$ is the length of the shortest path in X between x and y. These two distances on X are related by the following result (see e.g. [1]).

Theorem 3.1. *Let K be a compact subset of X. There exists a constant $r > 0$ such that for all $x, y \in K$ we have*

$$\mathrm{dist}(x, y) \leq \rho(x, y) \lesssim \mathrm{dist}(x, y)^r.$$

The first step to state volume estimate is the following result.

Proposition 3.2. *Let $\Gamma \subset \mathbb{B} \times \mathbb{B}$ and $X \subset \mathbb{B}$ be two analytic subsets with X locally irreducible and such that the first projection $\pi : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ defines a ramified covering of degree m from Γ to X. There exist constants $a > 0$ and $b \geq 1$ such that if $x, y \in X \cap \mathbb{B}_{1/2}$ then we can write*

$$\pi^{-1}(x) \cap \Gamma = \{x^1, \ldots, x^m\} \text{ and } \pi^{-1}(y) \cap \Gamma = \{y^1, \ldots, y^m\},$$

with $\mathrm{dist}(x, y) \geq a\mathrm{dist}(x^i, y^i)^{bm}$. Moreover, a increases with m but is independent of Γ and b depends only on X.

Proof. By Theorem 3.1, to establish the proposition, we can replace $\mathrm{dist}(x, y)$ by $\rho(x, y)$ on X. For $w \in X_{Reg}$, we can define the j-th Weierstrass polynomial, $k + 1 \leq j \leq 2k$, on $t \in \mathbb{C}$

$$P_j(t, w) = \prod_{z \in \pi^{-1}(w) \cap \Gamma} (t - z_j) = \sum_{l=0}^{m} a_{j,l}(w) t^l,$$

where $z = ((z_1, \ldots, z_k), (z_{k+1}, \ldots, z_{2k})) \in \mathbb{B} \times \mathbb{B}$. The coefficients $a_{j,l}$ are holomorphic on X_{Reg} and uniformly bounded by $m!$ since $\Gamma \subset \mathbb{B} \times \mathbb{B}$. As X is locally irreducible, they can be extended continuously to X (see e.g. [4]). It gives a continuous extension of the polynomials P_j to X with $P_j(z_j, \pi(z)) = 0$ if z is in Γ. Moreover, by Corollary 2.9 there exists $\alpha > 0$ such that the coefficients $a_{j,l}$ are uniformly α-Hölder continuous on $X \cap \mathbb{B}_{3/4}$ with respect to ρ.

We claim that there is a constant $a > 0$ such that if $x, y \in X \cap \mathbb{B}_{1/2}$ and $x' \in \mathbb{B}$ with $\widetilde{x} := (x, x') \in \Gamma$ then there is $\widetilde{y} \in \pi^{-1}(y) \cap \Gamma$ with

$$\rho(x, y) \geq a\mathrm{dist}(\widetilde{x}, \widetilde{y})^{bm},$$

where $b = 1/\alpha$. From this, the result follows exactly as in the end of the proof of [6, Lemma 4.3].

It remains to prove the claim. Fix $c > 0$ large enough and let $r = \rho(x, y)$. We can assume that r is non-zero and sufficiently small, otherwise the result is obvious. Since the covering has degree m, we can find an integer $2 \leq l \leq 4km$ such that for all $k + 1 \leq j \leq 2k$, no root of the polynomial $P_j(t, x)$ satisfies

$$c(l-1)r^{1/bm} \leq |\widetilde{x}_j - t| \leq c(l+1)r^{1/bm}.$$

It gives *a security ring* over x which does not intersect Γ. Using the regularity of P_j, it can be extend to a neighborhood of x. More precisely, for $\theta \in \mathbb{R}$ we define $\xi_j = \widetilde{x}_j + cle^{i\theta}r^{1/bm}$ and

$$G_{j,c,\theta}(w) = c^{-m+1}P_j(\xi_j, w).$$

The choice of l implies that

$$|G_{j,c,\theta}(x)| = c^{-m+1}|P_j(\xi_j, x)| \geq c^{-m+1} \prod_{z \in \pi^{-1}(x)} |\xi_j - z_j| \geq cr^{\alpha}.$$

Moreover, we deduce from the properties on the coefficients $a_{j,l}$ that the functions $G_{j,c,\theta}$ are uniformly α-Hölder continuous on $X \cap \mathbb{B}_{3/4}$ with respect to (j, c, θ). Hence, if c is large enough, they do not vanish on $B := \{z \in X \mid \rho(x, z) < 2r\}$ which contains y.

It implies that $P_j(t, w) \neq 0$ if w is in B and $|t - \widetilde{x}_j| = lcr^{1/bm}$. Therefore, if we denote by Σ the boundary of the polydisc of center x' and of radius $lcr^{1/bm}$, we have $\Gamma \cap (B \times \Sigma) = \varnothing$. Hence, since B is connected and contains y, by continuity there is a point $\widetilde{y}$ in $\pi^{-1}(y) \cap \Gamma$ with $|\widetilde{x}_j - \widetilde{y}_j| \leq lcr^{1/bm}$. This completes the proof of the claim. □

Remark 3.3. *Our proof shows that if we assume that $m \leq m_0$ for a fixed m_0, then a depends only on the Hölder constants (K, α) associated with $a_{j,l}$. Furthermore, if α is fixed then a is proportional to K. Note that without assumption on the constants a and b, Proposition 3.2 can be deduced from Theorem 2.1.*

The control of multiplicities of the covering gives the following more precise result.

Proposition 3.4. *Let* X, Γ *and* π *be as above. Let* $Z \subset X$ *be a proper analytic subset. Assume that the multiplicity at each point of* $\pi^{-1}(x)$ *is at most equal to* s *if* $x \in X \setminus Z$*. Then, there exist constants* $a > 0$, $b \geq 1$ *and* $N \geq 1$ *such that for all* $0 < t \leq 1/2$ *and* $x, y \in X \cap \mathbb{B}_{1/2}$ *with* $\mathrm{dist}(x, Z) > t$ *and* $\mathrm{dist}(y, Z) > t$*, we can write*

$$\pi^{-1}(x) \cap \Gamma = \{x^1, \ldots, x^m\} \text{ and } \pi^{-1}(y) \cap \Gamma = \{y^1, \ldots, y^m\},$$

with $\mathrm{dist}(x, y) \geq at^N \mathrm{dist}(x^i, y^i)^{bs}$.

