SCALED-FREE OBJECTS

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Abstract. Several functional analysts and C*-algebraists have been moving toward a categorical means of understanding normed objects. In this work, I address a primary issue with adapting these abstract concepts to functional analytic settings, the lack of free objects.

Using a new object, called a "crutched set", and associated categories, I devise generalized construction of normed objects as a left adjoint functor to a natural forgetful functor. Further, the universal property in each case yields a "scaled-free" mapping property, which extends previous notions of "free" normed objects. In particular, I construct the following types of scaled-free objects: Banach spaces, Banach algebras, C*-algebras, operator spaces, and operator algebras.

In subsequent papers, this scaled-free property, coupled with the associated functorial results, will give rise to a new view of presentation theory for C*-algebras, which inherits many properties and constructions from its algebraic counterpart.

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1. Introduction

The circle of ideas regarding free objects, particularly the notion of a pair of adjoint functors, is well-known in the literature of category theory and abstract algebra, such as resources [3] and [11]. However, free objects rarely exist in categories of normed objects.

To illustrate this, fix $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathbb{F}\mathbf{NVec}_1$ denote the category whose objects are normed $\mathbb{F}$-vector spaces and whose arrows are $\mathbb{F}$-linear transformations which are contractive. Also, let **Set** denote the category whose objects are sets and whose arrows are functions. Further, distinguish $\mathbb{O} := \{0\}$, the zero space.

Let $\mathscr{C}$ be a subcategory of $\mathbb{F}\mathbf{NVec}_1$. As every $V \in \mathrm{Ob}(\mathscr{C})$ is a set, there is a natural forgetful map to $\mathrm{Ob}(\mathbf{Set})$, where one regards V merely as a set. Similarly, given $V, W \in \mathrm{Ob}(\mathscr{C})$ and $\phi \in \mathscr{C}(V, W)$, ϕ is a function from V to W above all else. One can quickly check that these two associations define a functor $F_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}$, where one ignores all the algebraic and topological data from $\mathscr{C}$.

Proposition 1.1. *Let S be a nonempty set. If there is $V \in \mathrm{Ob}(\mathscr{C})$ such that $V \not\cong_{\mathbb{F}\mathbf{NVec}_1} \mathbb{O}$, then S has no reflection along $F_{\mathscr{C}}$.*

Proof. Assume for purposes of contradiction that S had a reflection (R, η) along $F_{\mathscr{C}}$. As $V \not\cong_{\mathbb{F}\mathbf{NVec}_1} \mathbb{O}$, there is some $v \in V$ with $\|v\|_V \neq 0$. For $n \in \mathbb{N}$, define $\phi_n \in \mathbf{Set}\,(S, F_{\mathscr{C}}V)$ by $\phi_n(s) := nv$, a constant function. Then, there must exist $\hat{\phi}_n \in \mathscr{C}(R, V)$ such that $F_{\mathscr{C}}\hat{\phi}_n \circ \eta = \phi_n$ for all $n \in \mathbb{N}$.

Define $r_s := \eta(s) \in R$. As each $\hat{\phi}_n$ is an arrow in $\mathbb{F}\mathbf{NVec}_1$, for all $n \in \mathbb{N}$ and $s \in S$,

$$\|r_s\|_R \geq \left\|\hat{\phi}_n\left(r_s\right)\right\|_V = \left\|\left(F_{\mathscr{C}}\hat{\phi}_n \circ \eta\right)(s)\right\|_V = \|\phi_n(s)\|_V = n\|v\|_V.$$

As $\|v\|_V \neq 0$, the right-hand side increases without bound. Hence, $\|r_s\|_R$ cannot have a finite value for any $s \in S$, which cannot occur in R. As such, this R is complete fiction.

□

This proposition has said something quite poignant. Unless one restricts to a trivial class of normed vector spaces, i.e. just isomorphic copies of $\mathbb{O}$, or considers an empty set of generators, there is no normed $\mathbb{F}$-vector space with the free mapping property, regardless of all other restrictions of object class or sets of contractive homomorphisms.

Since the free mapping property is a cornerstone to many constructions in pure algebra, particularly presentation theory, this is a most discouraging fact. Therefore, some sacrifice must be made to remedy the situation, which has spawned several avenues of research into generators and relations.

However, in the definition of "relation" itself, there has been much debate. In references [4] and [11], when one considers an algebraic context with a free object, a *relation* is simply an element of this free object. Yet, without a free object, how does one then define "relation"?

One response to this question for C*-algebras can be found in [12, p. 23]. However, while examples are shown of conditions which should be "relations", like norm bounds and *-polynomials, the actual criteria for such a condition are left nebulous.

All C*-algebraists agree that *-polynomials should be considered as "relations" like the algebraic case. Most also include norm bounds as "relations" since one can restrict to certain types of "admissable" representations to build a C*-algebra with a universal mapping property, as shown in [2]. However, the functional calculi are most agreeable with *-homomorphism, where applicable, as shown in [2] and [12]. Some conditions can force certain norm conditions as well, such as C*-algebras of graphs and other combinatorial objects in [7], and [19].

While all of these are examples of what a "relation" should be, a clear definition remains elusive. Hence, one returns to the base question of how to replace the free object in the picture of universal algebra for C*-algebras and *-homomorphisms. Within [9], a "free C*-algebra" is defined to be the *-monoid C*-algebra of the free *-monoid on a given set. Similarly, [6] stated that this algebra is "the closest one gets to free C*-algebras", though in [12, p. 25], it is noted that this algebra is clearly not free, corroborated by Proposition 1.1 of the present work.

One potential replacement is suggested in [10]. Though it does have a connection to a certain kind of freeness, this is not a C*-algebra. It is more closely related to the pro-C*-algebras developed in [18], created by changing categories to topological *-algebras over $\mathbb{C}$ and continuous *-homomorphisms.

There are more categorical approaches as well. In [13] and [14], a "C*-relation" is defined by considering a full subcategory of a comma category. However, this point of view obscures the classical picture, as well as the intuitive notion of a "relation" described above.

In a different direction, [15] considers a functor, unital C*-algebras and unital *-homomorphisms to groups and group homomorphisms by taking the unitary group. Here, the functor is shown to have a left adjoint, namely the group C*-algebra functor, and a few of its functorial properties are considered.

In [16] and [17], several unit ball functors are considered in an attempt to understand the categorical nature of the algebraic theory of C*-algebras and *-homomorphisms. In each case, a left adjoint exists, recreating the C*-algebraic structure. Further, they each explore the operations to build the equational theory. However, both recognize that the "free C*-algebra", again the universal C*-algebra of a set of contractions, is difficult to understand so this equational theory is very vague and unclear.

On the other hand, [8] introduced a new category of objects, sets with a positive-valued function, and used these objects to build a "free" C*-algebra, isomorphic again to the universal C*-algebra of a set of contractions. A substantial portion of this present work was completed before [8] came to the author's attention so there is some overlap.

The present work develops the same category of objects, but also generalizes it and identifies the properties of both. Using these categories, the present work builds not only C*-algebras, but other normed objects of interest. In particular, the same method can be used to construct Banach spaces, Banach algebras, operator spaces, and operator algebras with the analogous universal property. Further, the constructions generalize the work of [9], [16], and [17].

And, it is this "scaled-free" mapping property that is of interest. In subsequent papers, this mapping property, coupled with the associated functorial results, will be shown to give rise to a new view of presentation theory for C*-algebras, which retains many properties and constructions of its algebraic counterpart. In particular, a clear and intuitive definition of a "C*-algebraic relation" is forthcoming, analogous to the well-known algebraic definition from [4] and [11]. With this presentation theory, a Tietze transformation theorem for C*-algebras exists, analogous to the classical result of [20].

2. Crutched Sets

In this section, an object is defined and explored, creating a working environment for a forgetful functor. Since a normed object cannot be reconstructed if all its structure is stripped away, something more must be retained. Specifically, as Proposition 1.1 shows, the norm is the component causing the issue. Hence, the central notion taken here is that of a forgetful functor which strips away all data save two components: the underlying set and the norm. This object was previously introduced in [8], which also recognized this norm issue.

Here, the target categories of the forgetful functors of the present work are described and their properties explored. These properties and constructions are of interest as they starkly mirror those of categories of normed objects and distinguish both categories from **Set**. Further, the coproduct in Proposition 2.2.9 will be used in decomposition theorems and constructions not only in this work, but in subsequent papers.

2.1. Definitions & Basic Results.

The objective is to construct a category so that a forgetful functor from a category of normed objects and its homomorphisms will have a left adjoint. Explicitly, the objects will be a set with a "sizing" function.

Definition. A *crutched set* is a pair (S, f), where S is a set and f a function from S to $[0, \infty)$. The function f is called the *crutch function.* For $s \in S$, s is said to be *crutched* by the value $f(s)$, and $f(s)$ is the *crutch value* of s.

In Section 3, the nomenclature "crutched" becomes more clear, where this nonnegative-valued function supports, much like a crutch, the algebraic free construction to the construction of a normed object. Arguably, one could call this property "normed", but the author chooses not to use this term as there is no linearity assumed on f. Indeed, f is simply *any* set mapping from S to $[0, \infty)$. This object was also considered in [8].

Example 2.1.1. Given any normed vector space V, define $f_V : V \to [0, \infty)$ by $f_V(v) := \|v\|_V$, the norm function. Then, (V, f_V) is a crutched set.

Example 2.1.2. Let $(a_n)_{n\in\mathbb{N}} \subset [0, \infty)$. Define $f_{\vec{a}} : \mathbb{N} \to [0, \infty)$ by $f_{\vec{a}}(n) := a_n$. Then, $(\mathbb{N}, f_{\vec{a}})$ is a crutched set.

In many cases, it will be advantageous to regard a crutched set as a collection of tuples, an element of S and a nonnegative real value, rather than a set and a function. As such, it will be a common practice to write a crutched set as tuples or a sequence, like the previous example, when the set is countable.

Example 2.1.3. Let $S := \{s, t\}$ and $f : S \to [0, \infty)$ be a crutch function. Let $\lambda := f(x)$ and $\mu := f(y)$. Then, (S, f) can be also written as

$$\{(s, \lambda), (t, \mu)\}.$$

The arrows between two crutched sets should preserve the structure, specifically the crutch function. To that end, the following definitions are made purposefully analogous to the notion of linear continuity for normed structures.

Definition. Given two crutched sets (S, f) and (T, g), a function $\phi : S \to T$ is *bounded* if there is $M \geq 0$ such that for all $s \in S$, $g(\phi(s)) \leq Mf(s)$. This will be denoted $\phi : (S, f) \to (T, g)$. Let

$$\mathrm{crh}(\phi) := \inf \{M \in [0, \infty) : g(\phi(s)) \leq Mf(s) \forall s \in S\},$$

the *crutch bound* of ϕ. If $\mathrm{crh}(\phi) \leq 1$, ϕ is *constrictive.*

Similarly, use of existing terminology like "norm" or "contraction" is avoided, as there is no concept of linearity or distance in this setting. However, as these notions are analogous, familiar results follow immediately from definition.

First, the relationship between the crutch bound of a bounded function and the crutch functions of its domain and codomain directly mirrors the relationship between the norm of a bounded linear map and the norms of its domain and codomain.

Proposition 2.1.4. *Let (S, f) and (T, g) be crutched sets and $\phi : (S, f) \to (T, g)$ be bounded. Then, for all $s \in S$,*

$$g(\phi(s)) \leq \mathrm{crh}(\phi) f(s).$$

Proof. For $n \in \mathbb{N}$, there is $M_n \in \{M \in [0, \infty) : g(\phi(s)) \leq Mf(s) \forall s \in S\}$ such that $\mathrm{crh}(\phi) \leq M_n \leq \mathrm{crh}(\phi) + \frac{1}{n}$. For each $s \in S$,

$$g(\phi(s)) \leq M_n f(s) \leq \left(\mathrm{crh}(\phi) + \frac{1}{n}\right) f(s).$$

Letting $n \to \infty$, $g(\phi(s)) \leq \operatorname{crh}(\phi) f(s)$.

□

Observe that as a result, if $\phi : (S, f) \to (T, g)$ is constrictive, $g(\phi(s)) \leq f(s)$ for all $s \in S$. This is taken as definition for the maps considered in [8].

The above proposition immediately yields the following result regarding compositions of bounded functions, reflecting its counterpart for bounded linear maps.

