THE SPACE OF HEEGAARD SPLITTINGS

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ABSTRACT. For a Heegaard surface Σ in a closed orientable 3-manifold M, $\mathcal{H}(M, \Sigma) = \mathrm{Diff}(M)/\mathrm{Diff}(M, \Sigma)$ is the space of Heegaard surfaces equivalent to the Heegaard splitting (M, Σ). Its path components are the isotopy classes of Heegaard splittings equivalent to (M, Σ). We describe $H(M, \Sigma)$ in terms of $\mathrm{Diff}(M)$ and the Goeritz group of (M, Σ). In particular, for hyperbolic M each path component is a classifying space for the Goeritz group, and when the (Hempel) distance of (M, Σ) is greater than 3, each path component of $\mathcal{H}(M, \Sigma)$ is contractible. For splittings of genus 0 or 1, we determine the complete homotopy type (modulo the Smale Conjecture for M in the cases when it is not known).

Let M be a closed, orientable 3-manifold, not necessarily irreducible, and suppose that Σ is a Heegaard surface in M. The space $\mathcal{H}(M, \Sigma)$ of Heegaard splittings equivalent to (M, Σ) is defined to be the space of left cosets $\mathrm{Diff}(M)/\mathrm{Diff}(M, \Sigma)$, where $\mathrm{Diff}(M, \Sigma)$ is the subgroup of $\mathrm{Diff}(M)$ consisting of diffeomorphisms taking Σ onto Σ. In other words, this is the space of images of Σ under diffeomorphisms of M.

We will denote the homotopy groups $\pi_i(\mathcal{H}(M, \Sigma))$ by $\mathcal{H}_i(M, \Sigma)$. In particular, $\mathcal{H}_0(M, \Sigma)$ is the set of isotopy classes of Heegaard splittings equivalent to (M, Σ). In the present work, we focus on the groups $\mathcal{H}_i(M, \Sigma)$ for $i \geq 1$. (Note that $\mathcal{H}_i(M, \Sigma)$ is independent of the basepoint chosen, because $\mathrm{Diff}(M)$ acts transitively on $\mathcal{H}(M, \Sigma)$ and consequently any two path components are homeomorphic. We use the identity map 1_M, or more strictly speaking, the coset $1_M \mathrm{Diff}(M, \Sigma)$, as our implicit choice of basepoint of $\mathcal{H}(M, \Sigma)$.)

As one would expect, $\mathcal{H}(M, \Sigma)$ is closely related to $\mathrm{Diff}(M)$. When the genus of Σ is at least 2, we have a general result.

Theorem 1. *Suppose that Σ has genus at least 2. Then $\pi_q(\mathrm{Diff}(M)) \to \mathcal{H}_q(M, \Sigma)$ is an isomorphism for $q \geq 2$, and there are exact sequences*

$$1 \to \pi_1(\mathrm{Diff}(M)) \to \mathcal{H}_1(M, \Sigma) \to G(M, \Sigma) \to 1 \ ,$$
$$1 \to G(M, \Sigma) \to \mathrm{Mod}(M, \Sigma) \to \mathrm{Mod}(M) \to \mathcal{H}_0(M, \Sigma) \to 1 \ .$$

In Theorem 1, $\mathrm{Mod}(M)$ and $\mathrm{Mod}(M, \Sigma)$ denote the groups of path components of $\mathrm{Diff}(M)$ and $\mathrm{Diff}(M, \Sigma)$ respectively, and $G(M, \Sigma)$ is the *Goeritz group* of the Heegaard splitting, defined to be the kernel of $\mathrm{Mod}(M, \Sigma) \to \mathrm{Mod}(M)$. We remark that for most reducible M, $\pi_1(\mathrm{Diff}(M))$ is known to be non-finitely-generated [24], suggesting that $\mathcal{H}(M, \Sigma)$ has a complicated homotopy type in these cases.

When $\pi_1(M)$ is infinite, Theorem 1 applies to all cases except the genus-1 Heegaard surface in $S^1 \times S^2$. To state our result for that case, denote by LX the space of smooth free loops in a smooth manifold X, that is, the C^∞ maps from S^1 to X, with the C^∞ topology. There is a free involution $\alpha \colon LS^2 \to LS^2$ defined by $\alpha(\gamma) = \rho \circ \gamma$, where $\rho \colon S^2 \to S^2$ is the antipodal map. The quotient $LS^2/\langle \alpha \rangle$ can be identified with the connected component of the constant loop in $L\,\mathbb{RP}^2$.

Theorem 2. *For the unique genus-1 Heegaard surface Σ in $S^1 \times S^2$, $\mathcal{H}(S^1 \times S^2, \Sigma)$ is homotopy equivalent to $LS^2/\langle \alpha \rangle$.*

We remark that (at least when X has empty boundary) the inclusion function from LX to the space of all continuous free loops in X (with the compact-open topology) is a homotopy equivalence (see A. Stacey [31, Theorem 4.6]). The analogous statement holds for the space ΩX of smooth based free loops [31, Section 4.3]). The map $LX \to X$ given by evaluation at the basepoint is a locally trivial fibration [31, Corollary 4.8], with fiber ΩX. Since $LX \to X$ has an obvious section, the exact sequence of this fibration shows that $\pi_q(LX) \cong \pi_{q+1}(X) \oplus \pi_q(X)$ for all $q \geq 1$. The homology of LS^n was computed in W. Ziller [34, p. 21] (see also R. Cohen, J. Jones, and J. Yan [6]); for $n = 2$ it is $H_0(LS^2) \cong \mathbb{Z}$, $H_k(LS^2) \cong \mathbb{Z}$ for $k > 0$ odd, and $H_k(LS^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ for $k > 0$ even.

When $\pi_1(M)$ is infinite and M is irreducible, all Heegaard splittings of M have genus at least 2. In addition, apart from one case in which $\mathrm{Diff}(M)$ has not been fully determined, we know that $\mathrm{Diff}(M)$ has a very simple homotopy type. Theorem 1 becomes the following statement:

Corollary 1. *Suppose that M is irreducible and $\pi_1(M)$ is infinite, and that M is not a non-Haken infranilmanifold. Then $\mathcal{H}_i(M, \Sigma) = 0$ for $i \geq 2$, and there is an exact sequence*

$$1 \to Z(\pi_1(M)) \to \mathcal{H}_1(M, \Sigma) \to G(M, \Sigma) \to 1 .$$

Note that when the conclusion of Corollary 1 holds, each component of $\mathcal{H}(M, \Sigma)$ is aspherical, and if $\pi_1(M)$ is centerless, is a classifying space $K(G(M, \Sigma), 1)$ for the Goeritz group. As to the excluded cases in Corollary 1, a nilmanifold is a 3-manifold that is a quotient of Heisenberg space by a torsion-free lattice (topologically these are the S^1-bundles over the torus with nonzero Euler class), and an infranilmanifold is a finite quotient of a nilmanifold. Non-Haken infranilmanifolds are Seifert-fibered with base orbifold a 2-sphere with three cone points of types $(2, 4, 4)$, $(2, 3, 6)$, or $(3, 3, 3)$.

If the components of $\mathrm{Diff}(M)$ turn out to be homotopy equivalent to S^1 for these manifolds, as expected (see [27]), then Corollary 1 will hold without exclusion.

Corollary 1 applies whenever the (Hempel) distance $d(M, \Sigma)$ is greater than 3. Combined with various results from the literature, this provides a rather complete description of the homotopy type of $\mathcal{H}(M, \Sigma)$ for this case:

Corollary 2. *If $d(M, \Sigma) > 3$ then $\mathcal{H}(M, \Sigma)$ has finitely many components, each of which is contractible. In fact, the number of components of $\mathcal{H}(M, \Sigma)$ equals $|\mathrm{Mod}(M)|/|\mathrm{Mod}(M, \Sigma)|$, and if $d(M, \Sigma) > 2\,\mathrm{genus}(\Sigma)$, then $\mathcal{H}(M, \Sigma)$ is contractible.*

When $\pi_1(M)$ is finite, $\mathrm{Diff}(M)$ and $\mathcal{H}(M, \Sigma)$ can have more interesting homotopy types. For these cases, M admits an elliptic structure, that is, a Riemannian metric of constant sectional curvature 1, or equivalently M is a quotient of the standard round 3-sphere by a group of isometries acting freely. For elliptic 3-manifolds, the (Generalized) Smale Conjecture asserts that the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ of the subgroup of isometries of M is a homotopy equivalence. As we will discuss in Section 7 below, the Smale Conjecture is known for some cases, including S^3 and lens spaces other than $\mathbb{RP}^3$, but is open in general. Our computations of $\mathcal{H}_i(M, \Sigma)$ require this homotopy equivalence, and therefore must be regarded as modulo the Smale Conjecture for the unknown cases. In the statements of our remaining results, C_2 denotes a cyclic group of order 2.

Theorem 3. *For $n \geq 0$ let Σ_n be the unique Heegaard surface of genus n in S^3.*

(1) $\mathcal{H}(S^3, \Sigma_0) \simeq \mathbb{RP}^3$.
(2) $\mathcal{H}(S^3, \Sigma_1) \simeq \mathbb{RP}^2 \times \mathbb{RP}^2$,
(3) *For $n \geq 2$, $\mathcal{H}_i(S^3, \Sigma_n) \cong \pi_i(S^3 \times S^3)$ for $i \geq 2$, and there is a non-split exact sequence*

$$1 \to C_2 \to \mathcal{H}_1(S^3, \Sigma_n) \to G(S^3, \Sigma_n) \to 1 \ .$$

Theorem 4. *Let $L = L(m, q)$ be a lens space with $m \geq 2$ and $1 \leq q \leq m/2$. Assume, if necessary, that L satisfies the Smale Conjecture. For $n \geq 1$, let Σ_n be the unique Heegaard surface of genus n in L.*

(1) *If $q \geq 2$, then*
    (a) $\mathcal{H}(L, \Sigma_1)$ *is contractible.*
    (b) *For $n \geq 2$, $\mathcal{H}_i(L, \Sigma_n) = 0$ for $i \geq 2$, and there is an exact sequence*

$$1 \to \mathbb{Z} \times \mathbb{Z} \to \mathcal{H}_1(L, \Sigma_n) \to G(L, \Sigma_n) \to 1 \ .$$

(2) *If $m > 2$ and $q = 1$, then*
    (a) $\mathcal{H}(L, \Sigma_1) \simeq \mathbb{RP}^2$.