Proof. Let $t > 0$ and $x \in X \cap \mathbb{B}_{1/2}$ with $\mathrm{dist}(x, Z) > t$. We want to find a neighborhood $B \subset X$ of x such that each component of $\Gamma \cap \pi^{-1}(B)$ defines a ramified covering of degree at most equal to s over B.

We construct an analytic set Y associate to the multiplicities of Γ. Namely, first define $Y' \subset \Gamma^{s+1}$ by $\{z \in \Gamma^{s+1} \,|\, \pi \circ \tau_i(z) = \pi \circ \tau_j(z),\ 1 \leq i, j \leq s + 1\}$, where τ_i, $1 \leq i \leq s+1$, are the canonical projections of Γ^{s+1} onto Γ. For $i \neq j$ we set $A_{i,j} = \{z \in Y' \,|\, \tau_i(z) = \tau_j(z)\}$. Then, Y is defined by $\overline{Y' \setminus \cup_{i \neq j} A_{i,j}}$. The map $\pi_1 = \pi \circ \tau_1 : Y \to X$ defines a ramified covering. If x is generic in X then a point $z \in Y$ over $x \in X$ represents a family of $(s + 1)$ distinct points in $\pi^{-1}(x) \cap \Gamma$.

If π' denotes the second projection of $\mathbb{B} \times \mathbb{B}$ onto $\mathbb{B}$, consider the map $h : Y \to \mathbb{C}^{(s+1)^2}$ define by

$$h(z) = (\pi' \circ \tau_i(z) - \pi' \circ \tau_j(z))_{1 \leq i,j \leq s+1}.$$

By construction, $h(z) = 0$ means precisely that there is a point in Γ with multiplicity greater than s over $\pi_1(z)$. It implies that $\pi_1(h^{-1}(0)) \subset Z$. Hence, Theorem 2.1 implies there is a constant $M > 0$ such that if $\pi(z) \in X \cap \mathbb{B}_{1/2}$ then

$$\|h(z)\| \gtrsim \mathrm{dist}(z, h^{-1}(0))^M \gtrsim \mathrm{dist}(\pi_1(z), Z)^M. \tag{3.1}$$

Let $a, b > 0$ be the constants in Proposition 3.2. As in the proof of that proposition, we use the distance function ρ on X. Fix $\gamma > 0$ small enough and set $B = \{z \in X \,|\, \rho(x, z) < \gamma t^{Mbm})\}$. For $\widetilde{x} = (x, x')$ in Γ, we can choose $2 \leq l \leq 8m$ such that $\pi^{-1}(x) \cap \Gamma$ do not intersect the ring

$$r(l - 2) \leq \|\widetilde{x} - w\| \leq r(l + 2),$$

where $r = (a^{-1}\gamma)^{1/bm} t^M$. Hence, if H denotes the ball of center x' and of radius rl, we have $\mathrm{dist}(\pi^{-1}(x) \cap \Gamma, B \times \partial H) > r$. Therefore, by Proposition

3.2 $\Gamma \cap B \times \partial H = \varnothing$. It assures that π is proper on $\Gamma \cap B \times H$ and then defines a ramified covering.

Moreover, this covering has degree at most equal to s. Otherwise, according to the radius of H, we have

$$\min_{z \in \pi_1^{-1}(x)} \|h(z)\| \leq (s+1)^2 2rl,$$

which is in contradiction with (3.1) if γ is small enough, since $\text{dist}(x, Z) > t$ and $r = (a^{-1}\gamma)^{1/bm} t^M$.

Now, we want to apply Proposition 3.2 to this covering. But, in order to control the constants, we have to reduce B. Indeed, according to Remark 3.3, the constants of that proposition depend only on the Hölder continuity of the coefficients $a_{j,l}$ (and on the degree of the covering). These coefficients are bounded continuous weakly holomorphic functions defined on B then by Corollary 2.11 with $L = \overline{X \cap \mathbb{B}_{3/4}}$, they are $(Kt^{-M'}, \alpha)$-Hölder continuous on $B(x, t^{M'})$ for some $M' \geq 1$ large enough, $0 < \alpha \leq 1$ and $K > 0$ independent of x and t. Therefore, after a coordinates dilation by $t^{-M'}$ at x, we can apply Proposition 3.2 with the same exponent b and the second constant proportional to some power of t. Finally, let $y \in X$. We can assume that $\rho(x, y) \leq t^{M'}/2$, otherwise the proposition is obvious. Hence, the previous observation implies that

$$\pi^{-1}(x) \cap \Gamma \cap B \times H = \{x^1, \ldots, x^s\} \text{ and } \pi^{-1}(y) \cap \Gamma \cap B \times H = \{y^1, \ldots, y^s\},$$

with $\rho(x, y) \geq a' t^N \text{dist}(x^i, y^i)^{bs}$ where $a' > 0, b \geq 1$ and $N \geq 1$ are independent of x and t. More precisely, we can write $N = M' + N'$, where the contribution in $t^{M'}$ comes from the dilation and that in $t^{N'}$ comes from the estimate on Hölder continuity. The construction can be applied to each component of $\Gamma \cap \pi^{-1}(B)$. This gives the result. □

We now consider the dynamical context. Assume that $X \subset \mathbb{P}^k$ is an irreducible analytic set of dimensions l which is invariant by f. Denote by g the restriction of f to X. For x in X, we define *the local multiplicity* of g at x as the maximal number of points in $g^{-1}(z)$ which are near x for $z \in X$ close enough to $g(x)$. The local multiplicity is smaller than *the topological degree* i.e. the number of points in $g^{-1}(x)$ for x generic in X. In our case, the topological degree of g is equal to d^l, see [6].

There exists a finite covering $(U_i)_{i \in I}$ of X by open subsets of $\mathbb{P}^k$ such that $X \cap U_i$ can by decompose into locally irreducible components. Hence, we can apply Proposition 3.4 to the graph of g over each component of $X \cap U_i$. It gives the following corollary.

Corollary 3.5. *Let $\eta > 1$ and $Z \subset X$ be a proper analytic subset. Assume that the local multiplicity of g is less than η outside $g^{-1}(Z)$. Then there are constants $a > 0$, $b \geq 1$ and $N \geq 1$ such that if $0 < t < 1$ and x, y are two points outside Z_t where X is locally irreducible, then we can write*

$$g^{-1}(x) = \{x^1, \ldots, x^{d^l}\} \text{ and } g^{-1}(y) = \{y^1, \ldots, y^{d^l}\}$$

with $\operatorname{dist}(x, y) \geq at^N \operatorname{dist}(x^j, y^j)^{b\eta}$.

