A MARSTRAND THEOREM FOR SUBSETS OF INTEGERS

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ABSTRACT. We prove a Marstrand type theorem for a class of subsets of the integers. More specifically, after defining the counting dimension $D(E)$ of $E \subset \mathbb{Z}$ and the concepts of regularity and compatibility, we show that if $E, F \subset \mathbb{Z}$ are two regular compatible sets, then $D(E + \lfloor \lambda F \rfloor) \geq \min\{1, D(E) + D(F)\}$ for Lebesgue almost every $\lambda \in \mathbb{R}$. If in addition $D(E) + D(F) > 1$, then $E + \lfloor \lambda F \rfloor$ has positive upper-Banach density for Lebesgue almost every $\lambda \in \mathbb{R}$. The result has direct consequences when applied to arithmetic sets, such as the integer values of a polynomial with integer coefficients.

1. INTRODUCTION

The purpose of this paper is to prove a Marstrand type theorem for a class of subsets of the integers.

The well-known theorem of Marstrand [13] states the following: if $K \subseteq \mathbb{R}^2$ is a Borel set such that its Hausdorff dimension is greater than one, then, for almost every direction, its projection to $\mathbb{R}$ in the respective direction has positive Lebesgue measure. In other words, this means K is "fat" in almost every direction. When K is the product of two real subsets K_1, K_2, Marstrand's theorem can be stated in a more analytical form as: for Lebesgue almost every $\lambda \in \mathbb{R}$, the arithmetic sum $K_1 + \lambda K_2$ has positive Lebesgue measure. Many research is being made around this topic, mainly because the analysis of such arithmetic sums has applications in the theory of Homoclinic Birfurcations and also in Diophantine Approximations.

Given a subset $E \subset \mathbb{Z}$, let $d^*(E)$ denote its *upper-Banach density*[1]. A remarkable result in additive combinatorics is Szemerédi's theorem [16]. It asserts that if $d^*(E) > 0$, then E contains arbitrarily long arithmetics progressions. One can interpret this result by saying that density represents the correct notion of largeness needed to preserve finite configurations of $\mathbb{Z}$. On the other hand, Szemerédi's theorem cannot infer any property of subsets of zero upper-Banach density, and many of these sets are important. A class of examples are the integer values of a polynomial with integer coefficients and the prime numbers. These sets may, as well, contain combinatorially rich patterns. See, for example, references [3] and [9].

A set $E \subset \mathbb{Z}$ of zero upper-Banach density is characterized as occupying portions in intervals of $\mathbb{Z}$ that grow in a sublinear fashion as the length of the intervals grow. On the other hand, there still may exist some kind of growth speed. For example, the number of perfect squares on a interval of length n is about $n^{0.5}$. This exponent means, in some sense, a dimension of $\{n^2 ; n \in \mathbb{Z}\}$ inside $\mathbb{Z}$. In this article, we suggest a *counting dimension* $D(E)$ for E and establish the following Marstrand

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[1]See Section 2.2 for the proper definitions.

type result on the counting dimension of most arithmetic sums $E+\lfloor\lambda F\rfloor$ of a class of subsets $E,F\subset\mathbb{Z}$.

Theorem 1.1. *Let $E,F\subset\mathbb{Z}$ be two regular compatible sets. Then*

$$D(E+\lfloor\lambda F\rfloor)\geq\min\{1,D(E)+D(F)\}$$

for Lebesgue almost every $\lambda\in\mathbb{R}$. If in addition $D(E)+D(F)>1$, then $E+\lfloor\lambda F\rfloor$ has positive upper-Banach density for Lebesgue almost every $\lambda\in\mathbb{R}$.

The reader should make a parallel between the quantities $d^*(E)$ and $D(E)$ of subsets $E\subset\mathbb{Z}$ and Lebesgue measure and Hausdorff dimension of subsets of $\mathbb{R}$. It is exactly this association that allows Theorem 1.1 to be a Marstrand theorem for subsets of integers.

The notions of *regularity* and *compatibility* are defined in Subsections 4.1 and 4.2, respectively. Both are fulfilled for many arithmetic subsets of $\mathbb{Z}$, such as the integer values of a polynomial with integer coefficients. These subsets have special interest in Ergodic Theory and its connections with Combinatorics, due to ergodic theorems along these subsets [4], as well as its combinatorial implications. See [3] for the remarkable work of V. Bergelson and A. Leibman on the polynomial extension of Szemerédi's theorem. Theorem has a direct consequence when applied to this setting.

Theorem 1.2. *Let $p_i\in\mathbb{Z}[x]$ with degree d_i and let $E_i=(p_i(n))_{n\in\mathbb{Z}}$, $i=0,\ldots,k$. Then*

$$D(E_0+\lfloor\lambda_1E_1\rfloor+\cdots+\lfloor\lambda_kE_k\rfloor)\geq\min\left\{1,\frac{1}{d_0}+\frac{1}{d_1}+\cdots+\frac{1}{d_k}\right\}$$

for Lebesgue almost every $\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}^k$. If in addition $\sum_{i=0}^k {d_i}^{-1}>1$, then the arithmetic sum $E_0+\lfloor\lambda_1E_1\rfloor+\cdots+\lfloor\lambda_kE_k\rfloor$ has positive upper-Banach density for Lebesgue almost every $\lambda=(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}^k$.

The proof of Theorem 1.1 is based on the ideas developed in [11] and [12]. It relies on the fact that the cardinality of a regular subset of $\mathbb{Z}$ along an increasing sequence of intervals exhibits an exponential behavior ruled out by its counting dimension. As this holds for two regular subsets $E,F\subset\mathbb{Z}$, the compatibility assumption allows to estimate the cardinality of the arithmetic sum $E+\lfloor\lambda F\rfloor$ along the respective arithmetic sums of intervals and, finally, a double-counting argument estimates the size of the "bad" parameters for which such cardinality is small. Theorem 1.2 follows from Theorem 1.1 by a fairly simple induction.

The paper is organized as follows. In Section 2 we introduce the basic notations and definitions. Section 3 is devoted to the discussion of examples. In particular, the sets given by integer values of polynomial with integer coefficients are investigated in Subsection 3.1. In Section 4 we introduce the notions of regularity and compatibility. Subsection 4.3 provides a counterexample to Theorem 1.1 when the sets are no longer compatible. Finally, in Section 5 we prove Theorems 1.1 and 1.2. We also collect final remarks and further questions in Section 6.

2. Preliminaries

2.1. **General notation.** Given a set X, $|X|$ or $\#X$ denotes the cardinality of X. $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ the set of positive integers. We use Vinogradov notation to compare the asymptotic of functions.

Definition 2.1. Let $f, g : \mathbb{Z}$ or $\mathbb{N} \to \mathbb{R}$ be two real-valued functions. We say $f \ll g$ if there is a constant $C > 0$ such that

$$|f(n)| \leq C \cdot |g(n)| \,, \quad \forall\, n \in \mathbb{Z} \text{ or } \mathbb{N}.$$

If $f \ll g$ and $g \ll f$, we write $f \asymp g$.

For each $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x. For each $k \geq 1$, m_k denotes the Lebesgue measure of $\mathbb{R}^k$. For $k = 1$, let $m = m_1$. The letter I will always denote an interval of $\mathbb{Z}$:

$$I = (M, N] = \{M + 1, \ldots, N\}.$$

The *length* of I is equal to its cardinality, $|I| = N - M$.