(b) *For $n \geq 2$, $\mathcal{H}_i(L, \Sigma_n) \cong \pi_i(S^3)$ for $i \geq 2$, and there are exact sequences*

$$1 \to \mathbb{Z} \to \mathcal{H}_1(L, \Sigma_n) \to G(L, \Sigma_n) \to 1$$

for m odd, and

$$1 \to \mathbb{Z} \times C_2 \to \mathcal{H}_1(L, \Sigma_n) \to G(L, \Sigma_n) \to 1$$

for m even.

(3) *If $L = L(2,1)$, then*
  (a) $\mathcal{H}(L, \Sigma_1) \simeq \mathbb{RP}^2 \times \mathbb{RP}^2$.
  (b) *For $n \geq 2$, $\mathcal{H}_i(L, \Sigma_n) \cong \pi_i(S^3 \times S^3)$ for $i \geq 2$, and there is an exact sequence*

$$1 \to C_2 \times C_2 \to \mathcal{H}_1(L, \Sigma_n) \to G(L, \Sigma_n) \to 1 .$$

Theorem 5. *Let E be an elliptic 3-manifold, but not S^3 or a lens space. Assume, if necessary, that E satisfies the Smale Conjecture. Let Σ be a Heegaard surface in E.*

(1) *If $\pi_1(E) \cong D_{4m}^*$, or if E is one of the three manifolds with fundamental group either T_{24}^*, O_{48}^*, or I_{120}^*, then $\mathcal{H}_i(E, \Sigma) \cong \pi_i(S^3)$ for $i \geq 2$ and there is an exact sequence*

$$1 \to C_2 \to \mathcal{H}_1(E, \Sigma) \to G(E, \Sigma) \to 1 .$$

(2) *If E is not one of the manifolds in Case (1), that is, either $\pi_1(E)$ has a nontrivial cyclic direct factor, or $\pi_1(E)$ is a diagonal subgroup of index 2 in $D_{4m}^* \times C_n$ or of index 3 in $T_{48}^* \times C_n$, then $\mathcal{H}_i(E, \Sigma) = 0$ for $i \geq 2$, and there is an exact sequence*

$$1 \to \mathbb{Z} \to \mathcal{H}_1(E, \Sigma) \to G(E, \Sigma) \to 1 .$$

Theorems 1 and 2 are proven in Sections 3 and 4 respectively, and Corollaries 1 and 2 in Section 5. Theorem 3(1) is proven in Section 8. Theorem 3(2) and the (a) parts of Theorem 4 are proven in Section 10, and Theorem 3(3), the (b) parts of Theorem 4 and Theorem 5 in Section 11. The other sections provide auxiliary material used in the proofs.

1. Spaces of images, mapping class groups, and Goeritz groups

In this section and the next, we assume only that M is a closed manifold and Σ is a closed submanifold of positive codimension (although much of what we say extends to more general contexts).

A submanifold Σ' of M is called an *image* of Σ if there is a diffeomorphism of M carrying Σ onto Σ'. The images of Σ correspond to the left cosets $\mathrm{Diff}(M)/\mathrm{Diff}(M, \Sigma)$, since if $g, h \in \mathrm{Diff}(M)$, then $g(\Sigma) = h(\Sigma)$ if and only if $g^{-1}h \in \mathrm{Diff}(M, \Sigma)$. Therefore we call $\mathrm{Diff}(M)/\mathrm{Diff}(M, \Sigma)$ the *space of images equivalent to Σ*, and denote it by $\mathrm{Img}(M, \Sigma)$. In particular, when (M, Σ) is a Heegaard splitting of a closed 3-manifold, $\mathrm{Img}(M, \Sigma)$ is the space of Heegaard splittings $\mathcal{H}(M, \Sigma)$ as defined in the introduction.

A Fréchet space is a complete metrizable locally convex topological vector space. The topology of a Fréchet space is defined by a countable collection of seminorms such that $f_j \to f$ if $\|f_j - f\| \to 0$ for each of the seminorms. A Fréchet manifold is a (usually infinite-dimensional) manifold locally modeled on open subsets of a Fréchet space, with smooth (as maps of the Fréchet space) transition functions. Two convenient references for Fréchet spaces and Fréchet manifolds are R. Hamilton [11] and A. Kriegl and P. Michor [25].

The space of images $\operatorname{Img}(M, \Sigma)$ is a Fréchet manifold locally modeled on the sections close to the zero section from Σ to its normal bundle [11, Example 4.1.7]. It follows that $\operatorname{Img}(M, \Sigma)$ has the homotopy type of a CW-complex (see for example Section 2.1 of [18]).

The *mapping class group* $\operatorname{Mod}(M, \Sigma)$ *of the pair* (M, Σ) is defined to be the discrete group $\operatorname{Diff}(M, \Sigma)/\operatorname{diff}(M, \Sigma)$, where $\operatorname{diff}(M, \Sigma)$ (and in general, any space of isometries, diffeomorphisms, or imbeddings whose name begins with a small letter) is the connected component of the identity diffeomorphism. In particular, when Σ is empty, we write this as $\operatorname{Mod}(M)$ and it becomes the usual mapping class group. Note that we allow orientation-reversing diffeomorphisms, when M is orientable, so our $\operatorname{Mod}(M, \Sigma)$ is what is often called the extended mapping class group.

The *Goeritz group* of the pair (M, Σ) is the kernel $G(M, \Sigma)$ of the natural map $\operatorname{Mod}(M, \Sigma) \to \operatorname{Mod}(M)$. When Σ has codimension 1 and is two-sided in M, the *pure* Goeritz group $G_0(M, \Sigma)$ is defined to consist of the elements of $G(M, \Sigma)$ that do not interchange the sides of Σ. It is a subgroup of index at most 2 in $G(M, \Sigma)$.

To indicate the subgroup of orientation-preserving, we use a "+" subscript, as in $\operatorname{Diff}_+(M)$ or $\operatorname{Isom}_+(S^3)$.

2. Fibration theorems

In this section, we will obtain fibrations using a method of R. Palais [28] and J. Cerf [5], which is based on the following definition. Let X be a G-space and $x_0 \in X$. A *local cross-section* for X at x_0 is a map χ from a neighborhood U of x_0 into G such that $\chi(u)x_0 = u$ for all $u \in U$. By replacing $\chi(u)$ by $\chi(u)\chi(x_0)^{-1}$, one may always assume that $\chi(x_0) = 1_G$. If X admits a local cross-section at each point, it is said to admit local cross-sections.

A local cross-section $\chi_0 \colon U_0 \to G$ at a single point x_0 determines a local cross-section $\chi \colon gU_0 \to G$ at any point gx_0 in the orbit of x_0, by the formula $\chi(u) = g\chi_0(g^{-1}u)g^{-1}$, since then $\chi(u)(gx_0) = g\chi_0(g^{-1}u)g^{-1}gx_0 = g\chi_0(g^{-1}u)x_0 = gg^{-1}u = u$. In particular, if G acts transitively on X, then a local cross-section at any point provides local cross-sections at all points.

From [28] we have

Proposition 2.1. *Let G be a topological group and X a G-space admitting local cross-sections. Then any equivariant map of a G-space into X is locally trivial.*

In fact, when $\pi\colon Y \to X$ is G-equivariant, the local coordinates on $\pi^{-1}(U)$ are just given by sending the point $(u,z) \in U \times \pi^{-1}(x_0)$ to $\chi(u) \cdot z$.

We continue to assume only that M is a closed manifold and Σ is a closed submanifold of positive codimension. Clearly $\mathrm{Diff}(M)/\mathrm{Diff}(M,\Sigma)$ and $\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma)$ are $\mathrm{Diff}(M)$-spaces.

Theorem 2.2. $\mathrm{Diff}(M)/\mathrm{Diff}(M,\Sigma)$ *and* $\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma)$ *admit local* $\mathrm{Diff}(M)$ *cross-sections.*

Proof. We will argue for $\mathrm{Diff}(M)/\mathrm{Diff}(M,\Sigma) = \mathrm{Img}(M,\Sigma)$, since the case of $\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma)$ requires only trivial modifications. Since $\mathrm{Diff}(M)$ acts transitively, we need only find a local cross-section at $1_M\,\mathrm{Diff}(M,\Sigma)$.

Fix a Riemannian metric on M and a tubular neighborhood $N(\Sigma)$ determined by the exponential map $\mathrm{Exp}\colon \nu_{<\epsilon}(\Sigma) \to N(\Sigma) \subset M$, where $\nu_{<\epsilon}(\Sigma)$ is the space of normal vectors of Σ of length less than ϵ. For all g in a sufficiently small C^∞-neighborhood V of 1_M (in fact for all g sufficiently C^1-close to 1_M) in $\mathrm{Diff}(M)$, the tangent planes to $g(\Sigma)$ remain almost perpendicular to the tangent planes of the fibers of $N(\Sigma)$, and consequently $g(\Sigma)$ meets each normal fiber in $N(\Sigma)$ in exactly one point.

Denote by $\mathrm{Sect}(\Sigma, TM)$ the sections from Σ to the restriction of TM to Σ, and by Z the zero-section in $\mathrm{Sect}(\Sigma, TM)$ or in any other space of sections.

The image W of V in $\mathrm{Img}(M,\Sigma)$ is an open neighborhood of $1_M\,\mathrm{Diff}(M,\Sigma)$. Define $\Phi\colon W \to \mathrm{Sect}(\Sigma, TM)$ by putting $\Phi(g\,\mathrm{Diff}(M,\Sigma))(x)$ equal to the unique vector in $T_xM \cap \nu_{<\epsilon}(\Sigma)$ that exponentiates to $g(\Sigma) \cap \mathrm{Exp}(\nu_{<\epsilon}(x))$, where $\nu_{<\epsilon}(x)$ is the fiber of $\nu_\epsilon(\Sigma)$ at x. In particular, $\Phi(1_M\,\mathrm{Diff}(M,\Sigma)) = Z$, the zero section.

Lemma c from [28] provides a continuous linear map $k\colon \mathrm{Sect}(\Sigma, TM) \to \mathrm{Sect}(M, TM)$ such that for each $X \in \mathrm{Sect}(\Sigma, TM)$, $k(X)|_\Sigma = X$. In fact, k is defined just by using parallel translation to push each $X(x)$ to a vector at each of the points in the normal fiber at x, then multiplying by a smooth function that is 1 on Σ and is 0 off of $N(\Sigma)$.