From this, we obtain the following size estimate for image of balls which is crucial in the proof of our main result.

Corollary 3.6. *Let $\delta > 1$ and $E \subset X$ be a proper analytic subset. Denote by $\widetilde{E}$ the preimage of E by g. Assume that the local multiplicity is less than δ outside $\widetilde{E}$. There exist constants $0 < A \leq 1$, $b \geq 1$ and $N \geq 1$ such that if $0 < t \leq 1/2$, $r < t/2$ and $x \in X \setminus \widetilde{E}_t$, then $g(B_X(x, r))$ contains a ball of radius $At^N r^{b\delta}$. Moreover, b depends only on X.*

Proof. Fix $t > 0$ and $r < t/2$. As in the proof of Lemma 2.5 with $\widehat{Y} = X_{Sing} \cup g^{-1}(X_{Sing})$, possibly after replacing r by cr^l for some $c > 0$, we can assume that $B_X(x, r)$ and $g(B_X(x, r))$ are contained in X_{Reg}. The local multiplicity of f on X is bounded d^l. So, there exists $2 \leq i \leq 4d^l$, such that the ring $\{\frac{r(i-1)}{4d^l+1} \leq \operatorname{dist}(x, x') \leq \frac{r(i+1)}{4d^l+1}\}$ contains no preimage of $g(x)$. Thus, if $x' \in \partial B_X(x, r\frac{i}{4d^l+1})$, then

$$\operatorname{dist}(x', g^{-1}(g(x))) \geq \frac{r}{4d^l + 1}.$$

Moreover, we can apply Corollary 3.5 with $\eta = d^l$ and $Z = \varnothing$. Hence, there exists $a, b > 0$ such that $g(\overline{B_X(x, r\frac{i}{4d^l+1})}) \subset X \setminus E_{a(t/2)^{bd^l}}$. Therefore, we can apply once again Corollary 3.5 with $\eta = \delta$, $Z = E$ and $a(t/2)^{bd^l}$ instead of t. We get, for some constants $a' > 0$ and $N_0 \geq 1$

$$\operatorname{dist}(g(x'), g(x)) \geq a' \left(a \left(\frac{t}{2} \right)^{bd^l} \right)^{N_0} \left(\frac{r}{4d^l + 1} \right)^{b\delta},$$

and, since g is an open mapping near x

$$B_X(g(x), At^N r^{b\delta}) \subset g(B_X(x, r))$$

with $A = \frac{a^{N_0} a'}{2^{N_0 bd^l} (4d^l+1)^{b\delta}}$ and $N = N_0 bd^l$. □

Remark 3.7. *When X is smooth, the ball in $g(B_X(x, r))$ can be chosen centered at $g(x)$.*

4 Psh functions and exponential estimates

We refer to [4] for basics on currents and plurisubharmonic (psh for short) functions. Let T be a positive closed $(1,1)$-current of mass 1 on $\mathbb{P}^k$ with continuous local potentials. Let us recall briefly the associated notions of psh and weakly psh (wpsh for short) modulo T functions introduced in [6].

Let Y be an analytic space. A function $v : Y \to \mathbb{R} \cup \{-\infty\}$ is *wpsh* if it is psh on Y_{Reg} and for y in Y, we have $v(y) = \limsup v(z)$ with $z \in Y_{Reg}$ and $z \to y$. These functions coincide with psh functions if Y is smooth. On compact spaces, the notion is very restrictive. However, if X is an analytic subset of $\mathbb{P}^k$, we have the more flexible notion of *wpsh modulo T function* on X. Locally, it is the difference of a wpsh function on X and a potential of T. If X is smooth, we say that the function is psh modulo T.

Note that the restriction of a psh modulo T function to an analytic subset is either wpsh modulo T or equal to $-\infty$ on an irreducible component. If u is wpsh modulo T on X then $dd^c(u[X]) + T \wedge [X]$ is a positive closed current supported on X. On the other hand, if S is a positive closed $(1,1)$-current on $\mathbb{P}^k$ with mass 1, there is a psh modulo T function u on $\mathbb{P}^k$, unique up to a constant, such that $S = dd^c u + T$.

These notions bring good compactness properties which permit to obtain uniform estimates. We have the following statements established in [6].

Proposition 4.1. *Let (u_n) be a sequence of wpsh modulo T functions on X, uniformly bounded from above. Then there is a subsequence (u_{n_i}) satisfying one of the following properties:*

- *There is an irreducible component Y of X such that (u_{n_i}) converges uniformly to $-\infty$ on $Y \setminus X_{Sing}$.*
- *(u_{n_i}) converges in $L^p(X)$ to a wpsh modulo T function u for every $1 \leq p < +\infty$.*

In the last case, $\limsup u_{n_i} \leq u$ *on X with equality almost everywhere.*

It implies the following lemma.

Lemma 4.2. *Let $\mathscr{G}$ be a family of psh modulo T functions on $\mathbb{P}^k$ uniformly bounded from above. Assume that each irreducible component of X contains an analytic subset Y such that the restriction of $\mathscr{G}$ to Y is bounded in $L^1(Y)$. Then, the restriction of $\mathscr{G}$ to X is bounded in $L^1(X)$.*

A classical result of Hörmander [15] gives a uniform bound to $\exp(-u)$ in $L^1(\mathbb{B}_{1/2})$ for u in a class of psh functions in the unit ball of $\mathbb{C}^k$. Similar estimates can be obtained for compact families of quasi-psh functions. From now, we assume that T has (K, α)-Hölder continuous local potentials, with $0 < \alpha \leq 1$ and $K > 0$. In the rest of this section, we establish exponential estimates for psh modulo T functions in different situations. A key observation in our approach is that Hölder continuity allows us to work with non-compact families. In the sequel, we will apply these estimates to T the Green current associate to f. They allow us to control the volume of some sublevel sets of potentials of currents near exceptional sets.

As a consequence of classical Hörmander's estimate, we have the following lemma which will be of constant use. Here, ν denotes the standard volume form on $\mathbb{C}^k$ and T is seen as a fixed current on the unit ball $\mathbb{B}$ of $\mathbb{C}^k$. We assume that its admits a potential g which is (K, α)-Hölder continuous on $\mathbb{B}$.