Corollary 2.1.5. *Let (S, f), (T, g), and (U, h) be crutched sets and $\phi : (S, f) \to (T, g)$ and $\psi : (T, g) \to (U, h)$ be bounded. Then, $\psi \circ \phi : S \to U$ is bounded and*

$$\operatorname{crh}(\psi \circ \phi) \leq \operatorname{crh}(\psi) \operatorname{crh}(\phi).$$

Corollary 2.1.6. *Let (S, f), (T, g), and (U, h) be crutched sets and $\phi : (S, f) \to (T, g)$ and $\psi : (T, g) \to (U, h)$ be constrictive. Then, $\psi \circ \phi : (S, f) \to (U, h)$ is constrictive.*

Also, the computation of the crutch bound can be reformulated from an infimum to a supremum in a familiar way.

Proposition 2.1.7. *Let (S, f) and (T, g) be crutched sets and $\phi : (S, f) \to (T, g)$ be bounded. Then,*

$$\operatorname{crh}(\phi) = \sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right).$$

Proof. Let $L := \sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right)$. For all $s \notin f^{-1}(0)$,

$$0 \leq g(\phi(s)) \leq \operatorname{crh}(\phi) f(s)$$

so

$$0 \leq \frac{g(\phi(s))}{f(s)} \leq \operatorname{crh}(\phi).$$

Thus, $L \leq \operatorname{crh}(\phi)$.

For $s \notin f^{-1}(0)$, $\frac{g(\phi(s))}{f(s)} \leq L$ so $g(\phi(s)) \leq L f(s)$. For $s \in f^{-1}(0)$,

$$0 \leq g(\phi(s)) \leq \operatorname{crh}(\phi) f(s) = 0.$$

Then, $g(\phi(s)) = 0 = L f(s)$. Therefore, $\operatorname{crh}(\phi) \leq L$.

□

From this result, alternate criteria for boundedness can be devised.

Proposition 2.1.8. *Let (S, f) and (T, g) be crutched sets. A function $\phi : S \to T$ is bounded if and only if*

$$\sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right) < \infty$$

and $g(\phi(s)) = 0$ for all $s \in f^{-1}(0)$.

Proof. ($\Rightarrow$) This direction is the content of Proposition 2.1.7.

($\neg \Rightarrow \neg$) Assuming that ϕ is not bounded, then for each $M \geq 0$, there is $s_M \in S$ such that $g\left(\phi\left(s_M\right)\right) > Mf\left(S_M\right)$. If some $s_M \in f^{-1}(0)$, then

$$g\left(\phi\left(s_M\right)\right) > Mf\left(s_M\right) = 0.$$

If $s_M \notin f^{-1}(0)$ for all $M \geq 0$, then

$$\frac{g\left(\phi\left(s_M\right)\right)}{f\left(s_M\right)} > M$$

for every $M \geq 0$. Hence,

$$\sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right) = \infty.$$

□

Now, observe that the criterion on $f^{-1}(0)$ is necessary. Without linearity in ϕ, $f^{-1}(0)$ does not necessarily get mapped into $g^{-1}(0)$.

Example 2.1.9. Let V and W be normed vector spaces and $\phi : V \to W$ be a bounded linear function. Let f_V and f_W be crutch functions on V and W, respectively, defined as in Example 2.1.1. By Propositions 2.1.7 and 2.1.8, ϕ is a bounded function from (V, f_V) to (W, f_W) and $\operatorname{crh}(\phi) = \|\phi\|_{\mathcal{B}(V,W)}$.

Example 2.1.10. Given a crutched set (S, f), let $id_S : S \to S$ be given by $id_S(s) := s$, the identity function. Then, as $f \circ id_S = f$, id_S is constrictive with

$$\operatorname{crh}\left(id_S\right) = \begin{cases} 1, & S \neq f^{-1}(0), \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.1.11. Let $S := T := \mathbb{N}$. Define $f : S \to [0, \infty)$ by $f(n) := n$ and $g : T \to [0, \infty)$ by $g(n) := \dfrac{1}{n}$. Further, let $\phi : S \to T$ by $\phi(n) := n$. Then, for each $n \in S$,

$$\frac{g(\phi(n))}{f(n)} = \frac{\frac{1}{n}}{n} = \frac{1}{n^2} \leq 1,$$

meaning ϕ is bounded and $\operatorname{crh}(\phi) = 1$ by Propositions 2.1.7 and 2.1.8. In particular, ϕ is constrictive.

However, let $\psi : T \to S$ by $\psi(n) := n$, the inverse set map of ϕ. For $n \in T$,

$$\frac{f(\psi(n))}{g(n)} = \frac{n}{\frac{1}{n}} = n^2.$$

Thus, ψ is unbounded by Proposition 2.1.8.

2.2. **Category of Crutched Sets & Constrictive Maps.** Next, a detailed study is conducted of crutched sets and constrictive functions between them. This combination of objects and maps was considered previously in [8]. For notation, the symbol $\mathbf{CSet}_1$ will be used to denote the following data:

- $\mathrm{Ob}\left(\mathbf{CSet}_1\right) :=$ the class of all crutched sets;
- For $(S, f), (T, g) \in \mathrm{Ob}(\mathbf{CSet}_1)$, define

$$\mathbf{CSet}_1((S, f), (T, g)) := \left\{ \phi \in \mathbf{Set}(S, T) : \begin{array}{c} \phi \text{ constrictive from} \\ (S, f) \text{ to } (T, g) \end{array} \right\}.$$

Equipping this structure with function composition, the following results from Corollary 2.1.6 and Example 2.1.10.

Proposition 2.2.1. $\mathbf{CSet}_1$ *is a category.*

With this new structure defined, one considers some of its basic properties and constructions. Many of these will be very familiar to anyone with experience with **Set**. However, most interestingly, the basic constructions immediately resemble their counterparts in normed structures. This seems to indicate the dependency of these notions on the positive function, not algebraic structure or notions of linearity or distance.

To begin, consider the primary properties of constrictive mappings and criteria for isomorphism. This proposition makes precise the statements made in Remark 1.1.9 from [8].

Proposition 2.2.2. *Let (S, f) and (T, g) be crutched sets and $\phi : (S, f) \to (T, g)$ be constrictive. The following characterizations hold.*

1. *ϕ is a monomorphism in $\mathbf{CSet}_1$ iff ϕ is one-to-one;*
2. *ϕ is an epimorphism in $\mathbf{CSet}_1$ iff ϕ is onto;*
3. *ϕ is a section in $\mathbf{CSet}_1$ iff ϕ is one-to-one, $g \circ \phi = f$, and for all $t \notin \phi(S)$, there is $s_t \in S$ such that $f(s_t) \leq g(t)$;*
4. *ϕ is a retraction in $\mathbf{CSet}_1$ iff for all $t \in T$, $\phi^{-1}(t) \cap f^{-1}(g(t)) \neq \emptyset$;*
5. *ϕ is an isomorphism in $\mathbf{CSet}_1$ iff ϕ is one-to-one, onto, and $g \circ \phi = f$.*

Proof. (1) ($\Rightarrow$) Assume that ϕ is a monomorphism in $\mathbf{CSet}_1$. Let $s, \hat{s} \in S$ such that $\phi(s) = \phi(\hat{s})$. Let $U := \{0\}$ and $h(0) := \max\{f(s), f(\hat{s})\}$. Define $\alpha(0) := s$ and $\beta(0) := \hat{s}$. Clearly, α and β are both constrictive, and

$$(\phi \circ \alpha)(0) = \phi(s) = \phi(\hat{s}) = (\phi \circ \beta)(0).$$

Hence, $\alpha = \beta$, meaning $s = \alpha(0) = \beta(0) = \hat{s}$.

($\Leftarrow$) Assume ϕ is one-to-one. Then, for any crutched set (U, h), let $\alpha, \beta : (U, h) \to (S, f)$ be constrictive such that $\phi \circ \alpha = \phi \circ \beta$. For all $u \in U$, $\phi(\alpha(u)) = \phi(\beta(u))$. Since ϕ is one-to-one, $\alpha(u) = \beta(u)$, meaning $\alpha = \beta$.

(2) (⇒) Assume ϕ is an epimorphism in $\mathbf{CSet}_1$. Let $U := \{0,1\}$ and $h(u) := 0$. Define $\alpha, \beta : T \to U$ by $\alpha(t) := 0$ and

$$\beta(t) := \begin{cases} 0, & t \in \operatorname{ran}(\phi) \\ 1, & t \notin \operatorname{ran}(\phi) \end{cases} .$$

Clearly, α and β are both constrictive, and for all $s \in S$,

$$(\alpha \circ \phi)(s) = 0 = (\beta \circ \phi)(s).$$

Thus, $\alpha = \beta$, so for all $t \in T$, $\beta(t) = \alpha(t) = 0$. Therefore, $T = \operatorname{ran}(\phi)$.

(⇐) Assume ϕ is onto. Then, for any crutched set (U, h), let $\alpha, \beta : (T, g) \to (U, h)$ be constrictive such that $\alpha \circ \phi = \beta \circ \phi$. For all $t \in T$, there is some $s \in S$ such that $t = \phi(s)$. Thus, $\alpha(t) = (\alpha \circ \phi)(s) = (\beta \circ \phi)(s) = \beta(t)$, so $\alpha = \beta$.

(3) (⇒) Assume that ϕ is a section in $\mathbf{CSet}_1$. By definition, there is a constrictive $\psi : (T, g) \to (S, f)$ such that $\psi \circ \phi = id_S$. From basic function results, ϕ must be one-to-one. For all $s \in S$,

$$f(s) = (f \circ id_S)(s) = (f \circ \psi \circ \phi)(s) \leq (g \circ \phi)(s) \leq f(s)$$

so $f(s) = (g \circ \phi)(s)$, meaning $f = g \circ \phi$. Lastly, let $s_t := \psi(t)$ for each $t \in T$. Then, $f(s_t) \leq g(t)$.

(⇐) Assuming the result, define $\psi : T \to S$ by

$$\psi(t) := \begin{cases} s, & t = \phi(s), \\ s_t, & t \notin \phi(S). \end{cases}$$

As ϕ is one-to-one, this is a well-defined function, and $\psi \circ \phi = id_S$ by design. To prove ψ constrictive, observe that for $s \in S$,

$$f(\psi(\phi(s))) = f(s) = g(\phi(s))$$

and for $t \notin \phi(S)$,

$$f(\psi(t)) = f(s_t) \leq g(t).$$

(4) (⇒) Assume that ϕ is a retraction in $\mathbf{CSet}_1$. By definition, there is a constrictive $\psi : (T, g) \to (S, f)$ such that $\phi \circ \psi = id_T$. For $t \in T$, let $s_t := \psi(t)$. Observe that $\phi(s_t) = t$ and

$$g(t) = (g \circ id_S)(t) = (g \circ \phi \circ \psi)(t) \leq f(\psi(t)) = f(s_t) \leq g(t).$$

Thus, $g(t) = f(s_t)$, so $s_t \in \phi^{-1}(t) \cap f^{-1}(g(t))$.

(⇐) Assuming the result, let $s_t \in \phi^{-1}(t) \cap f^{-1}(g(t))$ and define $\psi : T \to S$ by $\psi(t) := s_t$. Then, $\phi \circ \psi = id_T$ by design. To prove ψ constrictive, observe that for all $t \in T$,

$$f(\psi(t)) = f(s_t) = g(t).$$

(5) (⇒) Assume that ϕ is an isomorphism in $\mathbf{CSet}_1$. By definition, there is a constrictive $\psi : (T, g) \to (S, f)$ such that $\psi \circ \phi = id_S$ and $\phi \circ \psi = id_T$. Then, ϕ is both a section and a retraction, in particular, an epimorphism. Hence, ϕ is one-to-one, onto, and $f = g \circ \phi$.

(⇐) Assuming the result, ϕ is an epimorphism since it is onto. Further, $T \setminus \phi(S) = \emptyset$ by this fact, meaning ϕ is furthermore a section. Hence, ϕ is an isomorphism.

□

It is of some note that each of the conditions in Item 5 are necessary. In particular, the condition $f = g \circ \phi$ is reminiscent of isometry in normed spaces. However, these constrictive functions are not linear so this condition alone does not imply even monomorphism, let alone isomorphism.

