EXISTENCE OF DOUBLING MEASURES VIA GENERALISED NESTED CUBES

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ABSTRACT. Using generalised dyadic cubes and the mass distribution principle, we give a straightforward new proof for the existence of doubling measures in complete doubling metric spaces.

1. Introduction and notation

A measure μ on a metric space (X, d) is called *doubling* if there is a constant $1 \le D < \infty$ such that

$$0 < \mu\big(B(x, 2r)\big) \le D\mu\big(B(x, r)\big) < \infty$$

for all $x \in X$ and $r > 0$. Here $B(x, r) = \{y \in X : d(x, y) \le r\}$ is a closed ball with centre x and radius r. We denote open balls by $U(x, r)$. By a measure we always mean a Borel regular outer measure. A metric space X has the *finite doubling property* if any ball $B(x, 2r) \subset X$ may be covered by finitely many balls of radius r. Furthermore, such a space is *doubling* if the number of the r-balls needed to cover $B(x, 2r)$ has an upper bound $N \in \mathbb{N}$ independent of x and r.

Let $\mathcal{D}(X)$ be the collection of all doubling measures on X. It is clear that if $\mathcal{D}(X) \ne \emptyset$, then X is doubling. The reverse implication is true if X is assumed to be complete. For compact doubling metric spaces this very important result was first proved by Vol'berg and Konyagin [13, 14]. Luukkainen and Saksman [10] generalised it to the complete case. A slightly simpler proof in the compact case can be found in Wu [15] (see also Heinonen [5]). Saksman [12] has constructed examples of domains Ω with $\mathcal{D}(\Omega) = \emptyset$ and the results of Csörnyei and Suomala [2] may be used to study whether $\mathcal{D}(X) \ne \emptyset$ for certain countable sets $X \subset \mathbb{R}$.

In this note, we give yet another proof for the fact that complete doubling metric spaces carry doubling measures. The doubling measures constructed in the papers [14, 15, 10] are all weak limits of approximating sequences of atomic measures. Our proof uses nested partitions of X and the mass distribution principle. It is purely a matter of taste, which of the two approaches is simpler and/or more natural; in fact, the main idea is the same in both of the approaches. One advantage in our proof is that it works directly also in the unbounded case. Thus the extra step in

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the unbounded case (see [10]) is not needed. Also, once the partitions have been fixed, our method can be used to construct doubling measures possessing certain measure theoretical self-similarity; See §4.

As a main technical tool, we use nested families of "cubes" sharing most of the good properties of dyadic (or r-adic) cubes of Euclidean spaces. The existence of such kind of nested partitions of metric spaces is also quite well known (see e.g. [9, 11, 1, 6]). In this note, we construct these generalised cubes in a more straightforward manner.

In what follows, the (metric) closure, interior and boundary of a set $A \subset X$ are denoted by $\overline{A}$, $\mathrm{int}(A)$ and ∂A, respectively.

2. Nested cubes in metric spaces with finite doubling property

Theorem 2.1. *If X is a metric space with the finite doubling property and $0 < r < \frac{1}{3}$, then there exists a collection $\{Q_{k,i} : k \in \mathbb{Z},\ i \in N_k \subset \mathbb{N}\}$ of Borel sets having the following properties:*

(i) $X = \bigcup_{i \in N_k} Q_{k,i}$ *for every* $k \in \mathbb{Z}$,
(ii) $Q_{k,i} \cap Q_{m,j} = \emptyset$ *or* $Q_{k,i} \subset Q_{m,j}$ *when* $k, m \in \mathbb{Z}$, $k \geq m$, $i \in N_k$ *and* $j \in N_m$,
(iii) for every $k \in \mathbb{Z}$ *and* $i \in N_k$ *there exists a point* $x_{k,i} \in X$ *so that*

$$U(x_{k,i}, cr^k) \subset Q_{k,i} \subset B(x_{k,i}, Cr^k)$$

where $c = \frac{1}{2} - \frac{r}{1-r}$ *and* $C = \frac{1}{1-r}$,
(iv) there exists a point $x_0 \in X$ *so that for every* $k \in \mathbb{Z}$ *there is* $i \in N_k$ *so that*

$$U(x_0, cr^k) \subset Q_{k,i},$$

(v) $\{x_{k,i} : i \in N_k\} \subset \{x_{k+1,i} : i \in N_{k+1}\}$ *for all* $k \in \mathbb{Z}$.

