ORTHOGONALLY SPHERICAL OBJECTS AND SPHERICAL FIBRATIONS

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Abstract. We introduce a relative version of the spherical objects of Seidel and Thomas [ST01]. Define an object E in the derived category $D(Z \times X)$ to be spherical over Z if the corresponding functor from $D(Z)$ to $D(X)$ gives rise to autoequivalences of $D(Z)$ and $D(X)$ in a certain natural way. Most known examples come from subschemes of X fibred over Z. This categorifies to the notion of an object of $D(Z \times X)$ orthogonal over Z. We prove that such an object is spherical over Z if and only if it possesses certain cohomological properties similar to those in the original definition of a spherical object. We then interpret this geometrically in the case when our objects are actual flat fibrations in X over Z.

1. Introduction

Let X be a smooth projective variety over $\mathbb{C}$ and $D(X)$ be the bounded derived category of coherent sheaves on X. Following certain developments in mirror symmetry Seidel and Thomas introduced in [ST01] the notion of a *spherical object*:

Definition 1.1 ([ST01])**.** An object E of $D(X)$ is *spherical* if:

(1) $\mathrm{Hom}^i_{D(X)}(E,E) = \begin{cases} \mathbb{C}, & \text{if } i = 0 \text{ or} \dim X, \\ 0, & \text{otherwise} \end{cases}$

(2) $E \simeq E \otimes \omega_X$ where ω_X is the canonical bundle of X.

The motivating idea came from considering Lagrangian spheres on a symplectic manifold. Given such a sphere one can associate to it a symplectic automorphism called the Dehn twist. Correspondingly:

Theorem ([ST01])**.** *Let $E \in D(X)$. The twist functor T_E is a cone we can associate to the natural transformation $E \otimes_{\mathbb{C}} \mathbf{R}\mathrm{Hom}_X(E, -) \xrightarrow{\text{eval}} \mathrm{Id}_{D(X)}$. If E is spherical, then T_E is an autoequivalence of $D(X)$.*

Spherical twists can be used to construct braid group actions on $D(X)$, as was indeed the main concern of [ST01]. They also deserve to be studied in their own right as some of the simplest examples of autoequivalences of $D(X)$ which are purely derived and do not come from autoequivalences of the underlying abelian category $\mathrm{Coh}(X)$. In fact, on smooth toric surfaces or on surfaces of general type whose canonical model has at worst A_n-singularities the whole of $\mathrm{Aut}\, D(X)$ is generated by spherical twists, lifts from $\mathrm{Aut}\,\mathrm{Coh}(X)$ and the shift ([IU05], [BP10]). In more complicated cases spherical twists are still an essential tool in studying the autoequivalences of $D(X)$ and stability conditions on it ([Bri08], [Bri09], [Bri06]).

In this paper we study a relative version of the construction above which deals not with a single object but with a family of objects in $D(X)$ over some base Z. A geometric picture to keep in mind is a subvariety D of X flatly fibred over Z. Even when the structure sheaf of D is not itself spherical in sense of [ST01] one may still produce an autoequivalence of $D(X)$ by exploiting the extra fibration structure which D possesses. We characterize the families of objects of $D(X)$ over Z for which this is possible and we do it terms of applicable cohomological criteria similar to Definition 1.1 above. Our study is a self-contained exercise in derived categories of coherent sheaves and doesn't use mirror symmetry or assume any knowledge of it. One should mention though that the original examples of these family twists were inspired by Kontsevich's proposal that the autoequivalences of $D(X)$ should correspond to loops in the moduli space of complex structures on its mirror, cf. [Hor99] (especially §4.1), [Hor05], [Sze01], [Sze04]. Maybe in the future our results could be used to construct further examples of this correspondence in a more general setting.

Consider an object E in the derived category $D(Z \times X)$ of the product of Z and X. We can view E as a *family of objects in $D(X)$ parametrised by Z* by considering the fibres of E over points of Z to be the derived pullbacks of E to the corresponding fibres of $X \times Z$ over Z:

$$\begin{array}{ccc} X & \xrightarrow{\iota_{X_p}} & Z \times X \\ \downarrow & & \downarrow{\scriptstyle \pi_Z} \\ \bullet & \xrightarrow[\iota_p]{} & Z \end{array} \qquad \forall\, p \in Z \quad E_p = \iota^*_{X_p} E$$

On the other hand, each object $E \in D(Z \times X)$ defines naturally a functor $\Phi_E \colon D(Z) \to D(X)$ called the *Fourier–Mukai transform with kernel E* (cf. [Huy06]) which sends each point sheaf $\mathcal{O}_p$ on Z to the fibre $E_p \in D(X)$. The interplay between these two points of view, moduli-theoretic and functorial, led to a string of celebrated results by Mukai, Bondal and Orlov, Bridgeland and others.

When Z is the point scheme Spec $\mathbb{C}$ the above formalism tells us to view an object $E \in D(X)$ as a functor $\Phi_E = E \otimes_{\mathbb{C}} (-)$ from $D(\mathbf{Vect})$ to $D(X)$. Then the functor $E \otimes_{\mathbb{C}} \mathbf{R}\operatorname{Hom}_X(E, -)$ is the composition of Φ_E with its right adjoint Φ_E^{radj} and the definition of the twist functor T_E given above amounts simply to T_E being a cone of the adjunction co-unit

$$\Phi_E \Phi_E^{\mathrm{radj}} \longrightarrow \mathrm{Id}_{D(X)}\,. \tag{1.1}$$

There is a subtlety involved here: taking cones, infamously, is not functorial in $D(X)$, so the cone of a morphism between two functors is not a priori well defined. However in [AL10] it is shown that for very general Z, X and $E \in D(Z \times X)$ we can represent the functors in (1.1) by Fourier–Mukai kernels and then represent the adjunction co-unit (1.1) by a natural morphism μ between these kernels. We can therefore define the twist functor T_E as the Fourier–Mukai transform whose kernel is the cone of μ and pose the following general problem:

Problem: Describe the objects E in $D(Z \times X)$ for which the twist T_E is an autoequivalence of $D(X)$.

A partial answer was provided by Horja in [Hor05] for smooth Z and X. He considers only those objects E of $D(Z \times X)$ which come from the derived category of a smooth subscheme of X flatly fibred over Z. For these he gives a cohomological criterion sufficient for the twist T_E to be an autoequivalence of $D(X)$. In [Ann07] Anno takes a different approach: she abstracts out the properties of the functors Φ_E defined by spherical objects of [ST01] and [Hor05] which are exploited in their proofs that the twists T_E are autoequivalences. In all these cases not only T_E is an autoequivalence, but this autoequivalence identifies naturally the left and right adjoints of Φ_E. And if this is true, then the same is true of the co-twist F_E *and vice versa*, where the co-twist F_E is the cone of the adjunction unit $\mathrm{Id}_{D(Z)} \to \Phi_E^{radj}\Phi_E$. In other words, one can show very generally that for a functor S between two triangulated categories:

$$\left\{ \begin{array}{c} F_S \text{ is an autoequivalence} \\ S^{\mathrm{radj}} \simeq F_S S^{\mathrm{ladj}}[1] \end{array} \right\} \quad \text{if and only if} \quad \left\{ \begin{array}{c} T_S \text{ is an autoequivalence} \\ S^{\mathrm{radj}} \simeq S^{\mathrm{ladj}} T_S[1] \end{array} \right\} \tag{1.2}$$

The functors which possess these equivalent properties are called *spherical functors*. We thereby define:

Definition (Definition 3.4)**.** An object $E \in D(Z \times X)$ is *spherical over Z if the corresponding Fourier–Mukai transform $\Phi_E \colon D(Z) \to D(X)$ is a spherical functor, in other words:*

1. *The co-twist F_E is an autoequivalence of $D(Z)$.*
2. *The natural map $\Phi^{ladj_E} \xrightarrow{(2.36)} F_E \Phi_E^{\mathrm{radj}}[1]$ is an isomorphism of functors.*

When $Z = $ Spec $\mathbb{C}$ this is equivalent to Definition 1.1 above (Example 3.5). It also explains why most of the examples over a non-trivial base Z came from subschemes of X fibred over Z. These are the cases when the autoequivalence F_E is of particularly nice form. Indeed, for such fibrations the Fourier–Mukai kernel of F_E must be supported on the diagonal Δ of $Z \times Z$ (Lemma 3.9), and an autoequivalence of $D(Z)$ is supported on Δ if and only if it is simply tensoring by some shifted line bundle $\mathcal{L}_E$ in $D(Z)$ (Prop. 3.7). This makes the Fourier–Mukai kernel of $\Phi_E^{radj}\Phi_E$, a certain $\mathbf{R}\mathcal{H}om$ complex, into an extension of $\Delta_* \mathcal{L}_E$ by $\Delta_* \mathcal{O}_X$. Pointwise, this turns into a familiar condition that a certain $\mathbf{R}\operatorname{Hom}$ complex is $\mathbb{C} \oplus \mathbb{C}[d]$ for some $d \in \mathbb{Z}$.

In Section 3 of the present paper we show that this argument can be made very general. Let Z and X be arbitrary schemes of finite type over an algebraically closed field k of characteristic 0. No assumptions of smoothness or projectivity are made. Instead we make two assumptions on the object $E \in D(Z \times X)$: E is perfect (locally quasi-isomorphic to a bounded complex of free sheaves) and the support of E is proper over Z and over X. These are necessary for Φ_E to have adjoints which are again Fourier–Mukai transforms. We then categorify the notion of "a subscheme of X fibred over Z". The graphs of such subschemes in $Z \times X$ are characterised by the property that their fibres over points of Z are mutually disjoint in X, and in derived categories the notion of disjointness is expressed by orthogonality - vanishing of all Hom's between two objects, so the objects we want are the objects in $D(Z \times X)$ which are *orthogonal* over Z, i.e. their fibres over points of Z are mutually orthogonal in $D(X)$. In Lemma 3.9 we show that E is orthogonal over Z if and only if the support of the Fourier–Mukai kernel of the co-twist F_E is contained in the diagonal Δ of $Z \times Z$. Hence F_E is an autoequivalence if and only if it is the functor of tensoring by some invertible (locally a shifted line bundle) object of $D(Z)$. We describe this object independently as the cone $\mathcal{L}_E$ of a natural morphism

$$\mathcal{O}_Z \xrightarrow{\text{Definition 3.6}} \pi_{Z*}\, \mathbf{R}\mathcal{H}om_{Z \times X}(\pi_{X*}\pi_X^* E, E) \qquad \pi_Z, \pi_X \text{ are projections } Z \times X \to Z, X \tag{1.3}$$

and show that, conversely, if $\mathcal{L}_E$ is invertible then $F_E \simeq (-) \otimes \mathcal{L}_E$ (Prop. 3.7). To check whether $\mathcal{L}_E$ is invertible we restrict (1.3) to points of Z, whence we obtain our main theorem.

Theorem (Theorem 3.1)**.** *Let Z and X be two separable schemes of finite type over k. Let E be a perfect object of $D(Z \times X)$ orthogonal over Z and proper over Z and X. Then E is spherical over Z if and only if:*

(1) *For every closed point $p \in Z$ such that the fibre E_p is not zero we have*

$$\mathbf{R}\operatorname{Hom}_X(\pi_{X*}E, E_p) = k \oplus k[d_p] \quad \text{for some } d_p \in \mathbb{Z}.$$

(2) *The canonical morphism α (see Definition 3.10) is an isomorphism:*

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

Interestingly, a similar statement can be made for kernels of Fourier–Mukai equivalences (Example 3.3).

The integer d_p in (1) is constant on every connected component of Z. The n-spherical objects in the literature are those with $d_p = -n$ for all $p \in Z$. We do regret the sign difference. For any Gorenstein point (z, x) in the support of E we have $d_{z,x} = -(\dim_x X - \dim_x Z)$ (Prop. 3.11). The canonical morphism α in (2) is the morphism of Fourier–Mukai kernels which represents the natural morphism $\Phi^{ladj} \to F_E \Phi^{radj}[1]$ in the definition of a spherical functor. Due to this indirect definition it may be very difficult, even in simple cases, to write α down explicitly and check that it is an isomorphism. We deal with this in §3.4 where we show that if $d_p < 0$ for every $p \in Z$ (or, more generally, if a certain cohomology vanishes) the condition (2) can be relaxed simply to the two objects being isomorphic via any isomorphism (Cor. 3.14).

In Section 4 we reconsider the case of flat fibrations. Let $\xi \colon D \hookrightarrow X$ be a subscheme with $\pi \colon D \to Z$ a flat and surjective map. We apply the results of Section 3 to $\mathcal{O}_D$ in $D(Z \times X)$. One of our goals is to understand what geometric properties a spherical fibration must possess. The two technical assumptions on the object E in Section 3 translate to the assumptions of the fibres of D over Z being proper and of $\mathcal{O}_D$ being a perfect object of $D(Z \times X)$. We first give the most general analogue of Theorem 3.1 which applies to any flat fibration D with the above properties (Theorem 4.1). Then we improve on it for the case when either the fibres of D are Gorenstein schemes or ξ is a Gorenstein map, noting in particular that for any spherical D these two conditions are, in fact, equivalent (Prop. 4.8). Finally, we treat the case when the immersion ξ is *regular*, i.e. locally on X the ideal of D is generated by a regular sequence. The key property for us is that the cohomology sheaves of $\xi^*\xi_*\mathcal{O}_D$ are then the vector bundles $\wedge^j \mathcal{N}^\vee$ where $\mathcal{N}$ is the normal sheaf of D in X. The object $\xi^*\xi_*\mathcal{O}_D$ is the key to computing the Ext complex in the condition (1) of Theorem 3.1 and therefore (1) can be deduced via a spectral sequence argument from fibre-wise vanishing of the cohomology of $\wedge^j \mathcal{N}$. In fact, the reverse implication can also be obtained if the complex $\xi^*\xi_*\mathcal{O}_D$ actually splits up as a direct sum of $\wedge^j \mathcal{N}^\vee[j]$. In [AC10] Arinkin and Caldararu had shown that for a smooth X this happens if and only if $\mathcal{N}$ extends to the first infinitesimal neighborhood of D in X, e.g. when D is carved out by a section of a vector bundle, or when D is the fixed locus of a finite group action, or when ξ can be split. For any regular immersion ξ we say that it is *Arinkin-Caldararu* if $\xi^*\xi_*\mathcal{O}_D$ splits up as the direct sum of its cohomology sheaves. Then:

Theorem (Theorem 4.2)**.** *Let D be a regularly immersed flat and perfect fibration in X over Z with proper fibres. Let $\mathcal{N}$ be the normal sheaf of D in X. Then D is spherical if for any closed point $p \in Z$ the fibre D_p is a connected Gorenstein scheme and*

(1) $H^i_{D_p}(\wedge^j \mathcal{N}|_{D_p}) = 0$ *unless* $i = j = 0$ *or* $i = \dim D_p$ *,* $j = \operatorname{codim}_X D$.

(2) $(\omega_{D/X})|_{D_p} \simeq \omega_{D_p}$ *where ω_{D_p} is the dualizing sheaf of D_p and $\omega_{D/X} = \wedge^{\operatorname{codim}_X D} \mathcal{N}$.*

Conversely, if D is spherical, then each fibre D_p is a connected Gorenstein scheme and (2) holds. And if ξ is an Arinkin-Caldararu immersion, then (1) also holds.

The *'If'* implication here lifts the result in [Hor05] to our more general setting and, in fact, the same argument works for any object in $D(D)$ and not just $\mathcal{O}_D$. The converse implication is new. Note that it implies that for any spherical fibration D we must have $H^i_{D_p}(\mathcal{O}_{D_p}) = 0$ for all $i > 0$, which agrees with the fact that in the known examples the fibres of spherical fibrations are Fano varieties.

Section 2 contains the preliminaries necessary for all of the above. In §2.2 we work out explicitly the morphisms of kernels which underly the left and right adjunction units of a general Fourier–Mukai functor. We need this to compute F_E since co-twist functors need to be defined as the cones of adjunction units. We get this result for free from the similar result for adjunction co-units in [AL10] using the Grothendieck duality arguments summarized in §2.1. We then review the formalism of spherical functors in Section §2.3.

Finally, in the Appendix we give an example of an orthogonally spherical object which is not a spherical fibration and which is a genuine complex and not just a shifted sheaf. It arises naturally when constructing an affine braid group action on (n,n)-fibre of the Grothendieck-Springer resolution of the nilpotent cone of $\mathfrak{sl}_{2n}(\mathbb{C})$. The authors hope that the tools developed in this paper will allow to construct more examples of orthogonally spherical objects which aren't sheaves and to study explicitly the derived autoequivalences which they induce.

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2. Preliminaries

Notation: Throughout the paper we define our schemes over the base field k which is assumed to be an algebraically closed field of characteristic 0. We also denote by **Vect** the category of finitely generated k-modules, or in other words - the category of finite-dimensional vector spaces over k. Given a fibre product $X_1 \times \cdots \times X_n$ we denote by π_i, the projection $X_1 \times \cdots \times X_n \to X_i$ onto the i-th component, unless specifically mentioned otherwise.

Let X be a scheme. We denote by $D_{\mathrm{qc}}(X)$, resp. $D(X)$, the full subcategory of the derived category of $\mathcal{O}_X$-**Mod** consisting of complexes with quasi-coherent, resp. bounded and coherent, cohomology. Given an object E in $D(\mathcal{O}_X\text{-}\mathbf{Mod})$ we denote by $\mathcal{H}^i(E)$ the i-th cohomology sheaf of E and by $E^\vee$ its derived dual, the object $\mathbf{R}\mathcal{H}om_X(E, \mathcal{O}_X)$.

All the functors in this paper are presumed to be derived until proven otherwise.

We therefore omit all the usual $\mathbf{R}$'s and $\mathbf{L}$'s. An exception is made for the derived bi-functor $\mathbf{R}\operatorname{Hom}_X(-,-)$ of the bi-functor $\operatorname{Hom}_X(-,-)$ which maps any $\mathcal{A}, \mathcal{B} \in \operatorname{Coh}(X)$ to the space $\operatorname{Hom}_X(\mathcal{A}, \mathcal{B})$ of the morphisms from $\mathcal{A}$ to $\mathcal{B}$ in $\operatorname{Coh}(X)$. This was done to distinguish for any $A, B \in D(X)$ the complex $\mathbf{R}\operatorname{Hom}_X(A, B)$ in $D(\mathbf{Vect})$ from the vector space $\operatorname{Hom}_{D(X)}(A, B)$ which is the space of the morphisms from A to B in $D(X)$. Another exception was made for the derived functor $\mathbf{R}\mathcal{H}om_X(-,-)$ of taking a sheaf of morphisms between two objects. This was done so that it still looks like a curly version of $\mathbf{R}\operatorname{Hom}_X(-,-)$.

All the categories we consider are most certainly 1-categories. However given a morphism $A \to B$ in a category we can consider it as a (trivial) commutative diagram. For two commutative diagrams of the same shape there is a well defined notion of them being isomorphic, e.g. in our case $A \to B$ is isomorphic to another diagram $A' \to B'$ if and only if there exist isomorphisms which make the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ A' & \longrightarrow & B' \end{array}$$

commute. Sometimes as an abuse of notation we describe this by saying that the morphism $A \to B$ is 'isomorphic' to the morphism $A' \to B'$. Clearly this imposes an equivalence relation on the set of morphisms in a given category. This equivalence relation is important in the context of a triangulated category because all the morphisms in the same equivalence class will have isomorphic cones.

2.1. **On duality theories.** The standard reference on Grothendieck-Verdier duality has for some time been [Har66]. There the duality theory is constructed by hand in a (comparatively) geometric and (comparatively) painful fashion. For a more modern and (comparatively) more elegant categorical approach which obtains the existence of the right adjoint to f_* by pure thought we can recommend the reader an excellent exposition in [Lip09]. Below we give a brief overview of the results we intend to use. Our approach relies heavily on notions of a perfect object in a derived category, both in an absolute sense and relative to a morphism. The reader may find this discussed at length in [Ill71b] and [Ill71a].