For $E \subset \mathbb{Z}$ and $\lambda \in \mathbb{R}$, λE denotes the set $\{\lambda n\,;\, n \in E\} \subset \mathbb{R}$ and $\lfloor \lambda E \rfloor$ the set $\{\lfloor \lambda n \rfloor\,;\, n \in E\} \subset \mathbb{Z}$.

2.2. **Counting dimension.**

Definition 2.2. The *upper-Banach density* of $E \subset \mathbb{Z}$ is equal to

$$d^*(E) = \limsup_{|I| \to \infty} \frac{|E \cap I|}{|I|}\,,$$

where I runs over all intervals of $\mathbb{Z}$.

Definition 2.3. The *counting dimension* or simply *dimension* of a set $E \subset \mathbb{Z}$ is equal to

$$D(E) = \limsup_{|I| \to \infty} \frac{\log |E \cap I|}{\log |I|}\,,$$

where I runs over all intervals of $\mathbb{Z}$.

Obviously, $D(E) \in [0, 1]$ and $D(E) = 1$ whenever $d^*(E) > 0$. The counting dimension allows the distinction between sets of zero upper-Banach density. Similar definitions to $D(E)$ have appeared in [2] and [8]. We now give another characterization for the counting dimension that is similar in spirit to the Hausdorff dimension of subsets of $\mathbb{R}$. Let α be a nonnegative real number.

Definition 2.4. The *counting α-measure* of $E \subset \mathbb{Z}$ is equal to

$$H_\alpha(E) = \limsup_{|I| \to \infty} \frac{|E \cap I|}{|I|^\alpha}\,,$$

where I runs over all intervals of $\mathbb{Z}$.

Clearly, $H_\alpha(E) \in [0, \infty]$. For a fixed $E \subset \mathbb{Z}$, the numbers $H_\alpha(E)$ are decreasing in α and one easily checks that

$$H_\alpha(E) = \infty \iff D(E) \geq \alpha\,,$$

which in turn implies the existence and uniqueness of $\alpha \geq 0$ such that

$$\begin{aligned} H_\beta(E) &= \infty \quad , \quad \text{if } 0 \leq \beta < \alpha, \\ &= 0 \quad , \quad \text{if } \beta > \alpha. \end{aligned}$$

The above equalities imply that $D(E) = \alpha$, that is, the counting dimension is exactly the parameter α where $H_\alpha(E)$ decreases from infinity to zero. Also, if $\beta > D(E)$, then

$$|E \cap I| \ll |I|^\beta, \tag{2.1}$$

where I runs over all intervals of $\mathbb{Z}$ and, conversely, if (2.1) holds, then $D(E) \leq \beta$.

Below, we collect basic properties of the counting dimension and α-measure.

(i) $E \subset F \Longrightarrow D(E) \leq D(F)$.
(ii) $D(E \cup F) = \max\{D(E), D(F)\}$.
(iii) For any $\lambda > 0$,

$$H_\alpha(\lfloor \lambda E \rfloor) = \lambda^{-\alpha} \cdot H_\alpha(E). \tag{2.2}$$

The first two are direct. Let's prove (iii). For any interval $I \subset \mathbb{Z}$, we have $|E \cap I| \asymp |\lfloor \lambda E \rfloor \cap \lfloor \lambda I \rfloor|$ and so

$$\begin{aligned} \frac{|E \cap I|}{|I|^\alpha} &\asymp \lambda^\alpha \cdot \frac{|\lfloor \lambda E \rfloor \cap \lfloor \lambda I \rfloor|}{|\lfloor \lambda I \rfloor|^\alpha} \\ \Longrightarrow \quad H_\alpha(E) &= \lambda^\alpha \cdot H_\alpha(\lfloor \lambda E \rfloor) \,. \end{aligned}$$

Remark 2.5. As $\lfloor -x \rfloor = -\lfloor x \rfloor$ or $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$, the sets $\lfloor -\lambda E \rfloor$ and $\lfloor \lambda E \rfloor$ have the same counting dimension. Also, $0 < H_\alpha(\lfloor -\lambda E \rfloor) < \infty$ if and only if $0 < H_\alpha(\lfloor \lambda E \rfloor) < \infty$. For these reasons, we assume from now on that $\lambda > 0$.

3. Examples

Example 1. Let $\alpha \in (0, 1]$ and

$$E_\alpha = \left\{ \left\lfloor n^{1/\alpha} \right\rfloor \,;\, n \in \mathbb{N} \right\}. \tag{3.1}$$

We infer that $0 < H_\alpha(E_\alpha) < 1$. To prove this, we make use of the inequality[2]

$$(x + y)^\alpha \leq x^\alpha + y^\alpha \,,\ x, y \geq 0 \,,$$

to conclude that

$$\frac{|E_\alpha \cap (M, N]|}{(N - M)^\alpha} \leq \frac{(N + 1)^\alpha - (M + 1)^\alpha}{(N - M)^\alpha} \leq 1.$$

This proves that $H_\alpha(E_\alpha) \leq 1$. On the other hand,

$$\frac{|E_\alpha \cap (0, N]|}{N^\alpha} \geq \frac{N^\alpha - 1}{N^\alpha} \asymp 1,$$

which establishes that $H_\alpha(E_\alpha) > 0$. As a consequence, $D(E_\alpha) = \alpha$.

Example 2. The set

$$E = \bigcup_{n \in \mathbb{N}} [n^n, (n + 1)^n] \cap E_{1-1/n}$$

has zero upper-Banach density and $D(E) = 1$.

Example 3. Consider a simple random walk $S = (S_n)_{n \geq 0}$ given by

$$S_0 = 0 \ ,\ S_n = X_1 + \cdots + X_n \,,\ n \geq 1 \,,$$

where $X_n = \pm 1$ are random variables with independent probabilities $(0.5, 0.5)$. If $E_S = \{n \in \mathbb{N} \,;\, S_n = 0\}$ is the set of zeroes of the random walk, then $D(E_S) \geq 0.5$ almost surely. This follows from estimates of Chung and Erdös [5] that, given $\varepsilon > 0$, for almost every S there is $N_0 \in \mathbb{N}$ such that

$$|E_S \cap (0, N]| > N^{0.5 - \varepsilon} \,,\quad \text{for every } N > N_0.$$

[2]For each $\alpha \in (0, 1]$, the function $x \mapsto x^\alpha$, $x \geq 0$, is concave and then, by Jensen's inequality, $(x + y)^\alpha \leq 2^{1-\alpha} \cdot (x + y)^\alpha \leq x^\alpha + y^\alpha$.

Example 4. Sets of zero upper-Banach density appear naturally in infinite ergodic theory. Let $(X, \mathcal{A}, \mu, T)$ be a sigma-finite measure-preserving system, with $\mu(X) = \infty$, and let $A \in \mathcal{A}$ have finite measure. Fixed $x \in A$, let $E = \{n \geq 1\,;\, T^n x \in A\}$. By Hopf's Ratio Ergodic Theorem, E has zero upper-Banach density almost surely. In many specific cases, its dimension can be calculated or at least estimated. See [1] for further details.

3.1. Polynomial subsets of $\mathbb{Z}$.

Definition 3.1. A *polynomial subset of* $\mathbb{Z}$ is a set $E = \{p(n)\,;\, n \in \mathbb{Z}\}$, where $p(x)$ is a polynomial with integer coefficients.

