A new characterization of Sobolev spaces on $\mathbb{R}^n$

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Abstract

In this paper we present a new characterization of Sobolev spaces on $\mathbb{R}^n$. Our characterizing condition is obtained via a quadratic multiscale expression which exploits the particular symmetry properties of Euclidean space. An interesting feature of our condition is that depends only on the metric of $\mathbb{R}^n$ and the Lebesgue measure, so that one can define Sobolev spaces of any order of smoothness on any metric measure space.

1 Introduction

In this paper we present a new characterization of the Sobolev spaces $W^{\alpha,p}$ on $\mathbb{R}^n$, where the smoothness index α is any positive real number and $1 < p < \infty$. Thus $W^{\alpha,p}$ consists of those functions $f \in L^p = L^p(\mathbb{R}^n)$ such that $(-\Delta)^{\alpha/2} f \in L^p$. Here Δ is the Laplacean and $(-\Delta)^{\alpha/2} f$ is defined on the Fourier transform side by $|\xi|^\alpha \hat{f}(\xi)$. If $0 < \alpha < n$ this means that f is a function in L^p which is the Riesz potential of order α of some other function g in L^p, namely $f = c_n 1/|x|^{n-\alpha} * g$. If α is integer, then $W^{\alpha,p}$ is the usual space of those functions in L^p such that all distributional derivatives up to order α are in L^p.

To convey a feeling about the nature of our condition we first discuss the case $\alpha = 1$. Consider the square function

$$S(f)^2(x) = \int_0^\infty \left| \frac{f_{B(x,t)} - f(x)}{t} \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n . \tag{1}$$

Here f is a locally integrable function on $\mathbb{R}^n$ and $f_{B(x,t)}$ denotes the mean of f on the open ball with center x and radius t. One should think of $\frac{f_{B(x,t)} - f(x)}{t}$ as a quotient of increments of f at the point x. Our characterization of $W^{1,p}$ reads as follows.

Theorem 1. *If $1 < p < \infty$, then the following are equivalent.*

(1) $f \in W^{1,p}$

(2) $f \in L^p$ and $S(f) \in L^p$.

If any of the above conditions holds then

$$\|S(f)\|_p \simeq \|\nabla f\|_p .$$

The symbol $A \simeq B$ means, as usual, that for some constant C independent of the relevant parameters attached to the quantities A and B we have $C^{-1} B \leq A \leq C B$.

Notice that condition (2) in Theorem 1 above is of a metric measure space character, because only involves integrals over balls. It can be used to define in any metric measure space X a notion of Sobolev space $W^{1,p}(X)$. It is not clear to the authors what are the relations of this space with other known notions of Sobolev space in a metric measure space, in particular with those of Hajlasz [H] or Shanmugalingam [S] (see also [HK]).

The proof of Theorem 1 follows a classical route (see [Str]). The relevant issue is the necessary condition. First, via a Fourier transform estimate we show that

$$\|S(f)\|_2 = c \, \|\nabla f\|_2 \,,$$

for good functions f. In a second step, we set up a singular integral operator T with values in $L^2(dt/t)$ such that

$$\|T(f)\|_{L^2(\mathbb{R}^n, L^2(dt/t))} = \|S(f)\|_2 \,.$$

The kernel of T turns out to satisfy Hormander's condition, so that we can appeal to a well known result of Benedek, Calderón and Panzone [GR, Theorem 3.4, p. 492] on vector valued Calderón-Zygmund Theory to conclude the proof. The major technical difficulty occurs in checking Hormander's condition.

The proof extends without pain to cover orders of smoothness α with $0 < \alpha < 2$. The square function $S(f)$ has to replaced by

$$S_\alpha(f)^2(x) = \int_0^\infty \left| \frac{f_{B(x,t)} - f(x)}{t^\alpha} \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n \,.$$

The result is then that, for $0 < \alpha < 2$, $f \in W^{\alpha,p}$ is equivalent to $f \in L^p$ and $S_\alpha(f) \in L^p$.

Notice that

$$S_\alpha(f)^2(x) = \int_0^\infty \left| \fint_{B(x,\,t)} \frac{f(y) - f(x)}{t^\alpha} \, dy \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n \,, \tag{2}$$

where the barred integral on a set stands for the mean over that set. Stricharzt ([Str]) used long ago the above square function for $0 < \alpha < 1$ to characterize $W^{\alpha,p}$. However the emphasis in [Str] was on a larger variant of $S_\alpha(f)$ in which the absolute value is inside the integral in y in (2). In the interval $1 \leq \alpha < 2$ putting the absolute value inside the integral destroys the characterization, because then one gives up the symmetry properties of $\mathbb{R}^n$. For instance, $S_\alpha(f)$ vanishes if f is a first degree polynomial.

There are in the literature square functions very close to (2) which characterize $W^{\alpha,p}$, for $0 < \alpha < 2$. For example, first differences of f may be replaced by second differences and the absolute value may be placed inside the integral ([Str] and [St, Chapter V]). The drawback with second differences is that they do not make sense in the setting of metric measure spaces. See also the paper by Dorronsoro [D].

We now proceed to explain the idea for the characterization of $W^{2,p}$. Take a smooth function f and consider its Taylor expansion up to order 2 around x

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + \sum_{|\beta|=2} \partial^\beta f(x)(y - x)^\beta + R \,, \tag{3}$$

where R is the remainder and β a multi-index of length 2. Our goal is to devise a square function which plays the role of $S_1(f)$ (see (2) for $\alpha = 1$) with respect to second order derivatives. The first remark is that the mean on $B(x,t)$ of the homogeneous polynomial of degree 1 in (3) is zero. Now, the homogeneous Taylor polinomial of degree 2 can be written as

$$\sum_{|\beta|=2} \frac{\partial_\beta f(x)}{\beta!}(y-x)^\beta = H(y-x) + \frac{1}{2n}\Delta f(x)\,|y-x|^2\,, \tag{4}$$

for a harmonic homogeneous polynomial H of degree 2. Hence the mean on $B(x,t)$ of the homogeneous Taylor polinomial of degree 2 is

$$\fint_{B(x,t)} \frac{1}{2n}\Delta f(x)\,|y-x|^2\,dy\,.$$

This suggests defining

$$S_2(f)(x)^2 = \int_0^\infty \left| \fint_{B(x,t)} \frac{\left(f(y)-f(x)-\frac{1}{2n}(\Delta f)_{B(x,t)}\,|y-x|^2\right)}{t^2}\,dy \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n\,. \tag{5}$$

We cannot replace $(\Delta f)_{B(x,t)}$ by $\Delta f(x)$ in the preceding definition, because the mean guarantees a little extra smoothness which one needs in a certain Fourier transform computation. Notice that, according to the remarks made before on the mean on the ball $B(x,t)$ of the homogeneous Taylor polynomials of degrees 1 and 2, in the expression above for $S_2(f)(x)$ one may add the missing terms to get the full Taylor polynomial of degree 2, except for the fact that $\Delta f(x)$ should be replaced by $(\Delta f)_{B(x,t)}$. Were f smooth enough, one could even add the homogeneous Taylor polynomial of degree 3, because it is odd (taking x as the origin) and thus its mean on $B(x,t)$ vanishes. This explains why whatever we can prove for $\alpha = 2$ will also extend to the range $2 < \alpha < 4$ by defining

$$S_\alpha(f)(x)^2 = \int_0^\infty \left| \fint_{B(x,t)} \frac{\left(f(y)-f(x)-\frac{1}{2n}(\Delta f)_{B(x,t)}\,|y-x|^2\right)}{t^\alpha}\,dy \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n\,. \tag{6}$$

Here is our second order theorem.

Theorem 2. *If $1 < p < \infty$, then the following are equivalent.*

(1) $f \in W^{2,p}$

(2) $f \in L^p$ and there exists a function $g \in L^p$ such that $S_2(f,g) \in L^p$, where the square function $S_2(f,g)$ is defined by

$$S_2(f,g)(x)^2 = \int_0^\infty \left| \fint_{B(x,t)} \frac{\left(f(y)-f(x)-g_{B(x,t)}\,|y-x|^2\right)}{t^2}\,dy \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n\,.$$

If $f \in W^{2,p}$ then one can take $g = \Delta f/2n$ and if (2) holds then necessarily $g = \Delta f/2n$, a. e.

If any of the above conditions holds then

$$\|S(f,g)\|_p \simeq \|\Delta f\|_p .$$

Notice that condition (2) in Theorem 2 only involves the Euclidean distance on $\mathbb{R}^n$ and integrals with respect to Lebesgue measure. Thus one may define a notion of $W^{2,p}(X)$ on any metric measure space X. For more comments on that see section 4.

Again the special symmetry properties of $\mathbb{R}^n$ play a key role. For instance, S_2 annihilates second order polynomials. Theorem 2 has a natural counterpart for smoothness indexes α satisfying $2 \le \alpha < 4$. The result states that a function $f \in W^{\alpha,p}$ if and only if $f \in L^p$ and there exists a function $g \in L^p$ such that $S_\alpha(f,g) \in L^p$, where

$$S_\alpha(f,g)(x)^2 = \int_0^\infty \left| ⨍_{B(x,t)} \frac{\left(f(y) - f(x) - g_{B(x,t)}\,|y-x|^2\right)}{t^\alpha}\, dy \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n .$$

We proceed now to state our main result, which covers all orders of smoothness and all p with $1 < p < \infty$. Before it is convenient to discuss the analogue of (4) for homogeneous polynomials of any even degree. Let P be a homogeneous polynomial of degree $2j$. Then P can be written as

$$P(x) = H(x) + \Delta^j P \,\frac{1}{L_j}\, |x|^{2j},$$

where $L_j = \Delta^j(|x|^{2j})$ and H satisfies $\Delta^j H = 0$. This follows readily from [St, 3.1.2, p. 69]. Considering the spherical harmonics expansion of $P(x)$ we see that $\int_{|x|=1} H(x)\, d\sigma = 0$, σ being the surface measure on the unit sphere, and thus that $\int_{|x|\le t} H(x)\, dx = 0$, $t > 0$. The precise value of L_j, which can be computed easily, will not be needed.

Theorem 3. *Given $\alpha > 0$ choose an integer N such that $2N \le \alpha < 2N+2$. If $1 < p < \infty$, then the following are equivalent.*

(1) $f \in W^{\alpha,p}$

(2) $f \in L^p$ and there exist functions $g_j \in L^p$, $1 \le j \le N$ such that $S_\alpha(f, g_1, g_2, \dots, g_N) \in L^p$, where the square function $S_\alpha(f, g_1, g_2, \dots, g_N)$ is defined by

$$S_\alpha(f, g_1, g_2, \dots, g_N)(x)^2 = \int_0^\infty \left| ⨍_{B(x,t)} \frac{R_N(y,x)}{t^\alpha}\, dy \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n ,$$

and $R_N(y,x)$ is

$$R_N(y,x) = f(y) - f(x) - g_1(x)\,|y-x|^2 + \cdots + g_{N-1}(x)\,|y-x|^{2(N-1)} + (g_N)_{B(x,t)} |y-x|^{2N} .$$

If $f \in W^{\alpha,p}$ then one can take $g_j = \Delta^j f / L_j$ and if (2) holds then necessarily $g_j = \Delta^j f / L_j$, a. e.