Now, define $\mathrm{TExp}\colon \mathrm{Sect}(M, TM) \to C^\infty(M, M)$, the space of smooth maps from M to M with the C^∞-topology, by $\mathrm{TExp}(X)(x) = \mathrm{Exp}(X(x))$. By Lemmas a and b of [28], TExp is continuous and maps a neighborhood of Z into $\mathrm{Diff}(M)$. On a neighborhood U of $1_M\,\mathrm{Diff}(M,\Sigma)$ contained in W and small enough so that $\mathrm{TExp}\circ k \circ \Phi(U) \subset \mathrm{Diff}(M)$, $\mathrm{TExp}\circ k \circ \Phi$ is a local cross-section. For if $g\,\mathrm{Diff}(M,\Sigma) \in U$, then by definition of Φ we have

$$\mathrm{Exp}\circ\Phi(g\,\mathrm{Diff}(M,\Sigma))(x) = g(\Sigma) \cap \mathrm{Exp}(\nu_{<\epsilon}(x))$$

for each $x \in \Sigma$. Therefore

$$\mathrm{Exp}\circ\Phi(g\,\mathrm{Diff}(M,\Sigma))\,1_M\,\mathrm{Diff}(M,\Sigma)(\Sigma) = \mathrm{Exp}\circ\Phi(g\,\mathrm{Diff}(M,\Sigma))(\Sigma) = g(\Sigma)$$

and consequently $\mathrm{TExp}\circ k \circ \Phi(g\,\mathrm{Diff}(M,\Sigma))1_M\,\mathrm{Diff}(M,\Sigma) = g\,\mathrm{Diff}(M,\Sigma)$.
$\square$

Proposition 2.1 and Theorem 2.2 give immediately

Corollary 2.3. *The quotient maps* $\mathrm{Diff}(M) \to \mathrm{Diff}(M)/\mathrm{Diff}(M,\Sigma)$ *and* $\mathrm{Diff}(M) \to \mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma)$ *are fibrations.*

Also, the natural map $\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma) \to \mathrm{Diff}(M)/\mathrm{Diff}(M,\Sigma)$ is $\mathrm{Diff}(M)$-equivariant, with fiber the discrete group $\mathrm{Diff}(M,\Sigma)/\mathrm{diff}(M,\Sigma) = \mathrm{Mod}(M,\Sigma)$, so we have

Corollary 2.4. *The natural map*

$$\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma) \to \mathrm{Diff}(M)/\mathrm{Diff}(M,\Sigma) = \mathrm{Img}(\Sigma)$$

is a covering map with fiber $\mathrm{Mod}(M,\Sigma)$.

Corollary 2.5. *For* $i \geq 2$, $\pi_i(\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma)) \to \pi_i(\mathrm{Img}(\Sigma))$ *is an isomorphism, and there is an exact sequence*

$$1 \to \pi_1(\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma)) \to \pi_1(\mathrm{Img}(M,\Sigma))$$
$$\to \mathrm{Mod}(M,\Sigma) \to \pi_0(\mathrm{Diff}(M)/\mathrm{diff}(M,\Sigma)) \to \pi_0(\mathrm{Img}(M,\Sigma)) \to 1 \ .$$

For later use, we include the following lemma.

Lemma 2.6. *The map* $\mathrm{Diff}(M,\Sigma) \to \mathrm{Diff}(\Sigma)$ *defined by restriction is a fibration over its image (which is a union of path components of* $\mathrm{Diff}(\Sigma)$*).*

Proof. Let $\mathrm{Imb}(\Sigma, M)$ be the space of all imbeddings of Σ into M that extend to diffeomorphisms of M. From [28], the map $\rho\colon \mathrm{Diff}(M) \to \mathrm{Imb}(\Sigma, M)$ defined by $\rho(f) = f|V$ is a fibration. We identify the image of $\mathrm{Diff}(M,\Sigma) \to \mathrm{Diff}(\Sigma)$ with the subspace of elements of $\mathrm{Imb}(\Sigma, M)$ that take Σ to Σ. Since $\mathrm{Diff}(M,\Sigma)$ is the full preimage of this subspace, over its image $\mathrm{Diff}(M,\Sigma) \to \mathrm{Diff}(\Sigma)$ is just the pullback fibration. $\square$

3. Heegaard splittings of genus at least 2

This section contains the proof of Theorem 1. We will use the following theorem of A. Hatcher [12, 16] and N. Ivanvov [19, 20]:

Theorem 3.1 (Hatcher, Ivanov)**.** *Let* M *be a Haken* 3*-manifold.*
  (i) *If* $\partial M \neq \emptyset$, *then* $\mathrm{diff}(M \ rel \ \partial M)$ *is contractible.*
  (ii) *If* M *is closed, then there is a homotopy equivalence* $(S^1)^k \to \mathrm{diff}(M)$, *where* k *is the rank of the center of* $\pi_1(M)$.

In [12], the results are stated for PL homeomorphisms, but the Smale Conjecture for S^3, also proven by Hatcher [15], gives the equivalence of these statements for the two categories (see [16]).

Proof of Theorem 1. Since the genus of Σ is at least 2, $\mathrm{diff}(\Sigma)$ is contractible [7]. From Lemma 2.6, there is a fibration

$$\mathrm{Diff}(M \ rel \ \Sigma) \cap \mathrm{diff}(M,\Sigma) \to \mathrm{diff}(M,\Sigma) \to \mathrm{diff}(\Sigma) \ .$$

Any two elements of $\mathrm{Diff}(M \ rel \ \Sigma) \cap \mathrm{diff}(M,\Sigma)$ are isotopic preserving Σ. Since $\pi_1(\mathrm{diff}(\Sigma))$ is trivial, they are isotopic relative to Σ. Therefore

$\mathrm{Diff}(M \operatorname{rel} \Sigma) \cap \mathrm{diff}(M, \Sigma) = \mathrm{diff}(M \operatorname{rel} \Sigma)$, which is contractible using Theorem 3.1(i), so the fibration shows that $\mathrm{diff}(M, \Sigma)$ is contractible.

By Corollary 2.3, the quotient map

$$\mathrm{Diff}(M) \to \mathrm{Diff}(M)/\mathrm{diff}(M, \Sigma)$$

is a fibration. Since it has contractible fiber, it is a homotopy equivalence. The assertions of Theorem 1 now follow from Corollary 2.5. $\square$

4. The case of $S^1 \times S^2$

In this section, we will prove Theorem 2. For more concise notation, we write M for $S^1 \times S^2$. In addition, we write the standard 2-sphere S^2 as $D_+^2 \cup D_-^2$, the upper and lower hemispheres, N or $+N$ for the center point of D_+^2, the north pole, and $-N$ for the south pole. The isometry group of S^2 is the orthogonal group $\mathrm{O}(3)$. By $\mathrm{O}(2)$ we denote the $\mathrm{O}(2)$-subgroup of $\mathrm{SO}(3)$ that preserves $D_+^2 \cap D_-^2$; its subgroup $\mathrm{SO}(2)$ preserves each of D_+^2 and D_-^2, while elements of $\mathrm{O}(2) - \mathrm{SO}(2)$ interchange D_+^2 and D_-^2. Since $\mathrm{SO}(3)$ acts transitively on S^2 and the stabilizer of N is $\mathrm{SO}(2)$, the space of cosets $\mathrm{SO}(3)/\mathrm{SO}(2)$ is homeomorphic to S^2.

In M define $T = S^1 \times D_+^2 \cap S^1 \times D_-^2$. It is a Heegaard surface in M, and the resulting splitting is called the *standard* genus-1 Heegaard splitting of M. The following must be well known, but we include a proof here.

Proposition 4.1. *Up to isotopy M has a unique Heegaard splitting for each positive genus.*

Proof. Assume first that the Heegaard splitting has genus 1. By Haken's Lemma [10] (see also [4, Lemma 1.1]), there is a 2-sphere S in M that meets each of the solid tori of the splitting in a single disk. It is easy to check that M contains a unique essential 2-sphere up to isotopy, so we may assume that S is a fiber and each solid torus of the splitting is a regular neighborhood of a loop crossing S in a single point. By the well-known light-bulb trick, such a loop is isotopic to a loop of the form $S^1 \times \{x\}$, so the Heegaard splitting is isotopic to the standard one.

Suppose now that the Heegaard splitting has genus $n > 1$, and apply Haken's Lemma as before to obtain a sphere that intersects each handlebody in a disk. Compressing the splitting along one of the two disks, then removing a neighborhood of the essential sphere, one of the handlebodies becomes a handlebody of genus $n - 1$, and the other a handlebody with two punctures. Filling in the punctures gives a Heegaard splitting of S^3 of genus $n - 1$. Waldhausen [33] showed that every positive genus Heegaard splitting of S^3 is a stabilization, which implies that the original Heegaard splitting of M was a stabilization. Inductively, the original splitting is obtained by repeated stabilization of the standard genus-1 splitting. $\square$

Proposition 4.1 shows, of course, that $\mathcal{H}(M, \Sigma)$ is connected for every Heegaard splitting of M.

Our proof of Theorem 2 will use the description of $\mathrm{Diff}(M)$ due to A. Hatcher [13, 14]. To set notation, define $R(M)$ to be the subgroup of $\mathrm{Diff}(S^1 \times S^2)$ consisting of the diffeomorphisms that take each $\{x\} \times S^2$ to some $\{y\} \times S^2$ by an element of the orthogonal group $\mathrm{O}(3)$ that depends on x, and where the diffeomorphism of S^1 sending each x to the corresponding y is an element of $\mathrm{O}(2)$.

As noted in [13, 14], $R(M)$ is homeomorphic (although not isomorphic) to the subgroup $\mathrm{O}(2) \times \mathrm{O}(3) \times \Omega\,\mathrm{SO}(3) \subset \mathrm{Diff}(M)$, where $\Omega\,\mathrm{SO}(3)$ denotes the space of smooth loops $\gamma\colon S^1 \to \mathrm{SO}(3)$ taking the basepoint $0 \in S^1 = \mathbb{R}\,/\,\mathbb{Z}$ to the identity rotation. The $\mathrm{O}(2)$-coordinate tells the effect of an element of $R(M)$ on the S^1-coordinate of $S^1 \times S^2$, the $\mathrm{O}(3)$-coordinate tells the effect on the S^2-coordinate of $\{0\} \times S^2$, and the element of $\Omega\,\mathrm{SO}(3)$ tells the deviation from being constant in the S^2-coordinate as the S^1-coordinate varies. More precisely, an element $(f, g, \gamma) \in R(M)$ acts on M by sending $(t, x) \in S^1 \times S^2$ to $(f(t), \gamma(t)(g(x)))$.