Let S be a Noetherian scheme. Let $\mathcal{FT}_S$ be the category of separated schemes of finite type over S whose morphisms are separated S-scheme maps of finite type. We have the following (relative) duality theory $D_{\bullet/S}$ for schemes in $\mathcal{FT}_S$: for any $X \xrightarrow{f} S$ let $D_{X/S}$ denote the functor $\mathbf{R}\mathcal{H}om\left(-, f^!\mathcal{O}_X\right)$ from $D(\mathcal{O}_X\text{-}\mathbf{Mod})$ to $D(\mathcal{O}_X\text{-}\mathbf{Mod})^{op}$. Here $(-)^!$ is the twisted inverse image pseudo-functor (see [Lip09], Theorem 4.8.1). It follows from [Ill71a], Corollary 4.9.2 that $D_{X/S}$ takes $D_{S\text{-}\mathrm{perf}}(X)$, the full subcategory of $D(\mathbf{Mod}\text{-}\mathcal{O}_X)$ consisting of objects perfect over S, to itself in the opposite category and the restriction is a self-inverse equivalence

$$D_{X/S}\colon \quad D_{S\text{-}\mathrm{perf}}(X) \xrightarrow{\sim} D_{S\text{-}\mathrm{perf}}(X)^{op}.$$

Now, given any two schemes X and Y in $\mathcal{FT}_S$ and any exact functor $F\colon D_{S\text{-}\mathrm{perf}}(X) \to D_{S\text{-}\mathrm{perf}}(Y)$ we define its dual under $D_{\bullet/S}$ to be the functor $D_{Y/S}\ F\ D_{X/S}\colon D_{S\text{-}\mathrm{perf}}(X) \to D_{S\text{-}\mathrm{perf}}(Y)$. The double-dual of a functor is then the functor itself and we say that F and $D_{Y/S}\ F\ D_{X/S}$ are *dual under* $D_{\bullet/S}$. The (contravariant) notion of a dual of a morphism of functors is defined accordingly. One can then easily see that if a functor has a left (resp. right) adjoint then $D_{\bullet/S}$ sends it to the right (resp. left) adjoint of its dual and interchanges the adjunction units with the adjunction co-units.

Let X be a scheme in $\mathcal{FT}_S$ and let E be a perfect (in an absolute sense) object of $D(\mathcal{O}_X\text{-}\mathbf{Mod})$. Then the functor $E \otimes (-)$ takes $D_{S\text{-}\mathrm{perf}}(X)$ to $D_{S\text{-}\mathrm{perf}}(X)$, its adjoint, both left and right, is the functor $E^\vee \otimes (-)$ and for any $F \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ we have ([AIL10], Lemma 1.4.6) a natural isomorphism

$$D_{X/S}(E \otimes F) \xrightarrow{\sim} E^\vee \otimes D_{X/S}F. \tag{2.1}$$

Therefore $E \otimes (-)$ and $E^\vee \otimes (-)$ are dual under $D_{\bullet/S}$ and $D_{\bullet/S}$ interchanges the adjunction unit $\mathrm{Id} \to E^\vee \otimes E \otimes (-)$ and the adjunction co-unit $E^\vee \otimes E\otimes \to \mathrm{Id}$.

Let $X \xrightarrow{f} Y$ be a proper map in $\mathcal{FT}_S$. Then f_* sends $D_{S\text{-}\mathrm{perf}}(X)$ to $D_{S\text{-}\mathrm{perf}}(Y)$. We have it from the sheafifed Grothendieck duality ([Lip09], Corollary 4.4.2) that for any $E \in D_{qc}(X)$ the natural map

$$D_{Y/S}(f_*E) \xrightarrow{\sim} f_*(D_{X/S}E). \tag{2.2}$$

is an isomorphism. It follows that f_* is self-dual under $D_{\bullet/S}$.

On the other hand, let $X \xrightarrow{f} Y$ be now any map in $\mathcal{FT}_S$ such that f^* takes $D_{S\text{-}\mathrm{perf}}(Y)$ to $D_{S\text{-}\mathrm{perf}}(X)$ (e.g. f is perfect or any f when $S = \mathrm{Spec}\ k$). We have ([Lip09], Excercise 4.2.3(e)) for any $E \in D_{qc}(Y)$ a natural isomorphism

$$D_{X/S}(f^*E) \xrightarrow{\sim} f^!(D_{Y/S}E) \tag{2.3}$$

It follows that f^* and $f^!$ are dual under $D_{\bullet/S}$. If f is proper, then f^* and $f^!$ are the left and the right adjoints of f_*, so them being dual under $D_{\bullet/S}$ is precisely equivalent to f_* being self-dual.

Even when f^* doesn't take S-perfect objects to S-perfect objects, it still follows immediately from the definitions of maps (2.2) and (2.3) in [Lip09] that for any $E \in D_{qc}(Y)$ the following diagram commutes

$$\begin{array}{ccc} D_{Y/S}(f_*f^*E) & \xrightarrow{(\mathrm{Id}\to f_*f^*)^{opp}} & D_{Y/S}E \\ {\scriptstyle (2.2)+(2.3)}\downarrow \simeq & \nearrow_{f_*f^!\to \mathrm{Id}} & \\ f_*f^!D_{Y/S}E & & \end{array} \tag{2.4}$$

i.e. $D_{\bullet/S}$ still send the adjunction unit $\mathrm{Id} \to f_*f^*$ to the adjunction co-unit $f_*f^! \to \mathrm{Id}$. We then also have:

Lemma 2.1. *Let $X \xrightarrow{f} Y$ be any map in $\mathcal{FT}_S$, let E be a perfect object in $D(\mathbf{Mod}\text{-}Y)$ and let F be an S-perfect object in $D(\mathbf{Mod}\text{-}Y)$. Then the natural map*

$$f^*E \otimes f^!F \to f^!(E \otimes F) \tag{2.5}$$

is an isomorphism.

Proof. By its definition the map (2.5) is the right adjoint with respect to f_* of the composition

$$f_*(f^*E \otimes f^!F) \xrightarrow{\text{inverse of projection formula map}} E \otimes f_*f^!F \xrightarrow{f_*f^!\to\mathrm{Id}} E \otimes F. \tag{2.6}$$

Using the duality isomorphism $D_{\bullet/S}D_{\bullet/S}F \simeq F$, isomorphisms (2.1)-(2.3) and (2.4), we can re-write (2.6) as

$$D_{\bullet/S}\ \left(E^\vee \otimes D_{\bullet/S}F \xrightarrow{\mathrm{Id}\to f_*f^*} E^\vee \otimes f_*f^*D_{\bullet/S}F \xrightarrow{\text{projection formula map}} f_*(f^*E \otimes f^*D_{\bullet/S}F)\right)^{\mathrm{opp}} \tag{2.7}$$

which by [AL10], Lemma 2.2 is the same map as

$$D_{\bullet/S}\ \left(E^\vee \otimes D_{\bullet/S}F \xrightarrow{\mathrm{Id}\to f_*f^*} f_*f^*(E^\vee \otimes D_{\bullet/S}F) \xrightarrow{f^*(\text{-}\otimes\text{-})\to f^*\otimes f^*} f_*\left(f^*E^\vee \otimes f^*D_{\bullet/S}F\right)\right)^{\mathrm{opp}}. \tag{2.8}$$

Using (2.1)-(2.3), (2.4) and $D_{\bullet/S}D_{\bullet/S}F \simeq F$ again, we deduce that (2.6) is the same map as

$$f_*(f^*E \otimes f^!F) \xrightarrow{f_*\alpha} f_*f^!(E \otimes F) \xrightarrow{f_*f^!\to\mathrm{Id}} E \otimes F \tag{2.9}$$

where the map α is isomorphic to

$$D_{\bullet/S}\ \left(f^*(E^\vee \otimes D_{\bullet/S}F) \xrightarrow{f^*(\text{-}\otimes\text{-})\to f^*\otimes f^*} f^*E^\vee \otimes f^*D_{\bullet/S}F\right)^{\mathrm{opp}}. \tag{2.10}$$

Since the right adjoint of (2.9) with respect to f_* is clearly α, we conclude that the natural map (2.5) is precisely the map α which is evidently an isomorphism. □

In the special case of $S = \text{Spec } k$ the category $\mathcal{FT}_k$ is simply the category of all schemes of finite type over k. For any such scheme X we have $D_{S\text{-perf}}(X) = D^b_{coh}(X)$. The resulting duality theory $D_{\bullet/k}$ is the usual duality theory of [Har66] with $D_{X/k}(\mathcal{O}_X)$ being dualizing complexes in sense of [Har66], Chapter V.

On the other hand we have the perfect duality theory which exists in the category of arbitrary schemes. Let X be a scheme and let DP_X denote the functor $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X)$ from $D(\mathcal{O}_X\text{-}\mathbf{Mod})$ to $D(\mathcal{O}_X\text{-}\mathbf{Mod})^{op}$, i.e. $\text{DP}_X(E) = E^\vee$. It is shown in [Ill71b], §7 that DP_X takes $D_{\text{perf}}(X)$, the full subcategory of $D(\mathbf{Mod}\text{-}\mathcal{O}_X)$ consisting of perfect objects, to itself in the opposite category and the restriction is a self-inverse equivalence

$$\text{DP}_X\colon \quad D_{\text{perf}}(X) \xrightarrow{\sim} D_{\text{perf}}(X)^{op}.$$

Then, given any two schemes X and Y, we define just as above the notions of a dual under DP of any functor $F\colon D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ and of any natural transformation between two such functors. Once again, the duality interchanges left adjoints with right adjoints and the adjunction units with the adjunction counits.

Let $X \xrightarrow{f} Y$ be any scheme map. Then f^* sends $D_{\text{perf}}(Y)$ to $D_{\text{perf}}(X)$ and we have ([Ill71b], Prps. 7.1.2) for any $E \in D_{\text{perf}}(Y)$ a natural isomorphism

$$\text{DP}_X f^* E \xrightarrow{\sim} f^* \text{DP}_Y E. \tag{2.11}$$

It follows that f^* is self-dual under DP.

Now let $X \xrightarrow{f} Y$ be any scheme map such that f_* sends $D_{\text{perf}}(X)$ to $D_{\text{perf}}(Y)$, e.g. a quasi-perfect map of concentrated schemes ([Lip09], §4.7). Then, since f^* is self-dual, the dual of f_* under DP is the left adjoint $f_\dagger$ of f^*. And when f is a separated finite-type perfect map of Noetherian schemes, we know ([AIL10], Lemma 2.1.10) that $f_\dagger(-)$ is naturally isomorphic to $f_*(f^!(\mathcal{O}_Y) \otimes -)$ in a way which makes the composition

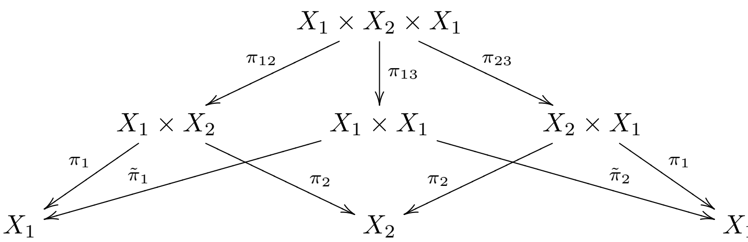
$$f_\dagger f^* \xrightarrow{\sim} f_*\left(f^!(\mathcal{O}_Y) \otimes f^*(-)\right) \xrightarrow{\sim} f_* f^! \xrightarrow{\text{the adjunction co-unit}} \text{Id}$$

be precisely the adjunction co-unit $f_\dagger f^* \to \text{Id}$.

2.2. **Adjunction units and Fourier–Mukai transforms.** The definition of a spherical functor S in [Ann07] demands that S has both a left adjoint L and a right adjoint R. It moreover demands us to work in the universe where taking cone of a morphism of functors is well-defined (see [Ann07], §1). In our case we can use a traditional choice of restricting ourselves to working only with the functors which are isomorphic to Fourier–Mukai transforms and only with the morphisms of functors which come from the morphisms of the corresponding Fourier–Mukai transforms. We still need, however, to demonstrate that S has both left and right adjoints and that these two adjoints and the four corresponding adjunction units and co-units belong to our chosen universe. That is - the adjoints are isomorphic to Fourier–Mukai transforms and the adjunction morphisms are all induced by morphisms of Fourier–Mukai kernels.

Partly this was achieved in §2.1 of [AL10]. We give a brief summary here. Quite generally, let X_1 and X_2 be two separated proper schemes of finite type over k and let E be a perfect object in $D(X_1 \times X_2)$. We have a commutative diagram of projection morphisms:

(2.12)



Let $\Phi_E\colon D(X_1) \to D(X_2)$ be the Fourier–Mukai transform $\pi_{2*}(E \otimes \pi_1^*(-))$ with kernel E, then:

1. A left adjoint Φ_E^{ladj} to Φ_E exists and is isomorphic to the Fourier–Mukai transform $\Psi_{E^\vee \otimes \pi_1^!(\mathcal{O}_{X1})}$ from $D(X_2)$ to $D(X_1)$.
2. A right adjoint Φ_E^{radj} to Φ_E exists and is isomorphic to the Fourier–Mukai transform $\Psi_{E^\vee \otimes \pi_2^!(\mathcal{O}_{X2})}$ from $D(X_2)$ to $D(X_1)$.
3. The adjunction co-unit $\Phi_E^{ladj}\Phi_E \to \text{Id}_{D(X_1)}$ is isomorphic to the morphism $\Theta_Q \to \Theta_{\mathcal{O}_\Delta}$ of Fourier–Mukai transforms $D(X_1) \to D(X_1)$ induced by the morphism $Q \to \mathcal{O}_\Delta$ of objects of $D(X_1 \times X_1)$ written down explicitly in [AL10], Theorem 2.1 (to which we refer the reader for all the details). An analogous statement holds for the adjunction co-unit $\Phi_E \Phi_E^{radj} \to \text{Id}_{D(X_2)}$ ([AL10], Cor. 2.5).
4. The condition of X_1 and X_2 being proper can be replaced by the condition of the support of E being proper over X_1 and over X_2 ([AL10], §2.2). If, moreover, E is actually a pushforward of an object in the derived category of a closed subscheme $X_1 \times X_2$ proper over X_1 and X_2, then there is

an alternative description of the morphisms of Fourier–Mukai kernels which produce the adjunction co-units $\Phi_E^{ladj}\Phi_E \to \mathrm{Id}_{D(X_1)}$ and $\Phi_E\Phi_E^{radj} \to \mathrm{Id}_{D(X_2)}$ ([AL10], Theorem 3.1 and Cor. 3.5).

What remains to be done is to obtain a similar result for the adjunction units $\mathrm{Id}_{D(X_1)} \to \Phi_E^{radj}\Phi_E$ and $\mathrm{Id}_{D(X_2)} \to \Phi_E\Phi_E^{ladj}$. Fortunately this can be obtained directly from the above results in [AL10] via the Grothendieck-Verdier duality in the following way.

The dual of the Fourier–Mukai transform

$$\Phi_E(-) = \pi_{2*}\left(E \otimes \pi_1^*(-)\right)$$

under the duality theory $D_{\bullet/k}$ (see Section 2.1) is the functor

$$\pi_{2*}\,\mathbf{R}\mathcal{H}om\left(E, \pi_1^!(-)\right). \tag{2.13}$$

There are two ways to view this functor. Firstly, via natural isomorphisms

$$\mathbf{R}\mathcal{H}om\left(E, \pi_1^!\mathcal{O}_{X1}\right) \otimes \pi_1^*(-) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om\left(E, \pi_1^!\mathcal{O}_{X1} \otimes \pi_1^*(-)\right) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om\left(E, \pi_1^!(-)\right) \tag{2.14}$$

we can identify (2.13) with the Fourier–Mukai transform $\Phi_{\mathbf{R}\mathcal{H}om(E,\pi_1^!\mathcal{O}_{X1})}$ from $D(X_1)$ to $D(X_2)$. Secondly, observe that (2.13) is the right adjoint Ψ_E^{radj} of the Fourier–Mukai transform Ψ_E from $D(X_2)$ to $D(X_1)$.

Taking this second point of view, it immediately follows that the dual of Φ_E^{radj} is Ψ_E and the dual of the adjunction unit

$$\mathrm{Id}_{D(X_1)} \to \Phi_E^{radj}\Phi_E \tag{2.15}$$

is the adjunction co-unit

$$\Psi_E\Psi_E^{radj} \to \mathrm{Id}_{D(X_1)}. \tag{2.16}$$

By [AL10], Corollary 2.4, the adjunction co-unit (2.16) is isomorphic to the morphism

$$\Theta_{\tilde{Q}} \to \Theta_{\mathcal{O}_\Delta} \tag{2.17}$$

of Fourier–Mukai tranforms $D(X_1) \to D(X_1)$ induced by the following morphism of objects of $D(X_1 \times X_1)$:

$$\tilde{Q} = \pi_{13*}\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\mathrm{Id}\to\Delta_*\Delta^*} \pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \tag{2.18}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \quad\simeq\quad \Delta_*\pi_{1*}\left(E^\vee \otimes E \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \tag{2.19}$$

$$\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{E^\vee\otimes E\otimes\to\mathrm{Id}} \Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \tag{2.20}$$

$$\Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\pi_{1*}\pi_1^!\to\mathrm{Id}} \Delta_*\mathcal{O}_{X_1}. \tag{2.21}$$

Identifying[1] the duals of $\Theta_{\tilde{Q}}$ and $\Theta_{\mathcal{O}_\Delta}$ under $D(\bullet/k)$ with $\Theta_{\mathbf{R}\mathcal{H}om(\tilde{Q},\tilde{\pi}_1^!\mathcal{O}_{X_1})}$ and $\Theta_{\mathbf{R}\mathcal{H}om(\mathcal{O}_\Delta,\tilde{\pi}_1^!\mathcal{O}_{X_1})}$, we see that the dual of (2.17) under $D_{\bullet/k}$ is the morphism of Fourier–Mukai transforms induced by the morphism

$$\mathbf{R}\mathcal{H}om(\mathcal{O}_\Delta, \tilde{\pi}_1^!\mathcal{O}_{X_1}) \to \mathbf{R}\mathcal{H}om(\tilde{Q}, \tilde{\pi}_1^!\mathcal{O}_{X_1}) \tag{2.22}$$

obtained by applying the relative dualizing functor $D_{X_1\times X_1/X_1} = \mathbf{R}\mathcal{H}om(-, \tilde{\pi}_1^!\mathcal{O}_{X_1})$ to (2.18) – (2.21).

Treating (2.18) – (2.21) as morphisms of functors in $\mathcal{O}_{X_1}$ and dualizing them with respect to the relative duality theory $D_{\bullet/X_1}$ as described in Section 2.1, we see that $D_{X_1\times X_1/X_1}$ applied to (2.18) – (2.21) yields:

$$\Delta_*D_{X_1/X_1}(\mathcal{O}_{X_1}) \xrightarrow{\mathrm{Id}\to\pi_{1*}\pi_1^*} \Delta_*\pi_{1*}\pi_1^*D_{X_1/X_1}(\mathcal{O}_{X_1}) \tag{2.23}$$

$$\Delta_*\pi_{1*}\pi_1^*D_{X_1/X_1}(\mathcal{O}_{X_1}) \xrightarrow{\mathrm{Id}\to E\otimes E^\vee\otimes} \Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^*D_{X_1/X_1}(\mathcal{O}_{X_1})\right) \tag{2.24}$$

$$\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^*D_{X_1/X_1}(\mathcal{O}_{X_1})\right) \quad\simeq\quad \pi_{13*}\Delta_*\Delta^!\left(\pi_{12}^*E \otimes \pi_{23}^*E^\vee \otimes \pi_{12}^!\pi_1^*D_{X_1/X_1}(\mathcal{O}_{X_1})\right) \tag{2.25}$$

$$\pi_{13*}\Delta_*\Delta^!\left(\pi_{12}^*E \otimes \pi_{23}^*E^\vee \otimes \pi_{12}^!\pi_1^*D_{X_1/X_1}\mathcal{O}_{X_1}\right) \xrightarrow{\Delta_*\Delta^!\to\mathrm{Id}} \pi_{13*}\left(\pi_{12}^*E \otimes \pi_{23}^*E^\vee \otimes \pi_{12}^!\pi_1^*D_{X_1/X_1}\mathcal{O}_{X_1}\right) \tag{2.26}$$

By the above (2.23)-(2.26) induces a morphism of the Fourier–Mukai transforms isomorphic to the dual of (2.17). Since (2.17) is itself isomorphic to the dual of $\mathrm{Id}_{X1} \to \Phi_E^{radj}\Phi_E$, we conclude that the morphism of transforms induced by (2.23)-(2.26) is isomorphic to $\mathrm{Id}_{X1} \to \Phi_E^{radj}\Phi_E$ Finally, since $D_{X_1/X_1}(\mathcal{O}_{X_1}) \simeq \mathcal{O}_{X_1}$ and $\pi_{12}^!\pi_1^*(\mathcal{O}_{X_1}) \simeq \pi_{23}^*\pi_2^!(\mathcal{O}_{X_2})$, we obtain:

---

[1] One has to be a little careful here since $\mathcal{O}_\Delta$, unlike $\tilde{Q}$, is not a perfect object of $D(X_1 \times X_1)$. However, both natural maps in the analogue of (2.14) are still isomorphisms so we can still make the same identification.