If any of the above conditions holds then

$$\|S_\alpha(f, g_1, \dots, g_N)\|_p \simeq \|(-\Delta)^{\alpha/2} f\|_p .$$

Again condition (2) in Theorem 2 only involves the Euclidean distance on $\mathbb{R}^n$ and integrals with respect to Lebesgue measure. Thus one may define a notion of $W^{\alpha,p}(X)$ for any positive α and any $1 < p < \infty$ on any metric measure space X. For previous notions of higher order Sobolev spaces on metric measure spaces see [LLW]. See section 4 for more on that.

The proof of Theorem 3 proceeds along the lines sketched before for $\alpha = 1$. First we use a Fourier transform computation to obtain the relation

$$\|S_\alpha(f, \Delta f/L_1, \dots, \Delta^N f/L_N)\|_2 = c\,\|(-\Delta)^{\alpha/2} f\|_2 \,.$$

Then we introduce a singular integral operator with values in $L^2(dt/t^{2\alpha+1})$ and we check that its kernel satisfies Hormander's condition.

The paper is organized as follows. In sections 1, 2 and 3 we prove respectively Theorems 1, 2 and 3. In this way readers interested only in first order Sobolev spaces may concentrate in section 1. Those readers interested in the main idea about jumping to orders of smoothness 2 and higher may read section 2. Section 3 is reserved to those interested in the full result. In any case the technical details for the proof of Theorem 1 are somehow different of those for orders of smoothness 2 and higher. The reason is that Hormander's condition involves essentially taking one derivative of the kernel and is precisely the kernel associated to the first order of smoothness that has minimal differentiability.

Our notation and terminology are standard. For instance, we shall adopt the usual convention of denoting by C a constant independent of the relevant variables under consideration and not necessarily the same at each occurrence.

If f has derivatives of order M for some non-negative integer M, then $\nabla^M f = (\partial^\beta f)_{|\beta|=M}$ is the vector with components the partial derivatives of order M of f and $|\nabla^M f|$ its Euclidean norm.

The Zygmund class on $\mathbb{R}^n$ consists of those continuous functions f such that, for some constant C,

$$|f(x+h) + f(x-h) - 2f(x)| \le C\,|h|, \quad x, h \in \mathbb{R}^n \,.$$

The basic example of a function in the Zygmund class which is not Lipschitz is $f(x) = |x| \log |x|$, $x \in \mathbb{R}^n$.

The Scharwtz class consists of those infinitely differentiable functions on $\mathbb{R}^n$ whose partial derivatives of any order decrease faster than any polynomial at ∞.

2 Proof of Theorem 1

The difficult part is the necessity of condition (2) and we start with this.

As a first step we show that

$$\|S_1(f)\|_2 = c\,\|\nabla f\|_2 \tag{7}$$

for a dimensional constant c. Set

$$\chi(x) = \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x)$$

and

$$\chi_t(x) = \frac{1}{t^n}\chi(\frac{x}{t}),$$

so that, by Plancherel,

$$\int_{\mathbb{R}^n} S_1(f)(x)^2\,dx = \int_0^\infty \int_{\mathbb{R}^n} |(f*\chi_t)(x) - f(x)|^2\ dx\ \frac{dt}{t^3}$$
$$= c\int_0^\infty \int_{\mathbb{R}^n} |\hat{\chi}(t\xi) - 1|^2 \left|\hat{f}(\xi)\right|^2\ d\xi\ \frac{dt}{t^3}.$$

Since $\hat{\chi}$ is radial, $\hat{\chi}(\xi) = F(|\xi|)$ for a certain function F defined on $[0,\infty)$. Exchange the integration in $d\xi$ and dt in the last integral above and make the change of variables $\tau = t\,|\xi|$. Then

$$\int_{\mathbb{R}^n} S_1(f)(x)^2\,dx = c\int_{\mathbb{R}^n}\int_0^\infty |(F(\tau) - 1|^2\ \frac{d\tau}{\tau^3}\ |\hat{f}(\xi)|^2|\xi|^2\,d\xi$$
$$= c\int_0^\infty |(F(\tau)-1|^2\ \frac{d\tau}{\tau^3}\ \ \|\nabla f\|_2^2$$

and (7) is reduced to showing that

$$\int_0^\infty |(F(\tau)-1|^2\ \frac{d\tau}{\tau^3} < \infty\,. \tag{8}$$

Set $B = B(0,1)$ and $e_1 = (1,0,\ldots,0) \in \mathbb{R}^n$. Then

$$F(t) = \hat{\chi}(te_1) = ⨍_B \exp(-\imath x_1 t)\,dx$$
$$= ⨍_B \left(1 - \imath x_1 t - \frac{1}{2}x_1^2 t^2 + \cdots\right) dx$$
$$= 1 - \frac{1}{2}\ ⨍_B x_1^2\,dx\ t^2 + \cdots,$$

which yields

$$F(t) - 1 = O(t^2), \quad \text{as} \quad t \to 0$$

and shows the convergence of (8) at 0.

Since $F(|\xi|) = \hat{\chi}(\xi)$ is the Fourier transform of an integrable function, $F(\tau)$ is a bounded function and so the integral (8) is clearly convergent at ∞.

We are left with the case of a general p between 1 and ∞. If $f \in W^{1,p}$, then $f = g*1/|x|^{n-1}$ for some $g \in L^p$ (with $1/|x|^{n-1}$ replaced by $\log|x|$ for $n = 1$). Set $I(x) = 1/|x|^{n-1}$. Then

$$f_{B(x,t)} - f(x) = (f*\chi_t)(x) - f(x) = (g*K_t)(x),$$

where

$$K_t(x) = (I*\chi_t)(x) - I(x) = ⨍_{B(x,t)} I(y)\,dy - I(x)\,. \tag{9}$$

If we let $T(g)(x) = (g * K_t)(x),\ \ x \in \mathbb{R}^n$, then one can rewrite $S_1(f)(x)$ as

$$S_1(f)(x) = \left(\int_0^\infty |(g * K_t)(x)|^2 \, \frac{dt}{t^3}\right)^{\frac{1}{2}} = \|Tg(x)\|_{L^2(dt/t^3)} \,.$$

Then (7) translates into

$$\int_{\mathbb{R}^n} \|Tg(x)\|^2_{L^2(dt/t^3)}\, dx = c\, \|g\|_2^2 \,,$$

and we conclude that T is an operator mapping isometrically $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n, L^2(dt/t^3))$. If the kernel $K_t(x)$ of T satisfies Hormander's condition

$$\int_{|x|\geq 2|y|} \|K_t(x-y) - K_t(x)\|_{L^2(dt/t^3)} \leq C, \quad y \in \mathbb{R}^n$$

then a well known result of Benedek, Calderón and Panzone on vector valued singular integrals (see [GR, Theorem 3.4, p. 492]) yields the L^p estimate

$$\int_{\mathbb{R}^n} \|Tg(x)\|^p_{L^2(dt/t^3)}\, dx \leq C_p\, \|g\|_p^p \,,$$

which can be rewritten as

$$\|S_1(f)\|_p \leq C_p\, \|\nabla f\|_p \,.$$

The reverse inequality follows from polarization from (7) by a well known argument ([GR, p. 507]) and so the proof of the necessary condition is complete. We are going to prove the following stronger version of Hormander's condition

$$\|K_t(x-y) - K_t(x)\|_{L^2(dt/t^3)} \leq C\, \frac{|y|}{|x|^{n+1}}, \quad y \in \mathbb{R}^n \,, \tag{10}$$

for almost all x satisfying $|x| \geq 2|y|$.

To prove (10) we deal separately with three intervals in the variable t.

Interval 1: $t < \frac{|x|}{3}$. From the definition of K_t in (9) we obtain

$$\nabla K_t(x) = (\nabla I * \chi_t)(x) - \nabla I(x) \,. \tag{11}$$

Notice that, in the distributions sense, the gradient of I is a constant times the vector valued Riesz transform, namely

$$\nabla I = -(n-1) p.v. \frac{x}{|x|^{n+1}} \,.$$

If $|x| \geq 2|y|$, then the segment $[x-y, x]$ does not intersect the ball $B(0, |x|/2)$ and thus

$$|K_t(x-y) - K_t(x)| \leq |y| \sup_{z\in[x-y,y]} |\nabla K_t(z)| \,. \tag{12}$$

If $t < |x|/3$ and $z \in [x-y, y]$, then $B(z,t) \subset \mathbb{R}^n \setminus B(0,|x|/6)$, and hence

$$\nabla K_t(z) = \fint_{B(z,\,t)} (\nabla I(w) - \nabla I(z))\, dw\,. \tag{13}$$

Taylor's formula up to order 2 for $\nabla I(w)$ around z yields

$$\nabla I(w) = \nabla I(z) + \nabla^2 I(z)(w-z) + O(\frac{|w-z|^2}{|x|^{n+2}})\,,$$

where $\nabla^2 I(z)(w-z)$ is the result of applying the matrix $\nabla^2 I(z)$ to the vector $w-z$. The mean value of $\nabla^2 I(z)(w-z)$ on $B(z,t)$ is zero, by antisymmetry, and thus, by (13),

$$|\nabla K_t(z)| \le C\,\frac{t^2}{|x|^{n+2}}$$

and so, by (12)

$$|K_t(x-y) - K_t(x)| \le C\,|y|\frac{t^2}{|x|^{n+2}}\,.$$

Integrating in t we finally get

$$\left(\int_0^{|x|/3} |K_t(x-y) - K_t(x)|^2\,\frac{dt}{t^3}\right)^{\frac{1}{2}} \le C\,\frac{|y|}{|x|^{n+2}}\left(\int_0^{|x|/3} t\,dt\right)^{\frac{1}{2}} = C\,\frac{|y|}{|x|^{n+1}}\,. \tag{14}$$

Interval 2: $|x|/3 < t < 2|x|$. The function $I * \chi_t$ is continuously differentiable on $\mathbb{R}^n \setminus S_t$, $S_t = \{x : |x| = t\}$, because its distributional gradient is given by $I * \nabla\chi_t$ and each component of $\nabla\chi_t$ is a Radon measure supported on S_t. The gradient of $I * \chi_t$ is given at each point $x \in \mathbb{R}^n \setminus S_t$ by the principal value integral

$$p.v.(\nabla I * \chi_t)(x) = -(n-1)p.v. \fint_{B(x,\,t)} \frac{y}{|y|^{n+1}}\,,$$

which exists for all such x. The difficulty in the interval under consideration is that it may happen that $|x| = t$ and then the gradient of $I * \chi_t$ has a singularity at such an x. We need the following estimate.