Lemma 4.3. *Let v be a psh modulo T function in $\mathbb{B}_t$ with $v \leq 0$ and $v(0) > -\infty$. Let $0 < s < -v(0)^{-1}$ and $t > 0$ such that $Kt^\alpha \leq s^{-1}$. There is a constant $c > 0$ independent of v, s and t such that*

$$\int_{\mathbb{B}_{t/2}} \exp(-\frac{sv}{2})\nu \leq ct^{2k}. \tag{4.1}$$

Proof. As v is psh modulo T, we have $v = v' - g$ with v' psh. We set $\widetilde{v}(z) = v'(z) - g(0) - Kt^\alpha$. Then $\widetilde{v}$ is psh in $\mathbb{B}_t$, $\widetilde{v}(0) = v(0) - Kt^\alpha \geq -2s^{-1}$ and $\widetilde{v} \leq v \leq 0$ because $g(z) - g(0) \leq Kt^\alpha$ on $\mathbb{B}_t$. By [15, Theorem 4.4.5] there exists $c > 0$ such that

$$\int_{\mathbb{B}_{1/2}} \exp(-\frac{s\widetilde{v}(tz)}{2})\nu \leq c,$$

thus, by a change of variables $z \mapsto tz$, we get

$$\int_{\mathbb{B}_{t/2}} \exp(-\frac{sv}{2})\nu \leq \int_{\mathbb{B}_{t/2}} \exp(-\frac{s\widetilde{v}}{2})\nu \leq ct^{2k}.$$

□

For the rest of the section, X always denotes an irreducible analytic subset of $\mathbb{P}^k$ of dimension l and v is a psh modulo T function in $\mathbb{P}^k$ with $v \leq 0$. In Section 6 we will extend the previous result to the neighborhood of X, where the condition at 0 is replaced by an integrability condition on X. For this purpose, we have to control the size of sublevel sets of v in X. This is the aim of the following global result.

Lemma 4.4. *For X as above, there exists $q_0 \geq 1$ with the following property. For $q > q_0$ we set $\epsilon = 2lq_0/q\alpha$ and take $M > 0$ and $s \geq 1$ such that $s^{1+\epsilon}\|v\|_{L^q(X)} \leq M$. Then, there exist constants $a, c > 0$ independent of v, q and s such that*

$$\int_X \exp(-asv)\omega^l \leq c. \tag{4.2}$$

If X is smooth, we can choose $q_0 = 1$.

Proof. First, assume that X is a compact smooth manifold with a volume form η. Since X has dimension l, for $t > 0$ we can cover it by balls $(B_i)_{i\in I}$ with $B_i := B_X(x_i, t)$ and such that $|I| \leq c' t^{-2l}$ for some $c' > 0$. Let $t = s^{-1/\alpha}$. As above, in each ball $B_X(x_i, 2t)$ we can write $v = v_i' - g_i$, where g_i is a local potential of T. Using local charts at x_i, we can identify $B_X(x_i, 2t)$ with $\mathbb{B}_{2t}$ in $\mathbb{C}^l$. We consider $\widetilde{v}_i(z) = s(v_i'(tz) - g_i(0))$. These functions are psh in $\mathbb{B}_2$. We show that they belong to a compact family, independent of v and s. Using a change of variables $z \mapsto tz$ and Hölder's inequality, we get

$$\begin{aligned}
\|\widetilde{v}_i\|_{L^1(\mathbb{B}_2)} &\leq \int_{\mathbb{B}_2} s|v(tz)|\nu + \int_{\mathbb{B}_2} s|g_i(tz) - g_i(0)|\nu \\
&\leq st^{-2l}\|v\|_{L^q(X)}|\mathbb{B}_2|^{1/p}t^{2l/p} + 2^\alpha K|\mathbb{B}_2| \\
&= s^{1+\epsilon}\|v\|_{L^q(X)}|\mathbb{B}_2|^{1/p} + 2^\alpha K|\mathbb{B}_2| \\
&\leq M|\mathbb{B}_2|^{1/p} + 2^\alpha K|\mathbb{B}_2| \leq M',
\end{aligned}$$

where p is the conjugate of q, $|\mathbb{B}_2|$ is the volume of $\mathbb{B}_2$ and M' is a positive constant. The family $\mathscr{U} = \{u \in PSH(\mathbb{B}_2) \,|\, \|u\|_{L^1(\mathbb{B}_2)} \leq M'\}$ is compact so there exists a constant $a > 0$ such that $\|\exp(-au)\|_{L^1(\mathbb{B})}$ is uniformly bounded for all $u \in \mathscr{U}$. Therefore, for $i \in I$

$$\int_{\mathbb{B}_t} \exp(-as(v_i'(z) - g_i(0)))\nu \lesssim t^{2l}.$$

Moreover, the Hölder continuity implies that $-sv(z) \leq K - s(v_i'(z) - g_i(x_i))$ in B_i. Hence, since $(B_i)_{i\in I}$ is a covering of X we obtain

$$\begin{aligned}
\int_X \exp(-asv)\eta &\leq \sum_{i\in I}\int_{B_i} \exp(-asv)\eta \\
&\leq \sum_{i\in I}\int_{B_i} \exp(a(K - s(v_i'(z) - g_i(x_i))))\eta \\
&\lesssim \sum_{i\in I} t^{2l} \leq c'.
\end{aligned}$$

This implies the lemma if X is smooth with $q_0 = 1$.

In the general case, we consider a desingularization $\pi : \widehat{X} \to X$ with a volume form η on $\widehat{X}$. The map π is surjective, then by Lemma 2.7, there exists $q_0 \geq 1$ such that

$$\|\widehat{v}\|_{L^{q/q_0}(\widehat{X},\eta)} \lesssim \|v\|_{L^q(X,\omega^l)}.$$

Moreover, $\pi^*(T)$ possesses α-Hölder local potentials and $\widehat{v} \leq 0$ is psh modulo $\pi^*(T)$. Therefore, this choice of q_0 allows us to apply the lemma on $\widehat{X}$ and get

$$\int_{\widehat{X}} \exp(-as\widehat{v})\eta \leq c.$$

The result follows since

$$\int_X \exp(-asv)\omega^l = \int_{\widehat{X}} \exp(-as\widehat{v})\pi^*(\omega^l) \leq \|h\|_\infty \int_{\widehat{X}} \exp(-as\widehat{v})\eta,$$

where we write $\pi^*(\omega^l) = h\eta$. □

The following estimate is a consequence of Lemma 4.3 and is related to the geometry of sublevel sets of psh modulo T functions. In Section 6, it will establish the existence of balls where we can apply our volume estimates.