Example 2.2.3. Let $S := \{0\}$, $f(0) := 1$, and $g(0) := 0$. Define $\phi : S \to S$ by $\phi(0) := 0$, a constrictive map. However, while ϕ is both monic and epic, it is not a section or retraction. This example concretely demonstrates the statement made in Remark 1.1.9 of [8] about monic and epic constrictions which are not sections or retractions.

Example 2.2.4. Let $S := \mathbb{N}$, $f(n) := 1$, $T := \{0\}$, and $g(0) := 1$. Define $\phi : S \to T$ by $\phi(n) := 0$, a constrictive map. Then, ϕ is a retraction and $g \circ \phi = f$, but it is not a monomorphism.

Similarly, define $\varphi : T \to S$ by $\varphi(0) := 1$, another constrictive map. Then, φ is section, but it is not an epimorphism.

Next, consider the standard universal constructions in $\mathbf{CSet}_1$.

Proposition 2.2.5. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ constrictive maps. Let*

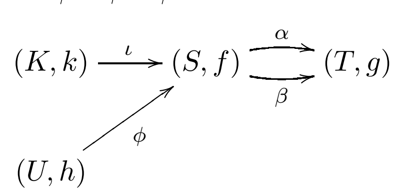
$$K := \{s \in S : \alpha(s) = \beta(s)\},$$

$k := f|_K$, and $\iota : K \to S$ by $\iota(s) := s$. Then,

$$\mathrm{Eq}_{\mathbf{CSet}_1}(\alpha, \beta) \cong_{\mathbf{CSet}_1} (K, k).$$

Further, given any $L \subseteq S$, define $l := f|_L$. Then, (L, l) can be realized as an equalizer of two parallel arrows from (S, f).

Proof. From definition, ι is constrictive and $\alpha \circ \iota = \beta \circ \iota$. To check the universal property, let (U, h) be a crutched set and $\phi : (U, h) \to (S, f)$ be constrictive such that $\alpha \circ \phi = \beta \circ \phi$.



Then, for all $u \in U$, $(\alpha \circ \phi)(u) = (\beta \circ \phi)(u)$. Hence, $\phi(u) \in K$ so define $\hat{\phi} := \phi|^K$, restricting its codomain. Since the crutch function is likewise restricted, $\hat{\phi}$ is constrictive. Also, $\phi = \iota \circ \hat{\phi}$ by expansion of codomain.

Assume that there was $\varphi : (U, h) \to (K, k)$ such that $\phi = \iota \circ \varphi$. Then, $\iota \circ \varphi = \iota \circ \hat{\phi}$ and as ι is one-to-one, $\varphi = \hat{\phi}$.

For (L, l), let $T := \{0, 1\}$ and $g(t) := 0$. Define $\alpha, \beta : S \to T$ by $\alpha(s) := 0$ and

$$\beta(s) := \begin{cases} 0, & s \in L \\ 1, & s \notin L \end{cases}.$$

Then,

$$\mathrm{Eq}_{\mathbf{CSet}_1}(\alpha, \beta) \cong_{\mathbf{CSet}_1} (K, k) = (L, l).$$

□

For a coequalizer, notice that the crutch function sharply reflects the quotient norm in normed algebraic structures.

Proposition 2.2.6. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ constrictive maps. Let*

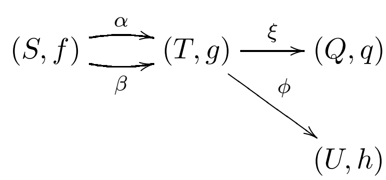
$$P := \{(\alpha(s), \beta(s)) \in T^2 : s \in S\}$$

and $\sim_P$ be the equivalence relation on T generated by P. Define $Q := T/\sim_P$, $q([t]) := \inf\{g(\tau) : \tau \sim_P t\}$, and $\xi : T \to Q$ by $\xi(t) := [t]$. Then,

$$\mathrm{Coeq}_{\mathbf{CSet}_1}(\alpha, \beta) \cong_{\mathbf{CSet}_1} (Q, q).$$

Further, given any equivalence relation $\sim$ on T, define $r : T/\sim \to [0, \infty)$ by $r([t]) := \inf\{g(\tau) : \tau \sim t\}$. Then, $(T/\sim, r)$ can be realized as a coequalizer of two parallel arrows to (T, g).

Proof. From definition, ξ is constrictive and $\xi \circ \alpha = \xi \circ \beta$. To check the universal property, let (U, h) be a crutched set and $\phi : (T, g) \to (U, h)$ be constrictive such that $\phi \circ \alpha = \phi \circ \beta$.



Consider $\sim_\phi := \{(t, \tau) \in T^2 : \phi(t) = \phi(\tau)\}$. Note that $\sim_\phi$ is an equivalence relation. Further, for all $s \in S$, $(\phi \circ \alpha)(s) = (\phi \circ \beta)(s)$. Hence, $P \subseteq \sim_\phi$ so $\sim_P \subseteq \sim_\phi$. Thus, if $t \sim_P \tau$, $t \sim_\phi \tau$, or rather, $\phi(t) = \phi(\tau)$. Hence, define $\hat{\phi} : Q \to U$ by $\hat{\phi}([t]) := \phi(t)$. By the above argument, this is well-defined. For all $t \in T$,

$$\left(\hat{\phi} \circ \xi\right)(t) = \hat{\phi}([t]) = \phi(t),$$

meaning $\hat{\phi} \circ \xi = \phi$. Now, for all $\tau \in [t]$,

$$\left(h \circ \hat{\phi}\right)([t]) = (h \circ \phi)(\tau) \le g(\tau).$$

Hence, $\left(h \circ \hat{\phi}\right)([t]) \le q([t])$, meaning $\hat{\phi}$ is constrictive.

Assume there was some other constrictive $\varphi : (Q, q) \to (U, h)$ such that $\phi = \varphi \circ \xi$. Then, for all $t \in T$,

$$\varphi([t]) = (\varphi \circ \xi)(t) = \phi(t) = \hat{\phi}([t]).$$

Thus, $\varphi = \hat{\phi}$.

For $(T/\sim, r)$, let $S :=\sim$ and $f(t,\tau) := \max\{g(t), g(\tau)\}$. Define $\alpha, \beta : S \to T$ by $\alpha(t,\tau) := t$ and $\beta(\tau) := \tau$. Then,

$$\mathrm{Coeq}_{\mathbf{CSet}_1}(\alpha,\beta) \cong_{\mathbf{CSet}_1} (Q,q) = (T/\sim, r).$$

□

The product in $\mathbf{CSet}_1$ should be compared to the ℓ^∞-sum of normed spaces.

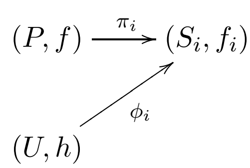
Proposition 2.2.7. *For an index set I, let (S_i, f_i) be crutched sets for $i \in I$. Define*

$$P := \left\{ \vec{s} \in \mathbf{Set}\left(I, \bigcup_{i\in I} S_i\right) : \vec{s}(i) \in S_i \forall i \in I, \sup\left\{f_i\left(\vec{s}(i)\right) : i \in I\right\} < \infty \right\},$$

$f : P \to [0,\infty)$ by $f(\vec{s}) := \sup\{f_i(\vec{s}(i)) : i \in I\}$, and $\pi_i : P \to S_i$ by $\pi_i(\vec{s}) := \vec{s}(i)$. Then,

$$\prod_{i\in I}^{\mathbf{CSet}_1} (S_i, f_i) \cong_{\mathbf{CSet}_1} (P, f).$$

Proof. From definition, π_i is constrictive for each $i \in I$. To check the universal property, let (U,h) be a crutched set and $\phi_i : (U,h) \to (S_i, f_i)$ be constrictive.



For each $u \in U$, observe that $(f_i \circ \phi_i)(u) \le h(u)$. Hence,

$$\sup\{(f_i \circ \phi_i)(u) : i \in I\} < \infty$$

so define $\phi : U \to P$ by $\phi(u)(i) := \phi_i(u)$. Then, $\pi_i \circ \phi = \phi_i$ for each $i \in I$. Also,

$$(f \circ \phi)(u) = \sup\{(f_i \circ \phi_i)(u) : i \in I\} \le h(u),$$

making ϕ constrictive.

Assume there was some other constrictive $\varphi : (U,h) \to (P,f)$ such that $\pi_i \circ \varphi = \phi_i$. Then, for each $i \in I$ and $u \in U$,

$$(\pi_i \circ \varphi)(u) = \phi_i(u)$$

Hence, $\varphi(u)(i) = \phi_i(u) = \phi(u)(i)$, meaning $\varphi(u) = \phi(u)$. Therefore, $\varphi = \phi$. □

Notice that there are times when a product of nontrivial objects is trivial.

Example 2.2.8. Define $S := \{0\}$ and $f_n(0) := n$ for all $n \in \mathbb{N}$. Then,

$$\prod_{n \in \mathbb{N}}^{\mathbf{CSet}_1} (S, f_n) \cong_{\mathbf{CSet}_1} \left(\emptyset, \mathbf{0}_{[0,\infty)}\right),$$

the empty set and the empty function into $[0, \infty)$. Further, the canonical projections from the product to S are empty functions, which are hardly onto mappings.

Dually, a coproduct of $\mathbf{CSet}_1$ is more closely related to the disjoint union in **Set**. This is of interest as it gives a canonical way of writing any crutched set in terms of singletons.

Proposition 2.2.9. *For an index set I, let (S_i, f_i) be crutched sets for $i \in I$. Define*

$$C := \left\{ (i, s) \in I \times \left(\bigcup_{i \in I} S_i \right) : s \in S_i \right\},$$

$f : C \to [0, \infty)$ by $f(i, s) := f_i(s)$, and $\rho_i : S_i \to C$ by $\rho_i(s) := (i, s)$. Then,

$$\coprod_{i \in I}^{\mathbf{CSet}_1} (S_i, f_i) \cong_{\mathbf{CSet}_1} (C, f).$$

Further, for any crutched set (T, g),

$$(T, g) \cong_{\mathbf{CSet}_1} \coprod_{t \in T}^{\mathbf{CSet}_1} \{(t, g(t))\}.$$

Proof. From definition, ρ_i is constrictive for each $i \in I$. To check the universal property, let (U, h) be a crutched set and $\phi_i : (S_i, f_i) \to (U, h)$ be constrictive.

$$\begin{array}{ccc} (C, f) & \xleftarrow{\rho_i} & (S_i, f_i) \\ & \swarrow_{\phi_i} & \\ (U, h) & & \end{array}$$

Define $\phi : C \to U$ by $\phi(i, s) := \phi_i(s)$. Then, for each $i \in I$ and $s \in S_i$,

$$(\phi \circ \rho_i)(s) = \phi(i, s) = \phi_i(s)$$

meaning $\phi \circ \rho_i = \phi_i$. Also,

$$(h \circ \phi)(i, s) = (h \circ \phi_i)(s) \leq f_i(s) = f(i, s)$$

so ϕ is constrictive.

Assume that there was some other constrictive $\varphi : (C, f) \to (U, h)$ such that $\varphi \circ \rho_i = \phi_i$. Then, for each $i \in I$ and $s \in S_i$,

$$\varphi(i, s) = (\varphi \circ \rho_i)(s) = \phi_i(s) = \phi(i, s).$$

Therefore, $\varphi = \phi$.

Given a crutched set (T, g), note that

$$\left\{(t,\tau) \in T \times \left(\bigcup_{t\in T}\{t\}\right) : \tau \in \{t\}\right\} = \{(t,t) : t \in T\} \cong_{\mathbf{Set}} T$$

and

$$f(t,t) = g|_{\{t\}}(t) = g(t).$$

Thus, the second result follows.

□

An empty product yields a terminal object, $\{(0,0)\}$, and the empty coproduct an initial object, $\left(\emptyset, \mathbf{0}_{[0,\infty)}\right)$. As $\mathbf{CSet}_1$ has all products and equalizers, all the other standard limit processes can be performed. Dually, colimit processes follow from the existence of all coproducts and coequalizers. Summarily, this may be stated as follows.

Corollary 2.2.10. *The category* $\mathbf{CSet}_1$ *is complete and cocomplete.*

However, **Set** also shares these completion properties. This is not unexpected as $\mathbf{CSet}_1$ adds relatively little structure to **Set**. Indeed, this is actually desired so as to remain close to the classical construction of a free object.