Lemma 1.

$$\left|p.v. \int_{B(x,\,t)} \frac{y}{|y|^{n+1}}\,dy\right| \le C\,\log\frac{|x|+t}{||x|-t|}, \quad x \in \mathbb{R}^n\,.$$

Proof. Assume without loss of generality that $x = (x_1, 0, \dots, 0)$. The coordinates y_j, $j \ne 1$, change sign under reflection around the y_1 axes. Hence

$$p.v. \int_{B(x,\,t)} \frac{y_j}{|y|^{n+1}}\,dy = 0, \quad 1 < j \le n\,.$$

Now, if $|x| < t$,

$$\left|p.v.\int_{B(x,t)} \frac{y_1}{|y|^{n+1}}\,dy\right| = \left|p.v.\int_{B(x,t)\setminus B(0,t-|x|)} \frac{y_1}{|y|^{n+1}}\,dy\right|$$
$$\leq C\int_{t-|x|}^{t+|x|} \frac{dt}{t} = C\,\log\frac{t+|x|}{t-|x|}\,.$$

If $|x| > t$,

$$\left|p.v.\int_{B(x,t)} \frac{y_1}{|y|^{n+1}}\,dy\right| = \left|\int_{B(x,t)} \frac{y_1}{|y|^{n+1}}\,dy\right|$$
$$\leq C\int_{|x|-t}^{|x|+t} \frac{dt}{t} = C\,\log\frac{|x|+t}{|x|-t}\,.$$

□

Assume without loss of generality that $y = (y_1, 0, \ldots, 0)$. The distributional gradient of $I * \chi_t$ is

$$-(n-1)p.v.\frac{y}{|y|^{n+1}} * \chi_t\,,$$

which is in L^2. Then $I * \chi_t \in W^{1,2}$ and consequently is absolutely continuous on almost all lines parallel to the first axes. Therefore

$$K_t(x-y) - K_t(x) = -\int_0^1 \nabla K_t(x-\tau y)\cdot y\,d\tau$$

for almost all x and

$$|K_t(x-y) - K_t(x)| \leq C\,\frac{|y|}{|x|^n}\int_0^1 \left(1 + \log\frac{|x-\tau y| + t}{||x-\tau y| - t|}\right) d\tau\,.$$

Hence

$$\left(\int_{|x|/3}^{2|x|} |K_t(x-y) - K_t(x)|^2\,\frac{dt}{t^3}\right)^{\frac{1}{2}} \leq C\,\frac{|y|}{|x|^{n+1}}\left(\int_{|x|/3}^{2|x|}\left(\int_0^1\left(1 + \log\frac{|x-\tau y| + t}{||x-\tau y| - t|}\right) d\tau\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$
$$= C\,\frac{|y|}{|x|^{n+1}}\,D\,,$$

where the last identity is a definition of D. Applying Schwarz to the inner integral in D and then changing the order of integration we get

$$D^2 \leq \int_0^1\left(\int_{|x|/3}^{2|x|}\left(1 + \log\frac{|x-\tau y| + t}{||x-\tau y| - t|}\right)^2 \frac{dt}{t}\right) d\tau\,.$$

For each τ make the change of variables

$$s = \frac{t}{|x - \tau y|}$$

to conclude that

$$D^2 \le \int_{2/9}^{4} \left(1 + \log \frac{1+s}{|1-s|}\right)^2 \frac{ds}{s}\,.$$

Interval 3: $2|x| \le t$. For each z in the segment $[x-y, y]$ we have $B(0, t/4) \subset B(z, t)$. Then, by (13),

$$\begin{aligned} \nabla K_t(z) &= -(n-1)\left(p.v.\frac{1}{|B(z,t)|}\int_{B(z,t)} \frac{w}{|w|^{n+1}}\,dw - \frac{z}{|z|^{n+1}}\right) \\ &= -(n-1)\left(\frac{1}{|B(z,t)|}\int_{B(z,t)\setminus B(0,t/4)} \frac{w}{|w|^{n+1}}\,dw - \frac{z}{|z|^{n+1}}\right) \end{aligned}$$

and so

$$|\nabla K_t(z)| \le C\,\frac{1}{|x|^n}, \quad z \in [x-y, y]\,.$$

Hence, owing to (12),

$$|K_t(x-y) - K_t(x)| \le C\,\frac{|y|}{|x|^n}$$

and thus

$$\left(\int_{2|x|}^{\infty} |K_t(x-y) - K_t(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}} \le C\,\frac{|y|}{|x|^n}\left(\int_{2|x|}^{\infty} \frac{dt}{t^3}\right)^{\frac{1}{2}} = C\,\frac{|y|}{|x|^{n+1}}\,,$$

which completes the proof of the strengthened form of Hormander's condition (10).

We turn now to prove that condition (2) in Theorem 1 is sufficient for $f \in W^{1,p}$. Let $f \in L^p$ satisfy $S_1(f) \in L^p$. Take an infinitely differentiable function $\phi \ge 0$ with compact support in $B(0,1)$, $\int \phi = 1$ and set $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$, $\epsilon > 0$. Consider the regularized functions $f_\epsilon = f * \phi_\epsilon$. Then f_ϵ is infinitely differentiable and $\|\nabla f_\epsilon\|_p \le \|f\|_p \|\nabla \phi_\epsilon\|_1$, so that $f_\epsilon \in W^{1,p}$. Thus, as we have shown before,

$$\|\nabla f_\epsilon\| \simeq \|S_1(f_\epsilon)\|_p\,.$$

We want now to estimate $\|S_1(f_\epsilon)\|_p$ independently of ϵ. Since

$$(f_\epsilon)_{B(x,t)} - f_\epsilon(x) = ((f * \chi_t - f) * \phi_\epsilon)(x)\,,$$

Minkowsky's integral inequality gives

$$S_1(f_\epsilon)(x) = \|(f_\epsilon)_{B(x,t)} - f_\epsilon(x)\|_{L^2(dt/t^3)} \le (S_1(f) * \phi_\epsilon)(x)\,,$$

and so $\|\nabla f_\epsilon\| \le C\,\|S_1(f)\|_p$, $\epsilon > 0$. For an appropriate sequence $\epsilon_j \to 0$ the sequences $\partial_k f_{\epsilon_j}$ tend in the weak $\star$ topology of L^p to some function $g_k \in L^p$, $1 \le k \le n$. On the other hand, $f_\epsilon \to f$ in L^p as $\epsilon \to 0$ and thus $\partial_k f_\epsilon \to \partial_k f$, $1 \le k \le n$ in the weak topology of distributions. Therefore $\partial_k f = g_k$ for all k and so $f \in W^{1,p}$.

3 Proof of Theorem 2

The difficult direction is (1) implies (2) and this is the first we tackle. We start by showing that if $f \in W^{2,2}$ then

$$\|S_2(f)\|_2 = c\,\|\Delta f\|_2 \tag{15}$$

where the square function $S_2(f)$ is defined in (5). To apply Plancherel in the x variable it is convenient to write the innermost integrand in (5) as

$$\begin{aligned}&⨍_{B(x,t)} \left(f(y) - f(x) - \left(⨍_{B(x,t)} \frac{\Delta f(z)}{2n}\, dz \right) |y-x|^2 \right) dy \\ &= ⨍_{B(0,t)} \left(f(x+h) - f(x) - \left(⨍_{B(0,t)} \frac{\Delta f(x+k)}{2n}\, dk \right) |h|^2 \right) dh\,.\end{aligned}$$

Applying Plancherel we get, for some dimensional constant c,

$$c\,\|S_2(f)\|_2^2 = \int_0^\infty \int_{\mathbb{R}^n} ⨍_{B(0,t)} \left(\exp(\imath \xi h) - 1 + \left(⨍_{B(0,t)} \exp(\imath \xi k)\, dk \right) \frac{|h|^2 |\xi|^2}{2n} \right) dh\, |\hat{f}(\xi)|^2\, d\xi\, \frac{dt}{t^5}\,.$$

Make appropriate dilations in the integrals with respect to the variables h and k to bring the integrals on $B(0,1)$. Then use that the Fourier transform of $\frac{1}{|B(0,1)|}\chi_{B(0,1)}$ is a radial function, and thus of the form $F(|\xi|)$ for a certain function F defined on $[0,\infty)$. The result is

$$c\,\|S_2(f)\|_2^2 = \int_{\mathbb{R}^n} \int_0^\infty \left| F(t\,|\xi|) - 1 + t^2 |\xi|^2 F(t\,|\xi|) \frac{1}{2n} ⨍_{B(0,1)} |h|^2\, dh \right|^2 \frac{dt}{t^5} |\hat{f}(\xi)|^2\, d\xi\,.$$

The change of variables $\tau = t\,|\xi|$ yields

$$c\,\|S_2(f)\|_2^2 = I\, \|\Delta f\|_2^2$$

where I is the integral

$$I = \int_0^\infty \left| F(\tau) - 1 + \tau^2 F(\tau) \frac{1}{2n} ⨍_{B(0,1)} |h|^2\, dh \right|^2 \frac{d\tau}{\tau^5}\,. \tag{16}$$

The only task left is to prove that the above integral is finite. Now, as $\tau \to 0$,

$$\begin{aligned} F(\tau) &= ⨍_{B(0,1)} \exp(\imath h_1 \tau)\, dh \\ &= ⨍_{B(0,1)} \left(1 + \imath h_1 \tau - \frac{1}{2} h_1^2 \tau^2 + \cdots \right) dh \\ &= 1 - \frac{1}{2} \left(⨍_{B(0,1)} h_1^2\, dh \right) \tau^2 + O(\tau^4)\,. \end{aligned}$$

Hence

$$F(\tau)-1+\tau^2F(\tau)\frac{1}{2n}\fint_{B(0,1)}|h|^2\,dh$$
$$=\left(-\frac{1}{2}\fint_{B(0,1)}h_1^2\,dh+\frac{1}{2n}\fint_{B(0,1)}|h|^2\,dh\right)\tau^2+O(\tau^4)=O(\tau^4)\,,$$

because clearly $\fint_{B(0,1)}|h|^2\,dh = n\fint_{B(0,1)}h_1^2\,dh$. Therefore the integral (16) is convergent at $\tau=0$.

To deal with the case $\tau\to\infty$ we recall that F can be expressed in terms of Bessel functions. Concretely, one has ([Gr, Appendix B.5, p. 429])

$$|B(0,1)|\,F(\tau)=\frac{J_{n/2}(\tau)}{|\tau|^{n/2}}\,.$$

The asymptotic behaviour of $J_{n/2}(\tau)$ gives the inequality

$$|F(\tau)|\le C\,\frac{1}{\tau^{\frac{n+1}{2}}}\le C\,\frac{1}{\tau}\,,$$

which shows that the integral (16) is convergent at ∞.

We turn our attention to the case $1<p<\infty$. Let $I_2(x)$ stand for the kernel defined on the Fourier transform side by

$$\hat{I}_2(\xi)=\frac{1}{|\xi|^2}\,.$$

In other words, I_2 is minus the standard fundamental solution of the Laplacean. Thus $I_2(x)=c_n\,1/|x|^{n-2}$ if $n\ge 3$, $I_2(x)=-\frac{1}{2\pi}\log|x|$ if $n=2$ and $I_2(x)=-\frac{1}{2}|x|$ if $n=1$. Given any $f\in W^{2,p}$ there exists $g\in L^p$ such that $f=I_2*g$ (indeed, $g=-\Delta f$). We claim that there exists a singular integral operator $T(g)$ taking values in $L^2(dt/t^5)$ such that

$$S_2(f)(x)=\|T(g)(x)\|_{L^2(dt/t^5)}\,. \tag{17}$$

Set

$$\chi(x)=\frac{1}{|B(0,1)|}\chi_{B(0,1)}(x)$$

and

$$\chi_t(x)=\frac{1}{t^n}\chi\left(\frac{x}{t}\right).$$

Then, letting $M=\fint_{B(0,1)}|z|^2\,dz$,

$$\fint_{B(x,t)}\left(f(y)-f(x)-\frac{1}{2n}(\Delta f)_{B(x,t)}\,|y-x|^2\right)dy=((I_2*\chi_t-I_2-\frac{M}{2n}\,t^2\,\chi_t)*g)(x)$$
$$=(K_t*g)(x)\,,$$

where

$$K_t(x) = (I_2 * \chi_t)(x) - I_2(x) - \frac{M}{2n}\, t^2\, \chi_t(x)\,.$$

Setting $T(g)(x) = (K_t * g)(x)$ we get (17) from the definition of $S_2(f)$ in (5). Then (15) translates into

$$\int_{\mathbb{R}^n} \|Tg(x)\|^2_{L^2(dt/t^5)}\, dx = c\, \|g\|_2^2\,,$$

and we conclude that T is an operator mapping isometrically $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n, L^2(dt/t^5))$, modulo the constant c. If the kernel $K_t(x)$ of T satisfies Hormander's condition

$$\int_{|x|\geq 2|y|} \|K_t(x-y) - K_t(x)\|_{L^2(dt/t^5)} \leq C, \quad y \in \mathbb{R}^n,$$

then a well known result of Benedek, Calderón and Panzone on vector valued singular integrals (see [GR, Theorem 3.4, p. 492]) yields the L^p estimate

$$\int_{\mathbb{R}^n} \|Tg(x)\|^p_{L^2(dt/t^5)}\, dx \leq C_p\, \|g\|_p^p\,,$$

which can be rewritten as

$$\|S_2(f)\|_p \leq C_p\, \|\Delta f\|_p\,.$$

The reverse inequality follows from polarization from (15) by a well known duality argument ([GR, p. 507]) and so the proof of the necessary condition is complete.