These are the sets we consider in Theorem 1.2. Their counting dimension is easily calculated as follows. Given $E, F \subset \mathbb{Z}$, let E and F be *asymptotic* if $E = \{\cdots < a_{-1} < a_0 < a_1 < \cdots\}$, $F = \{\cdots < b_{-1} < b_0 < b_1 < \cdots\}$ and there are $i, n_0 > 0$ such that

$$\left\{\begin{array}{ccccc} a_{n-i} & \leq & b_n & \leq & a_{n+i} \\ b_{n-i} & \leq & a_n & \leq & b_{n+i} \end{array}\right. , \quad \text{for every } n \geq n_0. \tag{3.2}$$

Denote this by $E \asymp F$.

Lemma 3.2. *If $E, F \subset \mathbb{Z}$ are asymptotic, then $D(E) = D(F)$. Also, for any $\alpha \geq 0$,*

$$0 < H_\alpha(E) < \infty \iff 0 < H_\alpha(F) < \infty. \tag{3.3}$$

Proof. Let $I = (M, N]$ be an interval and assume $E \cap I = \{a_{m+1}, a_{m+2}, \ldots, a_n\}$. By relation (3.2),

$$b_{m-i} \leq a_m \leq M < a_{m+1} \leq b_{m+i+1} \quad \text{and} \quad b_{n-i} \leq a_n \leq N < a_{n+1} \leq b_{n+i+1},$$

which imply the inclusions

$$\{b_{m+i+1}, \ldots, b_{n-i}\} \subset F \cap I \subset \{b_{m-i}, \ldots, b_{n+i+1}\}.$$

Then $|E \cap I| \asymp |F \cap I|$ and so

$$D(E) = \limsup_{|I| \to \infty} \frac{\log |E \cap I|}{\log |I|} = \limsup_{|I| \to \infty} \frac{\log |F \cap I|}{\log |I|} = D(F)\,.$$

The same asymptotic relation also proves (3.3). □

Let $E = \{p(n)\,;\, n \in \mathbb{Z}\}$, where $p(x) \in \mathbb{Z}[x]$ has degree d. Assuming p has leading coefficient $a > 0$, there is $i \geq 0$ such that $a \cdot (n-i)^d < p(n) < a \cdot (n+i)^d$ for large n and then $E \asymp aE_{1/d}$, where $E_{1/d}$ is defined as in (3.1). By Lemma 3.2, we get

$$D(E) = \frac{1}{d} \quad \text{and} \quad 0 < H_{1/d}(E) < \infty.$$

3.2. Cantor sets in $\mathbb{Z}$.

The famous ternary Cantor set of $\mathbb{R}$ is formed by the real numbers of $[0, 1]$ with only 0's and 2's on their base 3 expansion. In analogy to this, let $E \subset \mathbb{Z}$ be defined as

$$E = \left\{\sum_{n=0}^{N} a_n \cdot 3^n\,;\, N \in \mathbb{N} \text{ and } a_n = 0 \text{ or } 2\right\}. \tag{3.4}$$

The set E has been slightly investigated in [7]. There, A. Fisher proved that

$$H_{\log 2/\log 3}(E) > 0\,.$$

We prove below that $H_{\log 2/\log 3}(E) \leq 1$, which in particular gives that $D(E) = \log 2/\log 3$, as expected. Let $I = (M, N]$ be an interval of $\mathbb{Z}$. We can assume $M+1, N \in E$. Indeed, if $\tilde{I} = (\tilde{M}, \tilde{N}]$, where $\tilde{M}+1$ and $\tilde{N}$ are the smallest and largest elements of $E \cap I$, respectively, then

$$\frac{|E \cap I|}{|I|^\alpha} \leq \frac{|E \cap \tilde{I}|}{|\tilde{I}|^\alpha}\,.$$

Let $M+1, N \in E$, say

$$\begin{aligned} M+1 &= a_0 \cdot 3^0 + a_1 \cdot 3^1 + \cdots + a_{m-1} \cdot 3^m + 2 \cdot 3^m \\ N &= b_0 \cdot 3^0 + b_1 \cdot 3^1 + \cdots + b_{n-1} \cdot 3^n + 2 \cdot 3^n. \end{aligned}$$

We can also assume that $m < n$. If this is not the case, then the quotient $|E \cap I|/|I|^\alpha$ is again increased if we change I by $(M - 2 \cdot 3^n, N - 2 \cdot 3^n]$. In this setting,

$$N - M \geq 2 \cdot 3^n - (3^{m+1} - 2) \geq 3^n$$

and then

$$\frac{|E \cap I|}{|I|^{\log 2/\log 3}} \leq \frac{|E \cap (0, N]|}{|I|^{\log 2/\log 3}} \leq \frac{2^n}{(3^n)^{\log 2/\log 3}} = 1.$$

Because I is arbitrary, this gives that $H_{\log 2/\log 3}(E) \leq 1$. Observe also that the renormalization of $E \cap (0, 3^n)$ via the linear map $x \mapsto x/3^n$ generates a subset of the unit interval $(0,1)$ that is equal to the set of left endpoints of the remaining intervals of the n-th step of the construction. In other words, if $K = \bigcup_{n \in E} [n, n+1)$, then $K/3^n$ is exactly the n-th step of the construction of the ternary Cantor set of $\mathbb{R}$.

More generally, let us define a class of Cantor sets in $\mathbb{Z}$. Fix a basis $a \in \mathbb{N}$ and a binary matrix $A = (a_{ij})_{1 \leq i,j \leq a}$. Let

$$\Sigma_n(A) = \left\{(d_0 d_1 \cdots d_{n-1} d_n)\,;\, a_{d_{i-1} d_i} = 1,\ 1 \leq i \leq n\right\}$$

denote the set of admissible words of length n and $\Sigma^*(A) = \bigcup_{n \geq 0} \Sigma_n$ the set of all finite admissible words.

Definition 3.3. The *integer Cantor set* $E_A \subset \mathbb{Z}$ associated to the matrix A is the set

$$E_A = \{d_0 \cdot a^0 + \cdots + d_n \cdot a^n\,;\, (d_0 d_1 \cdots d_n) \in \Sigma^*(A)\}.$$

Remark 3.4. In [7], A. Fisher introduced another class of integer Cantor sets, called *random integer Cantor sets*.

Our definition was inspired on the fact that dynamically defined Cantor sets of the real line are homeomorphic to subshifts of finite type (see [11]), which is exactly what we did above, after truncating the numbers. The dimension of E_A, as in the inspiring case, depends on the Perron-Frobenius eigenvalue of A. Remember that the *Perron-Frobenius eigenvalue* is the largest eigenvalue $\lambda_+(A)$ of A. It has multiplicity one and maximizes the absolute value of the eigenvalues of A. Also, there is a constant $c = c(A) > 0$ such that

$$c^{-1} \cdot \lambda_+(A)^n \leq |\Sigma_n| \leq c \cdot \lambda_+(A)^n\,, \quad \text{for every } n \geq 0, \tag{3.5}$$

whose proof may be found in [10].