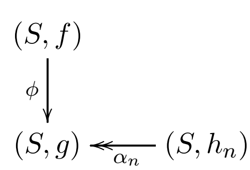
Yet, $\mathbf{CSet}_1$ is not isomorphic to **Set** as categories. To see this, recall that every object in **Set** is projective with respect to all epimorphisms in **Set**. However, the idea of constriction almost completely forbids this behavior in $\mathbf{CSet}_1$.

Proposition 2.2.11. *Let* (S, f) *be a crutched set.*

(1) (S, f) *is projective relative to all epimorphisms in* $\mathbf{CSet}_1$ *iff* $S = \emptyset$.
(2) (S, f) *is injective relative to all monomorphisms in* $\mathbf{CSet}_1$ *iff* $S \neq \emptyset$ *and* $f = 0$.

Proof. (1) ($\Leftarrow$) Assume that $S = \emptyset$. Then, $f = \mathbf{0}_{[0,\infty)}$. Let (T, g) and (U, h) be crutched sets and $\alpha : (U, h) \to (T, g)$ be an epimorphism. As $(\emptyset, \mathbf{0}_{[0,\infty)})$ is initial, there is only one function from it to any other crutched set, the empty function. Let $\mathbf{0}_T$ and $\mathbf{0}_U$ be the empty functions to T and U, respectively. Then, $\mathbf{0}_T = \alpha \circ \mathbf{0}_U$ trivially.

($\neg \Leftarrow \neg$) For purposes of contradiction, assume that $S \neq \emptyset$ and (S, f) is projective relative to all epimorphisms. For each $n \in \mathbb{N}$, define $g, h_n : S \to [0, \infty)$ by $g(s) := 0$ and $h_n(s) := n$. Also, let $\phi, \alpha_n : S \to S$ by $\phi(s) := \alpha_n(s) := s$. Then, consider the following diagram in $\mathbf{CSet}_1$ for each n.



Since α_n is onto and (S, f) projective to epimorphisms, there must be a constrictive $\phi_n : (S, f) \to (S, h_n)$ such that $\phi = \alpha_n \circ \phi_n$. Then, for each $s \in S$ and $n \in \mathbb{N}$,

$$s = \phi(s) = (\alpha \circ \phi_n)(s) = \phi_n(s)$$

and

$$n = (h_n \circ \phi_n)(s) \leq f(s).$$

Thus, f cannot have a finite value, contradicting that (S, f) was a crutched set.

(2) ($\Rightarrow$) Assume that (S, f) is injective relative to all monomorphisms. Let $\mathbf{0}_S : \emptyset \to S$ and $\mathbf{0}_{\{0\}} : \emptyset \to \{0\}$ be the empty functions into S and $\{0\}$, respectively. Consider the following diagram in $\mathbf{CSet}_1$.

$$\begin{array}{ccc} (S, f) & & \\ \uparrow{\scriptstyle \mathbf{0}_S} & & \\ (\emptyset, \mathbf{0}_{[0,\infty)}) & \underset{\mathbf{0}_{\{0\}}}{\rightarrowtail} & \{(0,0)\}. \end{array}$$

As (S, f) is injective relative to $\mathbf{0}_{\{0\}}$, there must be a constrictive map from $\{(0,0)\}$ to (S, f). Hence, there is a function from a nonempty set into S, forcing $S \neq \emptyset$.

Define $h : S \to [0, \infty)$ by $h(s) := 0$. Also, let $\phi, \alpha : S \to S$ by $\phi(s) := \alpha(s) := s$. Then, consider the following diagram in $\mathbf{CSet}_1$.

$$\begin{array}{ccc} (S, f) & & \\ \uparrow{\scriptstyle \phi} & & \\ (S, f) & \underset{\alpha}{\rightarrowtail} & (S, h). \end{array}$$

Then, there is a constriction $\hat{\phi} : (S, h) \to (S, f)$ such that $\phi = \hat{\phi} \circ \alpha$. Then, for each $s \in S$,

$$s = \phi(s) = \left(\hat{\phi} \circ \alpha\right)(s) = \hat{\phi}(s)$$

and

$$0 \leq f(s) = \left(f \circ \hat{\phi}\right)(s) \leq h(s) = 0.$$

($\Leftarrow$) Assume that $f = 0$ and $S \neq \emptyset$. Let (T, g) and (U, h) be crutched sets and $\alpha : (T, g) \to (U, h)$ be a monomorphism. Define $\hat{U} := \operatorname{ran}(\alpha)$ and observe that $\alpha|^{\hat{U}}$ is bijective. Given any $\phi : T \to S$, choose any $s_0 \in S$ and define $\hat{\phi} : U \to S$ by

$$\hat{\phi}(u) := \left\{ \begin{array}{ll} \phi(s), & u = \alpha(s) \\ s_0, & u \notin \hat{U} \end{array} \right. .$$

As α is one-to-one, this is a well-defined function. Clearly, $\phi = \hat{\phi} \circ \alpha$, and since $f = 0$, $\hat{\phi}$ is trivially constrictive.

□

There is precisely one isomorphism class of a projective object relative to all epimorphisms in $\mathbf{CSet}_1$, but $\mathbf{Set}$ has a proper class of such isomorphism classes. Hence, the distinction follows.

Corollary 2.2.12. *$\mathbf{CSet}_1$ and $\mathbf{Set}$ are not isomorphic as categories.*

To close this section on $\mathbf{CSet}_1$, this category can be used to extend the failure result of Proposition 1.1. As before, let $\mathscr{C}$ be a subcategory of $\mathbf{CSet}_1$. There is a natural forgetful map from $\mathrm{Ob}(\mathscr{C})$ to $\mathrm{Ob}(\mathbf{Set})$, where one strips away the crutch function. Similarly, given $(S, f), (T, g) \in \mathrm{Ob}(\mathscr{C})$ and $\phi \in \mathscr{C}((S, f), (T, g))$, $\phi \in \mathbf{Set}(S, T)$ by definition of $\mathbf{CSet}_1$. One can quickly check that these two associations define a functor $F_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}$, where one ignores all the numeric properties from $\mathscr{C}$.

Proposition 2.2.13. *Let S be a nonempty set. Assume that for each $n \in \mathbb{N}$, there is an object $(S_n, f_n) \in \mathrm{Ob}(\mathscr{C})$ with an element $s_n \in f_n^{-1}([n, \infty))$. Then, S has no reflection along $F_{\mathscr{C}}$.*

Proof. For purposes of contradiction, assume that S has a reflection $((R, f), \eta)$ along $F_{\mathscr{C}}$. For each $n \in \mathbb{N}$, define $\phi_n \in \mathbf{Set}\left(S, F_{\mathscr{C}} S_n\right)$ by $\phi_n(s) := s_n$, a constant function. Then, there is a unique $\hat{\phi}_n \in \mathscr{C}\left((R, f), (S_n, f_n)\right)$ such that $F_{\mathscr{C}} \hat{\phi}_n \circ \eta = \phi_n$ for all $n \in \mathbb{N}$.

For each $s \in S$, let $r_s := \eta(s) \in R$ and observe that for each $n \in \mathbb{N}$,

$$f\left(r_s\right) \geq f_n\left(\hat{\phi}_n\left(r_s\right)\right) = f_n\left(\left(F_{\mathscr{C}} \hat{\phi}_n \circ \eta\right)(s)\right) = f_n\left(\phi_n(s)\right) = f_n\left(s_n\right) \geq n.$$

Hence, $f\left(r_s\right)$ cannot have finite value for any $s \in S$, which cannot occur in (R, f). As such, this reflection is fiction.

□

In the case of Proposition 1.1, all the S_n were the same nontrivial normed $\mathbb{F}$-vector space and the s_n multiples of a nonzero vector. Thus, the above proposition genuinely resolves to Proposition 1.1 when $\mathscr{C}$ is a nontrivial subcategory of $\mathbb{F}\mathbf{NVec}_1$.

However, this generalization allows the elements of increasing size to come from different objects in $\mathscr{C}$, which seems to sour any possibility of classical free objects in most categories of interest. For example, the any subcategory of $\mathbf{CSet}_1$ containing the singleton crutched sets $\{(0, n)\}$ for $n \in \mathbb{N}$ cannot have a reflection along the forgetful functor for any nonempty set S.

2.3. **Category of Crutched Sets & Bounded Maps.** Likewise, crutched sets and bounded functions between them can be studied, comparing this structure to $\mathbf{CSet}_1$. For notation, the symbol $\mathbf{CSet}_\infty$ will be used to denote the following data:

- $\mathrm{Ob}\left(\mathbf{CSet}_\infty\right) :=$ the class of all crutched sets;
- For each $(S, f), (T, g)$ in $\mathrm{Ob}(\mathbf{CSet}_\infty)$, define

$$\mathbf{CSet}_\infty((S, f), (T, g)) := \left\{ \phi \in \mathbf{Set}(S, T) : \begin{array}{c} \phi \text{ bounded from} \\ (S, f) \text{ to } (T, g) \end{array} \right\}.$$

Equipping this structure with function composition, the following results from Corollary 2.1.5 and Example 2.1.10.

Proposition 2.3.1. $\mathbf{CSet}_\infty$ *is a category.*

With this new structure defined, one considers some of its basic properties and constructions. At first glance, $\mathbf{CSet}_\infty$ is very similar to $\mathbf{CSet}_1$, and most of its constructions are identical. However, there are some notable distinctions between the two, reminiscent of the differences between considering Banach spaces with bounded linear maps and contractive linear maps.

To begin, consider the primary properties of bounded mappings and criteria for isomorphism.

Proposition 2.3.2. *Let* (S, f) *and* (T, g) *be crutched sets and* $\phi : (S, f) \to (T, g)$ *be bounded. Define* $K := T \setminus \phi(S)$, $h := g|_K$, *and*

$$\lambda := \inf\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\}.$$

(1) ϕ *is a monomorphism in* $\mathbf{CSet}_\infty$ *iff* ϕ *is one-to-one;*
(2) ϕ *is an epimorphism in* $\mathbf{CSet}_\infty$ *iff* ϕ *is onto;*
(3) ϕ *is a section in* $\mathbf{CSet}_\infty$ *iff* ϕ *is one-to-one,* $\lambda > 0$, *and there is a bounded function* $\alpha : (K, h) \to (S, f)$;
(4) ϕ *is a retraction in* $\mathbf{CSet}_\infty$ *iff there are* $(s_t)_{t\in T} \subseteq S$ *such that* $\phi(s_t) = t$ *for all* $t \in T$, $f(s_t) = 0$ *for all* $t \in g^{-1}(0)$ *and*
$$\sup\left(\left\{\frac{f(s_t)}{g(t)} : t \notin g^{-1}(0)\right\} \cup \{0\}\right) < \infty;$$
(5) ϕ *is an isomorphism in* $\mathbf{CSet}_\infty$ *iff* ϕ *is one-to-one, onto, and* $\lambda > 0$.

Proof. (1) Use bounded α and β in the proof of Proposition 2.2.2.
(2) Use bounded α and β in the proof of Proposition 2.2.2.
(3) ($\Rightarrow$) Assume that ϕ is a section in $\mathbf{CSet}_\infty$. By definition, there is a bounded $\psi : (T, g) \to (S, f)$ such that $\psi \circ \phi = id_S$. From basic function results, ϕ must be one-to-one. Letting $\alpha := \psi|_K$, $\alpha : (K, h) \to (S, f)$ is bounded as ψ was. If there is $s \notin f^{-1}(0)$, observe that
$$0 < f(s) = (f \circ id_S)(s) = (f \circ \psi \circ \phi)(s) \le \mathrm{crh}(\psi)(g \circ \phi)(s)$$
so $\mathrm{crh}(\psi) \neq 0$ and
$$\frac{1}{\mathrm{crh}(\psi)} \le \frac{(g \circ \phi)(s)}{f(s)}.$$
Hence, $\lambda \ge \dfrac{1}{\mathrm{crh}(\psi)} > 0$.
If $S = f^{-1}(0)$, then $\lambda = \infty$ by convention.