Proposition 2.2. *Let X_1 and X_2 be two separated proper schemes of finite type over k and let E be a perfect object of $D(X_1 \times X_2)$. Then the adjunction unit $\mathrm{Id}_{X1} \to \Phi_E^{\mathrm{radj}} \Phi_E$ is isomorphic to the morphism of Fourier–Mukai transforms induced by the following morphism of objects of $D(X_1 \times X_1)$:*

$$\Delta_*(\mathcal{O}_{X_1}) \xrightarrow{\mathrm{Id} \to \pi_{1*}\pi_1^*} \Delta_* \pi_{1*} \pi_1^* (\mathcal{O}_{X_1}) \tag{2.27}$$

$$\Delta_* \pi_{1*} \pi_1^* (\mathcal{O}_{X_1}) \xrightarrow{\mathrm{Id} \to E \otimes E^\vee \otimes} \Delta_* \pi_{1*} \left(E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1})\right) \tag{2.28}$$

$$\Delta_* \pi_{1*} \left(E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1})\right) \quad \simeq \quad \pi_{13*} \Delta_* \Delta^! \left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_2^! (\mathcal{O}_{X_2})\right) \tag{2.29}$$

$$\pi_{13*} \Delta_* \Delta^! \left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_2^! (\mathcal{O}_{X_2})\right) \xrightarrow{\Delta_* \Delta^! \to \mathrm{Id}} \pi_{13*} \left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_2^! (\mathcal{O}_{X_2})\right) \tag{2.30}$$

In a similar fashion we also obtain:

Proposition 2.3. *Let X_1 and X_2 be two separated proper schemes of finite type over k and let E be a perfect object of $D(X_1 \times X_2)$. Then the adjunction unit $\mathrm{Id}_{X1} \to \Psi_E \Psi_E^{\mathrm{ladj}}$ is isomorphic to the morphism of Fourier–Mukai transforms induced by the following morphism of objects of $D(X_1 \times X_1)$:*

$$\Delta_*(\mathcal{O}_{X_1}) \xrightarrow{\mathrm{Id} \to \pi_{1*}\pi_1^*} \Delta_* \pi_{1*} \pi_1^* (\mathcal{O}_{X_1}) \tag{2.31}$$

$$\Delta_* \pi_{1*} \pi_1^* (\mathcal{O}_{X_1}) \xrightarrow{\mathrm{Id} \to E \otimes E^\vee \otimes} \Delta_* \pi_{1*} \left(E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1})\right) \tag{2.32}$$

$$\Delta_* \pi_{1*} \left(E \otimes E^\vee \otimes \pi_1^*(\mathcal{O}_{X_1})\right) \quad \simeq \quad \pi_{13*} \Delta_* \Delta^! \left(\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^* \pi_2^! (\mathcal{O}_{X_2})\right) \tag{2.33}$$

$$\pi_{13*} \Delta_* \Delta^! \left(\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^* \pi_2^! (\mathcal{O}_{X_2})\right) \xrightarrow{\Delta_* \Delta^! \to \mathrm{Id}} \pi_{13*} \left(\pi_{12}^* E^\vee \otimes \pi_{23}^* E \otimes \pi_{12}^* \pi_2^! (\mathcal{O}_{X_2})\right) \tag{2.34}$$

If X_1 and X_2 are not proper, but the support of E is proper over X_1 and X_2, one can still apply the above results via compactification, as described in [AL10], §2.2. If E is actually a pushforward of an object in the derived category of a closed subscheme $D \hookrightarrow X_1 \times X_2$ proper over both X_1 and X_2, one can also dualize Theorem 3.1 and Corollary 3.5 of [AL10] to obtain an alternative description of morphisms of kernels which induce both adjunction units. We leave this as an exercise for the reader.

We conclude with:

Lemma 2.4. *Let E_1 and E_2 be two objects of $D(X_1 \times X_2)$ and let α be a morphism from E_1 to E_2. Then α is an isomorphism if and only if the induced morphism of functors $\Phi_{E_1} \to \Phi_{E_2}$ is an isomorphism.*

Proof. The 'only if' statement is obvious. For the 'if' statement we use the fact that for any closed point $p \in X_1$ and any A in $D(X_1)$ we have a natural isomorphism $\Phi_A(\mathcal{O}_p) \xrightarrow{\sim} \iota_p^*(A)$ which is functorial in A. So if $\Phi_{E_1} \to \Phi_{E_2}$ is an isomorphism then the pullback of α to any closed point of X_1 is an isomorphism. This implies that the pullback of the cone α to any closed point of X_1 is 0. Therefore the cone of α is itself 0 and α is an isomorphism. □

2.3. **Twists, co-twists and spherical functors.** Let X_1 and X_2 be, as before, two separated proper schemes of finite type over k, let E be a perfect object in $D(X_1 \times X_2)$ and let Φ_E be the Fourier–Mukai transform from $D(X_1)$ to $D(X_2)$ with kernel E. Then, as explained in Section 2.2, the following is well-defined:

Definition 2.5. We define *the right twist T_E of Φ_E* to be the functorial cone of the adjunction co-unit $\Phi_E \Phi_E^{\mathrm{radj}} \to \mathrm{Id}_{D(X_2)}$ so that we have an exact triangle

$$\Phi_E \Phi_E^{\mathrm{radj}} \to \mathrm{Id}_{D(X_2)} \to T_E.$$

We define *the right co-twist T'_E of Φ_E* to be the functorial cone of the adjunction unit $\mathrm{Id}_{D(X_2)} \to \Phi_E \Phi_E^{\mathrm{ladj}}$ shifted by one to the right so that we have an exact triangle

$$T'_E \to \mathrm{Id}_{D(X_2)} \to \Phi_E \Phi_E^{\mathrm{ladj}}.$$

We define *the left twist F'_E of Φ_E* to be the functorial cone of the adjunction co-unit $\Phi_E^{\mathrm{ladj}} \Phi_E \to \mathrm{Id}_{D(X_1)}$ so that we have an exact triangle

$$\Phi_E^{\mathrm{ladj}} \Phi_E \to \mathrm{Id}_{D(X_1)} \to F'_E.$$

We define *the left co-twist F_E of Φ_E* to be the functorial cone of the adjunction unit $\mathrm{Id}_{D(X_1)} \to \Phi_E^{\mathrm{radj}} \Phi_E$ shifted by one to the right so that we have an exact triangle

$$F_E \to \mathrm{Id}_{D(X_1)} \to \Phi_E^{\mathrm{radj}} \Phi_E. \tag{2.35}$$

The following notion was introduced by Anno in [Ann07]:

Definition 2.6. We say that the Fourier–Mukai transform Φ_E is *a spherical functor* if the left co-twist F_E of Φ_E is an autoequivalence of $D(X_1)$ and if the composition

$$\Phi_E^{radj} \xrightarrow{\mathrm{Id}\to\Phi_E\Phi_E^{ladj}} \Phi_E^{radj}\Phi_E\Phi_E^{ladj} \xrightarrow{\Phi_E^{radj}\Phi_E\to F_E[1] \text{ in } (2.35)} F_E[1]\Phi_E^{ladj} \tag{2.36}$$

is an isomorphism of functors $D(X_2)\to D(X_1)$.

We then have the following key result:

Theorem 2.1 ([Ann07], Proposition 1)**.** *If Φ_E is spherical, then the left twist F'_E and the left co-twist F_E are mutually inverse autoequivalences of $D(X_1)$, while the right twist T_E and the right co-twist T'_E are mutually inverse autoequivalences of $D(X_2)$.*

When proving a functor to be spherical the reader may find the following lemma useful:

Lemma 2.7. *The composition (2.36) is the unique morphism $\Phi_E^{radj} \xrightarrow{\alpha} F_E[1]\Phi_E^{\mathrm{ladj}}$ which makes the following diagram commute:*

$$\begin{array}{ccc} \Phi_E^{radj}\Phi_E & \xrightarrow{(2.35)} & F_E[1] \\ \downarrow{\scriptstyle\alpha} & \nearrow{\scriptstyle adj.\ co\text{-}unit} & \\ F_E[1]\Phi_E^{ladj}\Phi_E & & \end{array} \tag{2.37}$$

Proof. We first show that the composition (2.36) makes (2.37) commute. Indeed, composing each term with Φ_E and composing the whole isomorphism with the adjunction co-unit $\Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}_{D(X_1)}$ we obtain

$$\Phi_E^{radj}\Phi_E \xrightarrow{\text{adj. unit}} \Phi_E^{radj}\Phi_E\Phi_E^{ladj}\Phi_E \xrightarrow{(2.35)} F_E[1]\Phi_E^{ladj}\Phi_E \xrightarrow{\text{adj. co-unit}} F_E[1]. \tag{2.38}$$

Since clearly the following square commutes

$$\begin{array}{ccc} \Phi_E^{radj}\Phi_E\Phi_E^{ladj}\Phi_E & \xrightarrow{(2.35)} & F_E[1]\Phi_E^{ladj}\Phi_E \\ {\scriptstyle\text{adj. co-unit}}\downarrow & & \downarrow{\scriptstyle\text{adj. co-unit}} \\ \Phi_E^{radj}\Phi_E & \xrightarrow[(2.35)]{} & F_E[1] \end{array} \tag{2.39}$$

the composition (2.38) equals to

$$\Phi_E^{radj}\Phi_E \xrightarrow{\text{adj. unit}} \Phi_E^{radj}\Phi_E\Phi_E^{ladj}\Phi_E \xrightarrow{\text{adj. co-unit}} \Phi_E^{radj}\Phi_E \xrightarrow{(2.35)} F_E[1] \tag{2.40}$$

and is therefore simply $\Phi_E^{radj}\Phi_E \xrightarrow{(2.35)} F_E[1]$, as required.

Conversely, let $\alpha\colon \Phi_E^{radj} \to F_E\Phi_E^{\mathrm{ladj}}[1]$ be a morphism which makes (2.37) commute. We then have a commutative diagram

$$\begin{array}{ccccc} \Phi_E^{radj} & \xrightarrow{\text{adj. unit}} & \Phi_E^{radj}\Phi_E\Phi_E^{ladj} & \xrightarrow{(2.35)} & F_E[1]\Phi_E^{ladj} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle=} \\ F_E[1]\Phi_E^{\mathrm{ladj}} & \xrightarrow[\text{adj. unit}]{} & F_E[1]\Phi_E^{\mathrm{ladj}}\Phi_E\Phi_E^{ladj} & \xrightarrow[\text{adj. co-unit}]{} & F_E[1]\Phi_E^{ladj} \end{array} \tag{2.41}$$

Since the bottom row is the identity morphism, we conclude that α equals to the morphism given by the top row, i.e. to the composition (2.36). ☐

3. Orthogonally spherical objects

Let Z and X be two separable schemes of finite type over k. Given a closed point p in Z we denote by ι_p the closed immersion $\operatorname{Spec} k \hookrightarrow Z$ and by ι_{Xp} the corresponding immersion $X \hookrightarrow Z\times X$:

$$\begin{array}{ccccc} X & \xhookrightarrow{\iota_{Xp}} & Z\times X & & \\ \downarrow{\scriptstyle\pi_k} & & \downarrow{\scriptstyle\pi_Z} & \searrow{\scriptstyle\pi_X} & \\ \operatorname{Spec} k & \xhookrightarrow{\iota_p} & Z & & X \end{array} \tag{3.1}$$

Given a perfect object E in $D(Z \times X)$ we define *the fibre* E_p *of* E *at* p to be the object $\iota_{Xp}^* E$ in $D(X)$. In this way we can think of any perfect object in $D(Z \times X)$ as a family of objects of $D(X)$ parametrised by Z. For all of our Fourier–Mukai transforms to take complexes with bounded coherent cohomologies to complexes with bounded coherent cohomologies and to be able to apply the results in Section 2.2 on the adjunctions units/co-units for Fourier–Mukai transforms, we assume throughout this section that either Z and X are both proper or that the support of the object E in $Z \times X$ is proper over both Z and X.

3.1. **Orthogonal objects.** Our first goal is to come up with a categorification of the notion of a subscheme D of X fibred over Z. Our motivating geometric example is the following setup:

Example 3.1. Let D be a flat fibration in X over Z with proper fibres. By this we mean a scheme D equipped with a morphism $\xi \colon D \hookrightarrow X$ which is a closed immersion and a morphism $\pi \colon D \to Z$ which is flat and proper. Denote by ι_D the map $D \hookrightarrow Z \times X$ given by the product of π and ξ. We set E to be the structure sheaf of the graph of D in $Z \times X$, that is - the object $\iota_{D*}\mathcal{O}_D$ in $D(Z \times X)$.

An arbitrary subscheme D' of $Z \times X$ is a graph of some subscheme D of X fibred over Z if and only if the fibres of D' over closed points of Z are disjoint as subschemes of X. In derived categories the notion of disjointness corresponds to the notion of orthogonality, that is, to the vanishing of all the Ext's between them. This motivates us to suggest as a categorification of the notion of a subscheme of X fibred over Z the following class of objects in $D(Z \times X)$:

Definition 3.2. Let E be a perfect object of $D(Z \times X)$. We say that E is *orthogonal over* Z if for any two distinct points p and q in Z the fibres E_p and E_q are orthogonal in $D(X)$, or in other words

$$\operatorname{Hom}^i_{D(X)}(E_p, E_q) = 0 \qquad \text{for all } i \in \mathbb{Z} \tag{3.2}$$

Since E is a perfect object we have $(E^\vee)_p = (E_p)^\vee$. So if E is orthogonal over Z, then its dual $E^\vee$ is also orthogonal over Z.

Any object whose support in $Z \times X$ is the graph of a subscheme of X fibred over Z is immediately orthogonal over Z - as all the Ext's between two objects with disjoint supports must vanish. Another class of examples comes from Fourier–Mukai equivalences:

Example 3.3. The kernel F of any fully faithful Fourier–Mukai transform $\Phi_F \colon D(Z) \xrightarrow{\sim} D(X)$ is orthogonal over Z, since for any $p \in Z$ the fibre F_p is precisely the image under Φ_F of the skyscraper sheaf $\mathcal{O}_p$. Moreover, we have also $\Phi_F^{radj}\Phi_F(\mathcal{O}_p) \simeq \pi_{Z*}\mathbf{R}\mathcal{H}om_{Z\times X}(F, \pi_X^!(F_p))$. Applying π_{k*}, where π_k is the structure morphism $Z \to \operatorname{Spec}\ k$, to the adjunction unit $\mathcal{O}_p \to \Phi_F^{radj}\Phi_F(\mathcal{O}_p)$ (which is an isomorphism as Φ_F is fully faithful) one obtains $\mathbf{R}\operatorname{Hom}_X(\pi_{X*}F, F_p) = k$. It is possible (e.g. using the same techniques as in the proof of Proposition 3.7 below) to show that the converse is also true, i.e. Φ_F is fully faithful if and only if F is orthogonal over Z and $\mathbf{R}\operatorname{Hom}_X(\pi_{X*}F, F_p) = k$ for all $p \in Z$. Suppose now that Φ_F is further an equivalence, then all its adjunction units and co-units are isomorphisms. By Lemma 2.4 the morphisms of Fourier–Mukai kernels which induce them are also isomorphisms. In particular, the isomorphism of functors

$$\Phi_F^{radj} \xrightarrow{\text{adj. unit}} \Phi_F^{radj}\Phi_F\Phi_F^{ladj} \xrightarrow{\text{inverse of adj. co-unit}} \Phi_F^{ladj} \tag{3.3}$$

must come from an isomorphism $F^\vee \otimes \pi_X^!(\mathcal{O}_X) \to F^\vee \otimes \pi_Z^!(\mathcal{O}_Z)$ of their Fourier–Mukai kernels. Conversely, any isomorphism $F^\vee \otimes \pi_X^!(\mathcal{O}_X) \to F^\vee \otimes \pi_Z^!(\mathcal{O}_Z)$ induces an isomorphism $\Phi_F^{radj} \xrightarrow{\sim} \Phi_F^{ladj}$. And, when X is connected, Φ_F being fully-faithful and Φ_F^{radj} being isomorphic to Φ_F^{ladj} imply together that Φ_F is an equivalence ([Bri99], Theorem 3.3). We conclude when X is connected the kernels of Fourier–Mukai equivalences are precisely the objects which are orthogonal over Z and for which

$$\mathbf{R}\operatorname{Hom}_X(\pi_{X*}F, F_p) = k \text{ for all } p \in Z \tag{3.4}$$

$$F^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq F^\vee \otimes \pi_Z^!(\mathcal{O}_Z) \tag{3.5}$$

Our main goal now is to show that the orthogonal objects which are one step up from that, in the sense that $\mathbf{R}\operatorname{Hom}_X(\pi_{X*}F, F_p) = k \oplus k[d]$ for some $d \in \mathbb{Z}$ and a similar condition to (3.5) holds, are the kernels of the spherical Fourier–Mukai transforms.

3.2. **Spherical objects.**

Definition 3.4. Let E be a perfect object of $D(Z\times X)$. We say that E is *spherical over* Z if the Fourier–Mukai transform $\Phi_E \colon D(Z) \to D(X)$ is a spherical functor in the sense of [Ann07] or, in other words, if:

1. The co-twist F_E is an autoequivalence of $D(Z)$.
2. The natural map $\Phi^{ladj_E} \to F_E\Phi_E^{radj}[1]$ is an isomorphism of functors.

If E is also orthogonal over Z we say further that E is *orthogonally spherical over* Z.

Example 3.5. The spherical objects introduced by Seidel and Thomas in [ST01] can be thought of as the objects spherical over Spec k. Indeed let $Z = \operatorname{Spec} k$ and let X be a smooth variety over k. Then π_X is an isomorphism which identifies Spec $k \times X$ with X. Under this identification $\pi_X^!(\mathcal{O}_X)$ becomes simply $\mathcal{O}_X$ and $\pi_k^!(k)$ becomes the dualizing complex $D_{X/k}$ which is isomorphic to $\omega_X[\dim X]$ since X is smooth. Therefore the Fourier–Mukai kernel of the right adjoint Φ_E^{radj} is $E^\vee$ and the Fourier–Mukai kernel of the left adjoint Φ_E^{ladj} is $E^\vee \otimes \omega_X[\dim X]$. The triple product Spec $k \times X \times$ Spec k is identified with X by the projection π_2 and under this identification the projection π_{13} becomes the map $\pi_k \colon X \to \operatorname{Spec} k$. Therefore the Fourier–Mukai kernel of the composition $\Phi_E^{\text{radj}}\Phi_E$ is

$$\pi_{k*}(E^\vee \otimes E) \simeq \pi_{k*}\,\mathbf{R}\,\mathcal{H}om_X(E,E) \simeq \mathbf{R}\operatorname{Hom}_X(E,E)$$

and by the results of Section 2.2 the adjunction unit $\operatorname{Id}_{D(\mathbf{Vect})} \to \Phi_E^{\text{radj}}\Phi_E$ comes from the natural morphism $k \to \mathbf{R}\operatorname{Hom}_X(E,E)$ of Fourier–Mukai kernels which sends 1 to the identity automorphism of E. Denote this morphism by γ.

The first condition for Φ_E to be a spherical functor is for the left co-twist F_E to be an autoequivalence of $D(\mathbf{Vect})$. The only autoequivalences of $D(\mathbf{Vect})$ are the shifts $(-)[d]$ by some $d \in \mathbb{Z}$ and their Fourier–Mukai kernels are $k[d]$. The Fourier–Mukai kernel of F_E is the shift by 1 to the left of the cone of $k \xrightarrow{\gamma} \mathbf{R}\operatorname{Hom}_X(E,E)$. If E is non-zero the morphism γ is non-zero and then F_E is an autoequivalence if and only if $\mathbf{R}\operatorname{Hom}_X(E,E)$ is $k \oplus k[d]$ for some $d \in \mathbb{Z}$. If this does hold then $F_E = (-)[d-1]$. If E is 0, then the cone of γ is k and therefore F_E is the identity functor $\operatorname{Id}_{D(\mathbf{Vect})}$. As a side note, observe that the object E is trivially isomorphic to its single fibre over the single closed point of Spec k. Hence one way of re-phrasing the above would be that F_E is an autoequivalence if and only if for every point p of the base such that the fibre E_p is non-zero we have $\mathbf{R}\operatorname{Hom}_X(\pi_{X*}E, E_p) = k \oplus k[d]$ for some $d \in \mathbb{Z}$.