Theorem 4.2 (A. Hatcher). *The inclusion $R(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence.*

Let $R(M, T)$ be the subgroup of $R(M)$ that takes T to T, that is, $R(M) \cap \mathrm{Diff}(M, T)$. Under the homeomorphism from $R(M)$ to $\mathrm{O}(2) \times \mathrm{O}(3) \times \Omega\,\mathrm{SO}(3)$, $R(M, T)$ corresponds to the subgroup $\mathrm{O}(2) \times (C_2 \times \mathrm{O}(2)) \times \Omega\,\mathrm{SO}(2)$. The C_2-factor of $C_2 \times \mathrm{O}(2)$ is generated by the reflection through the equator $D_+^2 \cap D_-^2$.

Proposition 4.3. *The inclusion $R(M, T) \to \mathrm{Diff}(M, T)$ is a homotopy equivalence.*

Proof. By Lemma 2.6, the restriction map $\mathrm{Diff}(M, T) \to \mathrm{Diff}(T)$ is a fibration over its image, which we will denote by $\mathrm{Diff}_0(T)$. Letting $R(T)$ denote the diffeomorphisms of $T = S^1 \times (D_+^2 \cap D_-^2)$ that send each $\{x\} \times (D_+^2 \cap D_-^2)$ to some $\{y\} \times (D_+^2 \cap D_-^2)$ by an element of $\mathrm{O}(2)$, and such that sending each $\{x\}$ to the corresponding $\{y\}$ is an element of $\mathrm{O}(2)$, we have a restriction map $R(M, T) \to R(T)$ that is a 2-fold covering projection.

We now have a commutative diagram

$$
\begin{array}{ccccc}
R(M \operatorname{rel} T) & \longrightarrow & R(M, T) & \longrightarrow & R(T) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle j} \\
\mathrm{Diff}(M \operatorname{rel} T) & \longrightarrow & \mathrm{Diff}(M, T) & \longrightarrow & \mathrm{Diff}_0(T)
\end{array}
$$

whose rows are fibrations and vertical maps are inclusions. The two components of $\mathrm{Diff}(M \operatorname{rel} T)$ are contractible, using Theorem 3.1, so the first vertical arrow is a homotopy equivalence. To complete the proof, it suffices to check that the third vertical arrow j is a homotopy equivalence.

Note first that $R(T)$ is homeomorphic to $\mathrm{O}(2) \times \mathrm{O}(2) \times \Omega\,\mathrm{SO}(2)$, compatibly with our homeomorphism from $R(M, T)$ to $\mathrm{O}(2) \times (C_2 \times \mathrm{O}(2)) \times \Omega\,\mathrm{SO}(2)$. A diffeomorphism of T lies in $\mathrm{Diff}_0(T)$ exactly when it preserves the circles

$\{t\} \times (D^2_+ \cap D^2_-)$ up to isotopy. These are exactly the diffeomorphisms isotopic to elements of $R(T)$, so j is surjective on path components. Since elements in different path components of $R(T)$ induce distinct outer automorphisms of $\pi_1(T)$, j is injective on path components. The composition of inclusions $\mathrm{SO}(2) \times \mathrm{SO}(2) \to r(T) \to \mathrm{diff}(T)$ is a well-known homotopy equivalence (see for example A. Gramain [9]). The components of $\Omega\,\mathrm{SO}(2)$ are contractible, so the inclusion $\mathrm{SO}(2) \times \mathrm{SO}(2) \to r(T)$ is a homotopy equivalence as well. Therefore $r(T) \to \mathrm{diff}(T)$ is a homotopy equivalence, and it follows that j is a homotopy equivalence on every path component of $R(T)$.   □

Proof of Theorem 2. By Proposition 4.1, we may use $\Sigma = T$ as our genus-1 Heegaard surface.

We have a commutative diagram whose vertical arrows are inclusions:

$$
\begin{array}{ccccc}
R(M,T) & \longrightarrow & R(M) & \longrightarrow & R(M)/R(M,T) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Diff}(M,T) & \longrightarrow & \mathrm{Diff}(M) & \longrightarrow & \mathrm{Diff}(M)/\mathrm{Diff}(M,T)
\end{array}
$$

By Corollary 2.3, the bottom row is a fibration. We claim that the top row is also a fibration. Since $R(M)$ acts transitively on $R(M)/R(M,T)$, it suffices to construct a local $R(M)$ cross-section at the coset $1_M R(M,T)$.

We will write X for $S^1 \times \{\pm N\}$, a union of two circles in M. Since $R(M,T)$ is exactly the subgroup of $R(M)$ that leaves X invariant, the image $r(X)$ of X under a coset $rR(M,T)$ is well-defined, and $rR(M,T) = sR(M,T)$ if and only if $r(X) = s(X)$.

For $w \in S^2 - \{-N\}$, let $\rho_w \in \mathrm{SO}(3)$ be the unique rotation with axis the cross product $w \times N$ that rotates w to N, and let ρ_N be the identity rotation. Now let U be the open set in $R(M)/R(M,T)$ consisting of the elements $rR(M,T)$ such that $r(X) \cap T = \emptyset$. When $rR(M,T) \in U$, $r(X)$ is contained in either the interior of $S^1 \times D^2_+$ or the interior of $S^1 \times D^2_-$, and $r(X)$ meets each $\{t\} \times D^2_+$ in a single point.

To define $\chi \colon U \to R(M)$, let $rR(M,T) \in U$, $r = (f,g,\gamma)$. For each $t \in S^1$, put $w_t = r(X) \cap (\{t\} \times D^2_+)$, $g_0 = \rho_{w_0}$, and $\delta(t) = \rho_{g_0(w_t)}$ (note that $g_0(w_t) \neq -N$, since this would say that $w_t = \rho_{w_0}^{-1}(-N) = -\rho_{w_0}^{-1}(N) = -w_0 \in M - (S^1 \times D^2_+)$). Since $\delta(0) = \rho_{g_0(w_0)} = \rho_N = 1_{S^2}$, $\delta \in \Omega\,\mathrm{SO}(3)$ and we can define $\chi(r) = (1,g_0,\delta)^{-1}$. To verify that χ is a local cross-section, we have $\chi(r)^{-1}(t,w_t) = (1,g_0,\delta)(t,w_t) = (t,\delta(t)(g_0(w_t))) = (t,\rho_{g_0(w_t)}(g_0(w_t))) = (t,N)$, so $\chi(r)^{-1}r \in R(M,T)$. That is, $\chi(r)(1_M r(M,T)) = rR(M,T)$, completing the proof of the claim.

By Theorem 4.2 and Proposition 4.3, the first and second vertical arrows of the diagram are homotopy equivalences. Therefore the third is a (weak) homotopy equivalence. To complete the proof, we will construct a homeomorphism $\phi \colon R(M)/R(M,T) \to LS^2/\langle \alpha \rangle$.

Define $\phi(rR(M,T))$ to be the element represented by the loop γ defined $\gamma(t) = \mathrm{proj}_{S^2}\, r(t,N)$. Note that although $\mathrm{proj}_{S^2}\, r(t,N)$ is not well-defined

on cosets as an element of LS^2, it is well-defined in $LS^2/\langle\alpha\rangle$, and clearly ϕ is continuous. Injectivity of ϕ follows using the fact that $R(M,T)$ is exactly the subgroup of $R(M)$ that preserves $S^1 \times \{\pm N\}$.

For surjectivity, it suffices to show that if $\tau\colon S^1 \to S^2$ is a smooth loop, then there exists $r_\tau \in R(M)$ such that $r_\tau(t,\tau(t)) = (t,N)$, since then we have $\phi(r_\tau^{-1}R(M,T)) = \tau$. To show r_τ exists, we will apply a sequence of elements of $R(M)$ whose composition moves each $(t,\tau(t))$ to (t,N).

First, there is an element $r = (1,g,1) \in R(M)$ such that that $r(0,\tau(0)) = (0,N)$, so we may assume that $\tau(0) = N$. Next, there exist $0 < \epsilon < 1/2$ and an element of the form $r = (1,1,\gamma)$ such that $r(t,\tau(t)) = (t,N)$ for $t \in [-\epsilon,\epsilon] \subset S^1$; for $t \in [-\epsilon,\epsilon]$, $r(t,x) = (t,\rho_{\tau(t)}(x))$, where ρ_w is as defined earlier in the proof where we were constructing a local $R(M)$ cross-section for $R(M) \to R(M)/R(M,T)$. So we may assume that $\tau(t) = N$ for $t \in [-\epsilon,\epsilon]$.

Regard τ as a path $I \to S^1 \to S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$. By the homotopy lifting property, τ lifts to a path $\delta\colon I \to \mathrm{SO}(3)$ with $\delta(0) = 1_{\mathrm{SO}(3)}$ and $\delta(t)(N) = \tau(t)$. In particular, $\delta(t)(N) = N$ for $t \in [0,\epsilon] \cup [1-\epsilon,1]$ so $\delta(t) \in \mathrm{SO}(2)$ for these t. Changing $\delta(t)$ by a smooth isotopy supported on $[0,\epsilon/2]\cup[1-\epsilon/2,1]$, we may assume that $\delta(t) = 1_{S^2}$ for $t \in [0,\epsilon/2]\cup[1-\epsilon/2,1]$. Consequently, δ defines an element $\delta\colon S^1 \to \mathrm{SO}(3)$ of the smooth loop space $\Omega\,\mathrm{SO}(3)$. Putting $r(t,x) = (t,\delta(t)(x))$, we have $\phi(rR(M,T))(t) = \delta(t)(N) = \tau(t)$. $\square$

5. THE IRREDUCIBLE CASE

In this section, we will prove Corollaries 1 and 2. For the manifolds in Corollary 1, the center $Z(\pi_1(M))$ is $\mathbb{Z}^k$ where $k = 3$ when M is the 3-torus and k is 0 or 1 otherwise. Moreover, $\mathrm{diff}(M) \simeq (S^1)^k$; for Haken manifolds this is Theorem 3.1(ii) above, and for hyperbolic M, $k = 0$ and it is D. Gabai's result [8] that the components of $\mathrm{Diff}(M)$ are contractible. When M is non-Haken and not hyperbolic, it is Seifert-fibered over a 2-orbifold O of nonpositive (orbifold) Euler characteristic $\chi^{orb}(O)$. When $\chi^{orb}(O) < 0$, that is, when M has an $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ or $\mathbb{H}^2 \times S^1$ geometric structure (see [29]), $\mathrm{diff}(M) \simeq (S^1)^k$ by [27]. When $\chi^{orb}(O) = 0$, M may be Haken, including all cases when M has a Euclidean geometric structure, or it may be a non-Haken infranilmanifold, excluded by hypothesis. In all the non-excluded cases, the isomorphism $\pi_1(\mathrm{diff}(M)) \to \mathbb{Z}^k$ is given explicitly by taking the trace at a basepoint of M of an isotopy from 1_M to 1_M that represents a given element of $\pi_1(\mathrm{diff}(M))$.