($\Leftarrow$) Assuming the conclusion, define $\psi : T \to S$ by

$$\psi(t) := \begin{cases} s, & t = \phi(s), \\ \alpha(t), & t \in K. \end{cases}$$

As ϕ is one-to-one, this is a well-defined function, and $\psi \circ \phi = id_S$ by design. To prove ψ bounded, note that for all $t \in K$,

$$f(\alpha(t)) \leq \text{crh}(\alpha)h(t) = \text{crh}(\alpha)g(t)$$

since α is bounded. If $t = \phi(s)$ for some $s \in S$, consider when $g(t) = 0$. If $f(s) \neq 0$, then $\lambda = 0$, contradicting the assumption. Thus, $f(s) = 0 \leq \text{crh}(\alpha)g(t)$.

If $S = f^{-1}(0)$, the proof is complete here as all $t = \phi(s)$ would satisfy $g(t) = 0$ by Proposition 2.1.8. If not, consider when $t = \phi(s)$ and $g(t) \neq 0$. By the converse of Proposition 2.1.8, $f(s) \neq 0$, meaning $\lambda \neq \infty$. Hence,

$$f(s) = \frac{f(s)}{(g \circ \phi)(s)} \cdot (g \circ \phi)(s) \leq \frac{1}{\lambda} g(t).$$

Therefore, for all $t \in T$,

$$f(\psi(t)) \leq \max\left\{\frac{1}{\lambda}, \text{crh}(\alpha)\right\} g(t),$$

meaning ψ is bounded.

(4) ($\Rightarrow$) Assume that ϕ is a retraction in $\mathbf{CSet}_\infty$. By definition, there is a bounded $\psi : (T, g) \to (S, f)$ such that $\phi \circ \psi = id_T$. For $t \in T$, let $s_t := \psi(t)$. Observe that $\phi(s_t) = t$. Also, by Proposition 2.1.8, $f(s_t) = 0$ for all $t \in g^{-1}(0)$ and

$$\sup\left(\left\{\frac{f(s_t)}{g(t)} : t \notin g^{-1}(0)\right\} \cup \{0\}\right) < \infty.$$

($\Leftarrow$) Assuming the result, define $\psi : T \to S$ by $\psi(t) := s_t$. Then, $\phi \circ \psi = id_T$ by design. Further, by Proposition 2.1.8, ψ is bounded.

(5) ($\Rightarrow$) Assume that ϕ is an isomorphism in $\mathbf{CSet}_\infty$. Then, there is a bounded $\psi : (T, g) \to (S, f)$ such that $\psi \circ \phi = id_S$ and $\phi \circ \psi = id_T$. Thus, ϕ is both a section and a retraction, in particular also an epimorphism. Hence, ϕ is one-to-one, onto, and $\lambda > 0$.

($\Leftarrow$) Assuming the result, ϕ is an epimorphism as it is onto. Further, $T \setminus \phi(S) = \emptyset$ by this fact, meaning ϕ is further a section. Hence, ϕ is an isomorphism.

□

Much like Proposition 2.2.2, each of the criteria in Item 5 are necessary. In particular, the infimum criterion is identical to the notion of "bounded below" for bounded linear maps, but like its "isometric" counterpart in Proposition 2.2.2, this fact alone does not imply monomorphism, let alone isomorphism. Examples 2.2.3 and 2.2.4 also demonstrate the necessity of the criteria in Item 5, but Example 2.1.11 demonstrates this bounded below idea in a less trivial way.

Next, equalizers for parallel arrows in $\mathbf{CSet}_\infty$ are computed precisely the same way they are in $\mathbf{CSet}_1$.

Proposition 2.3.3. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ bounded maps. Let*

$$K := \{s \in S : \alpha(s) = \beta(s)\},$$

$k := f|_K$, and $\iota : K \to S$ by $\iota(s) := s$. Then,

$$\mathrm{Eq}_{\mathbf{CSet}_\infty}(\alpha, \beta) \cong_{\mathbf{CSet}_\infty} (K, k).$$

Since the notions of equalizer in $\mathbf{CSet}_1$ and $\mathbf{CSet}_\infty$ determine the same object up to isomorphism in $\mathbf{CSet}_1$, the following definition seems very natural.

Definition. Given a crutched set (S, f), a *crutched subset* of (S, f) is a pair (K, k), where $K \subseteq S$ and $k = f|_K$.

Similarly, coequalizers for parallel arrows in $\mathbf{CSet}_\infty$ are also share the same structure as their $\mathbf{CSet}_1$ counterparts.

Proposition 2.3.4. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ bounded maps. Let*

$$P := \left\{(\alpha(s), \beta(s)) \in T^2 : s \in S\right\}$$

and $\sim_P$ be the equivalence relation on T generated by P. Define $Q := T/\sim_P$, $q([t]) := \inf\{g(\tau) : \tau \sim_P t\}$, and $\xi : T \to Q$ by $\xi(t) := [t]$. Then,

$$\mathrm{Coeq}_{\mathbf{CSet}_\infty}(\alpha, \beta) \cong_{\mathbf{CSet}_\infty} (Q, q).$$

Again, as the notions of coequalizer correspond between the two categories in question, the following definition appears sensical.

Definition. Given a crutched set (S, f) and an equivalence relation $\sim$ on S, the *crutched quotient set* of (S, f) by $\sim$ is (Q, q), where $Q := S/\sim$ and $q([t]) := \inf\{f(\tau) : \tau \sim t\}$.

Turning attention toward products, $\mathbf{CSet}_\infty$ begins to show more differences from $\mathbf{CSet}_1$. Computation of binary product in $\mathbf{CSet}_\infty$ is identical to its $\mathbf{CSet}_1$ counterpart.

Proposition 2.3.5. *Let (S_1, f_1) and (S_2, f_2) be crutched sets. Define*

$$P := S_1 \times S_2,$$

$f : P \to [0, \infty)$ by $f(s_1, s_1) := \max\{f_1(s_1), f_2(s_2)\}$, and $\pi_i : P \to S_i$ by $\pi_i(s_1, s_2) := s_i$ for $i = 1, 2$. Then,

$$(S_1, f_1) \prod^{\mathbf{CSet}_\infty} (S_2, f_2) \cong_{\mathbf{CSet}_\infty} (P, f).$$

As $\mathbf{CSet}_\infty$ has binary products and has a terminal object, namely $\{(0,0)\}$, it immediately has any finitary product by iteration of the binary product. However, $\mathbf{CSet}_\infty$ does not have arbitrary product objects. This is similar to the case of Banach spaces with bounded linear maps.

Example 2.3.6. For $n \in \mathbb{N}$, let $S_n := [0,\infty)$ and $f_n : S_n \to [0,\infty)$ by $f_n(\lambda) := \lambda$. Assume for purposes of contradiction that $(S_n, f_n)_{n\in\mathbb{N}}$ has a product (P, f) in $\mathbf{CSet}_\infty$. For $n \in \mathbb{N}$, define $\phi_n : S_1 \to S_n$ by $\phi_n(\lambda) := \lambda$, each a constrictive map with $\mathrm{crh}(\phi_n) = 1$. Then, there is a unique bounded function $\phi : (S_1, f_1) \to (P, f)$ such that $\phi_n = \pi_n \circ \phi$ for each $n \in \mathbb{N}$. By Proposition 2.1.5,

$$1 = \mathrm{crh}(\phi_n) \le \mathrm{crh}(\pi_n)\,\mathrm{crh}(\phi).$$

Thus, $\mathrm{crh}(\pi_n) \neq 0$.

Let $T := \{0\}$ and $g : T \to [0,\infty)$ by $g(0) := 1$. For $n \in \mathbb{N}$, define $\psi_n : T \to S_n$ by $\psi_n(0) := n\,\mathrm{crh}(\pi_n)$, each a bounded map with $\mathrm{crh}(\psi_n) = n\,\mathrm{crh}(\pi_n)$. Then, there is a unique bounded function $\psi : (T, g) \to (P, f)$ such that $\psi_n = \pi_n \circ \psi$ for all $n \in \mathbb{N}$. In this case,

$$n\,\mathrm{crh}(\pi_n) = \mathrm{crh}(\psi_n) \le \mathrm{crh}(\pi_n)\,\mathrm{crh}(\psi).$$

Hence, $n \le \mathrm{crh}(\psi)$ for all $n \in \mathbb{N}$, contradicting that ψ was bounded. Thus, $(S_n, f_n)_{n\in\mathbb{N}}$ cannot have a product in $\mathbf{CSet}_\infty$.

Therefore, as $\mathbf{CSet}_1$ and $\mathbf{Set}$ both have arbitrary products, $\mathbf{CSet}_\infty$ must be distinct from both.

Corollary 2.3.7. *$\mathbf{CSet}_\infty$ is not isomorphic to $\mathbf{Set}$ or $\mathbf{CSet}_1$ as categories.*

Similarly, $\mathbf{CSet}_\infty$ also has binary coproducts, computed just as in $\mathbf{CSet}_1$.

Proposition 2.3.8. *Let (S_1, f_1) and (S_2, f_2) be crutched sets. Define*

$$C := \{(i, s) \in \{1,2\} \times (S_1 \cup S_2) : s \in S_i\},$$

$f : C \to [0,\infty)$ by $f(i, s) := f_i(s)$, and $\rho_i : S_i \to C$ by $\rho_i(s) := (i, s)$ for $i = 1, 2$. Then,

$$(S_1, f_2) \coprod^{\mathbf{CSet}_\infty} (S_2, f_2) \cong_{\mathbf{CSet}_\infty} (C, f).$$

As $\mathbf{CSet}_\infty$ has binary coproducts and has an initial object, namely $\left(\emptyset, \mathbf{0}_{[0,\infty)}\right)$, it immediately has any finitary coproduct by iteration of the binary coproduct. However, just as with products, $\mathbf{CSet}_\infty$ does not have arbitrary coproduct objects.

Example 2.3.9. For $n \in \mathbb{N}$, define $S_n := \{0\}$ and $f_n : S_n \to [0,\infty)$ by $f_n(0) := 1$. Assume for purposes of contradiction that $(S_n, f_n)_{n\in\mathbb{N}}$ has a coproduct (C, f) in $\mathbf{CSet}_\infty$. For $n \in \mathbb{N}$, define $\phi_n : S_n \to S_1$ by $\phi_n(0) := 0$, each constrictive with $\mathrm{crh}(\phi_n) = 1$. Then, there is a unique bounded function $\phi : (C, f) \to (S_1, f_1)$ such that $\phi_n = \phi \circ \rho_n$ for each $n \in \mathbb{N}$. By Proposition 2.1.5,

$$1 = \mathrm{crh}(\phi_n) \le \mathrm{crh}(\phi)\,\mathrm{crh}(\rho_n).$$

Thus, $\operatorname{crh}(\rho_n) \neq 0$.

Let $T := \mathbb{N}$ and $g : T \to [0, \infty)$ by $g(n) := n \operatorname{crh}(\rho_n)$. Define $\psi_n : S_n \to T$ by $\psi_n(0) := n$, each a bounded map with $\operatorname{crh}(\psi_n) = n \operatorname{crh}(\rho_n)$. Then, there is a unique bounded function $\psi : (C, f) \to (T, g)$ such that $\psi_n = \psi \circ \rho_n$ for all $n \in \mathbb{N}$. In this case,

$$n \operatorname{crh}(\rho_n) = \operatorname{crh}(\psi_n) \leq \operatorname{crh}(\psi) \operatorname{crh}(\rho_n) .$$

Hence, $n \leq \operatorname{crh}(\psi)$ for all $n \in \mathbb{N}$, contradicting that ψ was bounded. Thus, $(S_n, f_n)_{n\in\mathbb{N}}$ cannot have a coproduct in $\mathbf{CSet}_\infty$.

Still, as $\mathbf{CSet}_\infty$ has all finitary products and equalizers, all finitary limit processes may be performed. Likewise, finitary colimit processes follow from finitary coproducts and coequalizers. In summary, these facts can be stated in the following way.

Corollary 2.3.10. *The category* $\mathbf{CSet}_\infty$ *is finitely complete and finitely cocomplete.*

To close the comparison between $\mathbf{CSet}_\infty$ and $\mathbf{CSet}_1$, the standard projective and injective objects can be completely characterized.

Proposition 2.3.11. *Let* (S, f) *be a crutched set.*

(1) (S, f) *is projective relative to all epimorphisms in* $\mathbf{CSet}_\infty$ *iff* $\operatorname{card}(S) < \aleph_0$ *and* $f(s) \neq 0$ *for all* $s \in S$.
(2) (S, f) *is injective relative to all monomorphisms in* $\mathbf{CSet}_\infty$ *iff* $S \neq \emptyset$ *and* $f = 0$.