By Lemma 2.7 the second condition for Φ_E to be spherical is an isomorphism $\alpha \colon E^\vee \xrightarrow{\sim} E^\vee \otimes \omega_X[\dim X + d]$ which makes the diagram (2.37) commute. If E is 0 then this condition is trivially satisfied, so assume otherwise. Since $E^\vee$ and $E^\vee \otimes \omega_X$ are bounded complexes with non-zero cohomologies in exactly the same degrees, the isomorphism α is only possible when $d = -\dim X$. On the other hand, the diagram (2.37) on the level of the corresponding Fourier–Mukai kernels is just

$$\begin{array}{ccc} k \oplus k[d] & \xrightarrow{0 \oplus \mathrm{Id}} & k[d] \\ \downarrow{\scriptstyle \alpha'} & \nearrow{\scriptstyle 0\oplus\mathrm{Id}} & \\ k \oplus k[d] & & \end{array} \tag{3.6}$$

where α' is the isomorphism induced by α. The diagram commutes if α' restricts to the identity morphism on the component $k[d]$ and we can achieve that by multiplying any given α by an appropriate scalar in k.

We conclude that E is spherical over Spec k if and only if either E is 0 or if $\mathbf{R}\operatorname{Hom}_X(E,E) = k \oplus k[-\dim X]$ and $E \simeq E \otimes \omega_X$, which is precisely the definition given in [ST01]. And since the base Spec k is a single point, any object spherical over Spec k is orthogonally spherical.

3.3. **A cohomological criterion for sphericity.** We now introduce the object in the derived category $D(Z)$ of the base Z which is the relative case version of the cone of the natural morphism $k \to \mathbf{R}\operatorname{Hom}_X(E,E)$ of the Example 3.5 where the base Z is just the single point Spec k:

Definition 3.6. For any perfect object E of $D(Z \times X)$ denote by $\mathcal{L}_E$ the object of $D(Z)$ which is the cone of the following composition of morphisms:

$$\mathcal{O}_Z \to \pi_{Z*}\mathcal{O}_{Z\times X} \to \pi_{Z*}\,\mathbf{R}\,\mathcal{H}om_{Z\times X}(E,E) \to \pi_{Z*}\,\mathbf{R}\,\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}E, E). \tag{3.7}$$

Here the first morphism is induced by the adjunction unit $\operatorname{Id}_{D(Z)} \to \pi_{Z*}\pi_Z^*$, the second by the adjunction unit $\operatorname{Id}_{D(Z\times X)} \to \mathbf{R}\,\mathcal{H}om(E, E \otimes -)$ and the third by the adjunction co-unit $\pi_X^*\pi_{X*} \to \operatorname{Id}_{D(Z\times X)}$.

Let p be any closed point of the base Z. Apply the pullback functor ι_p^* to the composition (3.7) to obtain a morphism $k \to \iota_p^*\pi_{Z*}\,\mathbf{R}\,\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}E, E)$. We have a sequence of natural isomorphisms:

$$\iota_p^*\pi_{Z*}\,\mathbf{R}\,\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}E, E) \xrightarrow{\text{base change iso. around (3.1)}} \pi_{k*}\iota_{Xp}^*\,\mathbf{R}\,\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}E, E) \tag{3.8}$$

$$\pi_{k*}\iota_{Xp}^*\,\mathbf{R}\,\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}E, E) \xrightarrow{\text{[Ill71b], Prps. 7.1.2}} \pi_{k*}\,\mathbf{R}\,\mathcal{H}om_X(\iota_{Xp}^*\pi_X^*\pi_{X*}E, \iota_{Xp}^*E) \tag{3.9}$$

$$\pi_{k*}\,\mathbf{R}\,\mathcal{H}om_X(\iota_{Xp}^*\pi_X^*\pi_{X*}E, \iota_{Xp}^*E) \xrightarrow{\pi_X\circ\iota_{Xp}=\mathrm{Id}_X} \pi_{k*}\,\mathbf{R}\,\mathcal{H}om_X(\pi_{X*}E, E_p) \simeq \mathbf{R}\operatorname{Hom}_X(\pi_{X*}E, E_p). \tag{3.10}$$

One can check that these natural isomorphisms identify ι_p^* (3.7) with the morphism

$$k \to \mathbf{R}\operatorname{Hom}_{D(X)}(\pi_{X*}E, E_p) \tag{3.11}$$

which sends 1 to the natural composition $\pi_{X*}E \xrightarrow{\sim} \iota_{X_p}^*\pi_X^*\pi_{X*}E \xrightarrow{\pi_X^*\pi_{X*}\to \mathrm{Id}} E_p$ where the isomorphism is due to the scheme map identity $\pi_X \circ \iota_{X_p} = \mathrm{Id}_X$. Thus we see that the pointwise restrictions of the morphism (3.7) give us a natural morphism $k \to \mathbf{R}\operatorname{Hom}_{D(X)}(\pi_{X*}E, E_p)$ for each fibre E_p of E over a closed point $p \in Z$. It turns out that for an orthogonal E the criterion for the left co-twist F_E of E to be an autoequivalence of $D(Z)$ is for the cone of each of these morphisms to be $k[d]$ for some $d \in \mathbb{Z}$:

Proposition 3.7. *Let E be a perfect object of $D(Z \times X)$ orthogonal over Z. The following are equivalent:*

(1) *For every closed point $p \in Z$ such that the fibre E_p is non-zero we have*

$$\mathbf{R}\operatorname{Hom}_{D(X)}(\pi_{X*}E, E_p) = k \oplus k[d_p] \quad \text{for some } d_p \in \mathbb{Z}.$$

(2) *The object $\mathcal{L}_E$ is an invertible object of $D(Z)$. That is - on every connected component of Z it is isomorphic to a shift of a line bundle (see [AIL10], §1.5 for more detail).*
(3) *The left co-twist F_E of the transform $\Phi_E \colon D(Z) \to D(X)$ is an autoequivalence of $D(Z)$.*

When the conditions above are satisfied:

- *Locally around any closed point $p \in Z$ we have $\mathcal{L}_E \simeq \mathcal{O}_Z[d_p]$ where*

$$d_p = \begin{cases} \text{the same integer as in (1)} & \text{if } E_p \neq 0 \\ 1 & \text{if } E_p = 0 \end{cases}$$

- *F_E is isomorphic the functor $\mathcal{L}_E \otimes (\text{-})[-1]$*

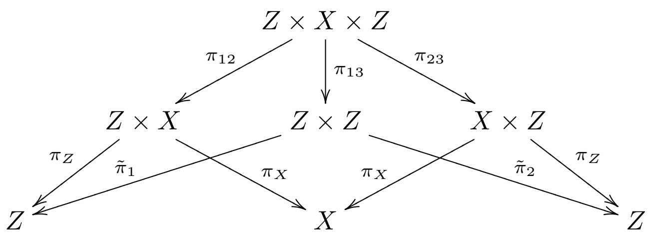
We see therefore that for the orthogonally spherical objects the geometric meaning of the object $\mathcal{L}_E$ defined rather abstractly above is that its restriction to each connected component of Z is a (shifted) line bundle which describes the autoequivalence of $D(Z)$ produced by taking the left co-twist of Φ_E.

To prove Proposition 3.7 we need two technical lemmas. Recall that by the Proposition 2.2 the adjunction unit $\mathrm{Id}_{D(Z)} \to \Phi_E^{radj}\Phi_E$ is isomorphic to morphism of the Fourier–Mukai transforms induced by the morphism

$$\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q = \pi_{13*}\left(\pi_{12}^*E \otimes \pi_{23}^*E^\vee \otimes \pi_{23}^*\pi_X^!\mathcal{O}_X\right) \tag{3.12}$$

of the objects of $D(Z \times Z)$. Here π_{ij} are the natural projection morphisms in the following commutative diagram:

(3.13)



Lemma 3.8. *Let E be a perfect object of $D(Z \times X)$ and let $p \in Z$ be a closed point. Then the following two morphisms in $D(Z)$ are isomorphic:*

$$\tilde{\pi}_{1*}\left(\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q\right) \tag{3.14}$$

and

$$\mathcal{O}_Z \xrightarrow{(3.7)} \pi_{Z*}\mathbf{R}\mathcal{H}om(\pi_X^*\pi_{X*}E, E) \tag{3.15}$$

Consequently, for every closed point $p \in Z$ the natural morphism $k \xrightarrow{(3.11)} \mathbf{R}\operatorname{Hom}_X(\pi_{X*}E, E_p)$ is isomorphic to $\pi_{k*}\left(\mathcal{O}_p \xrightarrow{adj.\ unit} \Phi_E^{radj}\Phi_E(\mathcal{O}_p)\right)$ and therefore $\pi_{k*}F_E(\mathcal{O}_p) \simeq \iota_p^*\mathcal{L}_E[-1]$.

Proof. For the first claim we need to show that $\mathcal{O}_Z \xrightarrow{(3.7)} \pi_{Z*}\,\mathbf{R}\mathcal{H}om(\pi_X^*\pi_{X*}E, E)$ is isomorphic to:

$$\tilde\pi_{1*}\Delta_*(\mathcal{O}_Z) \xrightarrow{\mathrm{Id}\to\pi_{Z*}\pi_Z^*} \tilde\pi_{1*}\Delta_*\pi_{Z*}\pi_Z^*(\mathcal{O}_Z) \tag{3.16}$$

$$\tilde\pi_{1*}\Delta_*\pi_{Z*}\pi_Z^*(\mathcal{O}_Z) \xrightarrow{\mathrm{Id}\to E\otimes E^\vee\otimes} \tilde\pi_{1*}\Delta_*\pi_{Z*}\left(E\otimes E^\vee\otimes\pi_Z^*(\mathcal{O}_Z)\right) \tag{3.17}$$

$$\tilde\pi_{1*}\Delta_*\pi_{Z*}\left(E\otimes E^\vee\otimes\pi_Z^*(\mathcal{O}_Z)\right) \simeq \tilde\pi_{1*}\pi_{13*}\Delta_*\Delta^!\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_X^!(\mathcal{O}_X)\right) \tag{3.18}$$

$$\tilde\pi_{1*}\pi_{13*}\Delta_*\Delta^!\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_X^!(\mathcal{O}_X)\right) \xrightarrow{\Delta_*\Delta^!\to\mathrm{Id}} \tilde\pi_{1*}\pi_{13*}\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_X^!(\mathcal{O}_X)\right) \tag{3.19}$$

By the scheme map identity $\tilde\pi_1\circ\Delta = \mathrm{Id}_Z$ we have $\tilde\pi_{1*}\Delta_* \simeq \mathrm{Id}_{D(Z)}$ and this identifies (3.16) and (3.17) with the first and the second morphisms in the composition (3.7). It remains to show that

$$\pi_{Z*}\,\mathbf{R}\mathcal{H}om_X(E,E) \xrightarrow{\pi_X^*\pi_{X*}\to\mathrm{Id}} \pi_{Z*}\,\mathbf{R}\mathcal{H}om(\pi_X^*\pi_{X*}E, E) \tag{3.20}$$

is isomorphic to

$$\tilde\pi_{1*}\pi_{13*}\Delta_*\Delta^!\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_X^!(\mathcal{O}_X)\right) \xrightarrow{\Delta_*\Delta^!\to\mathrm{Id}} \tilde\pi_{1*}\pi_{13*}\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_X^!(\mathcal{O}_X)\right). \tag{3.21}$$

By the scheme morphism identity $\tilde\pi_1\circ\pi_{13} = \pi_Z\circ\pi_{12}$ (see (3.13)) we have $\tilde\pi_{1*}\pi_{13*} \simeq \pi_{Z*}\pi_{12*}$. By the independent fibre square (see [Lip09], §3.10)

$$\begin{array}{ccc} Z\times X\times Z & \xrightarrow{\pi_{23}} & X\times Z \\ {\scriptstyle\pi_{12}}\downarrow & & \downarrow{\scriptstyle\pi_X} \\ Z\times X & \xrightarrow[\pi_X]{} & X \end{array} \tag{3.22}$$

we also have $\pi_{23}^*\pi_X^! \simeq \pi_{12}^!\pi_X^*$. We can therefore rewrite (3.21) as

$$\pi_{Z*}\pi_{12*}\Delta_*\Delta^!\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{12}^!\mathcal{O}_{Z\times X}\right) \xrightarrow{\Delta_*\Delta^!\to\mathrm{Id}} \pi_{Z*}\pi_{12*}\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{12}^!\mathcal{O}_{Z\times X}\right). \tag{3.23}$$

Now observe that π_{12}^*E is perfect, while $\pi_{23}^*E^\vee\otimes\pi_{12}^!\mathcal{O}_{Z\times X}$ is a tensor product of a perfect object and a π_{12}-perfect object and therefore itself π_{12}-perfect. Hence, even though Δ is not perfect, by Lemma 2.1 the natural map $\Delta^!(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{12}^!\mathcal{O}_{Z\times X}) \to \Delta^*(\pi_{12}^*E)\otimes\Delta^!(\pi_{23}^*E^\vee\otimes\pi_{12}^!\mathcal{O}_{Z\times X})$ is still an isomorphism. It therefore follows[2] from Lemma 2.3 of [AL10] that (3.23) is isomorphic to

$$\pi_{Z*}\left(E\otimes\pi_{12*}\Delta_*\Delta^!\left(\pi_{23}^*E^\vee\otimes\pi_{12}^!\mathcal{O}_{Z\times X}\right)\right) \xrightarrow{\Delta_*\Delta^!\to\mathrm{Id}} \pi_{Z*}\left(E\otimes\pi_{12*}\left(\pi_{23}^*E^\vee\otimes\pi_{12}^!\mathcal{O}_{Z\times X}\right)\right). \tag{3.24}$$

It remains to show that

$$\mathbf{R}\mathcal{H}om_X(E,\mathcal{O}_{Z\times X}) \xrightarrow{\pi_X^*\pi_{X*}\to\mathrm{Id}} \mathbf{R}\mathcal{H}om(\pi_X^*\pi_{X*}E,\mathcal{O}_{Z\times X}) \tag{3.25}$$

is isomorphic to

$$\pi_{12*}\Delta_*\Delta^!\,\mathbf{R}\mathcal{H}om\left(\pi_{23}^*E,\pi_{12}^!\mathcal{O}_{Z\times X}\right) \xrightarrow{\Delta_*\Delta^!\to\mathrm{Id}} \pi_{12*}\,\mathbf{R}\mathcal{H}om\left(\pi_{23}^*E,\pi_{12}^!\mathcal{O}_{Z\times X}\right). \tag{3.26}$$

Rewriting (3.25) and (3.26) in terms of the relative duality theory $D_{\bullet/Z\times X}$ (see Section 2.1) we obtain

$$D_{\bullet/Z\times X}\left(\pi_X^*\pi_{X*}E \xrightarrow{\pi_X^*\pi_{X*}\to\mathrm{Id}} E\right)^{\mathrm{opp}} \quad\text{and}\quad D_{\bullet/Z\times X}\left(\pi_{12*}\pi_{23}^*E \xrightarrow{\mathrm{Id}\to\Delta_*\Delta^*} \pi_{12*}\Delta_*\Delta^*\pi_{23}^*E\right)^{\mathrm{opp}}$$

respectively and these are isomorphic by [AL10], Lemma 2.4. This settles the first claim of this lemma.

For the second claim, we have a commutative fibre square

$$\begin{array}{ccc} Z & \xhookrightarrow{\iota_{p,Z}} & Z\times Z\,. \\ {\scriptstyle\pi_k}\downarrow & & \downarrow{\scriptstyle\tilde\pi_1} \\ \mathrm{Spec}\ k & \xhookrightarrow{\iota_p} & Z \end{array} \tag{3.27}$$

and for any $A\in D(Z\times Z)$ we have a standard isomorphism

$$\Phi_A(\mathcal{O}_p) \xrightarrow{\sim} \iota_{p,Z}^*A \tag{3.28}$$

[2] This lemma is stated in [AL10] for quasi-perfect scheme maps, but the proof works unchanged for any concentrated scheme maps as it never actually uses the fact that the map denoted in the lemma by χ_g is actually an isomorphism.

which is functorial in A. The adjunction unit morphism $\mathcal{O}_p \to \Phi_E^{radj}\Phi_E(\mathcal{O}_p)$ is isomorphic to the morphism $\Phi_{\Delta_*\mathcal{O}_Z}(\mathcal{O}_p) \to \Phi_Q(\mathcal{O}_p)$ induced by $\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q$ and is therefore isomorphic to $\iota_{p,Z}^*\left(\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q\right)$. By the base change around (3.27) we have $\pi_{k*}\iota_{p,Z}^* \simeq \iota_p^*\tilde{\pi}_{1*}$ and therefore $\pi_{k*}\left(\mathcal{O}_p \to \Phi_E^{radj}\Phi_E(\mathcal{O}_p)\right)$ is isomorphic to $\iota_p^*\tilde{\pi}_{1*}\left(\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q\right)$ and hence, by the first claim, to $\iota_p^*\left(\mathcal{O}_Z \xrightarrow{(3.7)} \pi_{Z*}\mathbf{R}\mathcal{H}om(\pi_X^*\pi_{X*}E, E)\right)$, which is precisely the natural morphism $k \xrightarrow{(3.11)} \mathbf{R}\operatorname{Hom}_X(\pi_{X*}E, E_p)$. This settles the second claim of the lemma and the last claim follows immediately by taking cones. ☐

Lemma 3.9. *Let E be a perfect object of $D(Z \times X)$. Then E is orthogonal over Z if and only if the support of the object Q is contained within the diagonal $\Delta \subset Z \times Z$. Consequently, if E is orthogonal over Z then for any closed point $p \in Z$ the object $F_E(\mathcal{O}_p)$, if non-zero, is supported at precisely the point p.*

Proof. Let q_1 and q_2 be closed points of Z, let $\bar{q} = (q_1, q_2)$ be the corresponding point of $Z \times Z$ and denote by $\iota_{\bar{q}}$ the closed embedding $\bar{q} \hookrightarrow Z \times Z$. From the standard spectral sequence $\mathbf{L}^i\,\iota_{\bar{q}}^*\mathcal{H}^jQ \Rightarrow \mathbf{L}^{i+j}\,\iota_{\bar{q}}^*Q$ it follows that $\bar{q} \in \operatorname{Supp}_{Z\times Z} Q$ if and only if $\iota_{\bar{q}}^*Q \neq 0$.