Proof of Corollary 1. All Heegaard splittings of M have genus at least 2, so we can apply Theorem 1. For $i \geq 2$, $\mathcal{H}_i(M,\Sigma) \cong \pi_i(\mathrm{Diff}(M))$, which is 0 since $\mathrm{diff}(M) \simeq (S^1)^k$, and there is an exact sequence

$$1 \to \pi_1(\mathrm{Diff}(M)) \to \mathcal{H}_1(M,\Sigma) \to G(M,\Sigma) \to 1 \ .$$

$\square$

We remark that in general, the exact sequence in Corollary 1 need not split. Suppose that M fibers over S^1 with fiber F and monodromy a diffeomorphism $h\colon F \to F$ of even order n, having at least two fixed points p and q. Let D_p and D_q be disjoint h-invariant disks about p and q respectively. Regard M as $F \times I/\sim$ where $(x,1) \sim (h(x),0)$. The diffeomorphisms $\widetilde{\phi}_t\colon F \times \mathbb{R} \to F \times \mathbb{R}$ defined by $\widetilde{\phi}_t(x,s) = (x, s+nt)$ induce diffeomorphisms $\phi_t\colon M \to M$ that are an isotopy from 1_M to 1_M with trace a primitive element of $Z(\pi_1(M))$. Now, let V be $\overline{F \times [0,1/2] - D_q \times [0,1/2]} \cup D_p \times [1/2,1]$ and W be $\overline{F \times [1/2,1] - D_p \times [1/2,1]} \cup D_q \times [0,1/2]$. These form a Heegaard splitting of M such that $\phi_{r/n}(V) = V$ for each integer r with $1 \le r \le n$. The loop sending t to $\phi_{t/n}$ for $0 \le t \le 1$ represents an element γ of $\mathcal{H}_1(M,\Sigma)$ such that γ^n is a generator σ of $Z(\pi_1(M)) \cong \mathbb{Z}$. If the exact sequence splits, then $\mathcal{H}_1(M,\Sigma)$ is a semidirect product $\mathbb{Z} \rtimes G(M,\Sigma)$. This maps surjectively onto $\mathbb{Z}/2 \times G(M,\Sigma)$, and $(\sigma,1)$ would be an even power in this quotient, which is impossible.

Proof of Corollary 2. All splittings of reducible 3-manifolds have distance $d(M,\Sigma) = 0$, and by J. Hempel [17] and A. Thompson [32]), $d(M,\Sigma) > 2$ implies that M is atoroidal and not Seifert-fibered, so M is hyperbolic. Corollary 1 shows that each component of $\mathcal{H}(M,\Sigma)$ is a $K(G(M,\Sigma),1)$-space. By [23], $d(M,\Sigma) > 3$ implies that $\mathrm{Mod}(M,\Sigma) \to \mathrm{Mod}(M)$ is injective, so $G(M,\Sigma)$ is trivial. Therefore the path components of $\mathcal{H}(M,\Sigma)$ are contractible.

D. Gabai [8] showed that the inclusion of the finite set of isometries into $\mathrm{Diff}(M)$ is a homotopy equivalence, so $\mathrm{Mod}(M,\Sigma)$ and hence $\mathcal{H}_0(M,\Sigma)$ are finite. In fact, the second exact sequence of Theorem 1 also shows that the number of components of $\mathcal{H}(M,\Sigma)$ equals $|\mathrm{Mod}(M)|/|\mathrm{Mod}(M,\Sigma)|$. When $d(M,\Sigma) > 2\,\mathrm{genus}(\Sigma)$, the main result of [23] shows that $\mathrm{Mod}(M,\Sigma) \to \mathrm{Mod}(M)$ is also surjective, so $\mathcal{H}(M,\Sigma)$ is contractible. $\qquad\square$

6. THE ISOMETRIES OF ELLIPTIC 3-MANIFOLDS

An elliptic 3-manifold is a closed 3-manifold E that admits a Riemannian metric of constant positive curvature; according to Perelman's celebrated work, this is equivalent to $\pi_1(E)$ being finite. We always assume that E is equipped with a metric of constant curvature 1, so is the quotient of S^3 by a finite group of isometries acting freely.

The elliptic 3-manifolds were completely classified long ago (see [26] for a discussion). The isometry groups of elliptic 3-manifolds have also been known for a long time. A detailed calculation was given in [26]. We will have to use some of the results and methodology of that work, so in the remainder of this section we review the necessary parts and set up some notation.

First we recall the beautiful description of SO(4) using quaternions. A nice reference for this is [29]. Fix coordinates on S^3 as $\{(z_0, z_1) \mid z_i \in \mathbb{C}, z_0\overline{z_0} + z_1\overline{z_1} = 1\}$. Its group structure as the unit quaternions can then

be given by writing points in the form $z = z_0 + z_1 j$, where $j^2 = -1$ and $jz_i = \overline{z_i}j$. The real part $\Re(z)$ is $\Re(z_0)$, and the imaginary part $\Im(z)$ is $\Im(z_0) + z_1 j$. The inverse of z is $\Re(z) - \Im(z) = \overline{z_0} - z_1 j$. The usual inner product on S^3 is given by $z \cdot w = \Re(zw^{-1})$.

The unique involution in S^3 is -1, and it generates the center of S^3. The pure imaginary unit quaternions P form the 2-sphere of vectors orthogonal to 1, and are exactly the elements of order 4. Consequently, P is invariant under conjugation by elements of S^3. Conjugation induces orthogonal transformations on P, defining a canonical 2-fold covering homomorphism $S^3 \to \mathrm{SO}(3)$ with kernel the center.

Left multiplication and right multiplication by elements of S^3 are orthogonal transformations of S^3, and there is a homomorphism $F \colon S^3 \times S^3 \to \mathrm{SO}(4)$ defined by $F(z, w)(q) = zqw^{-1}$. It is surjective and has kernel $\{(1, 1), (-1, -1)\}$. The center of $\mathrm{SO}(4)$ has order 2, and is generated by $F(1, -1)$, the antipodal map of S^3.

By S^1 we will denote the subgroup of points in S^3 with $z_1 = 0$, that is, all $z_0 \in S^1 \subset \mathbb{C}$. Let $\xi_k = \exp(2\pi i/k)$, which generates a cyclic subgroup $C_k \subset S^1$. The elements $S^1 \cup S^1 j$ form a subgroup $\mathrm{O}(2)^* \subset S^3$, which is exactly the normalizer of S^1 and of the C_k with $k > 2$. It is also the preimage in S^3 of the orthogonal group $\mathrm{O}(2) \subset \mathrm{SO}(3)$, under the 2-fold covering $S^3 \to \mathrm{SO}(3)$.

When H_1 and H_2 are groups, each containing -1 as a central involution, the quotient $(H_1 \times H_2)/\langle(-1, -1)\rangle$ is denoted by $H_1 \widetilde{\times} H_2$. In particular, $\mathrm{SO}(4)$ itself is $S^3 \widetilde{\times} S^3$, and contains the subgroups $S^1 \widetilde{\times} S^3$, $\mathrm{O}(2)^* \widetilde{\times} \mathrm{O}(2)^*$, and $S^1 \widetilde{\times} S^1$. The latter is isomorphic to $S^1 \times S^1$, but it is sometimes useful to distinguish between them. Finally, $\mathrm{Dih}(S^1 \times S^1)$ is the semidirect product $(S^1 \times S^1) \rtimes C_2$, where C_2 acts by complex conjugation in both factors.

There are 2-fold covering homomorphisms

$$\mathrm{O}(2)^* \times \mathrm{O}(2)^* \to \mathrm{O}(2)^* \widetilde{\times} \mathrm{O}(2)^* \to \mathrm{O}(2) \times \mathrm{O}(2) \to \mathrm{O}(2) \widetilde{\times} \mathrm{O}(2) \ .$$

Each of these groups is diffeomorphic to four disjoint copies of the torus, but they are pairwise nonisomorphic, as can be seen by examining their subsets of order 2 elements. Similarly, $S^1 \times S^3$ and $S^1 \widetilde{\times} S^3$ are diffeomorphic, but nonisomorphic.

The method used in [26] to calculate $\mathrm{Isom}(E)$ is straightforward. Let $G = \pi_1(E)$ imbedded as a subgroup of $\mathrm{SO}(4)$ so that $S^3/G = E$. An element $F(z, w)$ induces an isometry on E exactly when it lies in the normalizer $\mathrm{Norm}(G)$ of G in $\mathrm{O}(4)$, and this gives an isomorphism $\mathrm{Norm}(G)/G \cong \mathrm{Isom}(E)$. So for each G, one just needs to calculate $\mathrm{Norm}(G)$ and work out the quotient group $\mathrm{Norm}(G)/G$.