Proof. (1) ($\Leftarrow$) If $\operatorname{card}(S) = 0$, use the same proof as Proposition 2.2.11, considering bounded maps.

Assume $0 < \operatorname{card}(S) < \aleph_0$. Let (T, g) and (U, h) be crutched sets and $\alpha : (U, h) \to (T, g)$ be an epimorphism. Given a bounded function $\phi : (S, f) \to (T, g)$, consider the diagram below in $\mathbf{CSet}_\infty$.

$$\begin{array}{ccc} (S,f) & & \\ {\scriptstyle\phi}\big\downarrow & & \\ (T,g) & \underset{\alpha}{\twoheadleftarrow} & (U,h) \end{array}$$

For each $s \in S$, choose $u_s \in \alpha^{-1}(\phi(s))$, which is nonempty as α is onto. Define $\hat{\phi} : S \to U$ by $\phi(s) := u_s$. Note that $\phi = \alpha \circ \hat{\phi}$. Further, as $f(s) \neq 0$ for all $s \in S$ and S is a finite set,

$$\sup \left(\left\{ \frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0) \right\} \cup \{0\} \right) < \infty$$

so ϕ is bounded by Proposition 2.1.8.

($\neg \Leftarrow \neg$) For purposes of contradiction, assume first that there is $s_0 \in S$ such that $f(s_0) = 0$ and that (S, f) is projective relative to all epimorphisms. Define $g, h : S \to [0, \infty)$ by $g(s) := 0$ and $h(s) := 1$.

Also, let $\phi, \alpha : S \to S$ by $\phi(s) := \alpha(s) := s$. Then, consider the following diagram in $\mathbf{CSet}_\infty$.

$$\begin{array}{ccc} (S,f) & & \\ \phi \downarrow & & \\ (S,g) & \overset{}{\underset{\alpha}{\twoheadleftarrow}} & (S,h) \end{array}$$

Since α is onto and (S,f) projective to epimorphisms, there must be a bounded $\hat{\phi} : (S,f) \to (S,h)$ such that $\phi = \alpha \circ \hat{\phi}$. Then, for each $s \in S$,

$$s = \phi(s) = \left(\alpha \circ \hat{\phi}\right)(s) = \hat{\phi}(s)$$

so

$$1 = \left(h \circ \hat{\phi}\right)(s_0) \leq \operatorname{crh}\left(\hat{\phi}\right) f(s_0) = 0,$$

which is nonsense.

Assume instead that $\operatorname{card}(S) \geq \aleph_0$, that f is strictly positive, and that (S,f) is projective relative to all epimorphisms. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements in S. Further, let $T := \mathbb{N}$, $g, h : T \to [0,\infty)$ by $g(n) := nf(s_n)$ and $h(n) := 0$, $\alpha : T \to T$ by $\alpha(n) := n$, and $\phi : S \to T$ by

$$\phi(s) := \begin{cases} n, & s = s_n, \\ 1, & s \neq s_n. \end{cases}$$

Observe that ϕ and α are both bounded. Consider the following diagram in $\mathbf{CSet}_\infty$.

$$\begin{array}{ccc} (S,f) & & \\ \phi \downarrow & & \\ (T,h) & \underset{\alpha}{\twoheadleftarrow} & (T,g) \end{array}$$

By assumption, there is a bounded function $\hat{\phi} : (S,f) \to (T,g)$ such that $\phi = \alpha \circ \hat{\phi}$. Then, for each $n \in \mathbb{N}$,

$$n = \phi(s_n) = \left(\alpha \circ \hat{\phi}\right)(s_n) = \hat{\phi}(s_n)$$

and

$$nf(s_n) = g(n) = g\left(\hat{\phi}(s_n)\right) \leq \operatorname{crh}\left(\hat{\phi}\right) f(s_n).$$

Therefore, $n \leq \operatorname{crh}\left(\hat{\phi}\right)$ for all $n \in \mathbb{N}$, contradicting that $\hat{\phi}$ was bounded.

(2) Use the same proof as Proposition 2.2.11, considering bounded maps.

□

Notice that the inclusion of more maps between objects increased the number of projective objects, from one unique object in $\mathbf{CSet}_1$ to a countable family of isomorphism classes in $\mathbf{CSet}_\infty$. However, the number of injective objects remained unchanged.

To conclude discussion of $\mathbf{CSet}_\infty$, this category can also extend the failure results of Propositions 1.1 and 2.2.13. As before, let $\mathscr{C}$ be a subcategory of $\mathbf{CSet}_\infty$. There is a natural forgetful map from $\mathrm{Ob}(\mathscr{C})$ to $\mathrm{Ob}(\mathbf{Set})$, where one strips away the crutch function. Similarly, given $(S,f),(T,g) \in \mathrm{Ob}(\mathscr{C})$ and $\phi \in \mathscr{C}((S,f),(T,g))$, $\phi \in \mathbf{Set}(S,T)$ by definition of $\mathbf{CSet}_\infty$. One can quickly check that these two associations define a functor $F_\mathscr{C} : \mathscr{C} \to \mathbf{Set}$, where one ignores all the numeric properties from $\mathscr{C}$.

Proposition 2.3.12. *Let S be an infinite set. Assume that there is an object $(T,g) \in \mathrm{Ob}(\mathscr{C})$ with elements $t_n \in g^{-1}([n,\infty))$ for all $n \in \mathbb{N}$. Then, S has no reflection along $F_\mathscr{C}$.*

Proof. For purposes of contradiction, assume that S has a reflection $((R,f),\eta)$ along $F_\mathscr{C}$. Define $\phi : S \to T$ by $\phi(s) := t_1$. Then, there is a unique bounded $\hat{\phi} : (R,f) \to (T,g)$ such that $F_\mathscr{C}\hat{\phi} \circ \eta = \phi$. For each $s \in S$, define $r_s := \eta(s)$ and observe that

$$1 \leq g(t_1) = g(\phi(s)) = g\left(\left(F_\mathscr{C}\hat{\phi} \circ \eta\right)(s)\right) = g\left(\hat{\phi}(r_s)\right) \leq \mathrm{crh}\left(\hat{\phi}\right) f(r_s).$$

Thus, $f(r_s) \neq 0$.

Let $(s_j)_{j=1}^\infty \subseteq S$ be distinct. For each $j \in \mathbb{N}$, choose $n_j \in \mathbb{N}$ such that $n_j \geq j \cdot f\left(r_{s_j}\right)$. Define $\psi : S \to T$ by

$$\psi(s) := \begin{cases} t_{n_j}, & s = s_j, \\ t_1, & s \neq s_j. \end{cases}$$

Then, there is a unique $\hat{\psi} : (R,f) \to (T,g)$ such that $F_\mathscr{C}\hat{\psi} \circ \eta = \psi$. By Proposition 2.1.8,

$$\begin{aligned} \mathrm{crh}\left(\hat{\psi}\right) &\geq \frac{g\left(\hat{\psi}\left(r_{s_j}\right)\right)}{f\left(r_{s_j}\right)} \\ &= \frac{g\left(\left(F_\mathscr{C}\hat{\psi} \circ \eta\right)(s_j)\right)}{f\left(r_{s_j}\right)} \\ &= \frac{g\left(\psi\left(s_j\right)\right)}{f\left(r_{s_j}\right)} \\ &= \frac{g\left(t_{n_j}\right)}{f\left(r_{s_j}\right)} \\ &\geq \frac{n_j}{f\left(r_{s_j}\right)} \\ &\geq \frac{j \cdot f\left(r_{s_j}\right)}{f\left(r_{s_j}\right)} \\ &= j \end{aligned}$$

for all $j \in \mathbb{N}$. Then, $\hat{\psi}$ is unbounded, a contradiction. □

The above proposition does not have quite the impact that Propositions 1.1 and 2.2.13 had due to the loss of the constrictive property. To illustrate this, consider the following examples.

Example 2.3.13. Consider the entire category $\mathbf{CSet}_\infty$. Given a finite set S, let $R := S$, $\eta := id_S$, and $f : R \to [0,\infty)$ by $f(r) := 1$ for all $r \in R$. Given a crutched set (T, g) and $\phi \in \mathbf{Set}(S,T)$, define $\hat{\phi} : R \to T$ by $\hat{\phi} := \phi$. Observe that $\hat{\phi}$ is trivially bounded by Proposition 2.1.8, and $F_{\mathbf{CSet}_\infty}\hat{\phi} \circ \eta = \phi$ in an apparent way. Further, if $\varphi : (R, f) \to (T, g)$ such that $F_{\mathbf{CSet}_\infty}\varphi \circ \eta = \phi$, observe that for all $s \in S$,

$$\phi(s) = \left(F_{\mathbf{CSet}_\infty}\varphi \circ \eta\right)(s) = \varphi(s).$$

Thus, $\varphi = \hat{\phi}$. Hence, $((R, f), \eta)$ is a reflection of S along $F_{\mathbf{CSet}_\infty}$.

Example 2.3.14. Consider the category of $\mathbb{F}$-Banach spaces with bounded linear maps, $\mathbb{F}\mathbf{Ban}_\infty$. Given a finite set S, let $R := \ell^1(S)$ with its usual norm and $\eta : S \to R$ by $\eta(s) := \delta_s$, the point mass at $s \in S$. Given another $\mathbb{F}$-Banach space X and a set map $\phi : S \to X$, define $\hat{\phi} : R \to X$ by

$$\hat{\phi}(x) := \sum_{s\in S} x_s\phi(s),$$

where $x = \sum_{s\in S} x_s\delta_s$ is the decomposition of x with respect to the linear basis $(\delta_s)_{s\in S}$. A quick check shows that $\hat{\phi}$ is an $\mathbb{F}$-linear transformation, and since $\ell^1(S)$ is finite-dimensional, $\hat{\phi}$ is automatically continuous. Further,

$$\left(F_{\mathbb{F}\mathbf{Ban}_\infty}\hat{\phi} \circ \eta\right)(s) = \hat{\phi}\left(\delta_s\right) = \phi(s)$$

so $F_{\mathbb{F}\mathbf{Ban}_\infty}\hat{\phi} \circ \eta = \phi$.

If $\varphi : R \to X$ such that $F_{\mathbb{F}\mathbf{Ban}_\infty}\varphi \circ \eta = \phi$, observe that for all $s \in S$,

$$\phi(s) = \left(F_{\mathbb{F}\mathbf{Ban}_\infty}\varphi \circ \eta\right)(s) = \varphi\left(\delta_s\right).$$

Hence, $\varphi\left(\delta_s\right) = \hat{\phi}\left(\delta_s\right)$ so by linearity, $\varphi = \hat{\phi}$. Therefore, (R, η) is a reflection along $F_{\mathbb{F}\mathbf{Ban}_\infty}$.

What Proposition 2.3.12 has done is forbidden classical free objects generated by countable or larger sets in nontrivial categories of normed structures with bounded maps, i.e., copies of the zero space $\mathbb{0}$. Classical free objects may still exist for finite generation sets as shown in the above two examples, but this would require more particular attention to the type of structure.

3. Scaled-Free Constructions

With an understanding of $\mathbf{CSet}_1$ and $\mathbf{CSet}_\infty$, attention turns to modifying algebraic free constructions for normed objects. As such, this construction will be familiar to anyone who has studied universal algebraic objects. In particular, this method should be thought of as a generalization of the constructions done in [2] and [9] with the viewpoint of [16]. The use of the crutch function is analogous to the "$\mathcal{X}$-norms" in [10], but the universal objects created here are proper normed structures, as opposed to a general topological ones.

The modified construction shown in the present work is not entirely new, previously done for C*-algebras and LMC*-algebras within Section 1.3 of [8]. However, this presentation of the material explicitly carried the universal maps of both free *-semigroup and free *-algebra constructions throughout each result. The present work aims to streamline and generalize the construction for normed objects, moving directly from the original crutched set to the constructed structure.

3.1. **Banach Spaces.** Fix $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and consider the category of $\mathbb{F}$-Banach spaces with bounded $\mathbb{F}$-linear maps, $\mathbb{F}\mathbf{Ban}_\infty$. Most of the constructions done here will be prototypical for those that follow so this case will be considered in detail.