We have a commutative square:

(3.29)
$$\begin{array}{ccc} X & \xhookrightarrow{\iota_{X\bar{q}}} & Z \times X \times Z \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{13}} \\ \operatorname{Spec}\ k & \xhookrightarrow{\iota_{\bar{q}}} & Z \times Z \end{array}$$

Then:

(3.30)
$$\begin{aligned} \iota_{\bar{q}}^*Q = \iota_{\bar{q}}^*\pi_{13*}\left(\pi_{12}^*E \otimes \pi_{23}^*E^\vee \otimes \pi_{23}^*\pi_X^!\mathcal{O}_X\right) \simeq \\ \simeq \pi_{k*}\iota_{X\bar{q}}^*\left(\pi_{12}^*E \otimes \pi_{23}^*E^\vee \otimes \pi_{23}^*\pi_X^!\mathcal{O}_X\right) \simeq \\ \simeq \pi_{k*}\left(\mathbf{R}\mathcal{H}om(E_{q_2}, E_{q_1}) \otimes \iota_{Xq_2}^*\pi_X^!\mathcal{O}_X\right) \end{aligned}$$

We have a pair of independent fibre squares:

(3.31)
$$\begin{array}{ccccc} X & \xrightarrow{\iota_{X_{q_2}}} & Z \times X & \xrightarrow{\pi_X} & X \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi_k} \\ \operatorname{Spec}\ k & \xrightarrow[\iota_{q_2}]{} & Z & \xrightarrow[\pi_k]{} & \operatorname{Spec}\ k \end{array}$$

and so by the base change $\pi_X^!\mathcal{O}_X \simeq \pi_Z^*D_{Z/k}$, where $D_{Z/k}$ is the dualizing complex $\pi_k^!(k)$ on Z. Therefore $\iota_{X_{q_2}}^*\pi_X^!\mathcal{O}_X \simeq \iota_{X_{q_2}}^*\pi_Z^*D_{Z/k} \simeq \pi_k^*\iota_{q_2}^*D_{Z/k}$, and so finally:

(3.32)
$$\begin{aligned} \iota_{\bar{q}}^*Q = \pi_{k*}\left(\mathbf{R}\mathcal{H}om(E_{q_2}, E_{q_1}) \otimes \iota_{X_{q_2}}^*\pi_X^!\mathcal{O}_X\right) \simeq \\ \simeq \pi_{k*}\left(\mathbf{R}\mathcal{H}om(E_{q_2}, E_{q_1}) \otimes \pi_k^*\iota_{q_2}^*D_{Z/k}\right) \simeq \\ \simeq \pi_{k*}\mathbf{R}\mathcal{H}om(E_{q_2}, E_{q_1}) \otimes \iota_{q_2}^*D_{Z/k} \simeq \mathbf{R}\operatorname{Hom}_{D(X)}(E_{q_2}, E_{q_1}) \otimes \iota_{q_2}^*D_{Z/k} \end{aligned}$$

By [AIL10], Lemma 1.3.7 the support of any semi-dualizing (and, in particular, of any dualizing) complex on a noetherian scheme is the whole of the scheme. Therefore $\iota_{q_2}^*D_{Z/k}$ is non-zero for any $q_2 \in Z$. It then follows from (3.32) that $\iota_{\bar{q}}^*Q \neq 0$ if and only if $\operatorname{Hom}^i_{D(X)}(E_{q_2}, E_{q_1}) \neq 0$ for some $i \in \mathbb{Z}$. Therefore the support of Q in $Z \times Z$ consists precisely of all points (q_1, q_2) for which $\operatorname{Hom}^i_{D(X)}(E_{q_2}, E_{q_1}) \neq 0$ for some $i \in \mathbb{Z}$. Whence the assertion that E is orthogonal over Z if and only if the support of Q lies within the diagonal of $Z \times Z$.

For the second assertion, recall that $\Phi_E^{radj}\Phi_E(\mathcal{O}_p) \simeq \iota_{p,Z}^*Q$ and therefore $\iota_{p,Z}^*Q$ fits into an exact triangle

$$\mathcal{O}_p \to \iota_{p,Z}^*Q \to F_E(\mathcal{O}_p)[1]$$

in $D(Z)$. Since the support of $\mathcal{O}_p$ is p and the support of $\iota_{p,Z}^*Q$ lies within $\iota_{p,Z}^{-1}\operatorname{Supp}_{Z\times Z} Q \subseteq \iota_{p,Z}^{-1}\Delta = p$, it follows that the support of $F_E(\mathcal{O}_p)$ also lies within the point p. If the object $F_E(\mathcal{O}_p)$ is non-zero its support is closed and non-empty and must therefore be precisely p. ☐

Proof of Proposition 3.7. (1) $\Leftrightarrow$ (2): By [AIL10], Theorem 1.5.2 the object $\mathcal{L}_E$ is invertible if and only if for every closed point $p \in Z$ it is isomorphic in the neighborhood of p to $\mathcal{O}_Z[d_p]$ for some $d_p \in \mathbb{Z}$. This is equivalent to having $\iota_p^* \mathcal{L}_E = k[d_p]$. We have an exact triangle

$$k \xrightarrow{(3.11)} \mathbf{R}\operatorname{Hom}_{D(X)}(\pi_{X*}E, E_p) \to \iota_p^* \mathcal{L}_E \tag{3.33}$$

in $D(\mathbf{Vect})$. Hence $\iota_p^* \mathcal{L}_E = k[d_p]$ for some $d_p \in \mathbb{Z}$ is equivalent to either $\mathbf{R}\operatorname{Hom}_{D(X)}(\pi_{X*}E, E_p) = 0$ and $d_p = 1$ or to $\mathbf{R}\operatorname{Hom}_{D(X)}(\pi_{X*}E, E_p) = k \oplus k[d_p]$ and (3.11) not being the zero morphism. Therefore to establish (1) $\Leftrightarrow$ (2) and the first of the two assertions in the end it remains only to show that if (3.11) is the zero morphism then $E_p = 0$.

By Lemma 3.8 the morphism (3.11) is isomorphic to π_{k*} applied to the adjunction unit $\mathcal{O}_p \to \Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p)$. And this adjunction unit being zero is precisely equivalent to $E_p = \Phi_E(\mathcal{O}_p) = 0$. In Lemma 3.9 we've demonstrated that both $\mathcal{O}_p$ and $\Phi_E^{\mathrm{radj}} \Phi_E(\mathcal{O}_p)$ are supported at $p \in Z$. It suffices therefore to show the functor π_{k*} is faithful on the full subcategory $D_p(Z)$ of $D(Z)$ consisting of the complexes whose cohomology is supported at $p \in Z$. Indeed, let U be any open affine subset of Z containing p, let ι_U be the corresponding open immersion and observe that ι_{U*} restricts to an equivalence $\iota_{U*}\colon D_p(U) \xrightarrow{\sim} D_p(X)$ whose inverse is ι_U^*. On the other hand, $D_p(U) \xrightarrow{\pi_{k*}} D(\mathbf{Vect})$ decomposes as

$$D_p(U) \xrightarrow{\Gamma^*} D_p(\mathcal{O}_X(U)\text{-}\mathbf{Mod}) \xrightarrow{\text{forgetful}} D(\mathbf{Vect}) \tag{3.34}$$

Here Γ^* is the derived global sections functor and it is an equivalence since U is affine. The functor of forgetting the $\mathcal{O}_X(U)$-module structure is also faithful. The claim now follows.

(2) $\Leftrightarrow$ (3): The object $\mathcal{L}_E$ is invertible if and only if for every closed point $p \in Z$ we have $\iota_p^*(\mathcal{L}_E) = k[d_p]$ for some $d_p \in \mathbb{Z}$. By Lemma 3.8 we have $\iota_p^*(\mathcal{L}_E) = \pi_{k*} F_E(\mathcal{O}_p)[1]$. By Lemma 3.9 the object $F_E(\mathcal{O}_p)$ is contained within the full subcategory $D_p(Z)$ of $D(Z)$ consisting of the complexes whose cohomology is supported at p. Finally, the decomposition (3.34) makes it clear that the only object of $D_p(Z)$ whose image in $D(\mathbf{Vect})$ under π_{k*} is precisely k is the point sheaf $\mathcal{O}_p$. We conclude that $\mathcal{L}_E$ is invertible if and only if

$$\forall\, p \in Z, \quad F_E(\mathcal{O}_p) = \mathcal{O}_p[d] \qquad \text{for some } d \in \mathbb{Z}. \tag{3.35}$$

Suppose (3.35) holds. Let Q' be the Fourier–Mukai kernel of the left co-twist F_E. Since $\iota_{p,Z}^* Q' \simeq F_E(\mathcal{O}_p)$, by the semicontinuity theorem ([GD63], *Théorème 7.6.9*) the shift d in (3.35) is constant on every connected component of Z. Let U be such a connected component, then the spectral sequence argument of [Bri99], Lemma 4.3 to shows that the restriction of Q' to $U \times Z$ is the shift by d of a coherent sheaf flat over U, whose restriction to the fibre $\{p\} \times Z$ over every point $p \in U$ is precisely $\mathcal{O}_p$. Any such sheaf is necessarily a line bundle supported on the diagonal $U \hookrightarrow U \times Z$. We conclude that globally $Q' = \Delta_* \mathcal{L}'$ for some invertible object $\mathcal{L}'$ of $D(Z)$. This immediately implies that the corresponding Fourier–Mukai transform F_E is an equivalence.

Conversely, suppose F_E is an equivalence. Let p be any closed point of Z. As F_E is an equivalence we have $\operatorname{Hom}^{<0}_{D(Z)}(F_E(\mathcal{O}_p), F_E(\mathcal{O}_p)) = 0$ and $\operatorname{Hom}^{0}_{D(Z)}(F_E(\mathcal{O}_p), F_E(\mathcal{O}_p)) = k$. By Lemma 3.9 the support of $F_E(\mathcal{O}_p)$ is precisely p. Now the same spectral sequence argument as in Proposition 2.2 of [BO01] shows that $F_E(\mathcal{O}_p) = \mathcal{O}_p[d]$ for some $d \in \mathbb{Z}$.

For the second of the two assertions in the end: it follows from the definition of F_E that Q' is the object

$$\operatorname{Cone}\left(\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q\right)[-1]$$

of $D(Z \times Z)$. Therefore by Lemma 3.8 we have $\tilde{\pi}_{1*} Q' \simeq \mathcal{L}_E[-1]$. Above we've shown that $Q' = \Delta_* \mathcal{L}'$ for some invertible object $\mathcal{L}' \in D(Z)$ and since $\tilde{\pi}_{1*} \Delta_* \simeq \mathrm{Id}_{D(Z)}$ it follows that $\mathcal{L}' \simeq \mathcal{L}_E[-1]$. ☐

Suppose now that E satisfies the equivalent conditions of Proposition 3.7. Then the left co-twist F_E is an autoequivalence of $D(Z)$ whose Fourier–Mukai kernel is $\Delta_* \mathcal{L}_E[-1]$. There exists a unique morphism κ which completes the morphism (2.27)-(2.30) to an exact triangle

$$\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^! \mathcal{O}_X)\right) \xrightarrow{\kappa} \Delta_* \mathcal{L}_E \tag{3.36}$$

in $D(Z \times Z)$. Denote by κ_{FM} the corresponding natural transformation in the exact triangle

$$\mathrm{Id}_Z \xrightarrow{\text{adj. unit}} \Phi_E^{radj} \Phi_E \xrightarrow{\kappa_{FM}} F_E[1] \tag{3.37}$$

of Fourier–Mukai transforms $D(Z) \to D(Z)$.

The object E was defined to be spherical over Z if and only if Φ_E is a spherical functor. By the definition of a spherical functor Φ_E is spherical if and only if F_E is an autoequivalence and the natural composition

$$\Phi_E^{radj} \xrightarrow{\Phi_E^{radj}(\mathrm{Id}\to\Phi_E\Phi_E^{ladj})} \Phi_E^{radj}\Phi_E\Phi_E^{ladj} \xrightarrow{\kappa_{FM}} F_E[1]\Phi_E^{ladj} \tag{3.38}$$

is an isomorphism of functors. The Fourier–Mukai kernel of Φ_E^{radj} is $E^\vee \otimes \pi_X^!(\mathcal{O}_X)$ and the Fourier–Mukai kernel of $F_E[1]\Phi_E^{ladj}$ is

$$\pi_Z^*(\mathcal{L}_E) \otimes \big(E^\vee \otimes \pi_{X_1}^!(\mathcal{O}_Z)\big) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_E).$$

Proposition 2.3 affords us a canonical choice (2.31)-(2.34) of a morphism of Fourier–Mukai kernels which underlies the adjunction unit $\mathrm{Id} \to \Phi_E\Phi_E^{ladj}$, and we can therefore define:

Definition 3.10. Define α to be the morphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

of Fourier Mukai kernels which underlies the natural moprhism (3.38) if we choose (2.31)-(2.34) and κ as underlying morphisms of $\mathrm{Id} \to \Phi_E\Phi_E^{ladj}$ and κ_{FM}, respectively.

By Lemma 2.4 the composition (3.38) is an isomorphism if and only if the underlying morphism α is. We therefore obtain immediately the main result of this section:

Theorem 3.1. *Let X and Z be two separable schemes of finite type over k. Let E be a perfect object of $D(Z \times X)$ orthogonal over Z. Then E is spherical over Z if and only if*

(1) *For every closed point $p \in Z$ such that the fibre E_p is not zero we have*

$$\mathbf{R}\operatorname{Hom}_{D(X)}(\pi_{X*}E, E_p) = k \oplus k[d_p] \quad \textit{for some } d_p \in \mathbb{Z}.$$

(2) *The canonical morphism*

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_{X1}^!(\mathcal{L}_E) \qquad \textit{(see Definition 3.10)}$$

is an isomorphism.

Whenever E is orthogonally spherical over Z, the object $\mathcal{L}_E$ is invertible in $D(Z)$ and so locally around any closed point $p \in Z$ we have $\mathcal{L}_E \simeq \mathcal{O}_Z[d_p]$ for some $d_p \in \mathbb{Z}$. Over the locus where Z and X are not too degenerate this integer is precisely the difference in dimensions between X and Z:

Proposition 3.11. *Let E be an object of $D(Z \times X)$ orthogonally spherical over Z. Let $(p,q) \in Z \times X$ be a Gorenstein point in the support of E if such exists. Then*

$$d_p = -(\dim_q X - \dim_p Z)$$

Proof. If E is spherical over Z the canonical map

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

is an isomorphism. Let us restrict it to Spec $\mathcal{O}_{Z\times X,(p,q)}$. Since $\mathcal{O}_{Z\times X,(p,q)} = \mathcal{O}_{Z,p} \otimes \mathcal{O}_{X,q}$ is Gorenstein, the structure map Spec $\mathcal{O}_{Z\times X,(p,q)} \to$ Spec k is Gorenstein. Therefore the projections $\pi_{Z,p}$ and $\pi_{X,q}$ are Gorenstein, since we can filter Spec $\mathcal{O}_{Z\times X,(p,q)} \to$ Spec k through them

$$\begin{array}{ccc} \operatorname{Spec}\,\mathcal{O}_{Z\times X,(p,q)} & \xrightarrow{\pi_{X,q}} & \operatorname{Spec}\,\mathcal{O}_{X,q} \\ \downarrow{\scriptstyle \pi_{Z,q}} & & \downarrow \\ \operatorname{Spec}\,\mathcal{O}_{Z,p} & \longrightarrow & k \end{array} \tag{3.39}$$

and for perfect maps (and therefore for flat maps such as these) the composition of two maps is Gorenstein if and only if both composants are ([AF90], Prop. 2.3). Therefore

$$\pi_{Z,p}^!(\mathcal{O}_{Z,p}) = \mathcal{O}_{Z\times X,(p,q)}[\dim \mathcal{O}_{Z\times X,(p,q)} - \dim \mathcal{O}_{Z,p}] = \mathcal{O}_{Z,q}[\dim \mathcal{O}_{X,q}]$$
$$\pi_{X,q}^!(\mathcal{O}_{X,q}) = \mathcal{O}_{Z\times X,(p,q)}[\dim \mathcal{O}_{Z\times X,(p,q)} - \dim \mathcal{O}_{X,q}] = \mathcal{O}_{Z,q}[\dim \mathcal{O}_{Z,p}]$$

and so α restricts to Spec $\mathcal{O}_{Z\times X,(p,q)}$ as

$$E^\vee|_{\mathcal{O}_{Z\times X,(p,q)}}[\dim \mathcal{O}_{Z,p}] \xrightarrow{\sim} E^\vee|_{\mathcal{O}_{Z\times X,(p,q)}}[\dim \mathcal{O}_{X,q} + d_p]$$

Since (p,q) lies in the support of E, the restriction $E^\vee|_{\mathcal{O}_{Z\times X,(p,q)}}$ is a non-zero bounded complex. So

$$\dim \mathcal{O}_{Z,p} = \dim \mathcal{O}_{X,q} + d_p$$

whence the claim. ∎

3.4. **Concerning the canonical morphism α.** A reader who wasn't at all disturbed by the words "the canonical morphism α is an isomorphism" in Theorem 3.1 probably doesn't need to read this section.

However, to apply Theorem 3.1 to show that an object is spherical one needs to compute the canonical morphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

described in Definition 3.10 and show it to be an isomorphism. In all but few very simple examples computing this morphism directly, by computing the morphisms of the kernels underlying both terms of (3.38) and then composing them, is not a very pleasant endeavour.

Fortunately Lemma 2.7 gives us a different characterisation of α by telling us that α is the unique morphism from $E^\vee \otimes \pi_X^!(\mathcal{O}_X)$ to $E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$ such that the induced morphism α_{FM} of the corresponding Fourier–Mukai transforms makes the diagram

$$\begin{array}{ccc} \Phi_E^{radj}\Phi_E & \xrightarrow{\kappa_{FM}} & F_E[1] \\ \downarrow{\scriptstyle \alpha_{FM}} & \nearrow{\scriptstyle F_E[1](\Phi_E^{ladj}\Phi_E \to \mathrm{Id})} & \\ F_E[1]\Phi_E^{ladj}\Phi_E & & \end{array} \tag{3.40}$$

commute. Hence showing that α is an isomorphism is equivalent to exhibiting some isomorphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\sim} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$$

and then showing that it makes (3.40) commute.

We first use this to produce an alternative, more direct description of the morphism α:

Proposition 3.12. *Let E be a perfect object of $D(Z \times X)$ orthogonal over Z and suppose that $\mathcal{L}_E$ is an invertible object of $D(Z)$. Then the canonical morphism α of Definition 3.10 is precisely the image of the morphism κ defined by the exact triangle*

$$\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} \pi_{13*}\left(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!\mathcal{O}_X)\right) \xrightarrow{\kappa} \Delta_*\mathcal{L}_E.$$

under the following chain of isomorphisms between Hom *spaces:*

$$\begin{aligned} &\mathrm{Hom}_{D(Z\times Z)}\left(\pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!(\mathcal{O}_X))),\ \Delta_*\mathcal{L}_E\right) \xrightarrow{\textit{adjunction } (\Delta^*,\Delta_*)} \\ &\mathrm{Hom}_{D(Z)}\left(\Delta^*\pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!(\mathcal{O}_X))),\ \mathcal{L}_E\right) \xrightarrow{\Delta^*\pi_{13*}(\pi_{12}^*\text{-}\otimes\pi_{13}^*\text{-})\simeq\pi_{Z*}(\text{-}\otimes\text{-})} \\ &\mathrm{Hom}_{D(Z)}\left(\pi_{Z*}(E \otimes E^\vee \otimes \pi_X^!(\mathcal{O}_X)),\ \mathcal{L}_E\right) \xrightarrow{\textit{adjunction } (\pi_{Z*},\pi_Z^!)} \\ &\mathrm{Hom}_{D(Z\times X)}\left(E \otimes E^\vee \otimes \pi_X^!(\mathcal{O}_X),\ \pi_Z^!(\mathcal{L}_E)\right) \xrightarrow{\textit{adjunction } (E\otimes\text{-},E^\vee\otimes\text{-})} \\ &\mathrm{Hom}_{D(Z\times X)}\left(E^\vee \otimes \pi_X^!(\mathcal{O}_X),\ E^\vee \otimes \pi_Z^!(\mathcal{L}_E)\right) \end{aligned} \tag{3.41}$$

Proof. Let α' be the morphism which corresponds to κ under the chain of adjunctions and isomorphisms in (3.41). It follows from [AL10], Theorem 2.1 that the morphism underlying $F_E[1]\Phi_E^{ladj}\Phi_E \to F_E[1]$ is the composition

$$\pi_{13*}\left(\pi_{12}^*E \otimes \pi_{23}^*(E \otimes \pi_Z^!(\mathcal{L}_E))\right) \xrightarrow{\mathrm{Id}\to\Delta_*\Delta^*} \Delta_*\Delta^*\pi_{13*}\left(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^!(\mathcal{L}_E))\right) \tag{3.42}$$

$$\Delta_*\Delta^*\pi_{13*}\left(\pi_{12}^*E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^!(\mathcal{L}_E)\right) \xrightarrow{\Delta^*\pi_{13*}(\pi_{12}^*\text{-}\otimes\pi_{13}^*\text{-})\ \simeq\ \pi_{Z*}(\text{-}\otimes\text{-})} \Delta_*\pi_{Z*}\left(E \otimes (E^\vee \otimes \pi_Z^!(\mathcal{L}_E))\right) \tag{3.43}$$

$$\Delta_*\pi_{Z*}\left(E \otimes (E^\vee \otimes \pi_Z^!(\mathcal{L}_E))\right) \xrightarrow{E\otimes E^\vee\otimes\to\mathrm{Id}} \Delta_*\pi_{Z*}\pi_Z^!(\mathcal{L}_E) \tag{3.44}$$

$$\Delta_*\pi_{Z*}\pi_Z^!(\mathcal{L}_E) \xrightarrow{\pi_{1*}\pi_1^!\to\mathrm{Id}} \Delta_*\mathcal{L}_E \tag{3.45}$$

By Lemma 2.7 to establish that $\alpha = \alpha'$ it suffices to show that α' makes the diagram of the Fourier–Mukai kernel morphisms commute:

$$\begin{array}{ccc} \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^! \mathcal{O}_X)\right) & \xrightarrow{\kappa} & \Delta_* \mathcal{L}_E \\ \downarrow{\scriptstyle \pi_{13*}(\pi_{12}^*E \otimes \pi_{23}^*(\alpha'))} & \nearrow{\scriptstyle (3.42)-(3.45)} & \\ \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^! \mathcal{L}_E)\right) & & \end{array} \tag{3.46}$$

The morphism α' was defined as the unique morphism corresponding to κ under the chain of adjunctions and isomorphisms listed in (3.41). The reader may now readily check that applying these adjunctions and isomorphisms to the diagram (3.46) turns it into the diagram

$$\begin{array}{ccc} E^\vee \otimes \pi_X^!(\mathcal{O}_X) & \xrightarrow{\alpha} & E^\vee \otimes \pi_Z^!(\mathcal{L}_E) \\ \downarrow{\scriptstyle \alpha} & \nearrow{\scriptstyle \mathrm{Id}} & \\ E^\vee \otimes \pi_Z^!(\mathcal{L}_E) & & \end{array} \tag{3.47}$$

Since the diagram (3.47) clearly commutes, so must the diagram (3.46) which corresponds to it under several adjunctions and isomorphisms. □

Suppose now that we could show that the Hom space

$$\mathrm{Hom}_{D(Z\times X)}\Big(E^\vee \otimes \pi_X^!(\mathcal{O}_X), E^\vee \otimes \pi_Z^!(\mathcal{L}_E)\Big) \tag{3.48}$$

is a one-dimensional k-vector space and that α is a non-zero element in it. Then if any non-zero element of (3.48) were to be an isomorphism, so would be all its scalar multiples. Since the space (3.48) is one-dimensional, this would mean that all of its non-zero elements would have to be isomorphisms, including α. We could then replace the words "the canonical morphism α is an isomorphism" in Theorem 3.1 by words "there exists *some* isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$".