For convenient reference, we include two tables from [26]. Table 1 gives the isometry groups of the elliptic 3-manifolds with non-cyclic fundamental group. The first column shows the fundamental group of E, where C_m denotes a cyclic group of order m, and D_{4m}^*, T_{24}^*, O_{48}^*, and I_{120}^* are the binary

$\pi_1(E)$	E	$\mathrm{Isom}(E)$	$\mathcal{I}(E)$
$Q_8 = D_8^*$	quaternionic	$SO(3) \times S_3$	S_3
$Q_8 \times C_n$	quaternionic	$O(2) \times S_3$	$C_2 \times S_3$
$D_{4m}^*,\ m > 2$	prism	$SO(3) \times C_2$	C_2
$D_{4m}^* \times C_n,\ m > 2$	prism	$O(2) \times C_2$	$C_2 \times C_2$
index 2 diagonal	prism	$O(2) \times C_2$	$C_2 \times C_2$
T_{24}^*	tetrahedral	$SO(3) \times C_2$	C_2
$T_{24}^* \times C_n$	tetrahedral	$O(2) \times C_2$	$C_2 \times C_2$
index 3 diagonal	tetrahedral	$O(2)$	C_2
O_{48}^*	octahedral	$SO(3)$	$\{1\}$
$O_{48}^* \times C_n$	octahedral	$O(2)$	C_2
I_{120}^*	icosahedral	$SO(3)$	$\{1\}$
$I_{120}^* \times C_n$	icosahedral	$O(2)$	C_2

TABLE 1. The isometry group $\mathrm{Isom}(E)$ and its group of path components $\mathcal{I}(E)$ for the elliptic E with $\pi_1(E)$ not cyclic.

dihedral, tetrahedral, octahedral, and icosahedral groups of the indicated orders. The groups called index 2 and index 3 diagonal are certain subgroups of $D_{4m}^* \times C_n$ and $T_{24}^* \times C_n$ respectively. Table 2 gives the isometry groups of the elliptics with cyclic fundamental group. These are the 3-sphere $L(1,0)$, real projective space $L(2,1)$, and the lens spaces $L(m,q)$ with $m \geq 3$. Both tables give the full isometry group $\mathrm{Isom}(E)$, and the group $\mathcal{I}(E)$ of path components of $\mathrm{Isom}(E)$.

In S^3 there is a standard torus $T = \{z_0 + z_1 j \mid |z_0| = |z_1|\}$. It bounds two solid tori, V and W, where $|z_0| \leq |z_1|$ and $|z_0| \geq |z_1|$ respectively. In our work, certain isometries that preserve T will be useful.

(1) $\alpha = F(1, -1)$, the antipodal map. It preserves each of V and W.
(2) $\rho\colon z_0 + z_1 j \mapsto \overline{z_0} + z_1 j$. It is an orientation-reversing involution that preserves each of V and W.
(3) $\tau = F(j, j)\colon z_0 + z_1 j \mapsto \overline{z_0} + \overline{z_1} j$. It is an involution that restricts to a hyperelliptic involution on each of V and W.
(4) $\sigma_+ = F(i, ij)\colon z_0 + z_1 j \mapsto z_1 + z_0 j$. It is an involution that interchanges V and W.
(5) $\sigma_-\colon z_0 + z_1 j \mapsto z_1 + \overline{z_0} j$. It is an orientation-reversing isometry of order 4 that interchanges V and W.

The following relations among these isometries are easily checked:

(1) $\sigma_-^2 = \tau$.

m, q	$\mathrm{Isom}(L(m,q))$	$\mathcal{I}(L(m,q))$
$m = 1$ $(L(1,0) = S^3)$	$\mathrm{O}(4)$	C_2
$m = 2$ $(L(2,1) = \mathbb{RP}^3)$	$(\mathrm{SO}(3) \times \mathrm{SO}(3)) \rtimes C_2$	C_2
$m > 2$, m odd, $q = 1$	$\mathrm{O}(2)^* \widetilde{\times} S^3$	C_2
$m > 2$, m even, $q = 1$	$\mathrm{O}(2) \times \mathrm{SO}(3)$	C_2
$m > 2$, $1 < q < m/2$, $q^2 \not\equiv \pm 1 \bmod m$	$\mathrm{Dih}(S^1 \times S^1)$	C_2
$m > 2$, $1 < q < m/2$, $q^2 \equiv -1 \bmod m$	$(S^1 \widetilde{\times} S^1) \rtimes C_4$	C_4
$m > 2$, $1 < q < m/2$, $q^2 \equiv 1 \bmod m$, $\quad \gcd(m, q+1)\gcd(m, q-1) = m$	$\mathrm{O}(2) \widetilde{\times} \mathrm{O}(2)$	$C_2 \times C_2$
$m > 2$, $1 < q < m/2$, $q^2 \equiv 1 \bmod m$, $\quad \gcd(m, q+1)\gcd(m, q-1) = 2m$	$\mathrm{O}(2) \times \mathrm{O}(2)$	$C_2 \times C_2$

TABLE 2. Isometry groups of elliptic manifolds $L(m,q)$ with cyclic fundamental group.

(2) $\sigma_+\tau = \tau\sigma_+$, and $\rho\tau = \tau\rho$.
(3) $(\rho\sigma_+)^2 = \tau$, so ρ and σ_+ generate a dihedral group of order 8.
(4) $\sigma_+\sigma_-\sigma_+ = \sigma_-^{-1}$, so σ_+ and σ_- generate a dihedral group of order 8.

7. THE SMALE CONJECTURE

The original Smale Conjecture, proven by A. Hatcher [15], asserts that the inclusion $\mathrm{Isom}(S^3) \to \mathrm{Diff}(S^3)$ from the isometry group to the diffeomorphism group is a homotopy equivalence. The *Generalized Smale Conjecture* (henceforth just called the Smale Conjecture) asserts this for elliptic 3-manifolds.

N. Ivanov [21, 22] proved the Smale Conjecture for most of the elliptic 3-manifolds that contain one-sided Klein bottles, specifically:

(i) The lens spaces $L(4n, 2n - 1)$, $n \geq 2$
(ii) The quaternionic and prism manifolds for which $\pi_1(E)$ has a nontrivial cyclic direct factor.

The preprint [18] gives proofs of the Smale Conjecture for all lens spaces $L(m, q)$ with $m > 2$, and for all quaternionic and prism manifolds. Although the Smale Conjecture seems likely to hold for all elliptic 3-manifolds, no claim is currently asserted for the remaining cases. Perelman's methods do not seem to apply, at least in their current form (see [18, Section 1.4]).

8. Heegaard splittings of elliptic 3-manifolds: the genus-0 case

In this section we will prove Theorem 3(1), that is, the case when $M = S^3$ and Σ has genus 0.

Recall that $P \subset S^3$ is the 2-sphere orthogonal to 1. The stabilizer $\mathrm{Isom}_+(S^3, P)$ of P in $\mathrm{Isom}_+(S^3)$ is exactly the stabilizer of the pair $\{\pm 1\}$, which is the subgroup $\mathrm{O}(3) \subset \mathrm{SO}(4)$.

Lemma 8.1. *The inclusion* $\mathrm{Isom}(S^3, P) \to \mathrm{Diff}(S^3, P)$ *is a homotopy equivalence.*

Proof. Consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Isom}_+(S^3 \operatorname{rel} P) & \longrightarrow & \mathrm{Isom}_+(S^3, P) & \longrightarrow & \mathrm{Isom}(P) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Diff}_+(S^3 \operatorname{rel} P) & \longrightarrow & \mathrm{Diff}_+(S^3, P) & \longrightarrow & \mathrm{Diff}(P)
\end{array}
$$

in which the vertical maps are inclusions, and the rows are fibrations, the top row since it is a homomorphism of compact Lie groups, and the bottom row by Lemma 2.6. The right vertical arrow is a homotopy equivalence, by a theorem of S. Smale [30]. The left vertical arrow is a homotopy equivalence, since the Smale Conjecture for S^3 implies that $\mathrm{Diff}(D^3 \operatorname{rel} \partial D^3)$ is contractible. Therefore $\mathrm{Isom}_+(S^3, P) \to \mathrm{Diff}_+(S^3, P)$ is a homotopy equivalence. Since both $\mathrm{Isom}(S^3, P)$ and $\mathrm{Diff}(S^3, P)$ contain orientation-reversing elements, it follows that $\mathrm{Isom}(S^3, P) \to \mathrm{Diff}(S^3, P)$ is a homotopy equivalence. $\square$

Proof of Theorem 3(1). Since all 2-spheres (smoothly) imbedded in S^3 are isotopic, we may take Σ to be P. Consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Isom}(S^3, P) & \longrightarrow & \mathrm{Isom}(S^3) & \longrightarrow & \mathrm{Isom}(S^3)/\mathrm{Isom}(S^3, P) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Diff}(S^3, P) & \longrightarrow & \mathrm{Diff}(S^3) & \longrightarrow & \mathrm{Diff}(S^3)/\mathrm{Diff}(S^3, P)
\end{array}
$$

in which the vertical maps are inclusions. The rows are fibrations, the top row since it is a homomorphism of compact Lie groups, and the bottom row by Corollary 2.3. We have just seen that the left vertical arrow is a homotopy equivalence. The middle vertical arrow is the original Smale Conjecture, so we have

$$
\mathcal{H}(S^3, P) \simeq \mathrm{Isom}(S^3)/\mathrm{Isom}(S^3, P)
$$

$$
\simeq \mathrm{Isom}_+(S^3)/\mathrm{Isom}_+(S^3, P) = \mathrm{SO}(4)/\mathrm{O}(3) = \mathbb{RP}^3 \, ,
$$

the latter equality since $\mathrm{O}(3)$ is the stabilizer of $\{\pm 1\}$ under the transitive action of $\mathrm{SO}(4)$ on pairs of antipodal points in S^3. $\square$

9. Lens spaces

Our work on genus-1 splittings will require some information about lens spaces, which we recall in this section. Let L be the lens space $L(m, q)$, with $m \geq 2$ and q selected so that $1 \leq q \leq m/2$. We regard L as S^3/G_L, where $G_L \subset S^1 \widetilde{\times} S^1 \subset \mathrm{SO}(4)$ is the cyclic subgroup of order m generated by $\gamma_{m,q} = F(\xi_{2m}^{q+1}, \xi_{2m}^{q-1})$.

For each $n \geq 1$, L has a Heegaard surface Σ_n of genus n, and by a theorem of F. Bonahon [2] for $n = 1$ and Bonahon and J.-P. Otal [3] for $n \geq 2$, it is the unique Heegaard surface of this genus up to isotopy. Consequently, $\mathcal{H}(L, \Sigma_n)$ is path-connected.