As every $V \in \mathrm{Ob}\left(\mathbb{F}\mathbf{Ban}_\infty\right)$ is a set with a nonnegative function $f_V : V \to [0, \infty)$ by $f_V(v) := \|v\|_V$, there is a natural forgetful map to $\mathrm{Ob}\left(\mathbf{CSet}_\infty\right)$, where one regards V as a crutched set (V, f_V), ignoring all structure except the norm function. Similarly, given $V, W \in \mathrm{Ob}\left(\mathbb{F}\mathbf{Ban}_\infty\right)$ and $\phi \in \mathbb{F}\mathbf{Ban}_\infty(V, W)$, ϕ is firstly a function from V to W, and

$$\|\phi(v)\|_W \leq \|\phi\|_{\mathcal{B}(V,W)} \|v\|_V$$

for all $v \in V$ since ϕ is bounded. Hence, $\phi \in \mathbf{CSet}_\infty\left((V, f_V), (W, f_W)\right)$. One can quickly check that these two associations define a functor $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty} : \mathbb{F}\mathbf{Ban}_\infty \to \mathbf{CSet}_\infty$, where one ignores all data from $\mathbb{F}\mathbf{Ban}_\infty$ save the set and norm.

Now, fix a crutched set (S, f), thought of as a set of generators normed by their values under f. The objective is to build a reflection of (S, f) along $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$. To do so, define

$$V_S := \{v \in \mathbf{Set}(S, \mathbb{F}) : \mathrm{supp}(v) \text{ is finite}\},$$

all functions from S to $\mathbb{F}$ with finite support. Under point-wise addition and scalar multiplication, V_S is naturally an $\mathbb{F}$-vector space with $\mathbb{F}$-linear basis $(\delta_s)_{s \in S}$, the point masses. Let $\epsilon_S : S \to V_S$ by $\epsilon(s) := \delta_s$, the association of generators.

To continue the construction, one must norm V_S, which is where the numeric value of the crutch function arises. Observe that given any function $\phi : V_S \to T$ where (T, g) is a crutched set, $\phi \circ \epsilon_S : S \to T$. Thus, one can ask that this through-map be a bounded function from (S, f) to (T, g). In the case below, T will be a $\mathbb{F}$-Banach space regarded as a crutched set.

Lemma 3.1.1. *For each $v \in V_S$, define*

$$\mathscr{S}_v := \left\{ \frac{\|\phi(v)\|_W}{\operatorname{crh}(\phi \circ \epsilon_S)} : \begin{array}{l} W \in \operatorname{Ob}(\mathbb{F}\mathbf{Ban}_\infty), \\ \phi : V_S \to W \ \mathbb{F}\text{-linear}, \\ \operatorname{crh}(\phi \circ \epsilon_S) \neq 0, \infty \end{array} \right\} \cup \{0\}.$$

and $\rho_f : V_S \to [0, \infty)$ by $\rho_f(v) := \sup \mathscr{S}_v$. Then, ρ_f is a semi-norm on V_S.

Proof. Fix $v \in V_S$ and write it as $\sum_{j=1}^{n} \lambda_j \delta_{s_j}$.

Since $0 \in \mathscr{S}_v$, the main concern is the finiteness of its supremum. Given any $\phi : V_S \to W$ such that $\operatorname{crh}(\phi \circ \epsilon_S) \neq 0, \infty$, observe that

$$\begin{aligned} \frac{\|\phi(v)\|_W}{\operatorname{crh}(\phi \circ \epsilon_S)} &\leq \frac{1}{\operatorname{crh}(\phi \circ \epsilon_S)} \sum_{j=1}^{n} |\lambda_j| \left\|\phi\left(\delta_{s_j}\right)\right\|_W \\ &= \frac{1}{\operatorname{crh}(\phi \circ \epsilon_S)} \sum_{j=1}^{n} |\lambda_j| \left\|(\phi \circ \epsilon_S)(s_j)\right\|_W \\ &\leq \frac{1}{\operatorname{crh}(\phi \circ \epsilon_S)} \sum_{j=1}^{n} |\lambda_j| \operatorname{crh}(\phi \circ \epsilon_S) f(s_j) \\ &= \sum_{j=1}^{n} |\lambda_j| f(s_j), \end{aligned}$$

which is independent of W and ϕ. Thus, $\rho_f(v) < \infty$.

Now, for any $v, w \in V_S$ and $\lambda \in \mathbb{F}$, the following result, since $\|\cdot\|_W$ is an $\mathbb{F}$-Banach space norm and ϕ an $\mathbb{F}$-linear map.

$$\frac{\|\phi(v+w)\|_W}{\operatorname{crh}(\phi \circ \epsilon_S)} \leq \frac{\|\phi(v)\|_W}{\operatorname{crh}(\phi \circ \epsilon_S)} + \frac{\|\phi(w)\|_W}{\operatorname{crh}(\phi \circ \epsilon_S)} \leq \rho_f(v) + \rho_f(w),$$

$$\frac{\|\phi(\lambda v)\|_W}{\operatorname{crh}(\phi \circ \epsilon_S)} = |\lambda| \frac{\|\phi(v)\|_W}{\operatorname{crh}(\phi \circ \epsilon_S)}.$$

By taking suprema, ρ_f is a semi-norm on V_S.

□

To ensure that ρ_f is a norm, let $N_f := \{v \in V_S : \rho_f(v) = 0\}$, which is quickly seen to be an $\mathbb{F}$-subspace of V_S. Thus, V_S/N_f is a normed $\mathbb{F}$-vector space. Therefore, the completion, denoted $\mathcal{V}_{S,f}$, is an $\mathbb{F}$-Banach space. There is a canonical association $\eta_{S,f} : S \to \mathcal{V}_{S,f}$ by $\eta_{S,f}(s) := [\delta_s]$. This pair $(\mathcal{V}_{S,f}, \eta_{S,f})$ is a candidate for the reflection of S along $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$.

Theorem 3.1.2. *The pair $(\mathcal{V}_{S,f}, \eta_{S,f})$ is a reflection of (S, f) along $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$.*

Proof. For each $s \in S$, by the proof of Lemma 3.1.1,

$$\|\eta_{S,f}(s)\|_{\mathcal{V}_{S,f}} = \|[\delta_s]\|_{\mathcal{V}_{S,f}} \leq \rho_f\left(\delta_s\right) \leq f(s).$$

Thus, $\eta_{S,f}$ is constrictive from (S,f) to $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}\mathcal{V}_{S,f}$.

To check the universal property, let W be an $\mathbb{F}$-Banach space and $\phi \in \mathbf{CSet}_\infty\left((S,f), F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}W\right)$. Define $\hat{\phi} : V_S \to W$ by $\hat{\phi}\left(\delta_s\right) := \phi(s)$ and extend by $\mathbb{F}$-linearity, obtaining an $\mathbb{F}$-linear map.

Observe that for all $s \in S$,

$$\left(\hat{\phi} \circ \epsilon_S\right)(s) = \hat{\phi}(\delta_s) = \phi(s)$$

so $\operatorname{crh}\left(\hat{\phi} \circ \epsilon_S\right) = \operatorname{crh}(\phi) < \infty$ by assumption. If $\operatorname{crh}(\phi) \neq 0$,

$$\left\|\hat{\phi}(v)\right\|_W \leq \operatorname{crh}(\phi)\rho_f(v)$$

for all $v \in V$. If $\operatorname{crh}(\phi) = 0$, $\hat{\phi}$ is the immediately the zero map and will also satisfy the above inequality.

For all $v \in N_f$,

$$0 \leq \left\|\hat{\phi}(v)\right\|_W \leq \operatorname{crh}(\phi)\rho_f(v) = 0.$$

Thus, $N_f \subseteq \ker\left(\hat{\phi}\right)$ so there is an $\mathbb{F}$-linear map $\tilde{\phi} : V_S/N_f \to W$ by $\tilde{\phi}([v]) = \hat{\phi}(v)$. By the above inequality, $\left\|\tilde{\phi}([v])\right\|_W \leq \operatorname{crh}(\phi)\|[v]\|_{V_S/N_f}$ so $\tilde{\phi}$ is bounded and, therefore, continuous. Hence, $\hat{\phi}$ can be extended by continuity to $\varphi \in \mathbb{F}\mathbf{Ban}_\infty\left(\mathcal{V}_{S,f}, W\right)$. Observe that for each $s \in S$,

$$\left(F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}\varphi \circ \eta_{S,f}\right)(s) = F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}\varphi\left([\delta_s]\right) = \varphi\left([\delta_s]\right) = \tilde{\phi}\left([\delta_s]\right) = \hat{\phi}\left(\delta_s\right) = \phi(s).$$

Thus, $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}\varphi \circ \eta_{S,f} = \phi$.

Assume there was some other $\psi \in \mathbb{F}\mathbf{Ban}_\infty\left(\mathcal{V}_{S,f}, W\right)$ such that $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}\psi \circ \eta_{S,f} = \phi$. Then, for each $s \in S$,

$$\phi(s) = \left(F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}\psi \circ \eta_{S,f}\right)(s) = F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}\psi\left([\delta_s]\right) = \psi\left([\delta_s]\right)$$

Hence, $\psi = \varphi$ by $\mathbb{F}$-linearity and continuity.

□

Further, since (S,f) was arbitrary, the following functorial result is obtained.

Corollary 3.1.3. *There is a unique functor* $\mathbb{F}\,\mathrm{BanSp}_\infty : \mathbf{CSet}_\infty \to \mathbb{F}\mathbf{Ban}_\infty$ *such that* $\mathbb{F}\,\mathrm{BanSp}_\infty(S,f) = \mathcal{V}_{S,f}$, *and* $\mathbb{F}\,\mathrm{BanSp}_\infty \dashv F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$.

With this adjoint pair, consider its immediate properties. Interestingly, there is a tight relationship between the crutch bound of a bounded function and the norm of the extension.

Theorem 3.1.4 (Explicit Universal Property of $\mathbb{F}\operatorname{BanSp}_\infty \dashv F_{\mathbb{F}\mathbf{Ban}_\infty}^{\mathbf{CSet}_\infty}$)**.** *Let (S,f) be a crutched set and W be an $\mathbb{F}$-Banach space. For any bounded map $\phi:(S,f)\to F_{\mathbb{F}\mathbf{Ban}_\infty}^{\mathbf{CSet}_\infty}W$, there is a unique bounded $\mathbb{F}$-linear map $\hat{\phi}:\mathbb{F}\operatorname{BanSp}_\infty(S,f)\to W$ such that $\hat{\phi}\circ\eta_{S,f}=\phi$. Moreover,*

$$\operatorname{crh}(\phi)=\left\|\hat{\phi}\right\|_{\mathcal{B}(\mathbb{F}\operatorname{BanSp}_\infty(S,f),W)}.$$

Proof. All that remains to prove is the equality of the crutch bound on ϕ and the norm of $\hat{\phi}$. From the proof of Theorem 3.1.2,

$$\operatorname{crh}(\phi)\geq\left\|\hat{\phi}\right\|_{\mathcal{B}(\mathcal{V}_{S,f},W)}.$$

If $\operatorname{crh}(\phi)=0$, equality is immediate.

If $\operatorname{crh}(\phi)\neq 0$, then $S\neq f^{-1}(0)$. To prove equality in this case, the norms of the generators $[\delta_s]$ are first computed. From Theorem 3.1.2, $\|[\delta_s]\|_{\mathcal{V}_{S,f}}\leq f(s)$. Define $\psi:(S,f)\to\mathbb{F}$ by $\psi(s):=f(s)$, the crutch function itself. Observe that $\operatorname{crh}(\psi)=1$. By Theorem 3.1.2, there is a unique bounded $\mathbb{F}$-linear map $\hat{\psi}:\operatorname{BanSp}_\infty(S,f)\to\mathbb{F}$ such that $\hat{\psi}([\delta_s])=f(s)$. For all $s\in S$,

$$f(s)\leq\operatorname{crh}(\psi)\|[\delta_s]\|_{\mathcal{V}_{S,f}}\leq\|[\delta_s]\|_{\mathcal{V}_{S,f}},$$

forcing equality. Now, for all $s\notin f^{-1}(0)$,

$$\left\|\hat{\phi}\right\|_{\mathcal{B}(\mathcal{V}_{S,f},W)}\geq\frac{\left\|\hat{\phi}([\delta_s])\right\|_W}{\|[\delta_s]\|_{\mathcal{V}_{S,f}}}=\frac{\|\phi(s)\|_W}{f(s)},$$

giving

$$\operatorname{crh}(\phi)\leq\left\|\hat{\phi}\right\|_{\mathcal{B}(\mathcal{V}_{S,f},W)}.$$

□

This numeric condition actually shows that there is a second adjoint relationship here. Specifically, consider the category of $\mathbb{F}$-Banach spaces with contractive $\mathbb{F}$-linear maps, $\mathbb{F}\mathbf{Ban}_1$. Let $F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1}:\mathbb{F}\mathbf{Ban}_1\to\mathbf{CSet}_1$ be the restriction of $F_{\mathbb{F}\mathbf{Ban}_\infty}^{\mathbf{CSet}_\infty}$ to $\mathbb{F}\mathbf{Ban}_1$.