Proof. Fix a point $x_0 \in X$ and start by choosing a maximal collection of points $\{x_{0,i} : i \in N_0\} \subset X$ containing x_0 and having the property that $d(x_{0,i}, x_{0,j}) \geq 1$ if $i \neq j$. Next, for each $k \in \mathbb{N}$, let $\{x_{k,i} : i \in N_k\} \supset \{x_{k-1,i} : i \in N_{k-1}\}$ be a maximal collection of points having mutual distances at least r^k. If $k \in \mathbb{Z}$, $k < 0$, we let $\{x_{k,i} : i \in N_k\}$ be any maximal subcollection of $\{x_{k+1,i} : i \in N_{k+1}\}$ containing x_0 whose points have mutual distances at least r^{k-1}. It is then clear that

$$\{x_{k,i} : i \in N_k\} \subset \bigcup_{j \in N_{k-1}} B(x_{k-1,j}, r^{k-1}) \tag{2.1}$$

for all $k \in \mathbb{Z}$.

In the set of all possible pairs (k, i), $k \in \mathbb{Z}$, $i \in N_k$, consider a partial order $\prec$ that satisfies the following property: For each $k \in \mathbb{Z}$ and $i \in N_{k+1}$, we have $(k+1, i) \prec (k, j)$ if

$$j = \min\bigl\{h \in N_k : \mathrm{dist}(x_{k+1,i}, x_{k,h}) = \min_{l \in N_k} \mathrm{dist}(x_{k+1,i}, x_{k,l})\bigr\}.$$

Notice that $\min_{l\in N_k}\operatorname{dist}(x_{k+1,i},x_{k,l})$ exists, because X has the finite doubling property, and that j also exists and is unique. The sets $Q_{k,i}$ will be defined by using this partial order.

We first define the sets $Q_{0,i}$ for $i \in N_1$ as

$$Q_{0,i} = \overline{\{x_{l,j} \,:\, (l,j) \prec (0,i)\}} \setminus \bigcup_{j<i} Q_{0,j}.$$

For $k<0$ we define the sets $Q_{k,i}$ inductively as

$$Q_{k,i} = \bigcup_{(k+1,j)\prec(k,i)} Q_{k+1,j}$$

whereas for $k>0$, we let

$$Q_{k,i} = Q_{k-1,j} \cap \overline{\{x_{l,j} \,:\, (l,j) \prec (k,i)\}} \setminus \bigcup_{j<i} Q_{k,j},$$

where $(k,i) \prec (k-1,j)$. The defined sets are clearly Borel.

Let us check that the sets $Q_{k,i}$ satisfy the conditions (i)–(v). To see (i) it is enough to notice that

$$\bigcup_{i\in N_k} Q_{k,i} = \bigcup_{i\in N_k} \overline{\{x_{l,j} \,:\, (l,j) \prec (k,i)\}},$$

which is dense and closed in X. Conditions (ii) and (v) follow immediately from the construction. Let us next verify (iii). Since $d(x_{m,i},x_{m+1,j}) \le r^m$ if $(m+1,j) \prec (m,i)$ using (2.1), we get

$$Q_{k,i} \subset B\big(x_{k,i}, \sum_{m=k}^{\infty} r^m\big) = B\left(x_{k,i}, \tfrac{1}{1-r} r^k\right).$$

On the other hand, if $n>k$ and $(n,j) \not\prec (k,i)$, then $d(x_{k,i},x_{n,j}) \ge \frac{1}{2}r^k - \sum_{m=k+1}^{\infty} r^m = (\frac{1}{2} - \frac{r}{1-r})r^k$. This implies

$$U\left(x_{k,i}, (\tfrac{1}{2} - \tfrac{r}{1-r})r^k\right) \subset Q_{k,i}$$

and finishes the proof of (iii). Finally, the claim (iv) follows from (iii) since $x_0 \in \{x_{k,i} \,:\, i \in N_k\}$ for all $k \in \mathbb{Z}$. □

3. Existence of doubling measures

Theorem 3.1. *If $X \neq \emptyset$ is a complete doubling metric space, then $\mathcal{D}(X) \neq \emptyset$.*