Lemma 3.5. *If A is a binary $a \times a$ matrix, then*

$$D(E_A) = \frac{\log \lambda_+(A)}{\log a} \quad \text{and} \quad 0 < H_{\frac{\log \lambda_+(A)}{\log a}}(E_A) < \infty\,.$$

Proof. Let $I = (M, N]$. Again, we may assume $M+1, N \in E_A$, say

$$\begin{aligned} M+1 &= x_0 \cdot a^0 + \cdots + x_n \cdot a^n \\ N &= y_0 \cdot a^0 + \cdots + y_n \cdot a^n, \end{aligned}$$

with $y_n > x_n$. If $y_n \geq x_n + 2$, then

$$\left\{ \begin{array}{rcl} M+1 & \leq & (x_n+1) \cdot a^n \\ N & \geq & (x_n+2) \cdot a^n \end{array} \right. \Longrightarrow |I| \geq a^n$$

and, as $I \subset (0, a^{n+1})$, we have

$$\frac{|E_A \cap I|}{|I|^{\frac{\log \lambda_+(A)}{\log a}}} \leq \frac{|\Sigma_n|}{a^{\frac{n \log \lambda_+(A)}{\log a}}} \leq \frac{c \cdot \lambda_+(A)^n}{\lambda_+(A)^n} = c\,. \tag{3.6}$$

If $y_n = x_n + 1$, let $i, j \in \{0, 1, \ldots, n-1\}$ be the indices for which

(i) $x_i < a-1$ and $x_{i+1} = \cdots = x_{n-1} = a-1$,
(ii) $y_j > 0$ and $y_{j+1} = \cdots = y_{n-1} = 0$.

Then

$$\left\{ \begin{array}{rcl} M+1 & \leq & (x_n+1) \cdot a^n - a^i \\ N & \geq & (x_n+1) \cdot a^n + a^j \end{array} \right. \Longrightarrow |I| \geq a^i + a^j \geq a^{\max\{i,j\}}.$$

In order to $\sum_{k=0}^n z_k \cdot a^k$ belong to I, one must have $z_n = x_n$ or $z_n = x_n + 1$. In the first case $z_{i+1} = \cdots = z_{n-1} = a-1$ and in the second $z_{j+1} = \cdots = z_{n-1} = 0$. Then

$$|E_A \cap I| \leq |\Sigma_i| + |\Sigma_j| \leq 2c \cdot \lambda_+(A)^{\max\{i,j\}}$$

and so

$$\frac{|E_A \cap I|}{|I|^{\frac{\log \lambda_+(A)}{\log a}}} \leq \frac{2c \cdot \lambda_+(A)^{\max\{i,j\}}}{a^{\frac{\max\{i,j\} \log \lambda_+(A)}{\log a}}} = 2c\,. \tag{3.7}$$

Estimates (3.6) and (3.7) give $H_{\log \lambda_+(A)/\log a} < \infty$. On the other hand,

$$\frac{|E_A \cap (0, a^n]|}{a^{\frac{n \log \lambda_+(A)}{\log a}}} \geq \frac{c^{-1} \cdot \lambda_+(A)^{n-1}}{\lambda_+(A)^n} = c^{-1} \cdot \lambda_+(A)^{-1} \quad \Longrightarrow \quad H_{\log \lambda_+(A)/\log a} > 0\,,$$

which concludes the proof. □

By the above lemma, if $F \subset \{1, \ldots, a\}$ and $A = (a_{ij})$ is defined by $a_{ij} = 1$ iff $i, j \in F$, then $D(E_A) = \log |F| / \log a$, which extends the results about the ternary Cantor set (3.4).

In general, if E, F are subsets of $\mathbb{Z}$ such that $D(E) + D(F) > 1$, it is not true that $d^*(E+F) > 0$, because the elements of $E+F$ may have many representations as the sum of one element of E and other of F. This resonance phenomena decreases the dimension of $E+F$. Lemma 3.5 provides a simple example to this situation: if $E = E_A$ and $F = E_B$, where $A = (a_{ij})_{1 \leq i,j \leq 12}$, $B = (b_{ij})_{1 \leq i,j \leq 12}$ are defined by

$$a_{ij} = 1 \iff 1 \leq i, j \leq 4 \quad \text{and} \quad b_{ij} = 1 \iff 5 \leq i, j \leq 8,$$

then $D(E) + D(F) = 2 \log 4 / \log 12$, while $E + F = E_C$ for $C = (c_{ij})_{1 \leq i,j \leq 12}$ given by

$$c_{ij} = 1 \iff 1 \leq i, j \leq 11.$$

$E+F$ has counting dimension equal to $\log 11/\log 12$ and so $d^*(E+F)=0$. Theorem 1.1 proves that resonance is avoided if we are allowed to change the scales of the sets, multiplying one of them by a factor $\lambda\in\mathbb{R}$.

4. Regularity and compatibility

4.1. Regular sets.

Definition 4.1. A subset $E\subset\mathbb{Z}$ is *regular* if $0<H_{D(E)}(E)<\infty$.

By Lemmas 3.2 and 3.5, polynomial sets and Cantor sets are regular.

Definition 4.2. Given two subsets $E=\{\cdots<x_{-1}<x_0<x_1<\cdots\}$ and F of $\mathbb{Z}$, let $E*F$ denote the set

$$E*F=\{x_n\,;\,n\in F\}.$$

This is a subset of E whose counting dimension is at most $D(E)D(F)$. To see this, consider and arbitrary interval $I\subset\mathbb{Z}$. If $E\cap I=\{x_{i+1},\ldots,x_j\}$, then

$$(E*F)\cap I=\{x_n\,;n\in F\cap(i,j]\}\,.$$

Given $\alpha>D(E)$ and $\beta>D(F)$, the relation (2.1) guarantees that

$$\begin{aligned}|(E*F)\cap I| &= |F\cap(i,j]|\\ &\ll (j-i)^\beta\\ &= |E\cap I|^\beta\\ &\ll |I|^{\alpha\beta}\end{aligned}$$

and so $D(E*F)\le\alpha\beta$. Choosing α,β arbitrarily close to $D(E)$, $D(F)$, respectively, it follows that $D(E*F)\le D(E)D(F)$.

If E is regular, it is possible to choose F in such a way that $E*F$ is also regular and has dimension equal to $D(E)D(F)$. To this matter, choose disjoint intervals $I_n=(a_n,b_n]$, $n\ge 1$, such that

$$\frac{|E\cap I_n|}{|I_n|^{D(E)}}\gg 1$$

and let $E\cap I_n=\{x_{i_n+1}<x_{i_n+2}<\cdots<x_{j_n}\}$, where $i_n<j_n$. Let $\alpha\in[0,1]$ and

$$F=\bigcup_{n\ge 1}(E_\alpha+i_n)\cap(i_n,j_n]\,,$$

where E_α is defined as in (3.1). Then $D(F)=D(E_\alpha)=\alpha$ and

$$\begin{aligned}|(E*F)\cap I_n| &\ge |F\cap(i_n,j_n]|\\ &= |E_\alpha\cap(0,j_n-i_n]|\\ &\gg (j_n-i_n)^{D(F)}\\ &= |E\cap I_n|^{D(F)}\\ &\gg |I_n|^{D(E)D(F)},\end{aligned}$$

implying that

$$|(E*F)\cap I_n|\gg|I_n|^{D(E)D(F)}. \tag{4.1}$$

This proves the reverse inequality $D(E*F)\ge D(E)D(F)$. We thus established that, for a regular subset $E\subset\mathbb{Z}$ and $0\le\alpha\le D(E)$, there exists a regular subset $E'\subset E$ such that $D(E')=\alpha$. It is a harder task to prove that this holds even when E is not regular.