The standard torus $\{z_0 + z_1 j \mid |z_0| = |z_1|\} \subset S^3$ is invariant under the action of G_L, and its image under $S^3 \to S^3/G_L = L$ is a Heegaard torus in L. We denote the image by T, and the solid tori in L bounded by T by V and W.

In [2], F. Bonahon proved that every diffeomorphism of L preserves T up to isotopy, and used this to calculate the mapping class groups of lens spaces. To state the results, we first recall some notation from [2].

(1) For all (m, q), $\tau \gamma_{m,q} \tau = \gamma_{m,q}^{-1}$, so τ induces an orientation-preserving involution, also denoted by τ, which restricts to the hyperelliptic involution on each of V and W.

(2) When $q^2 = 1 \bmod m$, we have $\sigma_+ \gamma_{m,q} \sigma_+ = \gamma_{m,q}^q$, so σ_+ induces an orientation-preserving involution, also denoted by σ_+, that interchanges V and W.

(3) When $q^2 = -1 \bmod m$, we have $\sigma_- \gamma_{m,q} \sigma_-^{-1} = \gamma_{m,q}^{-q}$, so σ_- induces an orientation-reversing isometry, also denoted by σ_-, that interchanges V and W.

Theorem 9.1 (F. Bonahon). *The groups* $\mathrm{Mod}(L)$ *are as follows:*

(1) $\mathrm{Mod}(L(2, 1)) = C_2$ *generated by* σ_-.
(2) *If* $m > 2$ *and* $q = 1$, *then* $\mathrm{Mod}(L) = C_2$ *generated by* τ.
(3) *If* $m > 2$ *and* $q^2 = 1 \bmod m$ *but* $q \neq 1$, *then* $\mathrm{Mod}(L) = C_2 \times C_2$ *generated by* τ *and* σ_+.
(4) *If* $m > 2$ *and* $q^2 = -1 \bmod m$, *then* $\mathrm{Mod}(L) = C_4$ *generated by* σ_-.
(5) *If* $m > 2$ *and* $q^2 \neq \pm 1 \bmod m$, *then* $\mathrm{Mod}(L) = C_2$ *generated by* τ.

Note that Theorem 9.1 implies the well-known fact that $L(m, q)$ admits an orientation-reversing diffeomorphism if and only if $q^2 \equiv -1 \bmod m$. Since each of the elements τ, σ_+, and σ_- preserves T, Theorem 9.1 also implies

Corollary 9.2. $\mathrm{Mod}(L, T) \to \mathrm{Mod}(L)$ *is surjective.*

It is not difficult to compute $G(L, T)$, and then $\mathrm{Mod}(L, T)$ using the exact sequence $1 \to G(L, T) \to \mathrm{Mod}(L, T) \to \mathrm{Mod}(L) \to 1$. Since the proofs are not difficult and we will not need the results, we simply record them here:

Proposition 9.3. (1) *If* $q \neq 1$, *then* $G(L, T) = \{1\}$.

(2) *If $m > 2$ and $q = 1$, then $G(L, T) = C_2$, generated by σ_+, and $G_0(L, T) = \{1\}$.*
(3) *$G(L(2,1), T) = C_2 \times C_2$, generated by σ_+ and τ, and $G_0(L(2,1), T) = C_2$, generated by τ.*

Proposition 9.4.   (1) *If $q \neq 1$, then $\mathrm{Mod}(L, T) \to \mathrm{Mod}(L)$ is an isomorphism.*
(2) *If $m > 2$ and $q = 1$, then $\mathrm{Mod}(L, T) = C_2 \times C_2$, generated by σ_+ and τ.*
(3) *$\mathrm{Mod}(L(2,1), T) = D_8$, the dihedral group of order 8 generated by σ_+ and σ_-.*

10. Heegaard splittings of elliptic 3-manifolds: the genus-1 case

In this section we will prove Theorem 3(2) and the (a) statements in all three cases of Theorem 4. We will retain the notation of Section 9, so that L is the lens space $L(m, q)$ with $m \geq 2$ and $1 \leq q \leq m/2$, except that we now allow $L = L(1, 0)$, the 3-sphere. As in Section 9, L is regarded as S^3/G_L, where $G_L \subset S^1 \widetilde{\times} S^1 \subset \mathrm{SO}(4)$ is the subgroup generated by $\gamma_{m,q} = F(\xi_{2m}^{q+1}, \xi_{2m}^{q-1})$. In particular, $\gamma_{1,0} = F(-1, -1) = 1_{\mathrm{SO}(4)}$, and $\gamma_{2,1} = F(-1, 1) = \alpha$, the antipodal map.

Recall that $\mathrm{Isom}(L) = \mathrm{Norm}(G_L)/G_L$, where $\mathrm{Norm}(G_L)$ is the normalizer of G_L in $\mathrm{O}(4)$. Consequently, an element $F(z, w)$ in $\mathrm{Norm}(G_L) \cap \mathrm{SO}(4)$ induces an isometry on L, which we denote by $f(z, w)$.

We will need to know the groups $\mathrm{Norm}(G_L) \cap \mathrm{SO}(4)$, which we denote by $\mathrm{Norm}_+(G_L)$. In the following lemma, $\mathrm{Dih}(S^1 \widetilde{\times} S^1)$ denotes the subgroup of index 2 in $\mathrm{O}(2)^* \widetilde{\times} \mathrm{O}(2)^*$ generated by $S^1 \widetilde{\times} S^1$ and the involution $\tau = F(j, j)$, which acts by inversion on elements of $S^1 \widetilde{\times} S^1$. From [26] we have the following information.

Lemma 10.1.   (i) *For $m \leq 2$, $\mathrm{Norm}_+(G_L) = \mathrm{SO}(4)$.*
(ii) *For $m > 2$ and $q = 1$, $\mathrm{Norm}_+(G_L) = S^3 \widetilde{\times} \mathrm{O}(2)^*$.*
(iii) *For $m > 2$, $q > 1$ and $q^2 \equiv 1 \bmod m$, $\mathrm{Norm}_+(G_L) = \mathrm{O}(2)^* \widetilde{\times} \mathrm{O}(2)^*$.*
(iv) *For $m > 2$ and $q^2 \not\equiv 1 \bmod m$, $\mathrm{Norm}_+(G_L) = \mathrm{Dih}(S^1 \widetilde{\times} S^1)$.*

Proof. Part (i) is obvious. Part (ii) is found in Case III on p. 175 of [26], and Part (iii) is found in Case VI on p. 176 of [26]. Part (iv) is found in Cases IV and V on p. 175 of [26]. □

As in Section 9, T is the standard Heegaard torus for L. In particular, for $L = L(1, 0)$, T is $\{z_0 + z_1 j \mid |z_0| = |z_1|\}$. We will need to know the groups $\mathrm{Isom}_+(L, T)$.

Lemma 10.2.   (1) *$\mathrm{isom}(L, T) = (S^1 \widetilde{\times} S^1)/G_L$*
(2) *When $q^2 \equiv 1 \bmod m$, $\mathrm{Isom}_+(L, T) = (\mathrm{O}(2)^* \widetilde{\times} \mathrm{O}(2)^*)/G_L$.*
(3) *When $q^2 \not\equiv 1 \bmod m$, $\mathrm{Isom}_+(L, T) = \mathrm{Dih}(S^1 \widetilde{\times} S^1)/G_L$.*

Proof. Suppose first that $L = L(1,0) = S^3$. It is straightforward to check that $O(2)^* \widetilde{\times} O(2)^* \subset \mathrm{Isom}_+(S^3, T)$. Suppose that $F(z,w) \in \mathrm{Isom}_+(S^3, T)$. Now T is exactly the set of points equidistant from the two geodesics S^1 and $S^1 j$, in fact these are exactly the most distant points from it. Since $F(z,w)$ preserves T, it must preserve $S^1 \cup S^1 j$. A quick check shows that $z, w \in O(2)^*$ (starting with the case when $F(z,w)(1), F(z,w)(i) \in S^1$ and $F(z,w)(j), F(z,w)(ij) \in S^1 j$, we compute that either $(z,w) \in S^1 \times S^1$ or $(z,w) \in S^1 j \times S^1 j$, while when $F(z,w)(S^1) = S^1 j$, the previous case applies to $F(z,w)F(1,j)$ showing that $(z,w) \in S^1 \times S^1 j$ or $(z,w) \in S^1 j \times S^1$).

In general, we have

$$\mathrm{Isom}_+(L, T) = (\mathrm{Isom}_+(S^3, T) \cap \mathrm{Norm}(G_L))/G_L$$
$$= (O(2)^* \widetilde{\times} O(2)^* \cap \mathrm{Norm}(G_L))/G_L .$$

From Lemma 10.1, $O(2)^* \widetilde{\times} O(2)^* \cap \mathrm{Norm}(G_L)$ is $O(2)^* \widetilde{\times} O(2)^*$ when $q^2 \equiv 1 \bmod m$ and is $\mathrm{Dih}(S^1 \widetilde{\times} S^1)$ when $q^2 \not\equiv 1 \bmod m$. This establishes statements (2) and (3). The description of $\mathrm{isom}(L, T)$ in (1) follows directly. $\square$

Lemma 10.3. *The inclusion* $\mathrm{Isom}(L, T) \to \mathrm{Diff}(L, T)$ *is a homotopy equivalence.*

Proof. From Theorem 9.1, L admits an orientation-reversing diffeomorphism only when $q^2 \equiv -1 \bmod m$, in which case σ_- is an orientation-reversing element of $\mathrm{Isom}(L, T)$. That is, $\mathrm{Diff}(L, T)$ contains orientation-reversing elements if and only if $\mathrm{Isom}(L, T)$ does. Therefore it suffices to prove that the inclusion $k \colon \mathrm{Isom}_+(L, T) \to \mathrm{Diff}_+(L, T)$ is a homotopy equivalence.

We first check that k is injective on path components. By Lemma 10.2, $\mathrm{Isom}_+(L, T)$ has either two or four components, represented by 1_L, τ, and when there are four, $\sigma_+ = f(i, ij)$ and $\sigma_+ \tau = f(ij, -i)$. Of these, only the elements of $\mathrm{isom}(L, T)$ preserve the sides and are isotopic to the identity on T, so $\mathrm{Isom}_+(L, T) \cap \mathrm{diff}(L, T) = \mathrm{isom}(L, T)$.