Proof. Fix $0 < r \le \frac{1}{7}$ and let $\mathcal{Q} = \{Q_{k,i} : k \in \mathbb{Z},\ i \in N_k\}$, $\{x_{k,i} : k \in \mathbb{Z},\ i \in N_k\}$, and the constants $0<c<C<\infty$ be as in Theorem 2.1. Let

$$M_{k,i} = \#\{j \in N_{k+1} : Q_{k+1,j} \subset Q_{k,i}\} - 1. \tag{3.1}$$

Since X is doubling, it follows using Theorem 2.1 (iii) that there is $M \in \mathbb{N}$ such that $M_{k,i} \le M$ for every $k \in \mathbb{Z}$ and $i \in N_k$. Equip the set of all infinite words $\Sigma =$

$\{(i_k)_{k\in\mathbb{Z}} : Q_{k,i_k} \subset Q_{k-1,i_{k-1}}$ for all $k \in \mathbb{Z}\}$ with the usual ultrametric: the distance between two different words (i_k) and (j_k) is 2^{-n}, where n is the first index at which the words differ, $n = \min\{k : i_k \neq j_k\}$. Theorem 2.1 (ii) and (iv) guarantee that this metric is well defined. We also define cylinders $[k,i] = \{(j_n)_{n\in\mathbb{Z}} : j_k = i\}$ and set $\Sigma_* = \{[k,i] : k \in \mathbb{Z}$ and $i \in N_k\}$. Since X is complete, we may define a *projection* $\pi\colon \Sigma \to X$ by the relation $\{\pi\big((i_k)_{k\in\mathbb{Z}}\big)\} = \bigcap_{k\in\mathbb{Z}} \overline{Q_{k,i_k}}$. Now we clearly have $\pi([k,i]) = \overline{Q_{k,i}}$ for every $k \in \mathbb{Z}$ and $i \in N_k$.

We define a set function $\nu\colon \Sigma_* \to [0,\infty)$ by first choosing $0 < p < 1/(M+1)$, $i_0 \in N_0$ and setting $\nu([0,i_0]) = 1$ and then requiring that for every $k \in \mathbb{Z}$ and $i \in N_k$ we have

$$\nu([k+1,i]) = \begin{cases} p\nu([k,j]), & \text{if } Q_{k+1,i} \subset Q_{k,j} \text{ and } x_{k+1,i} \neq x_{k,j}, \\ (1 - M_{k,i}p)\nu([k,j]), & \text{if } Q_{k+1,i} \subset Q_{k,j} \text{ and } x_{k+1,i} = x_{k,j}. \end{cases} \tag{3.2}$$

We may now easily extend ν to a measure on Σ by setting

$$\nu(A) = \inf\{\sum_j \nu([k_j,i_j]) \,:\, A \subset \cup_j [k_j,i_j]\}.$$

for all $A \subset X$. The main reason for this to work is the fact that the cylinders $[k,j]$ are both open and closed (compact) in Σ. See also [4, §10]. It follows immediately from the construction that ν is a doubling measure on Σ.

Let $\mu = \pi\nu$ be the projected measure on X given by $\mu(A) = \nu(\pi^{-1}(A))$ for all $A \subset X$. It is then clear that we have the estimates

$$\mu\big(\mathrm{int}(Q_{k,i})\big) \leq \nu([k,i]) \leq \mu(\overline{Q_{k,i}}) \tag{3.3}$$

for every $k \in \mathbb{Z}$ and $i \in N_k$. Let us next show that this can be sharpened to

$$\mu(Q_{k,i}) = \nu([k,i]). \tag{3.4}$$

Fix $k \in \mathbb{Z}$, $i \in N_k$ and $(j_n)_{n\in\mathbb{Z}} \in \pi^{-1}(\partial Q_{k,i})$. Then $x := \pi\big((j_n)_{n\in\mathbb{Z}}\big) \in \overline{Q_{n,j_n}}$ for every $n \in \mathbb{Z}$. Theorem 2.1(iii) together with the fact $r \leq 1/7$ implies that

$$\overline{Q_{n+1,l_{n+1}}} \subset B(x_{n,j_n}, Cr^{n+1}) \subset U(x_{n,j_n}, cr^n) \subset Q_{n,j_n}$$