By Lemma 3.12 it is enough to show the Hom space

$$\mathrm{Hom}_{D(Z\times Z)}\Big(\ \pi_{13*}(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!(\mathcal{O}_X)))\ ,\ \Delta_*\mathcal{L}_E\Big) \tag{3.49}$$

to be one-dimensional and κ to be non-zero. In view of the exact triangle

$$\Delta_* \mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^! \mathcal{O}_X)\right) \xrightarrow{\kappa} \Delta_* \mathcal{L}_E$$

it isn't entirely an unreasonable thing to expect when $\mathcal{L}_E$ is a non-trivial shift of a line bundle. We make these ideas precise in:

Proposition 3.13. *Let E be a perfect object of $D(Z\times X)$ orthogonal over Z and suppose that $\mathcal{L}_E$ is an invertible object of $D(Z)$. Assume (for simplicity) that Z is connected, then $\mathcal{L}_E = L[d]$ for some line bundle $L \in \mathrm{Pic}\, Z$ and $d \in \mathbb{Z}$. Assume further that $d < 0$ or, more generally, that $d \neq 0, 1$ and*

$$\mathrm{Ext}^d_{Z\times Z}(\Delta_*\mathcal{O}_Z, \Delta_* L) = 0.$$

Then existence of any isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\sim} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$ implies that the canonical map α is an isomorphism.

Proof. Denote by Q the object $\pi_{13*}(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_X^!(\mathcal{O}_X)))$. We have an exact triangle in $D(Z\times Z)$:

$$\Delta_*\mathcal{O}_Z \xrightarrow{(2.27)-(2.30)} Q \xrightarrow{\kappa} \Delta_* L[d]$$

Denote by $\mathcal{H}^i$ the functor of taking i-th cohomology of a complex. Since $d \neq 0, 1$, the associated long exact sequence of cohomologies shows that the complex Q has exactly two non-zero cohomologies: $\Delta_*\mathcal{O}_Z$ in degree 0 and $\Delta_* L$ in degree $-d$. More precisely, it shows that the morphisms

$$\Delta_*\mathcal{O}_Z \xrightarrow{\mathcal{H}^0((2.27)-(2.30))} \mathcal{H}^0(Q) \quad \text{and} \quad \mathcal{H}^{-d}(Q) \xrightarrow{\mathcal{H}^{-d}(\kappa)} \Delta_* L$$

are isomorphisms. Use them from now to identify the spaces involved. Then, tautologically, the image of κ under the map

$$\mathrm{Hom}_{D(Z\times X)}(Q, \Delta_* L[d]) \xrightarrow{\mathcal{H}^{-d}(\text{-})} \mathrm{Hom}_{Z\times Z}(\Delta_* L, \Delta_* L) \xrightarrow{\sim} \mathrm{Hom}_Z(L, L) \xrightarrow{\sim} \Gamma(\mathcal{O}_Z) \tag{3.50}$$

is the element 1 of $\Gamma(\mathcal{O}_Z)$.

Claim: The map (3.50) *is injective.*
Clearly it suffices to show that the map $\mathcal{H}^{-d}$ in (3.50) is an isomorphism. Choose an injective resolution $I^\bullet$ of Δ_*. There is a standard spectral sequence associated to the filtration by columns of the total complex of the bicomplex $\mathrm{Hom}(Q^\bullet, I^\bullet)$:

$$E_2^{p,-q} = \mathrm{Ext}^p_{Z\times Z}(H^q(Q), \Delta_* L) \quad \Rightarrow \quad E_\infty^{p-q} = \mathrm{Hom}^{p-q}_{D(Z\times Z)}(Q, \Delta_* L).$$

We are interested in the space $\mathrm{Hom}_{D(Z\times X)}(Q, \Delta_* L[d])$ which is the limit E^d_∞ of the above spectral sequence. Since the complex Q has cohomology only in two degrees, there are only two potentially non-zero terms $E_2^{p,-q}$ with $p-q=d$. These are $E_2^{0,-d} = \mathrm{Hom}_{Z\times Z}(\Delta_* L, \Delta_* L)$ and $E_2^{d,0} = \mathrm{Ext}^d_{Z\times Z}(\Delta_*\mathcal{O}_Z, \Delta_* L)$. The space $E_2^{d,0}$ is 0 by assumption, so we have $E^d_\infty = E^{0,-d}_\infty$. But observe that there are no non-zero elements $E_2^{p,q}$ with $p<0$, and therefore at every page of the spectral sequence the incoming differential $E_r^{-r,-d+r-1} \to E_r^{0,-d}$ will be zero. Thefore we have natural inclusions $E^{0,-d}_{r+1} \hookrightarrow E^{0,-d}_r$ and the spectral sequence technology ensures that the natural map

$$\mathrm{Hom}_{D(Z\times X)}(Q, \Delta_* L[d]) \xrightarrow{\mathcal{H}^{-d}(\text{-})} E_2^{0,-d}$$

lifts along each of these inclusions. Let β denote the map we obtain at the limit:

$$(3.51)\qquad \begin{array}{ccc} \mathrm{Hom}_{D(Z\times X)}(Q, \Delta_* L[d]) & \xrightarrow{\mathcal{H}^{-d}(\text{-})} & E_2^{0,-d} \\ & \searrow^{\beta} & \uparrow \\ & & E_\infty^{0,-d} \end{array}$$

Since $E^{0,-d}_\infty$ is the only surviving component of E^d_∞ the map β is an isomorphism, and the claim follows.

If Z is proper the proof ends here, since then $\Gamma(\mathcal{O}_Z)$ is one-dimensional, and hence so is the space $\mathrm{Hom}_{D(Z\times X)}(Q, \Delta_* L[d])$. The argument outlined immediately before this Proposition would then complete the proof.

In general case, the argument is a bit more involved. Suppose, indeed, there exists an isomorphism α'

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\alpha'} E^\vee \otimes \pi_Z^!(\mathcal{L}_E).$$

Let Q' denote the object

$$\pi_{13*}(\pi_{12}^* E \otimes \pi_{23}^*(E^\vee \otimes \pi_Z^!(\mathcal{L}_E))),$$

then α' induces an isomorphism $Q \xrightarrow{\sim} Q'$ and therefore a Hom space isomorphism

$$(3.52)\qquad \mathrm{Hom}_{D(Z\times Z)}(Q', \Delta_* L[d]) \xrightarrow{\sim} \mathrm{Hom}_{D(Z\times Z)}(Q, \Delta_* L[d]) \xrightarrow{(3.50)} \Gamma(\mathcal{O}_Z)$$

Let $f \in \Gamma(\mathcal{O}_Z)$ be the image under (3.50) of the natural morphism (3.42)-(3.45) which underlies the morphism $\mathcal{L}_E \otimes \Phi_E^{ladj}\Phi_E \to \mathcal{L}_E \otimes \mathrm{Id}_Z$. Then the commutation condition of Lemma 2.7 is equivalent to saying that $f=1$, as $1 \in \Gamma(\mathcal{O}_Z)$ is the image of κ under (3.50).

It therefore suffices to show that f is invertible in $\Gamma(\mathcal{O}_Z)$, as then by Lemma 2.7 we would have $\alpha = \frac{1}{f}\alpha'$ and therefore an isomorphism. But if f isn't invertible, then there exists some closed point $p \in Z$ such that $f(p)=0$. But then the restriction of the morphism $Q' \xrightarrow{(3.42)-(3.45)} \Delta_* L[n]$ to the fibre of $Z \times Z \xrightarrow{\pi_1} Z$ over p would vanish. This means precisely that the induced morphism of Fourier–Mukai transforms vanishes on $\mathcal{O}_p$, and therefore

$$\Phi_E^{ladj}\Phi_E(\mathcal{O}_p) \xrightarrow{\text{adj. co-unit}} \mathcal{O}_p$$

is a zero map. Therefore $E_p = \Phi_E(\mathcal{O}_p) = 0$. But then by Proposition 3.7 we must have $d=1$, which contradicts our assumptions. ☐

Corollary 3.14. *Let E be a perfect object of $D(X)$ orthogonal over Z and suppose that $\mathcal{L}_E$ is an invertible object of $D(Z)$. Suppose that for any closed point $p \in Z$ we have $d_p < 0$ where d_p is the unique integer such that $\mathcal{L}_E \simeq \mathcal{O}_Z[d_p]$ locally around p.*

Then in Theorem 3.1 we can replace the condition that the canonical map α is an isomorphism by the condition that there exists an arbitrary isomorphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_E).$$

4. Spherical fibrations

The results of Section 3 are rather general and category-theoretic, owing to a rather general and category-theoretic nature of the objects it considers - arbitrary complexes in the derived category of the fibre product $Z \times X$. We now choose to restrict ourselves to a setup more geometric in its nature, and study what these results imply for the geometry involved.

Firstly, we assume Z and X to be abstract varieties. Previously we have assumed them to only be separated schemes of finite type over k, now we assume them to also be reduced and irreducible. Strictly speaking, neither assumption is essential for what we prove below. However without them the arguments would become more technically involved and the results - less concise.

Secondly, and more importantly, we restrict the range of objects we consider from arbitrary complexes in $D(Z \times X)$ to subschemes of X flatly fibred over Z.

4.1. **Flat and perfect fibrations with proper fibres.**

Definition 4.1. A *flat fibration D in X over Z* is a scheme D equipped with a closed immersion $\xi\colon D \hookrightarrow X$ and a flat surjective map $\pi\colon D \to Z$. For any closed point $p \in Z$ we denote by D_p the set-theoretic fibre of D over p:

(4.1)
$$\begin{array}{ccccc} D_p & \xhookrightarrow{\iota_{D_p}} & D & \xhookrightarrow{\xi} & X \\ \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle \pi} & & \\ \mathrm{Spec}\ k & \xhookrightarrow[\iota_p]{} & Z & & \end{array}$$

Denote by ι_D the map $D \hookrightarrow Z \times X$ given by the product of π and ι. We have $\xi = \pi_X \circ \iota_D$ and $\pi = \pi_Z \circ \iota_D$. Denote by ξ_p the composition $\xi \circ \iota_{D_p}$, it is the inclusion of the fibre D_p into X. Let E denote the object $\iota_{D*}\mathcal{O}_D$ of $D(Z \times X)$. We think of this object as representing D in the derived category $D(Z \times X)$.

The flatness of D over Z ensures that the category-theoretic notion of a fibre considered in the Section 3 coincides for D with the usual set-theoretic one:

Lemma 4.2. *Let D be a flat fibration in X over Z, let $\mathcal{E}$ be an object of $D(D)$ and let $E = \iota_{D*}\mathcal{E}$ be the corresponding object in $D(Z \times X)$. For any closed point $p \in Z$ denote by $\mathcal{E}_p$ the fibre $\iota^*_{D_p}\mathcal{E}$, then we have*

$$E_p \simeq \xi_{p*}\mathcal{E}_p$$

as objects of $D(X)$. In particular, when $\mathcal{E} = \mathcal{O}_D$, we have

$$E_p \simeq \xi_{p*}\mathcal{O}_{D_p}.$$

Proof. The fibre square in the diagram (4.1) decomposes into two fibre squares:

(4.2)
$$\begin{array}{ccccc} D_p & \xhookrightarrow{\iota_{D_p}} & D & & \\ \Big\downarrow{\scriptstyle \xi_p} & & \Big\downarrow{\scriptstyle \iota_D} & \searrow{\scriptstyle \xi} & \\ X & \xhookrightarrow{\iota_{X_p}} & Z \times X & \xrightarrow[\pi_X]{} & X \\ \Big\downarrow{\scriptstyle \pi_k} & & \Big\downarrow{\scriptstyle \pi_Z} & & \\ \mathrm{Spec}\ k & \xhookrightarrow[\iota_p]{} & Z & & \end{array}$$

The fibre E_p of E at p was defined to be the object $\iota^*_{X_p}E$ of $D(X)$. We have therefore $\mathcal{E}_p = \iota^*_{X_p}\iota_{D*}\mathcal{O}_D$. Consider the base change map

(4.3)
$$\iota^*_{X_p}\iota_{D*} \to \xi_{p*}\iota^*_{D_p}$$

for the top fibre square in the diagram (4.2). Applying it to $\mathcal{O}_D$ yields a morphism

$$E_p \to \xi_{p*}\mathcal{E}_p.$$

It suffices therefore to prove that the top fibre square in (4.2) is independent (see [Lip09], §3.10), since then the base change map (4.3) would be an isomorphism.

Observe that the bottom fibre square in (4.2) is independent since π_Z is a projection morphism and therefore flat. Also, the composition (4.1) of the two fibre squares in (4.2) is independent since π was assumed to be flat. And it can be easily seen that if a composition of two fibre squares is independent and the second square in the composition is independent, then the first square has to be independent as well. $\square$

In particular, this makes it clear that $\iota_{D*}\mathcal{O}_D$ is an object of $D(Z \times X)$ which is orthogonal over Z. Because for any two distinct points p and q of Z the fibres D_p and D_q are disjoint in X and therefore all Hom's between $\xi_{p*}\mathcal{O}_{D_p}$ and $\xi_{q*}\mathcal{O}_{D_q}$ vanish.

In Section 3 we had to make two technical assumptions on the object E of $D(Z \times X)$ that we were working with. These were necessary for all the adjoints of the Fourier–Mukai transform Φ_E to exist and to behave in a sensible way. The first assumption was that the support of E is proper over Z. The support of $\iota_{D*}\mathcal{O}_D$ in $Z \times X$ is the image of D under ι_D, so this assumption is equivalent to saying that the fibration morphism $\pi\colon D \to Z$ is proper. And π being proper is equivalent to all the fibres of D over closed points of Z being schemes proper over k ([GD61], *Corolaire 5.4.5)*.

The second assumption was that E is a perfect object of $D(Z \times X)$. This corresponds to $\iota_{D*}\mathcal{O}_D$ being perfect and we say that a fibration D is *perfect* if this holds. This condition can also be checked fibre per fibre, owing to the flatness of π:

Lemma 4.3. *Let D be a flat fibration in X over Z. Then it is perfect if and only if for every closed $p \in Z$ the object $\xi_{p*}\mathcal{O}_{D_p}$ is perfect in $D(X)$.*

Proof. We first claim that $\iota_{D*}\mathcal{O}_D$ is perfect relative to the morphism $\pi_Z\colon Z \times X \to Z$. There is a commutative diagram

(4.4)
$$\begin{array}{ccc} D & \xhookrightarrow{\iota_D} & Z \times X \\ {\scriptstyle\pi}\downarrow & \swarrow {\scriptstyle\pi_Z} & \\ Z & & \end{array}$$

and since ι_D is a closed immersion, and therefore proper, it takes π-perfect object to π_Z-perfect objects ([Ill71a], Proposition 4.8). By the definition of π being flat the structure sheaf $\mathcal{O}_D$ is π-flat and therefore most certainly π-perfect. We conclude that $\iota_{D*}\mathcal{O}_D$ is always π_Z-perfect.

By the fibre-wise criterion for perfection ([Ill71a], *Corollaire 4.6.1*) an object of $D(Z \times X)$ is globally perfect if and only if it is π_Z-perfect and its fibre over every closed point of Z is globally perfect in $D(X)$. And by Lemma 4.2 the fibre of $\iota_{D*}\mathcal{O}_D$ over any closed $p \in Z$ is precisely $\xi_{p*}\mathcal{O}_{D_p}$. The claim now follows. □

We have therefore several common scenarios in which $\iota_{D*}\mathcal{O}_D$ is perfect in $D(Z \times X)$:

Corollary 4.4. *Let D be a flat fibration in X over Z. Then any one of the following conditions is sufficient for D to be perfect:*

(1) *X is smooth.*
(2) *Z is smooth and $\xi\colon D \hookrightarrow Z$ is a regular immersion.*

Proof. By Lemma 4.3 it suffices to prove that for every closed $p \in Z$ the object $\xi_{p*}\mathcal{O}_{D_p}$ is perfect in $D(X)$.

If X is smooth, then every object of $D(X)$ is perfect and the claim follows trivially.

Assume now that Z is smooth and ξ is a regular immersion. To prove that $\xi_{p*}\mathcal{O}_{D_p}$ is perfect in $D(X)$ it suffices to prove that ξ_p is a regular immersion. This is because a regular immersion is both proper and perfect, and therefore takes perfect objects to perfect objects ([Ill71a], *Corollaire 4.8.1*). Recall now the commutative diagram (4.1). Smoothness of Z is equivalent to ι_p being a regular immersion for every closed point p of Z. Since π is faithfully flat, ι_p is a regular immersion if and only if ι_{D_p} is a regular immersion. Since ξ_p is the composition

$$D_p \xrightarrow{\iota_{D_p}} D \xrightarrow{\xi} X$$

and since a composition of two regular immersions is again a regular immersion, we conclude that ξ_p is indeed a regular immersion for every closed $p \in \mathbb{Z}$. □

Thus we arrive at the class of objects we want to work with: flat and perfect fibrations in X over Z with proper fibres. For such fibrations the results of Section 3 can be re-stated in a more natural way and improved upon. Our goal is to give a satisfying description of what does it mean for such fibrations to possess the following property:

Definition 4.5. Let D be a flat and perfect fibration in X over Z with proper fibres. We say that D is a *spherical fibration* if the object $E = \iota_{D*}\mathcal{O}_D$ is spherical over Z in $D(Z \times X)$.

So let D be a flat and perfect fibration in X with proper fibres and let $E = \iota_{D*}\mathcal{O}_D$ be the corresponding object in $D(Z \times X)$. Recall that the left co-twist F_E of the Fourier–Mukai transform Φ_E was defined as the cone of the adjunction unit $\mathrm{Id}_{D(Z)} \to \Phi_E\Phi_E^{radj}$ and that the first of the two conditions for E to be spherical was for Φ_E to be an autoequivalence of $D(Z)$.