Proposition 4.3. *Let $E \subset \mathbb{Z}$ and $0 \le \alpha \le 1$. If $H_\alpha(E) > 0$, then there exists a regular subset $E' \subset E$ such that $D(E') = \alpha$. In particular, for any $0 \le \alpha < D(E)$, there is $E' \subset E$ regular such that $D(E') = \alpha$.*

Proof. The idea is to apply a dyadic argument in E to decrease $H_\alpha(E)$ in a controlled way. Given an interval $I \subset \mathbb{Z}$ and a subset $F \subset \mathbb{Z}$, define

$$s_F(I) \doteq \sup_{J \subset I} \frac{|F \cap J|}{|J|^\alpha} \cdot$$

If $F = \{a_1, a_2, \ldots, a_k\} \subset I$, the dyadic operation of discarding the interior elements $a_2, \ldots, a_{k-1}$ of F alternately,

$$F = \{a_1, a_2, \ldots, a_k\} \rightsquigarrow F' = \{a_1, a_2, a_4, \ldots, a_{2\lceil k/2 \rceil - 2}, a_k\},$$

decreases $s_F(I)$ to approximately $s_F(I)/2$. More specifically, if $s_F(I) > 2$, then $s_{F'}(I) > 1/2$. Actually, for every interval $J \subset I$

$$\frac{|F' \cap J|}{|J|^\alpha} \le \frac{1}{2} \cdot \frac{|F \cap J| + 1}{|J|^\alpha} \le \frac{s_F(I)}{2} + \frac{1}{2} < s_F(I)$$

and, for J maximizing $s_F(I)$,

$$\frac{|F' \cap J|}{|J|^\alpha} \ge \frac{1}{2} \cdot \frac{|F \cap J| - 1}{|J|^\alpha} > 1 - \frac{1}{2 \cdot |J|^\alpha} \ge \frac{1}{2} \cdot$$

After a finite number of these dyadic operations, one obtains a subset $F' \subset F$ such that

$$\frac{1}{2} < s_{F'}(I) \le 2.$$

If $H_\alpha(E) < \infty$, there is nothing to do. Assume that $H_\alpha(E) = \infty$. We proceed inductively by constructing a sequence $F_1 \subset F_2 \subset \cdots$ of finite subsets of E contained in an increasing sequence of intervals $I_n = (a_n, b_n]$, $n \ge 1$, such that

(i) $1/2 < s_{F_n}(I_n) \le 3$;
(ii) there is an interval $J_n \subset I_n$ such that $|J_n| \ge n$ and

$$\frac{|F_n \cap J_n|}{|J_n|^\alpha} > \frac{1}{2} \cdot$$

Once these properties are fulfilled, the set $E' = \bigcup_{n \ge 1} F_n$ will satisfy the required conditions.

Take any $a \in E$ and $I_1 = \{a\}$. Assume I_n, F_n and J_n have been defined satisfying (i) and (ii). As $H_\alpha(E) = \infty$, there exists an interval J_{n+1} disjoint from $(a_n - |I_n|^{1/\alpha}, b_n + |I_n|^{1/\alpha}]$ for which

$$\frac{|E \cap J_{n+1}|}{|J_{n+1}|^\alpha} \ge (n+1)^{1-\alpha} \, .$$

This inequality allows to restrict J_{n+1} to a smaller interval of size at least $n+1$, also denoted J_{n+1}, such that

$$s_E(J_{n+1}) = \frac{|E \cap J_{n+1}|}{|J_{n+1}|^\alpha} \, . \tag{4.2}$$

Let $F'_{n+1} = E \cap J_{n+1}$ and apply the dyadic operation to F'_{n+1} until

$$\frac{1}{2} < s_{F'_{n+1}}(J_{n+1}) = \frac{|F'_{n+1} \cap J_{n+1}|}{|J_{n+1}|^\alpha} \le 2. \tag{4.3}$$

Let $I_{n+1} = I_n \cup K_n \cup J_{n+1}$ be the convex hull[3] of I_n and J_{n+1} and $F_{n+1} = F_n \cup F'_{n+1}$. Condition (ii) is satisfied because of (4.3). To prove (i), let I be a subinterval of I_{n+1}. We have three cases to consider.

- $I \subset I_n \cup K_n$: by condition (i) of the inductive hypothesis,

$$\frac{|F_{n+1} \cap I|}{|I|^\alpha} \leq \frac{|F_n \cap (I \cap I_n)|}{|I \cap I_n|^\alpha} \leq 3.$$

- $I \subset K_n \cup J_{n+1}$: by (4.3),

$$\frac{|F_{n+1} \cap I|}{|I|^\alpha} \leq \frac{|F'_{n+1} \cap (I \cap J_{n+1})|}{|I \cap J_{n+1}|^\alpha} \leq 2.$$

- $I \supset K_n$: as $|K_n| \geq |I_n|$,

$$\begin{aligned}
\frac{|F_{n+1} \cap I|}{|I|^\alpha} &= \frac{|F_n \cap (I \cap I_n)| + |F'_{n+1} \cap (I \cap J_{n+1})|}{(|I \cap I_n| + |K_n| + |I \cap J_{n+1}|)^\alpha} \\
&\leq \frac{|F_n \cap (I \cap I_n)|}{(|I \cap I_n| + |K_n|)^\alpha} + \frac{|F'_{n+1} \cap (I \cap J_{n+1})|}{(|I \cap J_{n+1}|)^\alpha} \\
&\leq \frac{|I_n|}{|K_n|^\alpha} + s_{F'_{n+1}}(J_{n+1}) \\
&\leq 3.
\end{aligned}$$

This proves condition (i) and completes the inductive step. □

By the above proposition and equation (2.2), for any $\alpha \in [0,1]$ and $h > 0$, there exists a regular subset $E \subset \mathbb{Z}$ such that $D(E) = \alpha$ and $H_\alpha(E) = h$. We'll use this fact in Subsection 4.3.

4.2. **Compatible sets.**

Definition 4.4. Two regular subsets $E, F \subset \mathbb{Z}$ are *compatible* if there exist two sequences $(I_n)_{n\geq 1}$, $(J_n)_{n\geq 1}$ of intervals with increasing lengths such that

(1) $|I_n| \asymp |J_n|$,
(2) $|E \cap I_n| \gg |I_n|^{D(E)}$ and $|F \cap J_n| \gg |J_n|^{D(F)}$.

The notion of compatibility means that E and F have comparable intervals on which the respective intersections obey the correct growth speed of cardinality. Some sets have these intervals in all scales.

Definition 4.5. A regular subset $E \subset \mathbb{Z}$ is *universal* if there exists a sequence $(I_n)_{n\geq 1}$ of intervals such that $|I_n| \asymp n$ and $|E \cap I_n| \gg |I_n|^{D(E)}$.

It is clear that E and F are compatible whenever one of them is universal and the other is regular. Every E_α is universal and the same happens to polynomial subsets, due to the asymptotic relation $E \asymp aE_{1/d}$ (see Subsection 3.1). In particular, any two polynomial subsets are compatible.

[3]Observe that $|K_n| \geq |I_n|^{1/\alpha}$.