Lemma 4.5. *For $s \geq 2$ set $F_s = \{x \in X \mid v(x) \leq -s^{-1}\}$. There are constants $\beta, c > 0$ independent of v and s such that if F_s contains no ball of radius $s^{-\beta}$ then*

$$\int_X \exp(-\frac{sv}{2})\omega^l \leq c.$$

Proof. We first consider the case where X is smooth. Let $t = 4^{-1}(Ks)^{-1/\alpha}$. As in the proof of the previous lemma, we cover X by balls $(B_i)_{i\in I}$ of radius t with $|I| \leq c't^{-2l}$, $c' > 0$. Assume there is no ball of radius t in F_s. Hence, for each $i \in I$ there exists x_i in B_i such that $v(x_i) > -s^{-1}$. The balls B'_i of center x_i and of radius $2t$ cover X. Thus

$$\int_X \exp(-\frac{sv}{2})\omega^l \leq \sum_{i\in I} \int_{B'_i} \exp(-\frac{sv}{2})\omega^l.$$

But, $s < -v(x_i)^{-1}$ and $K(4t)^\alpha \leq s^{-1}$ therefore we can apply Lemma 4.3 on each ball

$$\int_{B'_i} \exp(-\frac{sv}{2})\omega^l \lesssim t^{2l}.$$

Hence, we get

$$\int_X \exp(-\frac{sv}{2})\omega^l \lesssim \sum_{i\in I} t^{2l} \leq c',$$

which gives the result when X is smooth with $\beta > 1/\alpha$ such that $s^{-\beta} < t$.

If X is singular, we consider a desingularization $\pi : \widehat{X} \to X$. By Lemma 2.5, there exists $N \geq 1$ such that the image of a ball of radius r under π contains a ball of radius r^N. Hence, if β is large enough, the hypothesis on F_s assures there is no ball of radius t in $\widehat{F}_s = \pi^{-1}(F_s)$. Then, we can apply the lemma to $\widehat{v} = v \circ \pi$ which is psh modulo $\pi^*(T)$. We get

$$\int_X \exp(-\frac{sv}{2})\omega^l = \int_{\widehat{X}} \exp(-\frac{s\widehat{v}}{2})\pi^*(\omega^l) \leq c,$$

for some $c > 0$, since $\pi^*(\omega^l)$ is smooth. □

5 Exceptional sets

Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. The aim of this section is to construct two families $\mathscr{A}_\lambda$ and $\mathscr{B}_\lambda$ of analytic sets where the iterate sequence f^n has important local multiplicities. Let $X \subset \mathbb{P}^k$ be an irreducible invariant analytic set. Define $\kappa_{X,n}(x)$, or simply $\kappa_n(x)$ if no confusion is possible, as the local multiplicity of $f^n_{|X}$ at x. It is *a sub-multiplicative cocycles*, namely it is upper semi-continuous for the Zariski topology on X, $\min_X \kappa_n = 1$ and for any $m, n \geq 0$ and $x \in X$ we have the following sub-multiplicative relation

$$\kappa_{n+m}(x) \leq \kappa_m(f^n(x))\kappa_n(x).$$

The inequality may be strict when X is singular. Define

$$\kappa_{-n}(x) := \max_{y\in(f_{|X})^{-n}(x)} \kappa_n(y).$$

We recall the following theorem of Dinh [5], see also [10].

Theorem 5.1. *The sequence of functions $\kappa_{-n}^{1/n}$ converges pointwise to a function κ_-. Moreover, for every $\lambda > 1$, the level set $E_\lambda(X) = \{\kappa_- \geq \lambda\}$ is a proper analytic subset of X which is invariant under $f_{|X}$. In particular, κ_- is upper semi-continuous in the Zariski sense.*

For a generic endomorphism of $\mathbb{P}^k$, $E_\lambda(\mathbb{P}^k)$ is empty. In this case, Theorem 1.2 is already know in all bidegrees [7]. In our proof, we will proceed by induction, proving the the exponentially fast convergence on X if it is already established on each irreducible component of $E_\lambda(X)$. But, even if $E_\lambda(X)$ is invariant, its irreducible components are periodic and not invariant in general. Therefore, if X is only periodic, we define $E_\lambda(X)$ in the same way, replacing f by f^p and λ by λ^p, where p is a period of X. By Theorem 5.1, this definition is independent of the choice of p.

Fix $1 < \lambda < d$. Define the family $\mathscr{B}_\lambda$ of exceptional sets as follows. First, we set $\mathbb{P}^k \in \mathscr{B}_\lambda$. If X is in $\mathscr{B}_\lambda$, we add to $\mathscr{B}_\lambda$ all irreducible components of $E_\lambda(X)$. This family is finite and since the functions κ_- are upper semi-continuous in the Zariski sense, there exists $1 < \delta < \lambda$ such that $\mathscr{B}_\lambda = \mathscr{B}_\delta$, or equivalently $E_\lambda(X) = E_\delta(X)$ if $X \in \mathscr{B}_\lambda$. This will give us some flexibility in order to obtain estimates using an induction process.

As all elements of $\mathscr{B}_\lambda$ are periodic, they are invariant under some iterate f^{n_0}. Let us remark that it is sufficient to prove Theorem 1.2 for an iterate of f. Hence, we can assume that $n_0 = 1$, replacing f and λ by f^{n_0} and λ^{n_0}. Dinh also proved that $\kappa_{n_1} < \delta^{n_1}$ outside $(f_{|X})^{-n_1}(E_\lambda(X))$ for some $n_1 \geq 1$. Once again, we can assume that $n_1 = 1$.

The second family $\mathscr{A}_\lambda$, that takes place in Theorem 1.2, is defined as the set of minimal elements for the inclusion in $\mathscr{B}_\lambda$. This family is not empty and each element of $\mathscr{B}_\lambda$ contains at least one element of $\mathscr{A}_\lambda$. Note that no element of $\mathscr{A}_\lambda$ is contained in another one. These analytic sets play a special role in the next section, to start induction and to obtain compactness properties. When $\mathbb{P}^k$ is an element of $\mathscr{A}_\lambda$, it is the only element in $\mathscr{A}_\lambda$ and the exceptional set is empty. Otherwise, define the exceptional set as the union of all the elements of $\mathscr{A}_\lambda$.