for every $n \in \mathbb{Z}$, where $l_{n+1} \in N_{n+1}$ satisfies $x_{n+1,l_{n+1}} = x_{n,j_n}$. Hence $Q_{n+1,l_{n+1}} \cap \partial Q_{k,i} = \emptyset$ and so $[n+1,l_{n+1}] \subset [n,j_n] \setminus \pi^{-1}(\partial Q_{k,i})$ for every $n \in \mathbb{Z}$. This means that $\pi^{-1}(\partial Q_{k,i})$ is a porous subset of Σ and since ν is doubling, it now follows that $\mu(\partial Q_{k,i}) = \nu\big(\pi^{-1}(\partial Q_{k,i})\big) = 0$ (See e.g. [7, Proposition 3.4] for a proof of this elementary fact). Combining this with (3.3) implies (3.4).

It remains to show that μ is doubling. If $y \in X$ and $t > 0$, let $k \in \mathbb{Z}$ be such that $3r^k \leq t < 3r^{k-1}$. By Theorem 2.1(iii), the ball $B(y,t)$ contains $Q_{k,i}$ for some $i \in N_k$. On the other hand, the ball $B(y,2t)$ intersects $Q_{k,j}$ for at most $\widetilde{M}$ indices $j \in N_k$, where $\widetilde{M} < \infty$ depends only on r and the doubling constant N of X. Thus it suffices to show that there exists a constant $1 \leq \widetilde{C} < \infty$ so that

$$\mu(Q_{k,j}) \leq \widetilde{C}\mu(Q_{k,i}) \tag{3.5}$$

whenever $Q_{k,j} \cap B(y,2t) \neq \emptyset$. Fix $j \in N_k$ for which $Q_{k,j} \cap B(y,2t) \neq \emptyset$. We may assume that $i \neq j$ as otherwise (3.5) holds trivially. Observe that

$$d(x_{k,i}, x_{k,j}) \leq 3t + Cr^k < r^{k-3}. \tag{3.6}$$

Let m be the largest integer such that $Q_{k,i} \cup Q_{k,j} \subset Q_{m,l}$ for some $l \in N_m$. For each $m \leq n \leq k$ let $i_n, j_n \in N_n$ be the indices that satisfy $Q_{k,i} \subset Q_{n,i_n}$ and $Q_{k,j} \subset Q_{n,j_n}$. If $m < n \leq k-4$, it follows that

$$x_{n,j_n} \neq x_{n+1,j_{n+1}} \text{ and } x_{n,i_n} \neq x_{n+1,i_{n+1}} \tag{3.7}$$

as otherwise Theorem 2.1 implies (recall $r \leq \frac{1}{7}$)

$$d(x_{k,i}, x_{k,j}) > cr^n - Cr^{n+1} = \left(\tfrac{1}{2} - \tfrac{2r}{1-r}\right) r^n \geq r^{n+1} \geq r^{k-3}$$

contrary to (3.6). Now (3.4), (3.7), and (3.2) imply that $\mu(Q_{n,j_n}) = p\mu(Q_{n+1,j_{n+1}})$ and $\mu(Q_{n,i_n}) = p\mu(Q_{n+1,i_{n+1}})$ for $m < n \leq k-4$. Hence

$$\frac{\mu(Q_{k,j})}{\mu(Q_{k,i})} = \prod_{n=m}^{k-1} \frac{\mu(Q_{n+1,j_{n+1}})}{\mu(Q_{n,j_n})} \frac{\mu(Q_{n,i_n})}{\mu(Q_{n+1,i_{n+1}})} \leq p^{-4}$$

giving (3.5) and finishing the proof. □

4. Further remarks

1. It is tempting to try to define the measure μ in the above proof directly without going into the code space Σ. More precisely, first defining $\widetilde{\mu}(Q_{k,i})$ as $\nu([k,i])$ in (3.2) and then letting $\mu(A) = \inf\{\sum_j \widetilde{\mu}(Q_{k_j,i_j}) : A \subset \cup_j Q_{k_j,i_j}\}$ for $A \subset X$. Although it now follows from the proof of Theorem 3.1, it is not a priori clear that $\mu(Q_{k,i}) = \widetilde{\mu}(Q_{k,i})$ for all k and i. Observe that in the code space Σ this is not a problem since the cylinders $[k,i]$ are both open and compact.