4.3. **A counterexample of Theorem 1.1 for regular non-compatible sets.** In this subsection, we construct regular sets $E, F \subset \mathbb{Z}$ such that $D(E)+D(F)>1$ and $E+\lfloor \lambda F \rfloor$ has zero upper-Banach density for every $\lambda \in \mathbb{R}$. The idea is, in the contrast to compatibility, construct E and F such that the intervals $I, J \subset \mathbb{Z}$ for which $\dfrac{|E\cap I|}{|I|^{D(E)}}$ and $\dfrac{|F\cap J|}{|J|^{D(F)}}$ are bounded away from zero have totally different sizes.

Let $\alpha, \beta \in [1/2, 1)$. We define

$$E=\bigcup_{i\geq 1}(E_i\cap I_i) \quad \text{and} \quad F=\bigcup_{i\geq 1}(F_i\cap J_i)$$

such that the following conditions are satisfied:

(i) $E_i = \lfloor \mu_i E_i' \rfloor$, where $(E_i')_{i\geq 1}$ is a sequence of regular sets, each of them with dimension α, such that $H_\alpha(E_i') \to \infty$.

(ii) $F_i = \lfloor \nu_i F_i' \rfloor$, where $(F_i')_{i\geq 1}$ is a sequence of regular sets, each of them with dimension β, such that $H_\beta(F_i') \to \infty$.

(iii) $I_i = (A_i, B_i]$ and $J_i = (C_i, D_i]$, with[4]

$$A_i << B_i << C_i << D_i << A_{i+1}.$$

(iv) $\mu_{i+1} > \max\left\{(D_i - C_i)^2, i^{\frac{2}{1-\alpha}} \cdot (D_i - C_i)^{\frac{2\beta}{1-\alpha}}\right\}$.

(v) $\nu_i > \max\left\{(B_i - A_i)^2, i^{\frac{4}{1-\beta}} \cdot (B_i - A_i)^{\frac{2\alpha}{1-\beta}}\right\}$.

By (i) and (ii), we can restrict $(E_i')_{i\geq 1}$, $(F_i')_{i\geq 1}$ to subsequences in order to also have

(vi) $H_\alpha(E_i) = H_\beta(F_i) = 1/2$.

For any $\lambda \in \mathbb{R}$, the arithmetic sum $E+\lfloor \lambda F \rfloor$ is supported on the intervals $I_i + \lfloor \lambda J_j \rfloor$, $i, j \geq 1$. By condition (iii), if i, j are large enough, these intervals are pairwise disjoint. Consider an arbitrary interval $I = (M, N] \subset I_i + \lambda J_j$. Assume that $i \leq j$. For $a, a' \in I_i$ and $b, b' \in J_j$, one has

$$\begin{aligned} |(a+\lambda b) - (a' + \lambda b')| &\geq \lambda |b-b'| - |a-a'| \\ &\geq \lambda \nu_i - (B_i - A_i) \end{aligned}$$

which, by (v), is at least ${\nu_i}^{1/2}$ for large i. Then,

$$|I| < {\nu_i}^{1/2} \implies |(E+\lfloor \lambda F \rfloor)\cap I| \leq 1. \tag{4.4}$$

On the other hand, if $|I| \geq {\nu_i}^{1/2}$, (vi) gives that

$$\begin{aligned} |(E+\lfloor \lambda F \rfloor)\cap I| &= \sum_{a\in E\cap I_i} \left| J_j \cap \left(\frac{M+1-a}{\lambda}, \frac{N+1-a}{\lambda} \right] \right| \\ &\leq \sum_{a\in E\cap I_i} \left(\frac{|I|}{\lambda}\right)^\beta \\ &\leq \lambda^{-\beta} \cdot |I_i|^\alpha \cdot |I|^\beta \end{aligned}$$

and so, by (v),

$$\frac{|(E+\lfloor \lambda F \rfloor)\cap I|}{|I|} \leq \lambda^{-\beta} \cdot |I_i|^\alpha \cdot |I|^{\beta-1} \leq \lambda^{-\beta} \cdot |I_i|^\alpha \cdot \nu_i^{\frac{\beta-1}{2}} < \frac{1}{i} \tag{4.5}$$

for $i > \lambda^{-\beta}$.

[4] $x << y$ means that x is much smaller than y.

Assume now that $i > j$. The calculations are similar to the previous ones. In this case,

$$\begin{aligned}|(a+\lambda b)-(a'+\lambda b')| &\geq |a-a'|-\lambda|b-b'|\\ &\geq \mu_{j+1}-\lambda(D_j-C_j)\end{aligned}$$

is greater than ${\mu_{j+1}}^{1/2}$ if j is large. Then

$$|I| < {\mu_{j+1}}^{1/2} \implies |(E+\lfloor\lambda F\rfloor)\cap I| \leq 1. \tag{4.6}$$

If $|I| \geq {\mu_{j+1}}^{1/2}$, again by (vi) we have

$$\begin{aligned}|(E+\lfloor\lambda F\rfloor)\cap I| &= \sum_{b\in F\cap J_j}|I_i\cap(M+1-\lambda b, N+1-\lambda b]|\\ &\leq |J_j|^\beta\cdot|I|^\alpha\end{aligned}$$

and, by (iv), it follows that

$$\frac{|(E+\lfloor\lambda F\rfloor)\cap I|}{|I|} \leq |J_j|^\beta\cdot|I|^{\alpha-1} \leq |J_j|^\beta\cdot{\mu_{j+1}}^{\frac{\alpha-1}{2}} < \frac{1}{j}\,\cdot \tag{4.7}$$

The relations (4.4), (4.5), (4.6), (4.7) imply that $E+\lfloor\lambda F\rfloor$ has zero upper-Banach density.

5. Proofs of the Theorems

Let E, F be two regular compatible subsets of $\mathbb{Z}$. Fix a compact interval Λ of positive real numbers. Given distinct points $z=(a,b)$ and $z'=(a',b')$ of $E\times F$, let

$$\Lambda_{z,z'} = \{\lambda\in\Lambda\,;\, a+\lfloor\lambda b\rfloor = a'+\lfloor\lambda b'\rfloor\}\,.$$

Clearly, $\Lambda_{z,z'}$ is empty if $b=b'$. For $b\neq b'$, it is possible to estimate its Lebesgue measure, according to the

Lemma 5.1. *Let $z=(a,b)$ and $z'=(a',b')$ be distinct points of $\mathbb{Z}^2$. If $\Lambda_{z,z'}\neq\emptyset$, then*

(a) $m(\Lambda_{z,z'}) \ll |b-b'|^{-1}$ *and*
(b) $\min\Lambda\cdot|b-b'|-1 < |a-a'| < \max\Lambda\cdot|b-b'|+1$.

Proof. Assume $b>b'$ and let $\lambda\in\Lambda_{z,z'}$, say $a+\lfloor\lambda b\rfloor = n = a'+\lfloor\lambda b'\rfloor$. Then

$$\left\{\begin{array}{ccccc} n-a & \leq & \lambda b & < & n-a+1\\ n-a' & \leq & \lambda b' & < & n-a'+1\end{array}\right. \implies a'-a-1 < \lambda(b-b') < a'-a+1$$

and so

$$\Lambda_{z,z'} \subset \left(\frac{a'-a-1}{b-b'}\,,\,\frac{a'-a+1}{b-b'}\right),$$

which proves (a). The second part also follows from the above inclusion, as

$$\frac{a'-a-1}{b-b'} \leq \min\Lambda_{z,z'} \leq \max\Lambda \implies a'-a \leq \max\Lambda\cdot(b-b')+1$$

and

$$\frac{a'-a+1}{b-b'} \geq \max\Lambda_{z,z'} \geq \min\Lambda \implies a'-a \geq \min\Lambda\cdot(b-b')-1.$$

□

Observe that, as Λ is fixed throughout the rest of the proof, (b) implies that $|a-a'| \asymp |b-b'|$. We point out that, although ingenuous, Lemma 5.1 expresses the crucial property of transversality that makes the proof work, and all results related to Marstrand's theorem use a similar idea in one way or another.