We are going to prove the following stronger version of Hormander's condition

$$\|K_t(x-y) - K_t(x)\|_{L^2(dt/t^5)} \leq C\, \frac{|y|^{1/2}}{|x|^{n+1/2}}, \quad |x| \geq 2|y|\,. \tag{18}$$

For this we deal separately with the kernels $H_t(x) = (I_2 * \chi_t)(x) - I_2(x)$ and $t^2\,\chi_t(x)$. For $t^2\,\chi_t(x)$ we first remark that the quantity $|\chi_t(x-y) - \chi_t(x)|$ is non-zero only if $|x-y| < t < |x|$ or $|x| < t < |x-y|$, in which cases takes the value $1/c_n\, t^n$, $c_n = |B(0,1)|$. On the other hand, if $|x| \geq 2|y|$ then each z in the segment joining x and $x-y$ satisfies $|z| \geq |x|/2$. Assume that $|x-y| < |x|$ (the case $|x| < |x-y|$ is similar). Then

$$\begin{aligned}\left(\int_0^\infty (t^2\,(\chi_t(x-y) - \chi_t(x)))^2\, \frac{dt}{t^5}\right)^{\frac{1}{2}} &= c\left(\int_{|x-y|}^{|x|} \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\ &= c\left(\frac{1}{|x-y|^{2n}} - \frac{1}{|x|^{2n}}\right)^{\frac{1}{2}} \leq C\, \frac{|y|^{1/2}}{|x|^{n+1/2}}\,.\end{aligned}$$

We check now that H_t satisfies the stronger form of Hormander's condition. If $t < |x|/2$, then the origin does not belong to the ball $B(x-y,t)$ nor to the ball $B(x,t)$. Since I_2 is harmonic off the origin, the mean of I_2 on these balls is the value of I_2 at the center. Therefore $H_t(x-y) - H_t(x) = 0$ in this case.

If $t \geq |x|/2$, then

$$|H_t(x-y) - H_t(x)| \leq |y| \sup_{z \in [x-y,x]} |\nabla H_t(z)| \leq C \frac{|y|}{|x|^{n-1}} .$$

The last inequality follows from

$$\nabla H_t(z) = \fint_{B(z,t)} \nabla I_2(w)\, dw - \nabla I_2(z) ,$$

$|\nabla I_2(z)| \leq C\, 1/|z|^{n-1} \leq C\, 1/|x|^{n-1}$ and

$$|\fint_{B(z,t)} \nabla I_2(w)\, dw| \leq \fint_{B(z,t)} \frac{1}{|w|^{n-1}} \leq C \frac{1}{|z|^{n-1}} .$$

Therefore

$$\left(\int_0^\infty |H_t(x-y) - H_t(x)|^2 \frac{dt}{t^5} \right)^{\frac{1}{2}} \leq C \frac{|y|}{|x|^{n-1}} \left(\int_{|x|/2}^\infty \frac{dt}{t^5} \right)^{\frac{1}{2}} = C \frac{|y|}{|x|^{n+1}} .$$

We turn now to prove that condition (2) in Theorem 2 is sufficient for $f \in W^{2,p}$. Let f and g in L^p satisfy $S_2(f,g) \in L^p$. Take an infinitely differentiable function $\phi \geq 0$ with compact support in $B(0,1)$, $\int \phi = 1$ and set $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$, $\epsilon > 0$. Consider the regularized functions $f_\epsilon = f * \phi_\epsilon$ and $g_\epsilon = g * \phi_\epsilon$. Then f_ϵ is infinitely differentiable and $\|\Delta f_\epsilon\|_p \leq \|f\|_p \|\Delta\phi_\epsilon\|_1$, so that $f_\epsilon \in W^{2,p}$. Recalling that $M = \fint_{B(0,1)} |z|^2\, dz$, we get, by Minkowsky's integral inequality,

$$\begin{aligned} S_2(f_\epsilon, g_\epsilon)(x) &= \|(f_\epsilon * \chi_t)(x) - f_\epsilon(x) - (g_\epsilon * \chi_t)(x)\, M^2\, t^2\|_{L^2(dt/t^5)} \\ &= \|\left((f * \chi_t) - f - (g * \chi_t)\, M^2\, t^2 * \phi_\epsilon\right)(x)\|_{L^2(dt/t^5)} \\ &\leq (S_2(f,g) * \phi_\epsilon)(x) . \end{aligned}$$

Now we want to compare $(1/2n)\Delta f_\epsilon$ and g_ϵ. Define

$$D_\epsilon(x) = \left(\int_0^\infty M^2 \left| \frac{1}{2n}(\Delta f_\epsilon * \chi_t)(x) - (g_\epsilon * \chi_t)(x) \right|^2 \frac{dt}{t} \right)^{1/2} .$$

Then

$$\begin{aligned} D_\epsilon(x) &\leq S_\alpha(f_\epsilon)(x) + S_\alpha(f_\epsilon, g_\epsilon)(x) \\ &\leq S_\alpha(f_\epsilon)(x) + (S_\alpha(f,g)(x) * \phi_\epsilon)(x) , \end{aligned}$$

and thus $D_\epsilon(x)$ is an L^p function. In particular $D_\epsilon(x) < \infty$, for almost all $x \in \mathbb{R}^n$. Hence

$$|(1/2n)\Delta f_\epsilon(x) - g_\epsilon(x)| = \lim_{t\to 0} |(1/2n)(\Delta f_\epsilon * \chi_t)(x) - (g_\epsilon * \chi_t)(x)| = 0 ,$$

for almost all $x \in \mathbb{R}^n$, and so $(1/2n)\Delta f_\epsilon \to g$ in L^p as $\epsilon \to 0$. Since $f_\epsilon \to f$ in L^p as $\epsilon \to 0$, then $\Delta f_\epsilon \to \Delta f$ in the weak topology of distributions. Therefore $(1/2n)\Delta f = g$ and the proof is complete.

4 Proof of Theorem 3

The difficult direction in Theorem 3 is to show that condition (2) is necessary for $f \in W^{\alpha,p}$. The proof follows the pattern already described in the preceding sections. One introduces an operator T taking values in $L^2(dt/t^{2\alpha+1})$ and shows via a Fourier transform estimate that T sends $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n, L^2(dt/t^{2\alpha+1}))$ isometrically (modulo a multiplicative constant). The second step consists in showing that its kernel satisfies Hormander's condition, after which one appeals to a well known result of Benedek, Calderón and Panzone on vector valued singular integrals to finish the proof.

4.1 The fundamental solution of $(-\Delta)^{\alpha/2}$

Let I_α be the fundamental solution of $(-\Delta)^{\alpha/2}$, that is, I_α is a function such that $\hat{I}_\alpha(\xi) = |\xi|^{-\alpha}$ and is normalized prescribing some behavior at ∞. It is crucial for our proof to have an explicit expression for I_α. The result is as follows (see [ACL] or [MOPV, p. 3699]).

If α is not integer then

$$I_\alpha(x) = c_{\alpha,n}\, |x|^{\alpha-n}, \quad x \in \mathbb{R}^n\,, \tag{19}$$

for some constant $c_{\alpha,n}$ depending only on α and n.

The same formula works if α is an even integer and the dimension is odd or if α is an odd integer and the dimension is even.

The remaining cases, that is, α and n are even integers or α and n are odd integers are special in some cases. If $\alpha < n$ formula (19) still holds, but if α is of the form $n + 2N$, for some non-negative integer N, then

$$I_\alpha(x) = c_{\alpha,n}\, |x|^{\alpha-n}\, (A + B \log |x\,|), \quad x \in \mathbb{R}^n\,,$$

where $c_{\alpha,n}$, A and B are constants depending on α and n, and $B \neq 0$. Thus in this cases a logarithmic factor is present. For instance, if $\alpha = n$, then $I_\alpha(x) = B \log |x\,|$. If $n = 1$ and $\alpha = 2$, then $I_2(x) = -(1/2)\,|x\,|$ and there is no logarithmic factor.

4.2 The case $p = 2$

Given a positive real number α let N be the unique integer satisfying $2N \le \alpha < 2N + 2$. Define the square function associated with α by

$$S_\alpha(f)^2(x) = \int_0^\infty \left| \fint_{B(x,\,t)} \frac{\rho_N(y,x)}{t^\alpha}\, dy \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n\,, \tag{20}$$

where $\rho_N(y,x)$ is

$$f(y) - f(x) - \frac{1}{2n}\Delta f(x)\,|y-x|^2 - \cdots - \frac{1}{L_{N-1}}\Delta^{N-1} f(x)\,|y-x|^{2(N-1)} - \frac{1}{L_N}(\Delta^N f)_{B(x,\,t)}|y-x|^{2N}\,.$$

Recall that $L_j = \Delta^j(|x|^{2j})$ and that the role which the L_j play in Taylor expansions was discussed just before the statement of Theorem 3 in the introduction.

In this subsection we prove that

$$\|S_\alpha(f)\|_2 = c\,\|(-\Delta)^{\alpha/2}(f)\|_2 \,. \tag{21}$$

Our plan is to integrate in x in (20), interchange the integration in x and t and then apply Plancherel in x. Before we remark that making the change of variables $y = x + th$ we transform integrals on $B(x,t)$ in integrals on $B(0,1)$ and we get

$$\fint_{B(x,t)} \rho_N(y,x)\,dy = \fint_{B(0,1)} f(x+th)\,dh - \sum_{j=0}^{N-1} \frac{\Delta^j f(x)}{L_j}\, t^{2j} \fint_{B(0,1)} |h|^{2j}\,dh$$
$$- \fint_{B(0,1)} \Delta^N f(x+th)\,dh\; t^{2N} \fint_{B(0,1)} |h|^{2N}\,dh \,.$$

Now apply Plancherel in x, as explained before, and make the change of variables $\tau = t\,|\xi|$, where ξ is the variable in the frequency side. We obtain

$$\|S_\alpha(f)\|_2^2 = c\,I\,\|(-\Delta)^{\alpha/2} f\|_2^2 \,,$$

where

$$I = \int_0^\infty \left| F(\tau) - \sum_{j=0}^{N-1} (-1)^j \tau^{2j} \frac{M_j}{L_j} - (-1)^N \tau^{2N} \frac{M_N}{L_N}\, F(\tau) \right|^2 \frac{d\tau}{\tau^{2\alpha+1}} \,.$$

Here F is a function defined on $[0,\infty)$ such that $F(|\xi|)$ gives the Fourier transform of the radial function $\frac{1}{|B(0,1)|}\chi_{B(0,1)}$ at the point ξ, and we have introduced the notation $M_j = \fint_{B(0,1} |h|^{2j}\,dh$. We have to show that the integral I is finite.