Denote by $\mathcal{L}_D$ the object $\mathcal{L}_E$ of $D(Z)$, which was defined in Definition 3.6 to be the cone of a certain composition (3.7) of natural morphisms in $D(Z)$. This composition was later shown in Lemma 3.8 to be precisely the pushdown from $Z \times Z$ to Z via $\tilde{\pi}_{1*}$ of the composition (2.27)-(2.30) of morphisms in $D(Z \times Z)$. Recall that (2.27)-(2.30) is the composition of morphisms which defines on the level of Fourier–Mukai kernels the adjunction unit $\mathrm{Id}_{D(Z)} \to \Phi_E \Phi_E^{radj}$. Now, in [AL10], Section 3, we have demonstrated that whenever the object E of $D(Z \times X)$ is a pushforward of an object from some closed subscheme $Z \times X$, as is the case here, there exists a better, more economical decomposition of this morphism of Fourier–Mukai kernels than (2.27)-(2.30). It makes sense to assume that a pushdown of this more economical decomposition to Z via $\tilde{\pi}_1$ would produce a better description of the defining morphism of $\mathcal{L}_D$ than the composition (3.7). For the sake of simplicity we choose to state this better description directly and prove directly that it is isomorphic to the composition (3.7). An interested reader could check that dualising the composition in Corollary 3.5 of [AL10] in the way described in Section 2.2 of this paper and applying $\tilde{\pi}_{1*}$ would yield precisely the following:

Proposition 4.6. *Let D be a flat and perfect fibration in X with proper fibres and let $E = \iota_{D*}\mathcal{O}_D$ be the corresponding object in $D(Z \times X)$.*

Then the natural morphism in (3.7)

$$\mathcal{O}_Z \xrightarrow{\mathrm{Id}\to\pi_{Z*}\pi_Z^*} \pi_{Z*}\mathcal{O}_{Z\times X} \xrightarrow{\mathrm{Id}\to\mathbf{R}\mathcal{H}om(E,E\otimes-)} \pi_{Z*}\,\mathbf{R}\mathcal{H}om_{Z\times X}(E,E) \xrightarrow{(\pi_X^*\pi_{X*}\to\mathrm{Id})^{opp}} \pi_{Z*}\,\mathbf{R}\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}E,E)$$

is isomorphic to the natural morphism

$$\mathcal{O}_Z \xrightarrow{\mathrm{Id}\to\pi_*\pi^*} \pi_*\mathcal{O}_D \xrightarrow{(\xi^*\xi_*\to\mathrm{Id})^{opp}} \pi_*(\xi^*\xi_*\mathcal{O}_D)^\vee. \tag{4.5}$$

In particular, the object $\mathcal{L}_D$ is isomorphic to the cone of (4.5).

Proof. We have $\pi = \pi_Z \circ \iota_D$ and $\xi = \pi_X \circ \iota_D$. Decomposing $\mathrm{Id} \to \pi_*\pi^*$ as $\mathrm{Id} \to \pi_{Z*}\pi_Z^* \to \pi_{Z*}\iota_{D*}\iota_D^*\pi_Z^*$ we see that (4.5) is the composition of $\mathrm{Id} \to \pi_{Z*}\pi_Z^*$ with the image under π_{Z*} of

$$\mathcal{O}_{Z\times X} \xrightarrow{\mathrm{Id}\to\iota_{D*}\iota_D^*} \iota_{D*}\mathcal{O}_D \xrightarrow{\gamma(\mathcal{O}_D)} \iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\mathcal{O}_D,\mathcal{O}_D) \xrightarrow{(\xi^*\xi_*\to\mathrm{Id})^{\mathrm{opp}}} \iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\iota_D^*\pi_X^*\pi_{X*}\iota_{D*}\mathcal{O}_D,\mathcal{O}_D)$$

where given an object A we denote by $\gamma(A)$ the adjunction unit $\mathrm{Id} \to \mathbf{R}\mathcal{H}om(A, A \otimes \text{-})$. On the other hand (3.7) is the composition of $\mathrm{Id} \to \pi_{Z*}\pi_Z^*$ with the image under π_{Z*} of

$$\mathcal{O}_{Z\times X} \xrightarrow{\gamma(\iota_{D*}\mathcal{O}_D)} \mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D) \xrightarrow{(\pi_X^*\pi_{X*}\to\mathrm{Id})^{\mathrm{opp}}} \mathbf{R}\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D).$$

We claim that these two compositions are identified by

$$\iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\iota_D^*\pi_X^*\pi_{X*}\iota_{D*}\mathcal{O}_D,\mathcal{O}_D) \xrightarrow{\alpha(\iota_D)} \mathbf{R}\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D)$$

where $\alpha(\iota_D)$ is the natural bifunctorial isomorphism $\iota_{D*}\,\mathbf{R}\mathcal{H}om(\iota_D^*\text{-},\text{-}) \to \mathbf{R}\mathcal{H}om(\text{-},\iota_{D*}\text{-})$.

Dualizing the Proposition 3.1 of [AL10] under the relative duality theory $D_{\bullet/X}$ (where X is in the notation of loc. cit.) we see that the morphism

$$\mathcal{O}_{Z\times X} \xrightarrow{\gamma(\iota_{D*}\mathcal{O}_D)} \mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D)$$

is equal to the morphism

$$\mathcal{O}_{Z\times X} \xrightarrow{\mathrm{Id}\to\iota_{D*}\iota_D^*} \iota_{D*}\mathcal{O}_D \xrightarrow{\gamma(\mathcal{O}_D)} \iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\mathcal{O}_D,\mathcal{O}_D) \xrightarrow{\beta(\iota_D)} \mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D)$$

where given a scheme map f we denote by $\beta(f)$ the natural morphism $f_*\,\mathbf{R}\mathcal{H}om(,) \to \mathbf{R}\mathcal{H}om(f_*, f_*)$.

It remains to establish the commutativity of the diagram

$$\begin{array}{ccccc}
\iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\mathcal{O}_D,\mathcal{O}_D) & \xrightarrow{(\iota_D^*\iota_{D*}\to\mathrm{Id})^{\mathrm{opp}}} & \iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\iota_D^*\iota_{D*}\mathcal{O}_D,\mathcal{O}_D) & \xrightarrow{(\pi_X^*\pi_{X*}\to\mathrm{Id})^{\mathrm{opp}}} & \iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\iota_D^*\pi_X^*\pi_{X*}\iota_{D*}\mathcal{O}_D,\mathcal{O}_D) \\
\downarrow{\scriptstyle\beta(\iota_D)} & & & & \downarrow{\scriptstyle\alpha(\iota_D)} \\
\mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D) & & \xrightarrow[(\pi_X^*\pi_{X*}\to\mathrm{Id})^{\mathrm{opp}}]{} & & \mathbf{R}\mathcal{H}om_{Z\times X}(\pi_X^*\pi_{X*}\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D)
\end{array}$$

By the functoriality of $\alpha(\iota_D)$ it suffices to show that the diagram

$$\begin{array}{ccc}
\iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\mathcal{O}_D,\mathcal{O}_D) & \xrightarrow{(\iota_D^*\iota_{D*}\to\mathrm{Id})^{\mathrm{opp}}} & \iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\iota_D^*\iota_{D*}\mathcal{O}_D,\mathcal{O}_D) \\
\downarrow{\scriptstyle\beta(\iota_D)} & \swarrow{\scriptstyle\alpha(\iota_D)} & \\
\mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D) & &
\end{array} \tag{4.6}$$

commutes. But the isomorphism $\alpha(\iota_D)$ was defined as the composition

$$\iota_{D*}\,\mathbf{R}\mathcal{H}om(\iota_D^*\text{-},\text{-}) \xrightarrow{\beta(\iota_D)} \mathbf{R}\mathcal{H}om(\iota_{D*}\iota_D^*\text{-},\iota_{D*}\text{-}) \xrightarrow{(\mathrm{Id}\to\iota_{D*}\iota_D^*)^{\mathrm{opp}}} \mathbf{R}\mathcal{H}om(\text{-},\iota_{D*}\text{-})$$

and therefore we can re-write the diagram (4.6) as

$$\begin{array}{ccc}
\iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\mathcal{O}_D,\mathcal{O}_D) & \xrightarrow{(\iota_D^*\iota_{D*}\to\mathrm{Id})^{\mathrm{opp}}} & \iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\iota_D^*\iota_{D*}\mathcal{O}_D,\mathcal{O}_D) \\
\downarrow{\scriptstyle\beta(\iota_D)} & & \downarrow{\scriptstyle\beta(\iota_D)} \\
\mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D) & \xleftarrow{(\mathrm{Id}\to\iota_{D*}\iota_D^*)^{\mathrm{opp}}} & \mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\iota_D^*\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D)
\end{array}$$

By the functoriality of $\beta(\iota_D)$ it remains only to check that the diagram

$$\begin{array}{ccc}
\iota_{D*}\,\mathbf{R}\mathcal{H}om_D(\mathcal{O}_D,\mathcal{O}_D) & \xrightarrow[\beta(\iota_D)]{} & \mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D) \\
\downarrow{\scriptstyle\beta(\iota_D)} & & \downarrow{\scriptstyle(\iota_D^*\iota_{D*}\to\mathrm{Id})^{\mathrm{opp}}} \\
\mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D) & \xleftarrow{(\mathrm{Id}\to\iota_{D*}\iota_D^*)^{\mathrm{opp}}} & \mathbf{R}\mathcal{H}om_{Z\times X}(\iota_{D*}\iota_D^*\iota_{D*}\mathcal{O}_D,\iota_{D*}\mathcal{O}_D)
\end{array}$$

commutes, which it certainly does since

$$\iota_{D*}\mathcal{O}_D \xrightarrow{\mathrm{Id}\to\iota_{D*}\iota_D^*} \iota_{D*}\iota_D^*\iota_{D*}\mathcal{O}_D \xrightarrow{\iota_D^*\iota_{D*}\to\mathrm{Id}} \iota_{D*}\mathcal{O}_D$$

is an identity morphism. □

We have a direct analogue of Proposition 3.7 for our setting, giving us a criterion for determining when the left co-twist F_E is an autoequivalence:

Proposition 4.7. *Let D be a flat and perfect fibration in X over Z with proper fibres. The following are equivalent:*

(1) *For every closed point $p\in Z$ we have*

$$\mathbf{R}\,\mathrm{Hom}_{D(X)}(\xi_*\mathcal{O}_D,\xi_{p*}\mathcal{O}_{D_p}) = k\oplus k[d_p] \quad \text{for some } d_p\in\mathbb{Z}.$$

(2) *We have $\mathcal{L}_D \simeq L[d]$ for some $L\in\operatorname{Pic}Z$ and $d\in Z$.*
(3) *The left co-twist F_E is an autoequivalence of $D(Z)$.*

When the conditions above are satisfied $F_E\simeq(\text{-})\otimes\mathcal{L}_D[-1]$ and for any $p\in Z$ the integer d_p in (1) equals the integer d in (2).

Proof. Since $\xi=\pi_X\circ\iota_D$ we have $\pi_{X*}E=\pi_{X*}\iota_{D*}\mathcal{O}_D=\xi_*\mathcal{O}_D$. By Lemma 4.2 the categorical fibre E_p is precisely $\xi_{p*}\mathcal{O}_{D_p}$. Therefore (1) is equivalent to the item (1) of Proposition 3.7.

Since Z was assumed to be irreducible the invertible objects of $D(Z)$ are precisely shifts of line bundles ([AIL10], Theorem 1.5.2). Therefore (2) is equivalent to the item (2) of Proposition 3.7.

Hence the assertion of this Proposition follows directly from that of Proposition 3.7. □

One could simlarly re-state Theorem 3.1 word for word in our present setting, however under some mild non-degeneracy assumption on D we can apply the results of Section 3.4 to make a stronger and more geometric statement. Observe that since Z and X are abstract varieties they are generically non-singular. Hence the Gorenstein locus of $Z\times X$ is certaily dense in $Z\times X$ and our non-degeneracy assumption is that the graph of D doesn't lie completely outside this locus:

Theorem 4.1. *Let D be a flat and perfect fibration in X with proper fibres. Then D is spherical if:*

(1) *For any closed $p\in Z$ we have*

$$\mathbf{R}\,\mathrm{Hom}_X(\xi_*\mathcal{O}_D,\xi_{p*}\mathcal{O}_{D_p}) = k\oplus k[-(\dim X-\dim Z)]$$

(2) *There exists an isomorphism*

$$\iota_{D*}\xi^!(\mathcal{O}_X)\xrightarrow{\sim}\iota_{D*}\pi^!(\mathcal{L}_E)$$

If the graph of D in $Z\times X$ doesn't lie completely outside the Gorenstein locus of $Z\times X$, then the reverse implication also holds.

Proof. We have the following natural isomorphisms:

$$\iota_{D*}\xi^!(\mathcal{O}_X) \xrightarrow{\sim} \iota_{D*}\,\mathbf{R}\,\mathcal{H}om_{Z\times X}\big(\mathcal{O}_D, \iota_D^!\pi_X^!(\mathcal{O}_X)\big) \xrightarrow{\sim} \mathbf{R}\,\mathcal{H}om_{Z\times X}\big(\iota_{D*}\mathcal{O}_D, \pi_X^!(\mathcal{O}_X)\big) \xrightarrow{\sim} E^\vee \otimes \pi_X^!(\mathcal{O}_X)$$

where the second isomorphism is due to the sheafified Grothendieck duality and the third is due to $E = \iota_{D*}\mathcal{O}_D$ being perfect. Similarly we obtain $\iota_{D*}\pi^!(\mathcal{L}_E) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$. Therefore (2) is equivalent to there exists some isomorphism

$$E^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_E).$$

Suppose now that (1) and (2) hold. Then by Proposition 3.7 the assumption (1) implies that the left co-twist F_E is an autoequivalence and the object $\mathcal{L}_E$ is isomorphic to $\mathcal{L}[-(\dim X - \dim Z)]$ for some $L \in \operatorname{Pic}(Z)$. Since $\dim X - \dim Z > 0$ it follows by Proposition 3.13 that an existence of any isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \simeq E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$ implies that the canonical morphism α of Definition 3.10 is an isomorphism. We conclude that F_E is an autoequivalence, the canonical morphism α is an isomorphism and so E is spherical over Z.

Conversely, suppose that E is spherical over Z and that the graph of D doesn't lie completely outside the Gorenstein locus of $Z \times X$. Since E is spherical, then the left co-twist E is an autoequivalence and so by Proposition 3.7 the object $\mathcal{L}_E$ is isomorphic to $L[d]$ for some $L \in \operatorname{Pic} Z$ and $d \in \mathbb{Z}$. By the non-degeneracy assumption there exists a point $p \in D$ such that $\xi(p)$ is Gorenstein in X and $\pi(p)$ is Gorenstein in Z. Then by Proposition 3.11 we must have $d = -(\dim X - \dim Z)$. Applying Proposition 3.7 again yields the assertion (1). On the other hand, since E is spherical the canonical morphism α is an isomorphism $E^\vee \otimes \pi_X^!(\mathcal{O}_X) \xrightarrow{\sim} E^\vee \otimes \pi_Z^!(\mathcal{L}_E)$ and so the assertion (2) also holds. □

Recall now the notion of a Gorenstein map (see [AIL10], §2.4 or [AF90] for the local picture). A scheme map $f\colon S \to T$ is called Gorenstein if it is perfect and if $f^!(\mathcal{O}_T)$ is an invertible object of $D(S)$. If S is connected, this means that $f^!(\mathcal{O}_T)$ is a shift of some line bundle in $\operatorname{Pic} S$. We call this line bundle the *relative dualizing sheaf* and denote it by $\omega_{S/T}$. For any Gorenstein scheme S over k the *(global) dualizing sheaf* of S is the relative dualizing of $S \to \operatorname{Spec}\, k$ and we denote it simply by ω_S. Needless to say that for a smooth S this is precisely its canonical bundle.

In our setting: the map $\pi\colon D \to Z$ is faithfully flat, so it is Gorenstein if and only if its fibres are Gorenstein schemes ([AIL10], Prop. 2.5.10). On the other hand, the map ξ is the composition

$$D \xrightarrow{\iota_D} Z \times X \xrightarrow{\pi_X} X.$$

The closed immersion ι_D is perfect since $\iota_{D*}\mathcal{O}_D$ was assumed to be perfect ([Ill71a], Prop. 4.4). Hence ξ is perfect as it is a composition of two perfect maps. So ξ is Gorenstein if and only if $\xi^!(\mathcal{O}_X)$ is invertible.

If either π or ξ are Gorenstein, then we can re-state the second part of the Theorem 4.1 in terms of the line bundles involved and get rid of the non-degeneracy assumption on D:

Proposition 4.8. *Let D be a flat and perfect fibration in X over Z with proper fibres and assume that either the immersion $\xi\colon D \hookrightarrow X$ or the fibration $\pi\colon D \to X$ is Gorenstein. Then D is spherical if and only if:*

(1) *For any closed $p \in Z$ we have*

$$\mathbf{R}\operatorname{Hom}_X(\xi_*\mathcal{O}_D, \xi_{p*}\mathcal{O}_{D_p}) = k \oplus k[-(\dim X - \dim Z)]$$

By Proposition 4.7 this implies that $\mathcal{L}_D = L[-(\dim X - \dim Z)]$ for some $L \in \operatorname{Pic} Z$.

(2) *Both ξ and π are Gorenstein and there exists an isomorphism*

$$\omega_{D/X} \simeq \pi^* L \otimes \omega_{D/Z}$$

in $\operatorname{Pic} D$.

Proof. 'If': We have

$$\xi^!(\mathcal{O}_X) \simeq \omega_{D/X}[-(\dim X - \dim D)] \qquad \pi^!(\mathcal{O}_X) \simeq \omega_{D/Z}[\dim D - \dim Z]$$

and therefore the condition (2) implies to $\xi^!(\mathcal{O}_X) \simeq \pi^!(\mathcal{L}_E)$. Therefore D is spherical by Theorem 4.1.

'Only if': Suppose D is spherical, then arguing as in the proof of Theorem 4.1 we see that $\mathcal{L}_D$ is invertible and that the canonical morphism α induces an isomorphism

$$\iota_{D*}\xi^!(\mathcal{O}_X) \xrightarrow{\sim} \iota_{D*}\pi^!(\mathcal{L}_D). \tag{4.7}$$

Our assumptions imply that one of $\xi^!(\mathcal{O}_X)$ or $\pi^!(\mathcal{L}_D)$ is invertible, which means that (4.7) is an isomorphism of (shifted) coherent sheaves. But ι_{D*} is a closed immersion and therefore restricts to a fully faithful functor $\operatorname{Coh}(D) \to \operatorname{Coh}(Z \times X)$. Hence the isomorphism (4.7) lifts to an isomorphism

$$\xi^!(\mathcal{O}_X) \xrightarrow{\sim} \pi^!(\mathcal{L}_D). \tag{4.8}$$

Therefore $\xi^!(\mathcal{O}_X)$ and $\pi^!(\mathcal{L}_D)$ are both invertible, and hence both π and ξ are Gorenstein.

Since $\mathcal{L}_E$ is invertible it is of form $L[d]$ for some $L \in \operatorname{Pic} Z$ and $d \in \mathbb{Z}$. We can then re-write (4.8) as

$$\omega_{D/X}[-(\dim X - \dim D)] \simeq \pi^* L[d] \otimes \omega_{D/Z}[\dim D - \dim Z]$$

whence $d = -(\dim X - \dim Z)$ and the isomorphism (2). Finally, since $\mathcal{L}_D \simeq L[-(\dim X - \dim Z)]$ we can apply Proposition 4.7 to obtain the assertion (1). □

It is worth pointing out explicitly the following:

Corollary 4.9. *Let D be a spherical fibration in X over Z. Then $\xi\colon D \hookrightarrow X$ is a Gorenstein immersion if and only if all the fibres of $\pi\colon D \to Z$ are Gorenstein schemes.*

4.2. **Regularly immersed fibrations.** A most common type of a Gorenstein immersion is a regular immersion. These possess a number of properties which are rather useful to our cause. A detailed introduction can be found in [GD67], §16 and§19 and in [Ber71]. For now recall that a closed immersion $\iota\colon Y \hookrightarrow X$ of two schemes is called regular if the ideal sheaf $\mathcal{I}_Y$ of Y in X is locally generated by a regular sequence. It follows that locally on X the Koszul complex of Y is a resolution of the sheaf $\iota_* \mathcal{O}_Y$ by free sheaves. In particular, the co-normal sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ is a locally free sheaf on Y whose rank c is the codimension of Y in X. We denote by $\mathcal{N}_{Y/X}$ its dual $(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$, the normal sheaf of Y in X.