Let $(I_n)_{n\geq 1}$ and $(J_n)_{n\geq 1}$ be sequences of intervals satisfying the compatibility conditions of Definition 4.4. Associated to these intervals, consider, for each pair $(n,\lambda) \in \mathbb{N}\times\Lambda$, the set

$$N_n(\lambda) = \{(a,b),(a',b') \in (E\cap I_n)\times(F\cap J_n)\,;\, a+\lfloor\lambda b\rfloor = a'+\lfloor\lambda b'\rfloor\}$$

and, for each $n \geq 1$, the integral

$$\Delta_n = \int_\Lambda N_n(\lambda)dm(\lambda)\,.$$

By a double counting, one has the equality

$$\Delta_n = \sum_{z,z'\in(E\cap I_n)\times(F\times J_n)} m(\Lambda_{z,z'})\,. \tag{5.1}$$

Lemma 5.2. *Let $D(E)=\alpha$ and $D(F)=\beta$. In the above conditions,*

(a) If $\alpha+\beta<1$, then $\Delta_n \ll |I_n|^{\alpha+\beta}$.

(b) If $\alpha+\beta>1$, then $\Delta_n \ll |I_n|^{2\alpha+2\beta-1}$.

Proof. By equality (5.1),

$$\begin{aligned}
\Delta_n &= \sum_{(z,z')\in(E\cap I_n)\times(F\times J_n)} m(\Lambda_{z,z'})\\
&= \sum_{\substack{a\in E\cap I_n\\ b\in F\cap J_n}} \sum_{s=1}^{\ln|I_n|} \sum_{\substack{a'\in E\cap I_n\\ |a-a'|\asymp e^s}} \sum_{\substack{b'\in F\cap J_n\\ |b-b'|\asymp e^s}} m(\Lambda_{z,z'})\\
&\ll \sum_{\substack{a\in E\cap I_n\\ b\in F\cap J_n}} \sum_{s=1}^{\ln|I_n|} e^{-s}\cdot(e^s)^\alpha\cdot(e^s)^\beta\\
&= \sum_{\substack{a\in E\cap I_n\\ b\in F\cap J_n}} \sum_{s=1}^{\ln|I_n|} (e^s)^{\alpha+\beta-1}\\
&\ll |I_n|^{\alpha+\beta}\cdot\sum_{s=1}^{\ln|I_n|}\left(e^{\alpha+\beta-1}\right)^s
\end{aligned}$$

and then

$$\begin{aligned}
\Delta_n &\ll |I_n|^{\alpha+\beta}\cdot|I_n|^{\alpha+\beta-1} = |I_n|^{2\alpha+2\beta-1} &&, \text{ if } \alpha+\beta>1,\\
&\ll |I_n|^{\alpha+\beta}\cdot 1 = |I_n|^{\alpha+\beta} &&, \text{ if } \alpha+\beta<1.
\end{aligned}$$

□

Proof of Theorem 1. The proof is divided in three parts.

Part 1. $\alpha+\beta<1$: fix $\varepsilon>0$. By Lemma 5.2, the set of parameters $\lambda\in\Lambda$ for which

$$N_n(\lambda)\ll\frac{|I_n|^{\alpha+\beta}}{\varepsilon}\tag{5.2}$$

has Lebesgue measure at least $m(\Lambda)-\varepsilon$. We will prove that

$$\frac{|(E+\lfloor\lambda F\rfloor)\cap(I_n+\lfloor\lambda J_n\rfloor)|}{|I_n+\lfloor\lambda J_n\rfloor|^{\alpha+\beta}}\gg\varepsilon\tag{5.3}$$

whenever $\lambda\in\Lambda$ satisfies (5.2). For each $(m,n,\lambda)\in\mathbb{Z}^2\times\Lambda$, let

$$s(m,n,\lambda)=\#\left\{(a,b)\in(E\cap I_n)\times(F\cap J_n)\,;\,a+\lfloor\lambda b\rfloor=m\right\}.$$

Then

$$\sum_{m\in\mathbb{Z}}s(m,n,\lambda)=|E\cap I_n|\cdot|F\cap J_n|\asymp|I_n|^{\alpha+\beta}\tag{5.4}$$

and

$$\sum_{m\in\mathbb{Z}}s(m,n,\lambda)^2=N_n(\lambda)\ll\frac{|I_n|^{\alpha+\beta}}{\varepsilon}\cdot\tag{5.5}$$

The numerator in (5.3) is at least the cardinality of the set $S(n,\lambda)=\{m\in\mathbb{Z}\,;\,s(m,n,\lambda)>0\}$, because $(E+\lfloor\lambda F\rfloor)\cap(I_n+\lfloor\lambda J_n\rfloor)$ contains $S(n,\lambda)$. By the Cauchy-Schwarz inequality and relations (5.4), (5.5), we have

$$\begin{aligned}|S(n,\lambda)|&\geq\frac{\left(\displaystyle\sum_{m\in\mathbb{Z}}s(m,n,\lambda)\right)^2}{\displaystyle\sum_{m\in\mathbb{Z}}s(m,n,\lambda)^2}\\&\gg\frac{\left(|I_n|^{\alpha+\beta}\right)^2}{\dfrac{|I_n|^{\alpha+\beta}}{\varepsilon}}\\&=\varepsilon\cdot|I_n|^{\alpha+\beta}\end{aligned}$$

and so, as $|I_n+\lfloor\lambda J_n\rfloor|\asymp|I_n|$,

$$\frac{|(E+\lfloor\lambda F\rfloor)\cap(I_n+\lfloor\lambda J_n\rfloor)|}{|I_n+\lfloor\lambda J_n\rfloor|^{\alpha+\beta}}\gg\frac{|S(n,\lambda)|}{|I_n|^{\alpha+\beta}}\gg\varepsilon\,,$$

establishing (5.3).

For each $n\geq1$, let $G_\varepsilon^n=\{\lambda\in\Lambda\,;\,(5.3)\text{ holds}\}$. Then $m\left(\Lambda\backslash G_\varepsilon^n\right)\leq\varepsilon$, and the same holds for the set

$$G_\varepsilon=\bigcap_{n\geq1}\bigcup_{l=n}^{\infty}G_\varepsilon^l\,.$$

For each $\lambda\in G_\varepsilon$,

$$D_{\alpha+\beta}(E+\lfloor\lambda F\rfloor)\geq\varepsilon\implies D(E+\lfloor\lambda F\rfloor)\geq\alpha+\beta$$

and then, as the set $G=\bigcup_{n\geq1}G_{1/n}\subset\Lambda$ has Lebesgue measure $m(\Lambda)$, Part 1 is completed.