Using the series expansion of the exponential we see that, as $\tau \to 0$,

$$F(\tau) = \fint_{B(0,1)} \exp(\imath h_1 \tau)\,dh$$
$$= 1 + \cdots + (-1)^N \tau^{2N} \frac{1}{(2N)!} \fint_{B(0,1)} h_1^{2N}\,dh + O(\tau^{2N+2}) \,.$$

We need to compare $\fint_{B(0,1)} h_1^{2N}\,dh$ with $\fint_{B(0,1)} |h|^{2N}\,dh$. The linear functionals $P \to \Delta^{2j}(P)$ and $P \to \fint_{B(0,1)} P$, defined on the space H_{2j} of homogeneous polynomials of degree $2j$, have the same kernel. This follows from the discussion before the statement of Theorem 3 in the introduction. Therefore, for some constant c,

$$\Delta^{2j}(P) = c \fint_{B(0,1)} P, \quad P \in H_{2j} \,.$$

Taking $P(x) = |x|^{2j}$ we get $L_j = c \fint_{B(0,1)} |x|^{2j} = \,dx$, and taking $P(x) = x_1^{2j}$ we get $(2j)! = c \fint_{B(0,1)} x_1^{2j}\,dx$. Hence

$$\frac{1}{(2j)!} \fint_{B(0,1)} x_1^{2j}\,dx = \frac{1}{L_j} \fint_{B(0,1)} |x|^{2j}\,dx = \frac{M_j}{L_j} \,,$$

and thus, owing to the definition of I and the fact that $F(\tau) = 1 + O(\tau^2)$, as $\tau \to 0$,

$$I = \int_0^\infty O(\tau^{2(2N+2)}) \frac{d\tau}{\tau^{2\alpha+1}}, \quad \text{as } \tau \to 0 \,.$$

Then I is convergent at 0 because $\alpha < 2N + 2$.

We turn now to the case $\tau \to \infty$. Notice that the only difficulty is the last term in the integrand of I, because

$$\int_1^\infty \tau^{4j} \frac{d\tau}{\tau^{2\alpha+1}} < \infty, \quad 0 \le j \le N-1 \,,$$

provided $2N \le \alpha$. To deal with the term

$$\int_1^\infty |\tau^{2N} F(\tau)|^2 \frac{d\tau}{\tau^{2\alpha+1}} \tag{22}$$

we only need to recall F can be expressed in terms of Bessel functions. Concretely, one has ([Gr, Appendix B.5, p. 429])

$$|B(0,1)| \, F(\tau) = \frac{J_{n/2}(\tau)}{|\tau|^{n/2}} \,.$$

The asymptotic behaviour of $J_{n/2}(\tau)$ gives the inequality, as $\tau \to \infty$,

$$|F(\tau)| \le C \frac{1}{\tau^{\frac{n+1}{2}}} \le C \frac{1}{\tau} \,,$$

which shows that the integral (22) is convergent finite provided $2N \le \alpha$.

4.3 A vector valued operator and its kernel

Given $f \in W^{\alpha,p}$, there exists a function $g \in L^p$ such that $f = I_\alpha * g$. Indeed, $g = (-\Delta)^{\alpha/2}(f)$. Then

$$⨍_{B(x,t)} \left(f(y) - f(x) - \sum_{j=0}^{N-1} \frac{1}{L_j} \Delta^j f(x) \, |y-x|^{2j} - \frac{1}{L_N} (\Delta^N f)_{B(x,t)} \, |y-x|^{2N} \right) dy = (K_t * g)(x) \,,$$

where the kernel $K_t(x)$ is

$$K_t(x) = ⨍_{B(x,t)} \left(I_\alpha(y) - \sum_{j=0}^{N-1} \frac{1}{L_j} \Delta^j I_\alpha(x) \, |y-x|^{2j} - \frac{1}{L_N} (\Delta^N I_\alpha)_{B(x,t)} \, |y-x|^{2N} \right) dy \,. \tag{23}$$

Hence the square function associated with the smoothness index α is

$$S_\alpha(f)^2(x) = \int_0^\infty |(K_t * g)(x)|^2 \frac{dt}{t^{2\alpha+1}}, \quad x \in \mathbb{R}^n \,.$$

Define an operator T acting on functions $f \in L^2(\mathbb{R}^n)$ by

$$Tg(x) = (K_t * g)(x), \quad x \in \mathbb{R}^n .$$

The identity (21) in subsection 4.2 says that T takes values in $L^2(\mathbb{R}^n, L^2(dt/t^{2\alpha+1}))$ and, more precisely, that

$$\int_{\mathbb{R}^n} \|Tg(x)\|^2_{L^2(dt/t^{2\alpha+1})}\, dx = \|S_\alpha(f)\|_2^2 = c\, \|g\|_2^2 .$$

Therefore T is an operator mapping isometrically (modulo a multiplicative constant) $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n, L^2(dt/t^{2\alpha+1}))$ and we have an explicit expression for its kernel. If we can prove that $K_t(x)$ satisfies Hormander's condition

$$\int_{|x|\geq 2|y|} \|K_t(x-y) - K_t(x)\|_{L^2(dt/t^{2\alpha+1})}\, dx \leq C, \quad y \in \mathbb{R}^n ,$$

then the proof is finished by appealing to a well known result of Benedek, Calderón and Panzone ([GR, Theorem 3.4, p. 492]; see also [GR, p. 507]). In fact, we will show the following stronger version of Hormander's condition

$$\|K_t(x-y) - K_t(x)\|_{L^2(dt/t^{2\alpha+1})} \leq C\, \frac{|y|^\gamma}{|x|^{n+\gamma}}, \quad |x| \geq 2|y| , \tag{24}$$

for some $\gamma > 0$ depending on α and n.

The proof of (24) is lengthy. In the next subsection we will consider the case of small "increments" in t, namely $t < |x|/3$.

4.4 Hormander's condition: $t < |x|/3$

We distinguish two cases: $2N < \alpha < 2N+2$ and $\alpha = 2N$. To deal with the first case we set $K_t(x) = K_t^{(1)}(x) - K_t^{(2)}(x)$, where

$$K_t^{(1)}(x) = \fint_{B(x,t)} \left(I_\alpha(y) - \sum_{j=0}^{N} \frac{1}{L_j}\, \Delta^j I_\alpha(x)\, |y-x|^{2j} \right) dy \tag{25}$$

and

$$K_t^{(2)}(x) = \fint_{B(x,t)} \frac{1}{L_N} \left((\Delta^N I_\alpha)_{B(x,t)} - \Delta^N I_\alpha(x) \right) |y-x|^{2N}\, dy .$$

We first estimate $K_t^{(1)}$. To compute the gradient of $K_t^{(1)}$ we remark that

$$K_t^{(1)}(x) = (I_\alpha * \chi_t)(x) - \sum_{j=0}^{N} \frac{M_j}{L_j}\, t^{2j}\, \Delta^j I_\alpha(x) ,$$

where $M_j = \fint_{B(0,1)} |z|^{2j}\, dz$. Thus

$$\nabla K_t^{(1)}(x) = \fint_{B(x,t)} \left(\nabla I_\alpha(y) - \sum_{j=0}^{N} \frac{1}{L_j} \Delta^j (\nabla I_\alpha)(x)\, |y-x|^{2j} \right) dy.$$

Let $P_m(F,x)$ stand for the Taylor polynomial of degree m of the function F around the point x. Therefore

$$\nabla K_t^{(1)}(x) = \fint_{B(x,t)} \left(\nabla I_\alpha(y) - P_{2N+1}(\nabla I_\alpha, x)(y) \right) dy\,,$$

because the terms which have been added have zero integral on the ball $B(x,t)$, either because they are Taylor homogeneous polynomials of ∇I_α of odd degree or because they are the "zero integral part" of a Taylor homogeneous polynomial of ∇I_α of even degree (see the discussion before the statement of Theorem 3 in the introduction). Given x and y such that $|x| \geq 2|y|$, apply the formula above to estimate $\nabla K_t^{(1)}(z)$ for z in the segment from $x-y$ to y. The standard estimate for the Taylor remainder gives

$$|\nabla K_t^{(1)}(z)| \leq t^{2N+2} \sup_{w \in B(z,t)} |\nabla^{2N+3} I_\alpha(w)|\,.$$

Notice that if $z \in [x-y, y]$, $w \in B(z,t)$ and $t \leq |x|/3$, then $|w| \geq |x|/6$. Now, one has to observe that

$$|\nabla^{2N+3} I_\alpha(w)| \leq C\, |w|^{\alpha-n-2N-3}\,,$$

owing to the fact that possible logarithmic terms do not appear because the exponent $\alpha - n - 2N - 3 < -n-1$ is negative. By the mean value Theorem we then get

$$|K_t^{(1)}(x-y) - K_t^{(1)}(x)| \leq |y| \sup_{z\in[x-y,y]} |\nabla K_t^{(1)}(z)| \leq C\, |y|\, t^{2N+2}\, |x|^{\alpha-n-2N-3}\,.$$

Since

$$\left(\int_0^{|x|/3} t^{2(2N+2)}\, \frac{dt}{t^{2\alpha+1}} \right)^{1/2} = C\, |x|^{2N+2-\alpha}\,,$$

we obtain

$$\left(\int_0^{|x|/3} |K_t^{(1)}(x-y) - K_t^{(1)}(x)|^2\, \frac{dt}{t^{2\alpha+1}} \right)^{1/2} \leq C\, \frac{|y|}{|x|^{n+1}}\,, \tag{26}$$

which is the stronger form of Hormander's condition (24) with $\gamma = 1$ in the domain $t < |x|/3$.

We turn now to estimate $K_t^{(2)}$. Arguing as above we get

$$\nabla K_t^{(2)}(x) = C \fint_{B(x,t)} \left(\Delta^N(\nabla I_\alpha)(y) - P_1(\Delta^N(\nabla I_\alpha), x)(y) \right) |y-x|^{2N}\, dy\,.$$

If $z \in [x-y, y]$, then

$$|\nabla K_t^{(2)}(z)| \leq t^{2N+2} \sup_{w\in B(z,t)} |\nabla^{2N+3} I_\alpha(w)| \leq C\, t^{2N+2}\, |x|^{\alpha-n-2N-3}\,,$$

and so we get (26) with $K_t^{(1)}$ replaced by $K_t^{(2)}$ exactly as before.

Let us consider now the case $\alpha = 2N$. Since $\Delta^N I_{2N}$ is the Dirac delta at 0, $\Delta^N I_{2N}(x) = 0$. Hence $K_t(x) = K_t^{(1)}(x) - K_t^{(2)}(x)$, where $K_t^{(1)}$ is given by (25) with α replaced by $2N$ and

$$K_t^{(2)}(x) = \fint_{B(x,t)} \frac{1}{L_N}(\,(\Delta^N I_{2N})_{B(x,t)}\,|y-x|^{2N}\,dy = \frac{M_N}{L_N}\,t^{2N}\,(\Delta^N I_{2N})_{B(x,t)}\,.$$

The kernel $K_t^{(1)}$ is estimated as in the first case by just setting $\alpha = 2N$. The kernel $K_t^{(2)}$ requires a different argument.