To see that k is surjective on path components, let $f \in \mathrm{Diff}_+(L, T)$. If f interchanges the sides of T, then a well-known homology argument shows that $q^2 \equiv 1 \bmod m$. (Let μ and λ in $H_1(T)$ be a meridian and longitude for $T \subset V$ such that $m\lambda + q\mu$ is a meridian of $T \subset W$, and write $h = f|_T$. For $h_* \colon H_1(T) \to H_1(T)$, $h_*(\mu)$ is a meridian for $T \subset W$, so $h_*(\mu) = \epsilon(m\lambda + q\mu)$ for ϵ either 1 or -1. Writing $h_*(\lambda) = \epsilon(a\lambda + b\mu)$, $\det(h_*) = aq - mb$. Since h interchanges the sides of T, $\pm\mu = mh_*(\lambda) + qh_*(\mu)$, implying that $a = -q$ and hence $\det(h_*) = -q^2 - mb$. When f is orientation-preserving, h must be orientation-reversing, giving $q^2 \equiv 1 \bmod m$.) Since $q^2 \equiv 1 \bmod m$, σ_+ is an element of $\mathrm{Isom}_+(L, T)$, and composing it with f, we may assume that f preserves the sides of T. Since f must then preserve the meridian curves of both complementary tori up to isotopy, it is isotopic either to the identity on both solid tori or to the hyperelliptic involution on both. In the latter case, we may compose f with τ, an element of $\mathrm{Isom}_+(L, T)$, to assume that f is isotopic to the identity on both sides and therefore lies in

$\text{diff}(L, T)$. Therefore every path component of $\text{Diff}(L, T)$ contains elements of $\text{Isom}(L, T)$.

From Lemma 10.2, $\text{isom}(S^3, T)$ is a full $S^1 \times S^1$ subgroup of isometries in $\text{diff}(L, T)$, so k is a homotopy equivalence on each path component. $\square$

Proof of Theorem 3(2) and the (a) parts of Theorem 4. By a theorem of F. Waldhausen [33] for $m = 1$ and F. Bonahon [2] for $m \geq 2$, T is the unique Heegaard torus of L up to isotopy. So we may take $\Sigma_1 = T$.

Consider the commutative diagram

$$
\begin{array}{ccccc}
\text{Isom}(L, T) & \longrightarrow & \text{Isom}(L) & \longrightarrow & \text{Isom}(L)/\text{Isom}(L, T) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Diff}(L, T) & \longrightarrow & \text{Diff}(L) & \longrightarrow & \text{Diff}(L)/\text{Diff}(L, T) \ .
\end{array}
$$

whose vertical maps are inclusions. The rows are fibrations, the first since $\text{Isom}(L, T)$ is a closed subgroup of the Lie group $\text{Isom}(L)$, and the second by Corollary 2.3. Since L is assumed to satisfy the Smale Conjecture, the middle arrow is a homotopy equivalence. By Lemma 10.3, the first vertical arrow is a homotopy equivalence, so we have $\mathcal{H}(L, T) \simeq \text{Isom}(L)/\text{Isom}(L, T)$. Since L has an orientation-reversing isometry only when when $q^2 \equiv -1 \bmod m$, in which case $\sigma_- \in \text{Isom}(L, T)$ is an orientation-reversing isometry, we have

$$
\text{Isom}(L)/\text{Isom}(L, T) = \text{Isom}_+(L)/\text{Isom}_+(L, T)
$$
$$
= (\text{Norm}_+(G_L)/G_L)/(\text{O}(2)^* \widetilde{\times} \text{O}(2)^* \cap \text{Norm}(G_L))/G_L
$$
$$
= \text{Norm}_+(G_L)/(\text{O}(2)^* \widetilde{\times} \text{O}(2)^* \cap \text{Norm}(G_L)) \ .
$$

Using Lemma 10.1, we can now calculate $\mathcal{H}(L, T)$. When $m \leq 2$,

$$
\mathcal{H}(L, T) \simeq \text{SO}(4)/(\text{O}(2)^* \widetilde{\times} \text{O}(2)^*) = (S^3 \widetilde{\times} S^3)/(\text{O}(2)^* \widetilde{\times} \text{O}(2)^*)
$$
$$
= (S^3 \times S^3)/(\text{O}(2)^* \times \text{O}(2)^*) = S^3/\text{O}(2)^* \times S^3/\text{O}(2)^* = \mathbb{RP}^2 \times \mathbb{RP}^2 \ .
$$

When $m > 2$ and $q^2 = 1 \bmod m$, we have

$$
\mathcal{H}(L, T) \simeq (S^3 \widetilde{\times} \text{O}(2)^*)/(\text{O}(2)^* \widetilde{\times} \text{O}(2)^*)
$$
$$
= (S^3 \times \text{O}(2)^*)/(\text{O}(2)^* \times \text{O}(2)^*) = S^3/\text{O}(2)^* = \mathbb{RP}^2 \ .
$$

When $m > 2$ and $q^2 \neq 1 \bmod m$, we have

$$
\mathcal{H}(L, T) \simeq (\text{O}(2)^* \widetilde{\times} \text{O}(2)^*)/(\text{O}(2)^* \widetilde{\times} \text{O}(2)^*) \ ,
$$

a single point. $\square$

11. HEEGAARD SPLITTINGS OF ELLIPTIC 3-MANIFOLDS: GENUS 2 AND HIGHER

Proof of Theorem 3(3) and the (b) parts of Theorem 4. We continue to use the notation of Section 10. By results of Waldhausen [33] when $m = 1$ and Bonahon-Otal [3] for $m \geq 2$, L has unique Heegaard surface for each genus greater than 1, so $\mathcal{H}(L, \Sigma)$ is path-connected. Let Σ_n be a Heegaard surface

of genus $n \geq 2$. By Theorem 1, $\mathcal{H}_i(L, \Sigma_n) \cong \pi_i(\mathrm{Diff}(L))$, and there is an exact sequence

$$1 \to \pi_1(\mathrm{Diff}(L)) \to \mathcal{H}_1(L, \Sigma_n) \to G(L, \Sigma_n) \to 1 \ .$$

Since we are assuing that L satisfies the Smale Conjecture, the groups $\pi_i(\mathrm{Diff}(L)) \cong \pi_i(\mathrm{Isom}(L))$ for $i \geq 1$ can be found using Table 2.

Case I: $q \geq 2$

For $q \geq 2$, isom(L) is homeomorphic to $S^1 \times S^1$, so $\mathcal{H}_i(L, \Sigma_n) = 0$ for $i \geq 2$ and $\pi_1(\mathrm{Diff}(L))$ is $\mathbb{Z} \times \mathbb{Z}$.

Case II: $m > 2$ and $q = 1$

In this case, isom(L) is homeomorphic to $S^1 \times S^3$ or to $S^1 \times \mathrm{SO}(3)$ according as m is odd or even. So $\mathcal{H}_i(L, \Sigma_n) = \pi_i(S^3)$ for $i \geq 2$, while $\pi_1(\mathrm{Diff}(L))$ is $\mathbb{Z}$ or $\mathbb{Z} \times C_2$ according as m is odd or even.

Case III: $L = L(2, 1)$

In this case, isom(L) is homeomorphic to $\mathrm{SO}(3) \times \mathrm{SO}(3)$, so $\mathcal{H}_i(L, \Sigma_n) = \pi_i(S^3 \times S^3)$ for $i \geq 2$ and $\pi_1(\mathrm{Diff}(L))$ is $C_2 \times C_2$.

Case IV: $L = L(1, 0)$

In this case, isom(L) is $\mathrm{SO}(4)$, so $\mathcal{H}_i(L, \Sigma_n) = \pi_i(\mathrm{SO}(4)) \cong \pi_i(S^3 \times S^3)$ for $i \geq 2$ and $\pi_1(\mathrm{Diff}(L))$ is C_2. To see that the exact sequence in Theorem 3(3) does not split, observe that there is an isotopy J_t with $J_0 = 1_{S^3}$ and J_1 a hyperelliptic involution on Σ_n; J_t rotates through an angle π around an axis of symmetry of Σ_n. This defines an element of $\mathcal{H}_1(S^3, \Sigma_n)$ whose square is the generator of $\pi_1(\mathrm{Diff}(S^3))$. A normal C_2-subgroup is central, so if the exact sequence split we would have $\mathcal{H}_1(S^3, \Sigma_n) = \pi_1(\mathrm{Diff}(S^3)) \times \mathcal{G}(S^3, \Sigma_n)$, and the generator of $\pi_1(\mathrm{Diff}(S^3))$ could not be a square. $\square$

Proof of Theorem 5. Fix a Heegaard surface Σ in the elliptic 3-manifold E. Since E is not S^3 or a lens space, Σ has genus at least 2. By Theorem 1, $\mathcal{H}_i(E, \Sigma) \cong \pi_i(\mathrm{Diff}(E))$ for $i \geq 2$, and there is an exact sequence

$$1 \to \pi_1(\mathrm{Diff}(E)) \to \mathcal{H}_1(E, \Sigma) \to G(E, \Sigma) \to 1 \ .$$

Since E is assumed to satisfy the Smale Conjecture, $\pi_i(\mathrm{Diff}(M)) \cong \pi_i(\mathrm{Isom}(E))$.

Case I: $\pi_1(E) \cong D_{4m}^*$, or E is one of the three manifolds with fundamental group either T_{24}^*, O_{48}^*, or I_{120}^*.

Referring to Table 1, we see that isom(E) is homeomorphic to $\mathrm{SO}(3)$, so $\mathcal{H}_i(E, \Sigma) = \pi_i(S^3)$ for $i \geq 2$ and $\pi_1(\mathrm{isom}(E)) \cong C_2$.

Case II: E is not one of the manifolds in Case I, that is, either $\pi_1(E)$ has a nontrivial cyclic direct factor, or $\pi_1(E)$ is a diagonal subgroup of index 2 in $D_{4m}^* \times C_n$ or of index 3 in $T_{48}^* \times C_n$.

Again from Table 1, isom(E) is homeomorphic to S^1, so $\mathcal{H}_i(E, \Sigma) = 0$ for $i \geq 2$ and $\pi_1(\mathrm{isom}(E))$ is $\mathbb{Z}$. $\square$

For the manifolds in Theorem 5, M. Boileau and J.-P. Otal [1] have proven that there is a unique genus-2 Heegaard splitting up to isotopy, so in that case $\mathcal{H}(L, \Sigma)$ is known to be connected.

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