Part 2. $\alpha+\beta>1$: for a fixed $\varepsilon>0$, Lemma 5.2 implies the set of parameters $\lambda\in\Lambda$ for which

$$N_n(\lambda)\ll\frac{|I_n|^{2\alpha+2\beta-1}}{\varepsilon}\tag{5.6}$$

has Lebesgue measure at least $m(\Lambda)-\varepsilon$. In this case,

$$\begin{aligned} |S(n,\lambda)| &\geq \frac{\left(\sum\limits_{m\in\mathbb{Z}} s(m,n,\lambda)\right)^2}{\sum\limits_{m\in\mathbb{Z}} s(m,n,\lambda)^2} \\ &\gg \frac{\left(|I_n|^{\alpha+\beta}\right)^2}{\frac{|I_n|^{2\alpha+2\beta-1}}{\varepsilon}} \\ &= \varepsilon\cdot|I_n| \end{aligned}$$

and then

$$\frac{|(E+\lfloor\lambda F\rfloor)\cap(I_n+\lfloor\lambda J_n\rfloor)|}{|I_n+\lfloor\lambda J_n\rfloor|}\gg\frac{|S(n,\lambda)|}{|I_n|}\gg\varepsilon.$$

The measure-theoretical argument is the same as in Part 1.

Part 3. $\alpha+\beta=1$: let $n\geq 1$. Being regular, E has a regular subset $E_n\subset E$, also compatible[5] with F, such that $D(E)-1/n<D(E_n)<D(E)$. Then

$$1-\frac{1}{n}<D(E_n)+D(F)<1$$

and so, by Part 1, there is a full Lebesgue measure set Λ_n such that

$$D(E_n+\lfloor\lambda F\rfloor)\geq 1-\frac{1}{n}\ ,\quad \forall\,\lambda\in\Lambda_n.$$

The set $\Lambda=\bigcap_{n\geq1}\Lambda_n$ has full Lebesgue measure as well and

$$D(E+\lfloor\lambda F\rfloor)\geq 1,\quad \forall\,\lambda\in\Lambda.$$

□

Proof of Theorem 1.2. We also divide it in parts.

Part 1. $\sum_{i=0}^{k} {d_i}^{-1}\leq 1$: by Theorem 1.1,

$$D(E_0+\lfloor\lambda_1E_1\rfloor)\geq\frac{1}{d_0}+\frac{1}{d_1}\ ,\quad m-\text{a.e }\lambda_1\in\mathbb{R}.$$

To each of these parameters, apply Proposition 4.3 to obtain a regular subset $F_{\lambda_1}\subset E_0+\lfloor\lambda_1E_1\rfloor$ such that

$$D(F_{\lambda_1})=\frac{1}{d_0}+\frac{1}{d_1}\cdot$$

As E_2 is universal, another application of Theorem 1.1 guarantees that

$$D(F_{\lambda_1}+\lfloor\lambda_2E_2\rfloor)\geq\frac{1}{d_0}+\frac{1}{d_1}+\frac{1}{d_2}\ ,\quad m-\text{a.e }\lambda_2\in\mathbb{R}.$$

and them, by Fubini's theorem,

$$D(E_0+\lfloor\lambda_1E_1\rfloor+\lfloor\lambda_2E_2\rfloor)\geq\frac{1}{d_0}+\frac{1}{d_1}+\frac{1}{d_2}\ ,\quad m_2-\text{a.e }(\lambda_1,\lambda_2)\in\mathbb{R}^2.$$

[5]This may be assumed because of relation (4.1).

Iterating the above arguments, it follows that

$$D(E_0+\lfloor\lambda_1 E_1\rfloor+\cdots+\lfloor\lambda_k E_k\rfloor)\geq \frac{1}{d_0}+\cdots+\frac{1}{d_k}\ ,\quad m_k-\text{a.e }(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}^k.$$

Part 2. $\sum_{i=0}^k {d_i}^{-1}>1$: without loss of generality, we can assume

$$\frac{1}{d_0}+\cdots+\frac{1}{d_{k-1}}\leq 1<\frac{1}{d_0}+\cdots+\frac{1}{d_{k-1}}+\frac{1}{d_k}\cdot$$

By Part 1,

$$D(E_0+\lfloor\lambda_1 E_1\rfloor+\cdots+\lfloor\lambda_{k-1} E_{k-1}\rfloor)\geq \frac{1}{d_0}+\cdots+\frac{1}{d_{k-1}}$$

for $m_{k-1}-$ a.e $(\lambda_1,\ldots,\lambda_{k-1})\in\mathbb{R}^{k-1}$. To each of these $(\lambda_1,\ldots,\lambda_{k-1})\in\mathbb{R}^{k-1}$, let $F_{(\lambda_1,\ldots,\lambda_{k-1})}$ be a regular subset of $E_0+\cdots+\lfloor\lambda_{k-1}E_{k-1}\rfloor$ such that

$$D\left(F_{(\lambda_1,\ldots,\lambda_{k-1})}\right)=\frac{1}{d_0}+\cdots+\frac{1}{d_{k-1}}\cdot$$

Finnaly, because $D(F_{(\lambda_1,\ldots,\lambda_{k-1})})+D(E_k)>1$, Theorem 1.1 guarantees that

$$d^*\left(F_{(\lambda_1,\ldots,\lambda_{k-1}}+\lfloor\lambda_k E_k\rfloor\right)>0\ ,\quad m-\text{a.e }\lambda_k\in\mathbb{R}\,,$$

which, after another appication of Fubini's theorem, concludes the proof. □

6. Concluding remarks

We think there is a more specific way of defining the counting dimension that encodes the conditions of regularity and compatibility. A natural candidate would be a prototype of a Hausdorff dimension, where one looks to all covers, properly renormalized in the unit interval, and takes a lim inf. An alternative definition has appeared in [15].

Another interesting question is to consider arithmetic sums $E+nF$, where $n\in\mathbb{Z}$. These are genuine arithmetic sums, but we think very strong conditions on the sets E,F are needed to imply any result about $E+nF$.

The theory developed in this article may be extended without any problems to subsets of $\mathbb{Z}^k$. Given $E\subset\mathbb{Z}^k$, the *upper-Banach density* of E is equal to

$$d^*(E)=\limsup_{|I_1|,\ldots,|I_k|\to\infty}\frac{|E\cap(I_1\times\cdots\times I_k)|}{|I_1\times\cdots\times I_k|}\,,$$

where $I_1,\ldots,I_k$ run over all intervals of $\mathbb{Z}$, the *counting dimension* of E is

$$D(E)=\limsup_{|I_1|,\ldots,|I_k|\to\infty}\frac{\log|E\cap(I_1\times\cdots\times I_k)|}{\log|I_1\times\cdots\times I_k|}\,,$$

where $I_1,\ldots,I_k$ run over all intervals of $\mathbb{Z}$ and, for any α, the *counting α-measure* of E is

$$H_\alpha(E)=\limsup_{|I_1|,\ldots,|I_k|\to\infty}\frac{|E\cap(I_1\times\cdots\times I_k)|}{|I_1\times\cdots\times I_k|^\alpha}\,,$$

where $I_1,\ldots,I_k$ run over all intervals of $\mathbb{Z}$. The notions of regularity and compatibility are defined in an analogous manner. If $E,F\subset\mathbb{Z}^k$ are two regular compatible subsets, then

$$D(E+\lfloor\lambda F\rfloor)\geq\min\{k,D(E)+D(F)\}$$

for Lebesgue almost every $\lambda\in\mathbb{R}$. If in addition $D(E)+D(F)>k$, then $E+\lfloor\lambda F\rfloor$ has positive upper-Banach density for Lebesgue almost every $\lambda\in\mathbb{R}$.

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