6 Equidistribution speed

This section is devoted to the proof of Theorem 1.2. Fix an endomorphism f of algebraic degree $d \geq 2$ of $\mathbb{P}^k$, and denote by T its Green current. Recall that T is totally invariant i.e. $d^{-1} f^*(T) = T$, and has (K, α)-Hölder continuous local potentials for some $0 < \alpha \leq 1$, $K > 0$.

Fix $C > 0$ and $1 < \lambda < d$, and let $\mathscr{A}_\lambda$, $\mathscr{B}_\lambda$ be as in Section 5. Define $\mathscr{F}_\lambda(C)$ as the family of psh modulo T functions v on $\mathbb{P}^k$ such that $\max_{\mathbb{P}^k} v = 0$ and $\|v\|_{L^1(X)} \leq C$ for all $X \in \mathscr{A}_\lambda$. By construction of $\mathscr{A}_\lambda$, Lemma 4.2 implies that $\mathscr{F}_\lambda(C)$ is compact for each $C > 0$. Moreover, if X is an element of $\mathscr{B}_\lambda$, the restriction of $\mathscr{F}_\lambda(C)$ to X forms a family of wpsh modulo T functions on X which is relatively compact in $L^p(X)$ for every $1 \leq p < +\infty$.

If S is a positive closed $(1,1)$-current of mass 1, it is cohomologous to T. Hence, there exists a unique psh modulo T function u on $\mathbb{P}^k$ such that

$S = dd^c u + T$ and $\max_{\mathbb{P}^k} u = 0$. We call u the dynamical potential of S. As T is totally invariant, the dynamical potential of $S_n = d^{-n}(f^n)^*(S)$ is $u_n = d^{-n} u \circ f^n$.

Since $S_n - T$ is a continuous linear operator on $\mathscr{C}^0(\mathbb{P}^k)$ whose norm is bounded, by interpolation theory between Banach spaces we have

$$\|S_n - T\|_{\mathscr{C}^\beta} \lesssim \|S_n - T\|_{\mathscr{C}^2}^{\beta/2},$$

uniformly in S and n, see [18]. Consequently, in order to prove Theorem 1.2 we can assume that $\beta = 2$.

Moreover, it is easy to see that $\|dd^c\phi\|_\infty \lesssim \|\phi\|_{\mathscr{C}^2}$ for ϕ in $\mathscr{C}^2(\mathbb{P}^k)$. Therefore,

$$\begin{aligned} |\langle S_n - T, \phi\rangle| &= |\langle dd^c u_n, \phi\rangle| = |\langle u_n, dd^c\phi\rangle| \\ &\lesssim \|\phi\|_{\mathscr{C}^2} \|u_n\|_{L^1(\mathbb{P}^k)}. \end{aligned}$$

Hence, Theorem 1.2 is a direct consequence of the following theorem applied to $p = 1$ and $X = \mathbb{P}^k$.

Theorem 6.1. *For each $1 \leq p < +\infty$ and $X \in \mathscr{B}_\lambda$ there exists a constant $A_{X,p}$ such that for all $u \in \mathscr{F}_\lambda(C)$ and $n \geq 0$ we have*

$$\|u_n\|_{L^p(X)} \leq A_{X,p} \left(\frac{\lambda}{d}\right)^n,$$

where $u_n = d^{-n} u \circ f^n$.

As in Section 5, we can assume that each element of $\mathscr{B}_\lambda$ is invariant by f, and there is $1 < \delta < \lambda$ satisfying the following properties for all X in $\mathscr{B}_\lambda$:

- $E_\lambda(X) = E_\delta(X)$,
- $\kappa_{X,1} < \delta$ outside $\widetilde{E}_\lambda(X) = (f_{|X})^{-1}(E_\lambda(X))$.

Let X be an element $\mathscr{B}_\lambda$ of dimension l and $\lambda_1 > 0$ with $\delta < \lambda_1 < \lambda$. Assume that Theorem 6.1 is true on each irreducible component of $E_\lambda = E_\lambda(X)$ for λ_1 and all $p \geq 1$. To prove it on X, we consider the sublevel set $K_n = \{x \in X \,|\, u_n(x) \leq -s_n\}$ for a suitable constant s_n. Exponential estimates on $\widetilde{E}_\lambda$ will prove that its image by f^i, $0 \leq i \leq n$, cannot be concentrated near $\widetilde{E}_\lambda$. Therefore, volume estimates will imply that $f^n(K_n) = \{x \in X \,|\, u(x) \leq -d^n s_n\}$ is large if Theorem 6.1 is false on X. Hence, a good choice of s_n, allowed by the gap between λ_1 and λ, will give a contradiction.

We first fix some constants. In Corollary 3.6 the constant b depends only on X. Then, by replacing f by f^n and δ by δ'^n with $b\delta^n < \delta'^n < \lambda_1^n$, we can assume that $b = 1$. Let $0 < A \leq 1$, $N \geq 1$ be the other constants of Corollary 3.6. Fix $\lambda_2, \lambda_3 > 0$ such that

- $\delta < \lambda_1 < \lambda_2 < \lambda_3 < \lambda$,
- and $q > q_0$ large enough such that $\lambda_1/d < (\lambda_2/d)^{1+\epsilon}$ where ϵ and q_0 are defined in Lemma 4.4.

Multiplicities of $f_{|X}$ are controlled outside $\widetilde{E}_\lambda$. By induction hypothesis, we have a control of u_n on E_λ. We want to extend it to $\widetilde{E}_\lambda$. Let E be an irreducible component of E_λ. The restriction of f to each component of $(f_{|X})^{-1}(E)$ is surjective onto E. Therefore, we deduce from Lemma 2.7 that there exists $q' \geq 1$ such that

$$\|v \circ f\|_{L^q((f_{|X})^{-1}(E))} \lesssim \|v\|_{L^{qq'}(E)},$$

for all psh modulo T function v on $\mathbb{P}^k$. Hence, by induction hypothesis, there is a constant $M > 0$ such that $\|u_n\|_{L^q(\widetilde{E}_\lambda)} \leq M(\lambda_1/d)^n$ for $n \geq 1$. The next step is to obtain exponential estimates in a neighborhood of $\widetilde{E}_\lambda$.