Set

$$\chi(x) = \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x)$$

and

$$\chi_t(x) = \frac{1}{t^n}\chi(\frac{x}{t})\,,$$

so that, since $\Delta^N I_{2N}$ is the Dirac delta at the origin, $K_t^{(2)}$ is a constant multiple of $t^{2N}\chi_t$. We show now that this kernel satisfies the strong form of Hormander's condition. The quantity $|\chi_t(x-y) - \chi_t(x)|$ is non-zero only if $|x-y| < t < |x|$ or $|x| < t < |x-y|$, in which cases takes the value $1/c_n\,t^n$, $c_n = |B(0,1)|$. On the other hand, if $|x| \ge 2|y|$ then each z in the segment joining x and $x-y$ satisfies $|z| \ge |x|/2$. Assume that $|x-y| < |x|$ (the case $|x| < |x-y|$ is similar). Then

$$\begin{aligned}\left(\int_0^\infty (t^{2N}\,(\chi_t(x-y) - \chi_t(x)))^2\,\frac{dt}{t^{4N+1}}\right)^{\frac12} &= C\left(\int_{|x-y|}^{|x|}\frac{dt}{t^{2n+1}}\right)^{\frac12}\\ &= C\left(\frac{1}{|x-y|^{2n}} - \frac{1}{|x|^{2n}}\right)^{\frac12} \le C\,\frac{|y|^{1/2}}{|x|^{n+1/2}}\,,\end{aligned}$$

which is (24) with $\gamma = 1/2$.

4.5 Hormander's condition: $t \ge |x|/3$

The last term in the definition (23) of the kernel K_t is of the form

$$-\fint_{B(x,t)} \frac{1}{L_N}(\Delta^N I_\alpha)_{B(x,t)}\,|y-x|^{2N}\,dy = -\frac{M_N}{L_N}\,t^{2N}\,(\Delta^N I_\alpha)_{B(x,t)}\,,$$

and our first goal is to show that this kernel satisfies Hormander's condition in the domain $t \ge |x|/3$. Notice that

$$|\Delta^N I_\alpha(x)| \simeq \frac{1}{|x|^{n-\alpha+2N}}\,,$$

provided no logarithmic factors appear, which is the case if $\alpha - 2N < n$. Since $\alpha - 2N < 2$ the inequality $\alpha - 2N < n$ certainly holds whenever $n \ge 2$. Hence the only cases with logarithmic factors are $n = 1$ and $\alpha = 2N + 1$, which will be dealt with separately. In the previous subsection we dealt with the case $\alpha = 2N$ and so we can assume that

$2N < \alpha < 2N+2$, which implies that $\Delta^N I_\alpha$ is locally integrable. Given x and y with $|x| \geq 2|y|$ let D stand for the symmetric difference $(B(x-y,t)\setminus B(x,t)) \cup (B(x,t)\setminus B(x-y,t))$. It is easy to realize that $|D| \leq C\,|y|\,t^{n-1}$. We remind the reader of the following well known and easy to prove inequality.

Lemma. *Let E be a measurable subset of $\mathbb{R}^n$ and $0 < \beta < n$. Then*

$$\int_E \frac{dz}{|z|^{n-\beta}} \leq C\,|E|^{\beta/n},$$

where $|E|$ is the Lebesgue measure of E.

We have

$$t^{2N}\,|(\Delta^N I_\alpha)_{B(x-y,\,t)} - (\Delta^N I_\alpha)_{B(x,\,t)}| \leq C\,t^{2N-n} \int_D \frac{dz}{|z|^{n-\alpha+2N}}$$

$$\leq C\,t^{2N-n}\,(|y|\,t^{n-1})^{(\alpha-2N)/n} = |y|^{(\alpha-2N)/n}\,t^{\alpha-n-(\alpha-2N)/n}$$

and

$$\left(\int_{|x|/3}^{\infty} t^{2(\alpha-n-(\alpha-2N)/n)}\,\frac{dt}{t^{2\alpha+1}}\right)^{1/2} \simeq |x|^{-n-(\alpha-2N)/n}.$$

Combining the preceding two inequalities one gets Hormander's condition (24) with $\gamma = (\alpha-2N)/n$ in the domain $t \geq |x|/3$.

We have to investigate now the exceptional cases $n = 1$ and $\alpha = 2N+1$, in which no logarithmic factors appear. We have

$$I_\alpha(x) = |x|^{2N}\,(A + B\log|x|),$$

for some constants A and $B \neq 0$. Then

$$\Delta^N I_\alpha(x) = (A' + B'\log|x|)$$

and

$$\frac{d}{dx}\Delta^N I_\alpha(x) = C\,v.p.\frac{1}{x},$$

in the distributions sense. Thus, denoting by H the Hilbert transform and applying Schwarz inequality,

$$t^{2N}\,|(\Delta^N I_\alpha)_{B(x-y,\,t)} - (\Delta^N I_\alpha)_{B(x,\,t)}| = t^{2N}\,|\int_{x-y}^{x} H(\chi_t)(\tau)\,d\tau|$$

$$\leq t^{2N}\,|y|^{1/2}\,\|\chi_t\|_2 = t^{2N-1/2}\,|y|^{1/2}.$$

Since

$$\left(\int_{|x|/3}^{\infty} t^{2(2N-1/2)}\frac{dt}{t^{2\alpha+1}}\right)^{1/2} = C\,|x|^{2N-1/2-\alpha} = C\,|x|^{-1-1/2},$$

we obtain Hormander's condition (24) with $\gamma = 1/2$ in the domain $t \geq |x|/3$.

Our plan is to continue the proof by distinguishing three cases: $\alpha < n+1$, $\alpha = n+1$ and $\alpha > n+1$.

If $\alpha < n+1$, then all terms in the expression (23) defining K_t satisfy Hormander's condition in the domain $t \geq |x|/3$. Indeed, consider first the terms of the form $t^{2j}\,\Delta^j I_\alpha(x)$, $j \geq 0$. One has the gradient estimate

$$|t^{2j}\,\nabla\Delta^j I_\alpha(x)| \leq C\,t^{2j}\,|x|^{\alpha-n-2j-1}\,, \tag{27}$$

because no logarithmic factors appear, the reason being that the exponent $\alpha - n - 2j - 1 \leq \alpha - (n+1)$ is negative. Since

$$\left(\int_{|x|/3}^{\infty} t^{2(2j)}\,\frac{dt}{t^{2\alpha+1}}\right)^{1/2} = C\,|x|^{2j-\alpha}\,,$$

we get Hormander's condition with $\gamma = 1$ in the domain $t \geq |x|/3$.

It remains to look at the first term $\fint_{B(x,t)} I_\alpha(y)\,dy$ in (23). Its gradient can be easily estimated as follows

$$\left|\fint_{B(x,t)} \nabla I_\alpha(y)\,dy\right| \leq C \fint_{B(x,t)} |y|^{\alpha-n-1}\,dy\,.$$

Notice that there are no logarithmic factors precisely because $\alpha < n+1$. The integrand in the last integral is locally integrable if and only if $\alpha > 1$. Assume for the moment that $1 < \alpha < n+1$. Then

$$\left|\fint_{B(x,t)} \nabla I_\alpha(y)\,dy\right| \leq C\,t^{-n+\alpha-1}\,.$$

Since

$$\left(\int_{|x|/3}^{\infty} t^{2(\alpha-n-1)}\,\frac{dt}{t^{2\alpha+1}}\right)^{1/2} = C\,|x|^{-n-1}\,,$$

we get Hormander's condition with $\gamma = 1$ in the domain $t \geq |x|/3$. The case $\alpha = 1$ has been treated in section 1, so we can assume that $0 < \alpha < 1$. In this case, denoting by D the symmetric difference between $B(x,t)$ and $B(x-y,t)$, we obtain

$$\begin{aligned} |\fint_{B(x-y,t)} I_\alpha(y)\,dy - \fint_{B(x,t)} I_\alpha(y)\,dy| &\leq C\,t^{-n} \int_D \frac{dy}{|y|^{n-\alpha}} \\ &\leq C\,t^{-n}\,(t^{n-1}\,|y|)^{\alpha/n} = C\,t^{\alpha-n-\alpha/n}\,|y|^{\alpha/n}\,. \end{aligned}$$

Since

$$\left(\int_{|x|/3}^{\infty} t^{2(\alpha-n-\alpha/n)}\,\frac{dt}{t^{2\alpha+1}}\right)^{1/2} = C\,|x|^{-n-\alpha/n}\,,$$

we get Hormander's condition with $\gamma = \alpha/n$ in the domain $t \geq |x|/3$.

We tackle now the case $\alpha = n+1$. Since α and n are integers with different parity no logarithmic factor will appear in I_α. Thus $I_\alpha(x) = C\,|x|$. The proof above shows that the terms $t^{2j}\,\Delta^j I_\alpha(x)$ appearing in the expression (23) of the kernel K_t still work for $j \geq 1$. The remaining term is

$$\fint_{B(x,\,t)} (I_\alpha(y) - I_\alpha(x))\ dy$$

and its gradient is estimated by remarking that the function $|x|$ satisfies a Lipschitz condition. We obtain

$$\left| \fint_{B(x,\,t)} (\nabla I_\alpha(y) - \nabla I_\alpha(x))\ dy \right| \leq C\,.$$

But clearly

$$\left(\int_{|x|/3}^{\infty} \frac{dt}{t^{2\alpha+1}} \right)^{1/2} = C\,|x|^{-\alpha} = C\,|x|^{-n-1}\,,$$

which completes the argument.

We turn our attention to the case $\alpha > n+1$. There is a unique positive integer M such that $-1 < \alpha - n - 2M < 1$. The part of K_t which has to be estimated is

$$H_t(x) = \fint_{B(x,\,t)} \left(I_\alpha(y) - \sum_{j=0}^{N-1} \frac{1}{L_j}\,\Delta^j I_\alpha(x)\,|y-x|^{2j} \right) dy\,.$$

We split H_t into two terms according to M, that is, $H_t = H_t^{(1)} - H_t^{(2)}$, where

$$H_t^{(1)}(x) = \fint_{B(x,\,t)} \left(I_\alpha(y) - \sum_{j=0}^{M-1} \frac{1}{L_j}\,\Delta^j I_\alpha(x)\,|y-x|^{2j} \right) dy$$

and

$$H_t^{(2)}(x) = \fint_{B(x,\,t)} \left(\sum_{j=M}^{N-1} \frac{1}{L_j}\,\Delta^j I_\alpha(x)\,|y-x|^{2j} \right) dy\,.$$

The estimate of each of the terms in $H_t^{(2)}$ is performed as we did for the case $\alpha < n+1$. The gradient estimate is exactly (27). Now no logarithmic factors appear because the exponent $\alpha - n - 2j - 1 \leq \alpha - n - 2M - 1 < 0$ is negative. The rest is as before.