It quickly follows (see [Har66], §III.7) that

$$\iota^!(\mathcal{O}_X) = \wedge^c \mathcal{N}_{Y/X}[-c]$$

i.e. the relative dualizing sheaf $\omega_{Y/X}$ is the line bundle $\wedge^c \mathcal{N}_{Y/X}[-c]$. We can also compute the cohomology sheaves of the object $\iota^* \iota_* \mathcal{O}_Y$ ([Ber71], Proposition 2.5) to obtain

$$\mathcal{H}^{-i}(\iota^* \iota_* \mathcal{O}_Y) = \wedge^i \mathcal{N}^\vee_{Y/X} \qquad \forall\, i \in \mathbb{Z}.$$

Let now A be any object of $D(Y)$. By projection formula we have

$$\iota_*(\iota^* \iota_* A) \simeq \iota_* A \otimes \iota_* \mathcal{O}_Y \simeq \iota_*(A \otimes \iota^* \iota_* \mathcal{O}_Y)$$

and since ι_* is fully faithful on the underlying abelian categories of coherent sheaves it follows that the cohomology sheaves of $\iota^* \iota_* A$ are isomorphic to those of $A \otimes \iota^* \iota_* \mathcal{O}_Y$.

It is an interesting question: when does $\iota^* \iota_* \mathcal{O}_Y$ actually splits up as a direct sum of its cohomology sheaves:

$$\iota^* \iota_* \mathcal{O}_Y \xrightarrow{\sim} \bigoplus_i \wedge^i \mathcal{N}^\vee_{Y/X}[i] \tag{4.9}$$

Clearly, this is true in case a global Koszul resolution of Y in X exists, i.e. when Y is carved out in X by a section of vector bundle. For smooth X the general answer was provided by Arinkin and Caldararu in [AC10]: $\iota^* \iota_* \mathcal{O}_Y$ is isomorphic to $\bigoplus_i \wedge^i \mathcal{N}^\vee_{Y/X}[i]$ if and only if the normal sheaf $\mathcal{N}_{Y/X}$ extends to the first infinitesimal neighborhood of Y in X. The examples of when this condition holds include: when Y is carved out by a section of a vector bundle, when the immersion $\iota\colon Y \hookrightarrow X$ can be split and when Y is the fixed locus of a finite group action in X.

For arbitrary schemes we make the following definition, since the authors of [AC10] clearly couldn't:

Definition 4.10. Let Y and X be a pair of schemes and let $\iota\colon Y \to X$ be a regular immersion. We say that ι is an *Arinkin-Caldararu immersion* if $\iota^* \iota_* \mathcal{O}_Y$ is isomorphic to $\bigoplus_i \wedge^i \mathcal{N}^\vee_{Y/X}[i]$ in $D(X)$.

Back to our setting: we say that a fibration D in X over Z is *regularly immersed* if $\xi\colon D \hookrightarrow X$ is a regular immersion. Knowing the cohomology sheaves of $\xi^* \xi_* \mathcal{O}_D$ allows us to use a spectral sequence argument to reduce the condition

$$\mathbf{R}\operatorname{Hom}_X(\xi_* \mathcal{O}_D, \xi_{p*} \mathcal{O}_{D_p}) = k \oplus k[-(\dim X - \dim Z)]$$

to a statement on the vanishing of the sheaf cohomology of $\wedge^i \mathcal{N}$ on D_p. And if ξ is actually Arinkin-Caldararu and $\xi^* \xi_* \mathcal{O}_D$ breaks up as a sum of its cohomologies, then no spectral sequence is necessary and we immediately obtain an implication running both ways:

Theorem 4.2. *Let D be a regularly immersed flat and perfect fibration in X over Z with proper fibres. Let c be the codimension of D in X, let d be the dimension of the fibres of D and let $\mathcal{N}$ be the normal sheaf of D in X.*

Then D is spherical if for any closed point $p \in Z$ the fibre D_p is a connected Gorenstein scheme and

1. *$H^i_{D_p}(\wedge^j \mathcal{N}|_{D_p}) = 0$ unless $i = j = 0$ or $i = d$, $j = c$.*
2. *$(\omega_{D/X})|_{D_p} \simeq \omega_{D_p}$ where ω_{D_p} is the dualizing sheaf of D_p.*

Conversely, if D is spherical, then each fibre D_p is a connected Gorenstein scheme and (2) holds. And if ξ is an Arinkin-Caldararu immersion, then (1) also holds.

First we need the following lemma which gives a global version of the fibre-wise conditions of Theorem 4.2:

Lemma 4.11. *Let D be a regularly immersed flat and perfect fibration in X over Z with proper fibres. Assume that for any closed point $p \in Z$ the fibre D_p is a connected Gorenstein scheme.*

Then having for every closed point $p \in Z$

$$H^i_{D_p}(\wedge^j \mathcal{N}|_{D_p}) = 0 \text{ unless } i = j = 0 \text{ or } i = d\,,\ j = c \tag{4.10}$$
$$\omega_{D/X}|_{D_p} = \omega_{D_p}$$

is equivalent to having

$$\pi_* \mathcal{O}_D = \mathcal{O}_Z, \quad \pi_* \wedge^j \mathcal{N} = 0 \text{ for all } 0 < j < c, \quad \pi_* \omega_{D/X} = L[d], \tag{4.11}$$
$$\omega_{D/X} = \pi^* L \otimes \omega_{D/Z}$$

for some $L \in \operatorname{Pic} Z$. In particular, (4.10) implies that $H^0_{D_p}(\mathcal{O}_{D_p}) \simeq H^d_{D_p}(\omega_{D/X}|_{D_p}) \simeq k$.

Proof. By flat base change around the square

$$\begin{array}{ccc} D_p & \xhookrightarrow{\iota_{D_p}} & D \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} k & \xhookrightarrow[\iota_p]{} & Z \end{array}$$

we have a functorial isomorphism $\iota_p^* \pi_* \simeq \pi_{k*} \iota_{D_p}^*$. Since $H^i_{D_p}(\wedge^j \mathcal{N}|_{D_p})$ is precisely the i-th cohomology of $\pi_{k*}\iota^*_{D_p}(\wedge^j \mathcal{N})$, we see that restricting (4.11) to any closed $p \in Z$ by ι_p^* immediately gives (4.10).

Conversely, assume that (4.10) holds for every closed $p \in Z$. By the Grothendieck duality on D_p we have

$$H^d_{D_p}(\omega_{D/X}|_{D_p}) \simeq H^d_{D_p}(\omega_{D_p}) \simeq H^0_{D_p}(\mathcal{O}_{D_p})$$

and since D_p is proper and connected we have $H^0_{D_p}(\mathcal{O}_{D_p}) \simeq k$. Therefore by the same base change we obtain that for every closed $p \in Z$ we have

$$\begin{aligned} &\iota_p^* \pi_*\ \mathcal{O}_Z \simeq k \\ &\iota_p^* \pi_*\ \wedge^j \mathcal{N}|_{D_p} = 0 \qquad \text{for all } 0 < j < c \\ &\iota_p^* \pi_*\ \omega_{D/X} \simeq k[-d] \end{aligned}$$

Therefore $\pi_* \wedge^j \mathcal{N}$ vanishes for $0 < j < c$, while $\pi_* \mathcal{O}_D \simeq L'$ and $\pi_* \omega_{D/X}[d] \simeq L$ for some $L', L \in \operatorname{Pic} Z$. But then we must have $L' \simeq \mathcal{O}_Z$ since the adjunction unit $\mathcal{O}_Z \to \pi_* \pi^* \mathcal{O}_Z$ gives a nowhere vanishing morphism $\mathcal{O}_Z \to L'$ of line bundles. This is because by flat base change the restriction of the adjunction unit $\mathcal{O}_Z \to \pi_* \pi^* \mathcal{O}_Z$ to any $p \in Z$ is the adjunction unit $k \to \pi_{k*} \pi_k^*\, k$ which certainly doesn't vanish.

Similarly, observe that by the sheafified Grothendieck duality

$$L \simeq \pi_* \omega_{D/X}[d] \simeq \pi_*\, \mathbf{R}\mathcal{H}om(\omega^{-1}_{D/X} \otimes \omega_{D/Z}, \omega_{D/Z}[d]) \simeq \Big(\pi_*(\omega^{-1}_{D/X} \otimes \omega_{D/Z})\Big)^\vee .$$

Therefore the adjunction co-unit $\pi^* \pi_* (\omega^{-1}_{D/X} \otimes \omega_{D/Z}) \to (\omega^{-1}_{D/X} \otimes \omega_{D/Z})$ gives a nowhere vanishing line bundle morphism $\pi^* L^\vee \to \omega^{-1}_{D/X} \otimes \omega_{D/Z}$, whence the final assertion that $\omega_{D/X} \simeq \pi^* L \otimes \omega_{D/Z}$. □

Proof of Theorem 4.2. 'If' direction: Since ξ_p is the composition $D_p \xrightarrow{\iota_{D_p}} D \xrightarrow{\xi} X$ we have by adjunction

$$\mathbf{R}\operatorname{Hom}_X(\xi_* \mathcal{O}_D, \xi_{p*} \mathcal{O}_{D_p}) \simeq \mathbf{R}\operatorname{Hom}_D(\xi^* \xi_* \mathcal{O}_D, \iota_{D_p *} \mathcal{O}_{D_p}).$$

There is a standard spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_D(\mathcal{H}^{-j}(\xi^* \xi_* \mathcal{O}_{\mathcal{D}}), \iota_{D_p *} \mathcal{O}_{D_p}) \quad \Rightarrow \quad E_\infty^{i+j} = \operatorname{Hom}^{i+j}_{D(D)}(\xi^* \xi_* \mathcal{O}_{\mathcal{D}}, \iota_{D_p *} \mathcal{O}_{D_p})$$

and since for any $j \in \mathbb{Z}$ we have $\mathcal{H}^{-j}(\xi^* \xi_* \mathcal{O}_{\mathcal{D}}) = \wedge^j \mathcal{N}^\vee$, it follows by adjunction that

$$E_2^{i,j} \simeq \operatorname{Ext}^i_D(\wedge^j \mathcal{N}^\vee, \iota_{D_p *} \mathcal{O}_{D_p}) \simeq \operatorname{Ext}^i_{D_p}(\wedge^j \mathcal{N}^\vee|_{D_p}, \mathcal{O}_{D_p}) \simeq H^i_{D_p}(\wedge^j \mathcal{N}|_{D_p}).$$

The fibers of D are proper and connected, so $H^0_{D_p}(\mathcal{O}_{D_p}) \simeq k$. We also have

$$H^d_{D_p}(\omega_{D/X}|_{D_p}) \simeq H^d_{D_p}(\omega_{D_p}) \simeq H^0_{D_p}(\mathcal{O}_{D_p}) \simeq k.$$

by the assumption (2) and the Grothendieck duality.

So by the assumption (1) and by Lemma 4.11 all $E_2^{i,j}$ are zero except for

$$E_2^{0,0} = H^0_{D_p}(\mathcal{O}_{D_p}) \simeq k \text{ and } E_2^{d,c} = H^d_{D_p}(\omega_{D/X}|_{D_p}) \simeq k.$$

Since $d+c = \dim X - \dim Z \neq 0$ the convergence of the spectral sequence tells us precisely that

$$\mathbf{R}\operatorname{Hom}_X(\xi_*\mathcal{O}_{\mathcal{D}}, \xi_{p*}\mathcal{O}_{D_p}) \simeq k \oplus k[-(\dim X - \dim Z)].$$

Then, by Propositions 4.7 and 4.6, we have $\mathcal{L}_D \simeq L[c+d]$ for some $L \in \operatorname{Pic} Z$ and there is an exact triangle

$$\mathcal{O}_Z \to \pi_*\mathbf{R}\mathcal{H}om(\xi^*\xi_*\mathcal{O}_D, \mathcal{O}_D) \to L[-(c+d)].$$

Since $c+d > 0$ it follows that the $(c+d)$-th cohomology sheaf of the complex $\pi_*\mathbf{R}\mathcal{H}om(\xi^*\xi_*\mathcal{O}_D, \mathcal{O}_D)$ is isomorphic to L. On the other hand, we can compute this cohomology sheaf via a spectral sequence similar to the one above and the computation gives $\pi_*\omega_{D/X}[d]$. Therefore by Lemma 4.11 we have $\omega_{D/X} \simeq \pi^*L \otimes \omega_{D/Z}$. We can therefore conclude by Proposition 4.8 that D is spherical.

'Only If' direction:

Conversely, suppose D is spherical. By Proposition 4.8 the fibres of π are Gorenstein schemes and we have

$$\mathbf{R}\operatorname{Hom}^i_X(\xi_*\mathcal{O}_{\mathcal{D}}, \xi_{p*}\mathcal{O}_{D_p}) \simeq k \oplus k[-(\dim X - \dim Z)]$$

for each fibre D_p. On the other hand, the same spectral sequence as before shows that the 0-th cohomology of the complex $\mathbf{R}\operatorname{Hom}^i_X(\xi_*\mathcal{O}_{\mathcal{D}}, \xi_{p*}\mathcal{O}_{D_p})$ is isomorphic to $H^0_{D_p}(\mathcal{O}_{D_p})$. Therefore $H^0_{D_p}(\mathcal{O}_{D_p} = k)$ and so the fibers D_p are connected. Also, by Proposition 4.8 we have $\mathcal{L}_D \simeq L[-(\dim X - \dim Z)]$ for some $L \in \operatorname{Pic} Z$ and $\omega_{D/X} \simeq \pi^*L \otimes \omega_{D/Z}$. Restricting this to every fiber gives the assertion (2).

Finally, suppose that ξ is Arinkin-Caldararu. Then $\xi^*\xi_*\mathcal{O}_D \simeq \bigoplus_i \wedge^i\mathcal{N}^\vee[-i]$, so

$$\mathbf{R}\operatorname{Hom}_X(\xi_*\mathcal{O}_D, \xi_{p*}\mathcal{O}_{D_p}) \simeq \mathbf{R}\operatorname{Hom}_D(\xi^*\xi_*\mathcal{O}_D, \iota_{D_p*}\mathcal{O}_{D_p}) \simeq$$
$$\simeq \bigoplus_i \mathbf{R}\operatorname{Hom}_D(\wedge^i\mathcal{N}^\vee, \iota_{D_p*}\mathcal{O}_{D_p})[i] \simeq \bigoplus_i \mathbf{R}\operatorname{Hom}_{D_p}(\mathcal{O}_{D_p}, \wedge^i\mathcal{N})[i]$$

and we see that the assertion (1) is equivalent to

$$\mathbf{R}\operatorname{Hom}_X(\xi_*\mathcal{O}_D, \xi_{p*}\mathcal{O}_{D_p}) \simeq k \oplus k[-(\dim X - \dim Z)].$$

□

Appendix A. An example

It is well-known that the derived category $D(T^*Fl_n)$ where Fl_n is the full flag variety for some Lie algebra $\mathfrak{g}$ carries an action of the affine braid group [KT07], [Bez06]. It is shown in [KT07] that the action of the usual braid group Br_n is by spherical twists T_i, $i = 1, \ldots, n-1$ in spherical functors $S_i\colon D(T^*\mathcal{P}_i) \to D(T^*Fl_n)$, where $\mathcal{P}_i$ are the partial flag varieties with the space of dimension i missing from the flag. The functor S_i is obtained as the composition $\iota_*\pi^*$, where $\iota\colon D_i \hookrightarrow T^*Fl_n$ is the embedding of the divisor $D_i = Fl_n \times_{\mathcal{P}_i} T^*\mathcal{P}_i$, and $\pi\colon D_i \to T^*\mathcal{P}_i$ is a $\mathbb{P}^1$-bundle. The Fourier–Mukai kernel of S_i is an example of a spherical fibration, being the structure sheaf of $D_i \subset T^*Fl_n \times T^*\mathcal{P}_i$ where D_i embeds into T^*Fl_n and is fibered over $T^*\mathcal{P}_i$.

Recall that the usual braid group is generated by $n-1$ "crossings" $t_1, \ldots, t_{n-1}$, with the relations $t_it_{i+1}t_i = t_{i+1}t_it_{i+1}$. The affine braid group is generated by the same $t_1, \ldots, t_{n-1}$, plus a "rotation" generator r (if the affine braid group is viewed as the group of braids in an annulus, this generator shifts strands, say, counterclockwise). The relations then are $rt_ir^{-1} = t_{i+1}$ and $r^2t_nr^{-2} = t_1$. One can add one more "crossing" $r^{-1}t_1r = t_0 = t_n = rt_{n-1}r^{-1}$, keeping the relations $t_it_{i+1}t_i = t_{i+1}t_it_{i+1}$. In the above affine braid group action on $D(T^*Fl_n)$ the action of the functor corresponding to t_n is not known to have an interpretation as a spherical twist. This can be mended in a specific case, and the relative spherical object that induces the twist will not be a structure sheaf of a subscheme. For the details and proofs, please see [Ann08].

Let $\mathfrak{g}$ be $\mathfrak{sl}_n(\mathbb{C})$. Consider the Grothendieck-Springer resolution $\pi : T^*Fl_n \to \mathcal{N}$ of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. It is also well-known [Bez06] that this action of the affine braid group on $D(T^*Fl_n)$ is local with respect to $\mathcal{N}$ and thus can be transferred to any $U \subset T^*Fl_n$ that is a preimage of some $S \subset \mathcal{N}$. Let z_{2n} be a nilpotent element of $\mathfrak{sl}_{2n}(\mathbb{C})$, with two Jordan blocks of rank n, let $\mathcal{S}_{2n} \subset \mathcal{N}$ be a transversal slice to the orbit of z_{2n} under the adjoint action of $\mathrm{SL}_{2n}(\mathbb{C})$, and let $\mathcal{U}_{2n} \subset T^*Fl_{2n}$ be the preimage of $\mathcal{S}_{2n}$ under the resolution π. This $\mathcal{U}_{2n}$ is a complex symplectic variety of complex dimension $2n$. The preimage $\mathcal{X}_{2n}$ of z_{2n} is a projective variety of dimension n. It is a union of smooth components intersecting normally. For simplicity, denote the derived category $D_{\mathcal{X}_{2n}}(\mathcal{U}_{2n})$ by $\mathcal{D}_{2n}$.

This geometric setting (the choice of $z_{2n} \in \mathcal{N}$) is special, since for all i the preimages of the Springer fiber $\mathcal{S}_{2n}$ in $T^*\mathcal{P}_i$ are isomorphic to $\mathcal{U}_{2n-2}$ (see [Ann08], Section 4), hence the functors S_i may be viewed as having the same source $\mathcal{D}_{2n-2}$. Thus, there are $n-1$ spherical functors $S_i : \mathcal{D}_{2n-2} \to \mathcal{D}_{2n}$ such that the twists T_i in them generate the (usual) braid group action on $\mathcal{D}_{2n}$. Moreover, there is an autoequivalence $R : \mathcal{D}_{2n} \to \mathcal{D}_{2n}$ (constructed in [Ann08], section 4.1) that corresponds to the affine generator r (see above). The remaining twist T_n can be obtained by conjugating T_1 or T_{2n-1} by R.

It is proven in [Ann08], that the generator T_n is indeed a twist in some functor $S_n : \mathcal{D}_{2n-2} \to \mathcal{D}_{2n}$. In fact, S_n is isomorphic to RS_1 or $R^{-1}S_{2n-1}$. The remarkable thing about S_n is that being a composition of S_1 (or S_{2n-1}) and an autoequivalence of $\mathcal{D}_{2n}$, it retains many properties of S_i's. In particular, its kernel $\mathcal{K} \in D(\mathcal{U}_{2n-2} \times \mathcal{U}_{2n})$ is orthogonally spherical over $\mathcal{U}_{2n-2}$. At the same time $\mathcal{K}$ is a genuine object of the derived category $D(\mathcal{U}_{2n-2} \times \mathcal{U}_{2n})$, that is, not isomorphic to the direct sum of its cohomology sheaves. It may be seen in the computation carried out in [Ann08], section 7.2, for $n = 2$; in this case $\mathcal{U}_{2n-2} = \mathcal{U}_2 \simeq T^*\mathbb{P}^1$, and while the image of $\mathcal{O}_{\mathbb{P}^1}$ is a sheaf on $\mathcal{U}_4$, the image of $\mathcal{O}_{\mathbb{P}^1}(-1)$ is not. If $\mathcal{K}$ was actually a spherical fibration, that is, a structure sheaf of some $D \subset \mathcal{U}_4$ fibered over $\mathcal{U}_2$, this would not be possible.

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