Lemma 6.2. *There exist constants $c, \eta \geq 1$ and $n_0 \geq 1$ such that if $n \geq n_0$ then for all $u \in \mathscr{F}_\lambda(C)$ we have*

$$\int_{\widetilde{E}_{\lambda,t_n}} \exp(-(d/\lambda_2)^n u_n)\omega^l \leq c,$$

where $t_n = (\lambda_2/d)^{n\eta}$.

Proof. Let E be an irreducible component of $\widetilde{E}_\lambda$ of dimension i. According to the choice of q, we can find $\lambda_2' < \lambda_2$ such that $\lambda_1/d < (\lambda_2'/d)^{1+\epsilon}$. Hence

$$\|u_n\|_{L^q(E)}(d/\lambda_2')^{(1+\epsilon)n} \leq M(\lambda_1/d)^n(d/\lambda_2')^{(1+\epsilon)n} \leq M.$$

and by Lemma 4.4 with $s = (d/\lambda_2')^n$ we have

$$\int_E \exp(-a'(d/\lambda_2')^n u_n)\omega^i \leq c',$$

for some constants $a', c' > 0$. Therefore, if we set $\rho_n = (\lambda_2/d)^n$, the volume in E of $F_n = \{x \in E \,|\, u_n(x) \leq -\rho_n\}$ is smaller than $c' \exp(-a'(\lambda_2/\lambda_2')^n)$. In particular, F_n contains no ball of radius $\rho_n^{2/\alpha}$ for n large enough.

If X is smooth then set $t_n = \rho_n^{1/\alpha}$. As in Lemma 4.5, for n large enough, we can find a covering of E_{t_n} by balls with center in E and of radius $2t_n$ on which Lemma 4.3 holds. Hence, we get

$$\int_{E_{t_n}} \exp(-au_n/\rho_n)\omega^l \leq c,$$

for some $a, c > 0$. The same argument with λ_2 slightly smaller shows that we can choose $a = 1$. We conclude the proof by summing on all irreducible components of $\widetilde{E}_\lambda$.

When X is singular, we consider a desingularization $\pi : \widehat{X} \to X$. In order to establish the estimate near E, we proceed inductively as follows. Assume that there exists a triplet (A, a, θ) with $a > 0$, $\theta \geq 1$ and an analytic set $A \subset E$ such that

$$\int_{E_{t^\theta} \setminus A_{t^{1/\theta}}} \exp(-au_n/\rho_n)\omega^l$$

is uniformly bounded in $n \geq 0$ for $t \leq \rho_n$. Then, using the properties of the elements of $\mathscr{B}_\lambda$ and dynamical arguments, we claim that a similar estimate holds if we substitute (A, a, θ) by some (A', a', θ') with $\dim(A') < \dim(A)$. It will give the result for η large enough after less than l steps since $\dim(E) < l$.

More precisely, let V be an irreducible component of A with maximal dimension. We distinguish two cases, according to whether V is in $\mathscr{B}_\lambda$ or not. In the first case, we know that for all $p \geq 1$, $\|u_n\|_{L^p(V)} \lesssim (\lambda_1/d)^n$. We set $\widehat{V} := \pi^{-1}(V)$. We denote by $\widehat{V}_1$ the union of all components of $\widehat{V}$ which are mapped onto V and by $\widehat{V}_2$ the union of the other components of $\widehat{V}$. Therefore, Lemma 2.7 implies that

$$\|\widehat{u}_n\|_{L^p(\widehat{V}_1)} \lesssim (\lambda_1/d)^n,$$

for all $p \geq 1$, where $\widehat{u}_n = u_n \circ \pi$. Hence, the smooth version of the lemma implies that

$$\int_{\widehat{V}_{1,\rho_n^{1/\alpha}}} \exp(-a'\widehat{u}_n/\rho_n)\pi^*(\omega^l)$$

is uniformly bounded for $a' > 0$ small enough. Moreover, by Lemma 2.6, there exists a constant $\theta' \geq 1$ such that $\pi(\widehat{V}_{1,t})$ contains $V_{t^{\theta'}} \setminus V_{2,t^{1/2}}$, where $V_2 = \pi(\widehat{V}_2)$. It gives the desired result near V, since $\dim(V_2) < \dim(V)$.

From now, we can assume that no irreducible component of A with maximal dimension belong to $\mathscr{B}_\lambda$ (in particular $A \neq E$). Let V denote the union of all irreducible components of A with maximal dimension. In particular, these components are not totally invariant for $f_{|E}$, therefore there exist an analytic set $Z \subset E$ containing no component of V and an integer $m \geq 1$ such that $f^m(Z) = V$. We set $Z' = Z \cap A$. The assumption on A and θ implies that if $t \leq \rho_n$ then

$$\int_{Z_{t^\theta} \setminus A_{t^{1/\theta}}} \exp(-au_n/\rho_n)\omega^l$$

is bounded uniformly on n. By Corollary 2.2, $Z_{t_\theta}\cap A_{t^{1/\theta}}$ is contained in $Z'_{t^{1/\theta''}}$ for some $\theta''>\theta$. So,

$$\int_{Z_{t^{\theta''}}\setminus Z'_{t^{1/\theta''}}}\exp(-au_n/\rho_n)\omega^l$$

is bounded uniformly on n. Fix a constant $B>1$ large enough. We deduce from Corollary 3.6 applied to $\mathbb{P}^k$ that for all $t>0$ $f^m(Z_t)$ contains $V_{B^{-1}t^{d^{mk}}}$. Moreover, since f^m is Lipschitz, $f^m(Z'_t)$ is contained in V'_{Bt}, where $V'=f^m(Z')$. So, we have

$$f^m(Z_{t^{\theta''}}\setminus Z'_{t^{1/\theta''}})\supset V_{t^{\theta'}}\setminus V'_{t^{1/\theta'}},$$

for $t>0$ small enough and $\theta'>\theta''$ large enough. It follows that

$$\begin{aligned}\int_{V_{t^{\theta'}}\setminus V'_{t^{1/\theta'}}}\exp(-a'u_n/\rho_n)\omega^l &\le \int_{Z_{t^{\theta''}}\setminus Z'_{t^{1/\theta''}}}\exp(-a'\frac{u_{n+m}\lambda_2^m}{\rho_{n+m}})(f^m_{|X})^*(\omega^l)\\ &\lesssim \int_{Z_{t^{\theta''}}\setminus Z'_{t^{1/\theta''}}}\exp(-a'\frac{u_{n+m}\lambda_2^m}{\rho_{n+m}})\omega^l, \qquad (6.1)\end{aligned}$$

since $(f^m_{|X})^*(\omega^l)\lesssim\omega^l$. Moreover, for a' same enough the right-hand side in (6.1) is bounded uniformly on n and $\dim(V')=\dim(Z')<\dim(V)$ since Z contains no component of V. This together with the estimate outside A prove the claim with $A'=V'$. □