To estimate $H_t^{(1)}$ we distinguish three cases: $-1 < \alpha - n - 2M < 0$, $0 < \alpha - n - 2M < 1$ and $\alpha - n - 2M = 0$. In the first case we write the gradient of $H_t^{(1)}$ as

$$\begin{aligned}\nabla H_t^{(1)}(x) &= \fint_{B(x,\,t)} \left(\nabla I_\alpha(y) - \sum_{j=0}^{M-1} \frac{1}{L_j}\,\Delta^j (\nabla I_\alpha)(x)\,|y-x|^{2j} \right) dy \\ &= \fint_{B(x,\,t)} (\nabla I_\alpha(y) - P_{2M-2}(\nabla I_\alpha, x)(y)\,)\ dy\,,\end{aligned}$$

where P_{2M-2} is the Taylor polynomial of degree $2M-2$ of ∇I_α around the point x. As before, the added terms have zero integral on $B(x,t)$ either because they are homogeneous

Taylor polynomials of odd degree or the "zero integral part" of homogeneous Taylor polynomials of even degree. Now fix y in $B(x,t)$ but not in the half line issuing from x and passing through the origin. Define a function g on the interval $[0,1]$ by

$$g(\tau) = \nabla I_\alpha(x+\tau(y-x)) - P_{2M-2}(\nabla I_\alpha, x)(x+\tau(y-x)), \quad 0 \le \tau \le 1\,.$$

Since $g^{j)}(0) = 0$, $0 \le j \le 2M-2$,

$$\begin{aligned}\nabla I_\alpha(y) - P_{2M-2}(\nabla I_\alpha, x)(y) &= g(1) - \sum_{j=0}^{2M-2} \frac{g^{j)}(0)}{j!} \\ &= \int_0^1 \frac{(1-\tau)^{2M-2}}{(2M-2)!}\, g^{2M-1)}(\tau)\, d\tau\,,\end{aligned}$$

by the integral form of Taylor's remainder. The obvious estimate for the derivative of g of order $2M-1$ is

$$|g^{2M-1)}(\tau)| \le |\nabla^{2M-1}\nabla I_\alpha(x+\tau(y-x))||y-x|^{2M-1} \le C\, \frac{t^{2M-1}}{|x+\tau(y-x)|^{n-(\alpha-2M)}}\,.$$

Since we are in the first case α is not integer and thus no logarithmic factors exist. Moreover $0 < n-(\alpha-2M) < 1$, which implies that and that $1/|z|^{n-(\alpha-2M)}$ is locally integrable in any dimension. Therefore

$$\begin{aligned}|\nabla H_t^{(1)}(x)| &\le C \int_0^1 \left(t^{2M-1-n} \int_{B(x,t)} \frac{dy}{|x+\tau(y-x)|^{n-(\alpha-2M)}} \right) d\tau\,, \\ &= C\, t^{2M-1-n} \int_0^1 \left(\int_{B(x,t\,\tau)} \frac{dz}{|z|^{n-(\alpha-2M)}} \right) \frac{d\tau}{\tau^n} \\ &\le C\, t^{2M-1-n} \int_0^1 (t\,\tau)^{\alpha-2M} \frac{d\tau}{\tau^n} \\ &= t^{\alpha-n-1} \int_0^1 \frac{d\tau}{\tau^{n-(\alpha-2M)}} = C\, t^{\alpha-n-1}\,.\end{aligned}$$

Since

$$\left(\int_{|x|/3}^\infty t^{2(\alpha-n-1)} \frac{dt}{t^{2\alpha+1}} \right)^{1/2} = C\,|x|^{-n-1}\,,$$

we get Hormander's condition with $\gamma = 1$ in the domain $t \ge |x|/3$.

Let us consider the second case: $0 < \alpha-n-2M < 1$. This time we express the gradient of $H_t^{(1)}$ by means of a Taylor polynomial of degree $2M-1$:

$$\begin{aligned}\nabla H_t^{(1)}(x) &= \fint_{B(x,\,t)} \left(\nabla I_\alpha(y) - \sum_{j=0}^{M-1} \frac{1}{L_j}\, \Delta^j(\nabla I_\alpha)(x)\, |y-x|^{2j} \right) dy \\ &= \fint_{B(x,\,t)} (\nabla I_\alpha(y) - P_{2M-1}(\nabla I_\alpha, x)(y)\,)\; dy\,.\end{aligned}$$

Using again the integral form of the Taylor remainder of the function g, with P_{2M-2} replaced by P_{2M-1}, we obtain

$$
\begin{aligned}
|\nabla H_t^{(1)}(x)| &\le C \int_0^1 \left(t^{2M-n} \int_{B(x,t)} \frac{dy}{|x+\tau(y-x)|^{n-(\alpha-2M-1)}} \right) d\tau\,, \\
&= C\, t^{2M-n} \int_0^1 \left(\int_{B(x,t\,\tau)} \frac{dz}{|z\,|^{n-(\alpha-2M-1)}} \right) \frac{d\tau}{\tau^n} \\
&\le C\, t^{2M-n} \int_0^1 (t\,\tau)^{\alpha-2M-1} \frac{d\tau}{\tau^n} \\
&= t^{\alpha-n-1} \int_0^1 \frac{d\tau}{\tau^{n-(\alpha-2M-1)}} = C\, t^{\alpha-n-1}\,,
\end{aligned}
$$

from which we get the desired estimate as before.

We turn now to the last case left, $\alpha = n + 2M$, with M a positive integer. In this case

$$
I_\alpha(x) = C\,|x|^{2M}\,(A + B \log|x\,|), \quad x \in \mathbb{R}^n, \quad B \neq 0\,,
$$

where A, B and C are constants depending on n and M. We also have

$$
\Delta^{M-1} I_\alpha(x) = C\,|x|^2\,(A_1 + B_1 \log|x\,|), \quad x \in \mathbb{R}^n
$$

and

$$
\nabla\Delta^{M-1} I_\alpha(x) = C\,x\,(A_2 + B_2 \log|x\,|), \quad x \in \mathbb{R}^n\,.
$$

In particular $\nabla\Delta^{M-1} I_\alpha$ is in the Zygmund class on $\mathbb{R}^n$. We have

$$
\begin{aligned}
\nabla H_t^{(1)}(x) &= \fint_{B(x,t)} \left(\nabla I_\alpha(y) - \sum_{j=0}^{M-2} \frac{1}{L_j} \Delta^j (\nabla I_\alpha)(x)\,|y-x|^{2j} - \frac{1}{L_{M-1}} \Delta^{M-1}(\nabla I_\alpha)(x)\,|y-x|^{2M-2} \right) dy \\
&= \fint_{B(x,t)} \left(\nabla I_\alpha(y) - P_{2M-3}(\nabla I_\alpha, x)(y) - \frac{1}{L_{M-1}} \Delta^{M-1}(\nabla I_\alpha)(x)\,|y-x|^{2M-2} \right) dy\,.
\end{aligned}
$$

Introduce the function g as above, with P_{2M-2} replaced by P_{2M-3}, so that

$$
\begin{aligned}
\nabla I_\alpha(y) - P_{2M-3}(\nabla I_\alpha, x)(y) &= g(1) - \sum_{j=0}^{2M-3} \frac{g^{j)}(0)}{j!} \\
&= \int_0^1 (2M-2)(1-\tau)^{2M-3} \frac{g^{2M-2)}(\tau)}{(2M-2)!}\, d\tau\,.
\end{aligned}
$$

Now

$$\frac{g^{2M-2)}(\tau)}{(2M-2)!} = \sum_{|\beta|=2M-2} \left(\frac{\partial^\beta \nabla I_\alpha(x+\tau(y-x))}{\beta!}\right)(y-x)^\beta$$

$$= \sum_{|\beta|=2M-2} \left(\partial^\beta \nabla I_\alpha(x+\tau(y-x)) - \partial^\beta \nabla I_\alpha(x)\right) \frac{(y-x)^\beta}{\beta!}$$

$$+ \sum_{|\beta|=2M-2} \left(\frac{\partial^\beta \nabla I_\alpha(x)}{\beta!}\right)(y-x)^\beta .$$

The last term in the preceding equation is the homogeneous Taylor polynomial of degree $2M-2$ of the vector ∇I_α around the point x. It is then equal to a homogeneous polynomial of the same degree with zero integral on $B(x,t)$ plus $\frac{1}{L_{M-1}} \Delta^{M-1}(\nabla I_\alpha)(x)\,|y-x|^{2M-2}$ (by the discussion before the statement of Theorem 3 in the introduction). Hence

$$\int_{B(x,t)} \left(\sum_{|\beta|=2M-2} \left(\frac{\partial^\beta \nabla I_\alpha(x)}{\beta!}\right)(y-x)^\beta - \frac{1}{L_{M-1}} \Delta^{M-1}(\nabla I_\alpha)(x)\,|y-x|^{2M-2} \right) dy = 0\,,$$

and therefore, remarking that $\int_0^1 (2M-2)(1-\tau)^{2M-3}\,d\tau = 1$,

$$\nabla H_t^{(1)}(x)$$
$$= \fint_{B(x,t)} \int_0^1 (2M-2)(1-\tau)^{2M-3} \left(\sum_{|\beta|=2M-2} \left(\partial^\beta \nabla I_\alpha(x+\tau(y-x)) - \partial^\beta \nabla I_\alpha(x)\right) \frac{(y-x)^\beta}{\beta!} \right) d\tau\, dy\,.$$

Thus

$$|\nabla H_t^{(1)}(x)| \le C \int_0^1 \sum_{|\beta|=2M-2} \left| \fint_{B(x,t)} \left(\partial^\beta \nabla I_\alpha(x+\tau(y-x)) - \partial^\beta \nabla I_\alpha(x)\right) \frac{(y-x)^\beta}{\beta!}\, dy \right| d\tau\,.$$

Making the change of variables $h = \tau(y-x)$ the integral in y above becomes

$$J = \tau^{-|\beta|} \fint_{B(0,t\tau)} \left(\partial^\beta \nabla I_\alpha(x+h) - \partial^\beta \nabla I_\alpha(x)\right) \frac{h^\beta}{\beta!}\, dh\,,$$

which is invariant under the change of variables $h' = -h$, because $|\beta|$ is even. Hence

$$2J = \tau^{-|\beta|} \fint_{B(0,t\tau)} \left(\partial^\beta \nabla I_\alpha(x+h) + \partial^\beta \nabla I_\alpha(x-h) - 2\,\partial^\beta \nabla I_\alpha(x)\right) \frac{h^\beta}{\beta!}\, dh\,.$$

Now we claim that $\partial^\beta \nabla I_\alpha$ is in the Zygmund class for $|\beta| = 2M-2$. This follows from the fact that the Zygmund class in invariant under homogeneous smooth Calderón -Zygmund operators, Δ^{M-1} is an elliptic operator and $\Delta^{M-1}\nabla I_\alpha$ is in the Zygmund class. Hence

$$|J| \le C\,\tau^{-|\beta|} \fint_{B(0,t\tau)} |h|^{1+|\beta|}\, dh \le C\, t^{2M-1}\,\tau\,.$$

Thus

$$|\nabla H_t^{(1)}(x)| \leq C\, t^{2M-1}\,.$$

Since

$$\left(\int_{|x|/3}^{\infty} t^{2(2M-1)} \frac{dt}{t^{2\alpha+1}}\right)^{1/2} = C\,|x|^{-n-1}\,,$$

we get Hormander's condition with $\gamma = 1$ in the domain $t \geq |x|/3$.

4.6 The sufficient condition

In this section we prove that condition (2) in Theorem 3 is sufficient for $f \in W^{\alpha,p}$. Let $f, g_1, \ldots, g_N \in L^p$ satisfy $S_\alpha(f, g_1, \ldots, g_N) \in L^p$. Take an infinitely differentiable function $\phi \geq 0$ with compact support in $B(0,1)$, $\int \phi = 1$ and set $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon}),\ \epsilon > 0$. Consider the regularized functions $f_\epsilon = f * \phi_\epsilon$, $g_{j,\epsilon} = g_j * \phi_\epsilon$, $1 \leq j \leq N$. We want to show first that the infinitely differentiable function f_ϵ is in $W^{\alpha,p}$. We have $(-\Delta)^{\alpha/2} f_\epsilon = f * (-\Delta)^{\alpha/2}\phi_\epsilon$. We need a lemma.