2. The authors of the articles [14, 10] prove not only the existence of doubling measures but also the existence of α-homogeneous measures for each α strictly larger than the Assouad dimension of X (see e.g. [5] for the definitions). It is an easy exercise to check that given such α, if $r > 0$ is small enough and $p = r^{-\beta}$ in the proof of Theorem 3.1 above, where β is between α and the Assouad dimension of X, then the measure μ will be α-homogeneous.

3. Let us briefly discuss how a method from [8] can be used to bound the upper local dimensions

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{t \downarrow 0} \frac{\log \mu(B(x,t))}{\log t}$$

for the measure μ constructed in the proof of Theorem 3.1. We define the local L^q-spectrum $\tau_q(\mu, x)$ for $0 < q < 1$ as in [8, Theorem 4.4] by setting

$$\tau_q(\mu, x) = \lim_{t \downarrow 0} \liminf_{k \to \infty} \frac{\log \sum_{Q \in \mathcal{Q}_k(x,t)} \mu(Q)^q}{k \log r},$$

where $\mathcal{Q}_k(x,t) = \{Q_{k,j} \cap B(x,t) : j \in N_k\}$. Fix $k \in \mathbb{Z}$ and $i \in N_k$ and let $M_{k,i}$ be as in (3.1). Since $M_{k,i} \le M$ and $p \le \frac{1}{M_{k,i}+1}$, we get $M_{k,i}\,p^q + (1 - M_{k,i}\,p)^q \le Mp^q + (1 - Mp)^q$ for all $0 < q < 1$. Using (3.2) and (3.5) this implies

$$\sum_{Q_{k+1,j} \subset Q_{k,i}} \mu(Q_{k+1,j})^q = \mu(Q_{k,i})^q \left(M_{k,i}\,p^q + (1 - M_{k,i}\,p)^q\right) \le \mu(Q_{k,i}) \left(Mp^q + (1 - Mp)^q\right).$$

Using this estimate recursively leads to $\tau_q(\mu, x) \ge \log(Mp^q + (1 - Mp)^q)/\log r$ for all $x \in X$ and $0 < q < 1$. Combining this with [8, Theorem 4.2] gives

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \le \lim_{q \uparrow 1} \frac{\tau_q(\mu, x)}{q - 1} \le \frac{Mp \log p + (1 - Mp)\log(1 - Mp)}{\log r} \tag{4.1}$$

for μ-almost all $x \in X$.

As the upper bound in (4.1) can be made arbitrarily small by choosing $p > 0$ small enough in (3.2), the above discussion shows that for any $\varepsilon > 0$, there is a doubling measure μ on X such that $\overline{\dim}_{\mathrm{loc}}(\mu, x) \le \varepsilon$ for μ-almost all $x \in X$. In particular, such a μ has full measure on a set of packing dimension at most ε, see [3, §10.1]. We remark that this result was known before only for the Hausdorff dimension, see Wu [15].

4. Using precisely the same idea as in the proof of Theorem 3.1, one can define more general "self-similar" type doubling measures on X. Suppose for instance, that our space X is such that the number of descendants of each cube $Q_{k,i}$ is at least $n \in \mathbb{N}$. Let $p_1, \ldots, p_n > 0$ be positive numbers with $\sum_{i=1}^n p_i = 1$ and fix $0 < p < M^{-2}$. Instead of (3.2), we distribute the measure of $Q_{k,i}$ among the descendants in the following way: For each $1 \le m \le n$ choose $j_m \in N_{k+1}$ so that $Q_{k+1,j_m} \subset Q_{k,i}$ and $j_m \ne j_l$ when $m \ne l$. Define $\mu(Q_{k+2,j}) = p\mu(Q_{k,i})$ if $Q_{k+2,j} \subset Q_{k,i}$ and $x_{k+2,j} \notin \{x_{k+1,j_1}, \ldots x_{k+1,j_n}\}$. Then divide the rest of the measure of $\mu(Q_{k,i})$ among the "central subcubes" of $Q_{k,j_1}, \ldots Q_{k,j_n}$ according to the probabilities $p_1, \ldots, p_n$.

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