From now, we fix $p\ge 1$ and for u in $\mathscr{F}_\lambda(C)$ denote by $\mathscr{N}(u)=\{n\ge 1\,|\,\|u_n\|_{L^p(X)}\ge(\lambda/d)^n\}$ and by $\mathscr{N}$ the union of $\mathscr{N}(u)$ for all u. Our goal is to prove that $\mathscr{N}$ is finite, which will imply Theorem 6.1. For this purpose, we have the following result.

Lemma 6.3. *There are constants $n_1\ge 1$ and $\beta\ge 1$ such that if n is in $\mathscr{N}(u)$ with $n\ge n_1$ then $K_n=\{x\in X\mid u_n(x)\le-(\lambda_3/d)^n\}$ contains a ball of radius $(\lambda_3/d)^{\beta n}$.*

Proof. Since $x^p\lesssim\exp(x)$ if $x\ge 0$, we deduce from the assumption on $\|u_n\|_{L^p(X)}$ that

$$\begin{aligned}(\lambda/\lambda_3)^n &\lesssim \Big(\int_X(-(d/\lambda_3)^nu_n)^p\omega^l\Big)^{1/p}\\ &\lesssim\Big(\int_X\exp(-(d/\lambda_3)^nu_n/2)\omega^l\Big)^{1/p}. \qquad (6.2)\end{aligned}$$

On the other hand, let β be the constant in Lemma 4.5. For n sufficiently large we have $(d/\lambda_3)^n \geq 2$. Hence, Lemma 4.5 with $s = (d/\lambda_3)^n$ imply that K_n has to contain a ball of radius $(\lambda_3/d)^{n\beta}$, otherwise the right-hand side of (6.2) would be bounded uniformly on n, which is a impossible since $\lambda_3 < \lambda$. □

We can now complete the proof of the main theorem.

End of the proof of Theorem 6.1. If $B \subset X$ is a Borel set then $|B|$ denotes its volume with respect to the measure ω^l. As we have already seen, the volume of a ball of radius r in X is larger than $c'r^{2l}$, $0 < c' \leq 1$. Therefore, observe that if x is in $\widetilde{E}_{\lambda,t_n/2}$ then $|\widetilde{E}_{\lambda,t_n} \cap B_X(x,r)| = |B_X(x,r)| \geq c'(r/2)^{2l}$ for $r < t_n/2$.

From now, assume in order to obtain a contradiction that $\mathscr{N}$ is infinite. Consider $u \in \mathscr{F}_\lambda(C)$ and $n \in \mathscr{N}(u)$ large enough. Fix also β large enough. So, we have $(\lambda_3/d)^{\beta n} < t_n/4$ and

$$c \exp(-(\lambda_3/\lambda_2)^n) \leq c'(A^\delta (t_n/2)^{N\delta} (\lambda_3/d)^{\beta n}/2)^{2l}, \tag{6.3}$$

where c is defined in Lemma 6.2. Let $r_0 = (\lambda_3/d)^{\beta n}$, and for $1 \leq i \leq n$ let $r_i = A(t_n/2)^N r_{i-1}^\delta$. We will prove by induction that for $0 \leq i \leq n$, $f^i(K_n) = \{x \in X \,|\, u_{n-i}(x) \leq -\lambda_3^n/d^{n-i}\}$ contains a ball B_i of radius r_i.

Since β is large, Lemma 6.3 implies that the assertion is true for $i = 0$. Let $0 \leq i \leq n-1$ and assume the property is true for i. We deduce from Lemma 6.2 that

$$\int_{\widetilde{E}_{\lambda,t_{n-i}}} \exp(-(d/\lambda_2)^{n-i} u_{n-i}) \omega^l \leq c,$$

and in particular

$$|\widetilde{E}_{\lambda,t_n} \cap B_i| \leq |\widetilde{E}_{\lambda,t_{n-i}} \cap f^i(K_n)| < c \exp(-(\lambda_3/\lambda_2)^n \lambda_2^i),$$

since $t_n \leq t_{n-i}$. This and (6.3) imply that

$$|B_i| \geq c' r_i^{2l} > 2^{2l} |\widetilde{E}_{\lambda,t_n} \cap B_i|,$$

since $r_i \geq (A^\delta (t_n/2)^{N\delta} r_0)^{\delta^i}$ and $\delta < \lambda_2$. Consequently, the center of B_i is not in $\widetilde{E}_{\lambda,t_n/2}$ and by Corollary 3.6, $f(B_i) \subset f^{i+1}(K_n)$ contains a ball B_{i+1} of radius $r_{i+1} = A(t_n/2)^N r_i^\delta$. Note that we already reduced the problem to the case where the constant b in Corollary 3.6 is equal to 1.

Therefore, for all n in $\mathscr{N}(u)$ sufficiently large, the volume of $f^n(K_n) = \{x \in X \,|\, u(x) \leq -\lambda_3^n\}$ is greater than $D^{n\delta^n}$, with $0 < D < 1$ independent of u and n. This contradicts the inequality $\delta < \lambda_3$. Indeed, since $\mathscr{F}_\lambda(C)$ is bounded in $L^q(X)$, by Lemma 4.4 there exists $a' > 0$ such that

$$\int_X \exp(-a'u)\omega^l$$

is uniformly bounded for u in $\mathscr{F}_\lambda(C)$

Hence, $\mathscr{N}$ is finite and in particular bounded by some $n_2 \geq 1$. We conclude using the fact that the restriction of $\cup_{n=0}^{n_2} d^{-n}(f^n)^*(\mathscr{F}_\lambda(C))$ to X is a relatively compact family of wpsh modulo T functions and then bounded in $L^p(X)$. Therefore, we have

$$\|u_n\|_{L^p(X)} \lesssim \left(\frac{\lambda}{d}\right)^n,$$

if $n \leq n_2$ and thus for every $n \geq 0$ by the definition of $\mathscr{N}$. □

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