Lemma 2.

(i) If φ is a function in the Scharwtz class and α any positive number, then $(-\Delta)^{\alpha/2}\varphi$ belongs to all L^q spaces , $1 \leq q \leq \infty$.

(ii) If $f \in L^p, 1 \leq p \leq \infty$, then $(-\Delta)^{\alpha/2} f$ is a tempered distribution.

Proof. Set $\psi = (-\Delta)^{\alpha/2}\varphi$. If $\alpha = 2m$ with m a positive integer, then $\psi = (-\Delta)^m\varphi$ is in the Scharwtz class and so the conclusion in (i) follows. If $\alpha = 2m + 1$, then

$$\psi = (-\Delta)^{1/2}(-\Delta)^m\varphi = -\imath \sum_{j=1}^{n} R_j(\,\partial_j\,(-\Delta)^m\varphi\,)\,,$$

where R_j are the Riesz transforms, that is, the Calderón-Zygmund operators whose Fourier multiplier is $\xi_j/|\xi|$. It is clear from the formula above that ψ is infinitely differentiable on $\mathbb{R}^n$ and so the integrability issue is only at ∞. Since $\partial_j\,(-\Delta)^m\varphi\,)$, has zero integral, one has, as $x \to \infty$, $|\psi(x)| \leq C\,|x|^{-n-1}$, and so the conclusion follows.

Assume now that $m - 1 < \alpha < m$, for some positive integer m. Thus

$$\hat{\psi}(\xi) = |\xi|^\alpha \hat{\varphi}(\xi) = |\xi|^m \hat{\varphi}(\xi)\frac{1}{|\xi|^{m-\alpha}}\,.$$

If m is even, of the form $m = 2M$ for some positive integer M, then

$$\psi = \Delta^M\varphi * I_{m-\alpha}\,,$$

where $I_{m-\alpha}(x) = C\,|x|^{m-\alpha-n}$. Hence ψ is infinitely differentiable on $\mathbb{R}^n$. Since $\Delta^M\varphi$ has zero integral, $|\psi(x)| \leq C\,|x|^{m-\alpha-n-1}$, as $x \to \infty$. But $\alpha - m + 1 > 0$ and thus ψ is in all L^q spaces.

If m is odd, of the form $m = 2M + 1$ for some non-negative integer M, then

$$\psi = -\imath \sum_{j=1}^{n} R_j(\partial_j \Delta^M \varphi) * I_{m-\alpha} .$$

Again ψ is infinitely differentiable on $\mathbb{R}^n$ and, since $R_j(\partial_j \Delta^M \varphi)$ has zero integral (just look at the Fourier transform and remark that it vanishes at the origin), we get $|\psi(x)| \leq C\,|x|^{m-\alpha-n-1}$, as $x \to \infty$, which completes the proof of (i).

To prove (ii) take a function φ in the Schwartz class. Let q be the exponent conjugate to p. Define the action of $(-\Delta)^{\alpha/2} f$ on the Schwartz function φ as $\langle f, (-\Delta)^{\alpha/2}\varphi\rangle$. By part (i) and Hölder's inequality one has

$$|\langle(-\Delta)^{\alpha/2} f, \varphi\rangle| = |\langle f, (-\Delta)^{\alpha/2}\varphi\rangle| \leq C\,\|f\|_p\,\|(-\Delta)^{\alpha/2}\varphi\|_q ,$$

which completes the proof of (ii). □

Let us continue the proof of the sufficiency of condition (2). By the lemma $(-\Delta)^{\alpha/2}\phi_\epsilon$ is in L^1 and so

$$\|(-\Delta)^{\alpha/2} f_\epsilon\|_p = \|f * (-\Delta)^{\alpha/2}\phi_\epsilon\|_p \leq \|f\|_p\,\|(-\Delta)^{\alpha/2}\phi_\epsilon\|_1 .$$

Hence $f_\epsilon \in W^{\alpha,p}$.

Next, we claim that

$$S_\alpha(f_\epsilon, g_{1,\epsilon}, \ldots, g_{N,\epsilon})(x) \leq (S_\alpha(f, g_1, \ldots, g_N) * \phi_\epsilon)(x), \quad x \in \mathbb{R}^n . \tag{28}$$

One has

$$S_\alpha(f, g_1, \ldots, g_N)(x) = \|(f * \chi_t)(x) - f(x) - \sum_{j=1}^{N-1} M_j\, g_j(x)\, t^{2j} - M_N\,(g_N * \chi_t)(x)\, t^{2N}\|_{L^2(dt/t^{2\alpha+1})} ,$$

where $M_j = \fint_{B(0,1)} |z|^{2j}\, dz$. Minkowsky's integral inequality now readily yields (28).

Set

$$D_\epsilon(x) = \|\sum_{j=1}^{N-1} M_j\,(\frac{\Delta^j f_\epsilon(x)}{L_j} - g_{j,\epsilon}(x))\, t^{2j} - M_N \left((\frac{\Delta^N f_\epsilon}{L_N} - g_{N,\epsilon}) * \chi_t\right)(x)\, t^{2N}\|_{L^2(dt/t^{2\alpha+1})} .$$

By (28)

$$\begin{aligned} D_\epsilon(x) &\leq S_\alpha(f_\epsilon)(x) + S_\alpha(f_\epsilon, g_{1,\epsilon}, \ldots, g_{N,\epsilon})(x) \\ &\leq S_\alpha(f_\epsilon)(x) + (S_\alpha(f, g_1, \ldots, g_N) * \phi_\epsilon)(x) , \end{aligned}$$

and so $D_\epsilon \in L^p$. In particular, $D_\epsilon(x)$ is finite for almost all $x \in \mathbb{R}^n$. Thus

$$\liminf_{t\to 0} \left|\sum_{j=1}^{N-1} M_j\,(\frac{\Delta^j f_\epsilon(x)}{L_j} - g_{j,\epsilon}(x))\, t^{2j} - M_N \left((\frac{\Delta^N f_\epsilon}{L_N} - g_{N,\epsilon}) * \chi_t\right)(x)\, t^{2N}\right| t^{-\alpha} = 0 ,$$

for almost all $x \in \mathbb{R}^n$. It is easy to conclude that the only way this may happen is whenever

$$\frac{\Delta^j f_\epsilon(x)}{L_j} = g_{j,\epsilon}(x), \quad 1 \le j \le N\,,$$

for almost all $x \in \mathbb{R}^n$. Hence

$$\frac{\Delta^j f_\epsilon}{L_j} \to g_j, \quad 1 \le j \le N\,,$$

in L^p as $\epsilon \to 0$. Since $f_\epsilon \to f$ in L^p as $\epsilon \to 0$,

$$\Delta^j f_\epsilon \to \Delta^j f, \quad 1 \le j \le N\,,$$

in the weak topology of tempered distributions. Hence

$$\frac{\Delta^j f}{L_j} = g_j, \quad 1 \le j \le N\,.$$

We claim now that the functions f_ϵ are uniformly bounded in $W^{\alpha,p}$. Indeed, by the proof of necessity of condition (2) and by (28),

$$\begin{aligned} \|(-\Delta)^{\alpha/2} f_\epsilon\|_p &\simeq \|S_\alpha(f_\epsilon, \Delta f_\epsilon/L_1, \ldots, \Delta^N f_\epsilon/L_N)\|_p \\ &\le \|S_\alpha(f, \Delta f/L_1, \ldots, \Delta^N f/L_N)\|_p < \infty. \end{aligned}$$

Hence there exist a function $h \in L^p$ and a sequence $\epsilon_j \to 0$ as $j \to \infty$ such that

$$(-\Delta)^{\alpha/2} f_{\epsilon_j} \to h \quad \text{as} \quad j \to \infty$$

in the weak $\star$ topology of L^p. On the other hand, by Lemma 2, $(-\Delta)^{\alpha/2} f$ is a tempered distribution and so

$$(-\Delta)^{\alpha/2} f_\epsilon \to (-\Delta)^{\alpha/2} f \quad \text{as} \quad \epsilon \to 0$$

in the weak topology of tempered distributions. Therefore $(-\Delta)^{\alpha/2} f = h \in L^p$ and the proof is complete.

5 Final remarks

Let (X, d, μ) be a metric measure space, that is, X is a metric space with distance d and μ is a Borel measure on X. We assume that the support of μ is X. Then, given $\alpha > 0$ and $1 < p < \infty$, we can define the Sobolev space $W^{\alpha,p}(X)$ as follows. Let N be the unique integer such that $2N \le \alpha < 2N + 2$. Given locally integrable functions $f, g_1, \ldots, g_N$ define a square function by

$$S_\alpha(f, g_1, g_2, \ldots, g_N)(x)^2 = \int_0^D \left| \fint_{B(x,t)} \frac{R_N(y,x)}{t^\alpha}\, d\mu(y) \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n\,,$$

where D is the diameter of X and $R_N(y, x)$ is

$$R_N(y, x) = f(y) - f(x) - g_1(x)\, d(y, x)^2 + \cdots - g_{N-1}(x)\, d(y, x)^{2(N-1)} - (g_N)_{B(x,t)} d(y, x)^{2N} .$$

Here the barred integral stands for the mean with respect to μ on the indicated set, $B(x, t)$ is the open ball with center x and radius t and $g_{B(x,t)}$ is the mean of the function g on $B(x, t)$.

We say that a function f belongs to the Sobolev space $W^{\alpha,p}(X)$ provided $f \in L^p(\mu)$ and there exist functions $g_1, g_2, \ldots, g_N \in L^p(\mu)$ such that $S_\alpha(f, g_1, g_2, \ldots, g_N) \in L^p(\mu)$.

We have seen in the previous sections that this definition yields the usual Sobolev spaces if $X = \mathbb{R}^n$ is endowed with the Euclidean distance and μ is Lebesgue measure. One can prove with some effort that the same is true if $\mathbb{R}^n$ is replaced by a half-space. Very likely this should also work for smoothly bounded domains, but we have not gone that far.

There are many interesting questions one may ask about these new Sobolev spaces. For instance, how do they compare, for $\alpha = 1$, to the known first order Sobolev spaces, notably those introduced by Hajlasz in [H] or the Newtonian spaces of [S] ? For higher orders of smoothness one would like to compare them with those introduced by Liu, Lu and Wheeden in [LLW]. One may also wonder about their intrinsic properties, namely, about versions of the Sobolev imbedding theorem, the Poincaré inequality and so on.

For the Sobolev imbedding theorem the following remark might be useful. In $\mathbb{R}^n$ the L^p space can be characterized by means of the following "zero smoothness" square function:

$$S_0(f)^2(x) = \int_0^\infty \left| f_{B(x,t)} - f_{B(x,2t)} \right|^2 \frac{dt}{t}, \quad x \in \mathbb{R}^n .$$

The result is then that a locally integrable function f is in L^p if and only if $S_0(f) \in L^p$. The proof follows the pattern described several times in this paper. One first deals with the case $p = 2$ via a Fourier transform computation. Then one introduces a $L^2(dt/t)$-valued operator T such that

$$\|T(f)\|_{L^2(\mathbb{R}^n, L^2(dt/t))} = c\, \|S_0(f)\|_2$$

and one shows that its kernel satisfies Hormander's condition.

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