Then, Theorem 3.1.4 yields the following results immediately.

Corollary 3.1.5. *The pair $(\mathcal{V}_{S,f},\eta_{S,f})$ is a reflection of (S,f) along $F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1}$.*

Corollary 3.1.6. *There is a unique functor $\mathbb{F}\operatorname{BanSp}_1:\mathbf{CSet}_1\to\mathbb{F}\mathbf{Ban}_1$ such that $\mathbb{F}\operatorname{BanSp}_1(S,f)=\mathcal{V}_{S,f}$, and $\mathbb{F}\operatorname{BanSp}_1\dashv F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1}$.*

Corollary 3.1.7 (Explicit Universal Property of $\mathbb{F}\operatorname{BanSp}_1\dashv F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1}$)**.** *Let (S,f) be a crutched set and W be an $\mathbb{F}$-Banach space. For any constrictive map $\phi:(S,f)\to F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1}W$, there is a unique contractive $\mathbb{F}$-linear map $\hat{\phi}:\mathbb{F}\operatorname{BanSp}_1(S,f)\to W$ such that $\hat{\phi}\circ\eta_{S,f}=\phi$. Moreover,*

$$\operatorname{crh}(\phi)=\left\|\hat{\phi}\right\|_{\mathcal{B}(\mathbb{F}\operatorname{BanSp}_1(S,f),W)}.$$

By means of this constrictive version, two other useful incarnations of the universal property can be determined. In many applications, a crutch function may not be readily available or gleaned from context, but this is not a horrible impediment. For any particular $\mathbb{F}$-Banach space W, one can "steal" its norm to fabricate a crutch function.

Corollary 3.1.8 (Norm-Stealing Form). *Let S be a set and W be an $\mathbb{F}$-Banach space. For any function $\phi : S \to W$, define $f_\phi : S \to [0,\infty)$ by $f_\phi(s) := \|\phi(s)\|_W$. Then, there is a unique contractive $\mathbb{F}$-linear $\hat{\phi} : \mathbb{F}\,\mathrm{BanSp}_1\,(S, f_\phi) \to W$ such that $\hat{\phi}\,([\delta_s]) = \phi(s)$ for all $s \in S$.*

Yet, the next form of the universal property motivates the name of this paper.

Corollary 3.1.9 (Scaled-Free Mapping Property). *Let (S, f) be a crutched set and W be an $\mathbb{F}$-Banach space. For any function $\phi : S \to W$, there is a unique contractive $\mathbb{F}$-linear map $\hat{\phi} : \mathbb{F}\,\mathrm{BanSp}_1(S, f) \to W$ such that for all $s \in S$,*

$$\|\phi(s)\|_W \cdot \hat{\phi}\,([\delta_s]) = f(s) \cdot \phi(s).$$

Proof. Let $V := \mathbb{F}\,\mathrm{BanSp}_1(S, f)$ and define $\varphi : S \to W$ by

$$\varphi(s) := \begin{cases} \dfrac{f(s)}{\|\phi(s)\|_W}\phi(s), & \|\phi(s)\|_W \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that for all $s \in S$, $\|\varphi(s)\|_W \leq f(s)$, making φ constrictive from (S, f) to $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1} W$. By Theorem 3.1.4, there is a unique contractive $\mathbb{F}$-linear map $\hat{\phi} : V \to W$ such that for all $s \in S$,

$$\hat{\phi}\,([\delta_s]) = \varphi(s) = \begin{cases} \dfrac{f(s)}{\|\phi(s)\|_W}\phi(s), & \|\phi(s)\|_W \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For all $s \in S$ satisfying $\|\phi(s)\|_W \neq 0$, a multiplication yields the desired equality. In the case that $\|\phi(s)\|_W = 0$, then $\phi(s) = 0$ so

$$\|\phi(s)\|_W \cdot \hat{\phi}\,([\delta_s]) = 0 = f(s) \cdot \phi(s).$$

$\square$

This particular version of the universal property is termed the *scaled-free mapping property*, because an element $s \in S$ can be mapped anywhere in W, and $[\delta_s]$ is sent to a nonnegative scalar multiple of this location.

Further, in the next example, observe that given any $\mathbb{F}$-Banach space, one can find some crutched set (S, f) such that $\mathbb{F}\,\mathrm{BanSp}_1(S, f)$ maps surjectively and contractively onto it. This is one of the desired properties for the free object for most algebraic settings.

Example 3.1.10. Given an $\mathbb{F}$-Banach space W, let $S := W$, the underlying set of W, and $f : S \to [0, \infty)$ by $f(s) := \|s\|_W$. Define $\phi : S \to W$ by $\phi(s) := s$, the identity map. Trivially, ϕ is a constriction from (S, f) to $F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1} W$. By Theorem 3.1.4, there is a unique contractive $\mathbb{F}$-linear map $\hat{\phi} : \mathbb{F}\,\mathrm{BanSp}_1(S, f) \to W$ such that $\hat{\phi}([\delta_s]) = \phi(s)$ for all $s \in S$. Then, for all $w \in W$, $w = \hat{\phi}([\delta_w])$. Hence, $\hat{\phi}$ is surjective.

Example 3.1.11. Given an $\mathbb{F}$-Banach space W, let $S := \{s \in W : \|s\|_W = 1\}$, the hypersphere of W, and $f : S \to \{1\}$ be constant. Define $\phi : S \to W$ by $\phi(s) := s$, the inclusion map. Trivially, ϕ is a constriction from (S, f) to $F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1} W$. By Theorem 3.1.4, there is a unique contractive $\mathbb{F}$-linear map $\hat{\phi} : \mathbb{F}\,\mathrm{BanSp}_1(S, f) \to W$ such that $\hat{\phi}([\delta_s]) = \phi(s)$ for all $s \in S$. Then, for all $b \in W \setminus \{0\}$, $w = \|w\|_W \cdot \hat{\phi}\left(\left[\delta_{\frac{1}{\|w\|_W} w}\right]\right)$. Also, $\hat{\phi}(0) = 0$. Hence, $\hat{\phi}$ is surjective.

Moreover, $\mathbb{F}\,\mathrm{BanSp}_1(S, f)$ is a very familiar space.

Theorem 3.1.12. *Given a crutched set (S, f), let $T := S \setminus f^{-1}(0)$. Then,*

$$\mathbb{F}\,\mathrm{BanSp}_1(S, f) \cong_{\mathbb{F}\mathbf{Ban}_1} \ell^1_{\mathbb{F}}(T).$$

Proof. Let $X := \ell^1_{\mathbb{F}}(T)$ and $\theta : S \to X$ by

$$\theta(s) := \begin{cases} f(s)\delta_s, & s \in T, \\ 0, & s \notin T. \end{cases}$$

Showing that (X, θ) satisfies the universal property of Corollary 3.1.5 is sufficient for the isomorphism. To this end, let W be an $\mathbb{F}$-Banach space and $\phi : S \to F_{\mathbb{F}\mathbf{Ban}_1}^{\mathbf{CSet}_1} W$ be constrictive. Letting $Y := \mathrm{span}\{\delta_s : s \in T\}$ be the $\mathbb{F}$-linear span of the point masses δ_s in X, define $\hat{\phi} : Y \to W$ by $\hat{\phi}(\delta_s) := \dfrac{1}{f(s)}\phi(s)$ and extending by $\mathbb{F}$-linearity, obtaining an $\mathbb{F}$-linear map. However, notice that for any finite subset $E \subseteq T$ and scalars $(\lambda_s)_{s\in E}$,

$$\begin{aligned} \left\|\hat{\phi}\left(\sum_{s\in E}\lambda_s\delta_s\right)\right\|_W &\leq \sum_{s\in E}|\lambda_s|\left\|\hat{\phi}(\delta_s)\right\|_W \\ &= \sum_{s\in E}\frac{|\lambda_s|}{f(s)}\|\phi(s)\|_W \\ &\leq \sum_{s\in E}|\lambda_s| \\ &= \left\|\sum_{s\in E}\lambda_s\delta_s\right\|_X. \end{aligned}$$

Therefore, $\hat{\phi}$ is contractive and, therefore, continuous. Since Y is norm-dense in X, $\hat{\phi}$ can be extended to all of X. Also, θ is a constrictive map from (S, f) to $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1} X$ satisfying for all $s \in T$,

$$\left(F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\hat{\phi} \circ \theta\right)(s) = F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\hat{\phi}\left(f(s)\delta_s\right) = \hat{\phi}\left(f(s)\delta_s\right) = f(s)\hat{\phi}\left(\delta_s\right) = \phi(s).$$

For $s \notin T$, note that $0 \leq \|\phi(s)\|_W \leq f(s) = 0$. Thus,

$$\left(F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\hat{\phi} \circ \theta\right)(s) = F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\hat{\phi}\left(0\right) = \hat{\phi}\left(0\right) = 0 = \phi(s).$$

Therefore, $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\hat{\phi} \circ \theta = \phi$.

If $\psi : X \to W$ is a contractive $\mathbb{F}$-linear map satisfying $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\psi \circ \theta = \phi$, then for all $s \in T$,

$$\phi(s) = \left(F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\psi \circ \theta\right)(s) = F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}\psi\left(f(s)\delta_s\right) = \psi\left(f(s)\delta_s\right) = f(s)\psi\left(\delta_s\right)$$

so $\psi = \hat{\phi}$ by $\mathbb{F}$-linearity and continuity.

□

This characterization should be compared to the well-known unit ball functor. Explicitly, the functor $U_{\mathbb{F}} : \mathbb{F}\mathbf{Ban}_1 \to \mathbf{Set}$ by associating a Banach space with its closed unit ball and a contraction with its restriction to the unit ball. As shown in [1], every set S has a reflection along this functor, namely $\ell^1_{\mathbb{F}}(S)$.

However, with the functor $U_{\mathbb{F}}$, the norm has been hardcoded by the choice of the unit ball. That is, any element of S *must* be sent to an element of norm at most 1.

What the construction here has done is allowed the norms of generators to vary, encoding the numeric data in the crutch function rather than the choice of a subset. Indeed, the f in Theorem 3.1.4 is fixed prior to construction, but has no restriction otherwise. In particular, it need not be constant or bounded.

Also, the functors $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$ and $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Ban}_1}$ only remove structure, not altering the underlying set in any way. This aspect seems to give a more natural "forgetful" feel like the classical situation of algebraic free objects.

Theorem 3.1.12 states that the properties of the unit ball functor are recovered via this more general construction and are actually extended to the case of bounded $\mathbb{F}$-linear maps by Theorem 3.1.4. Arguably, one can choose to scale all generators to norm 1, but in some cases, it may be preferable to let individual generators have different crutched values.

3.2. **Banach Algebras.** Consider the category of $\mathbb{F}$-Banach algebras with bounded $\mathbb{F}$-algebra homomorphisms, $\mathbb{F}\mathbf{BanAlg}_\infty$. Let

$$F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{BanAlg}_\infty} : \mathbb{F}\mathbf{BanAlg}_\infty \to \mathbf{CSet}_\infty$$

be the restriction of $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$ to $\mathbb{F}\mathbf{BanAlg}_\infty$.

As in the previous section, one would like to build a reflection along this forgetful functor for any given crutched set. However, due to the introduction of multiplication, this is not possible except in trivial cases.

Proposition 3.2.1. *Let (S, f) be a crutched set such that $S \neq f^{-1}(0)$. Then, (S, f) has no reflection along $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{BanAlg}_\infty}$.*

Proof. For purposes of contradiction, assume that (S, f) has a reflection (R, η) along $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{BanAlg}_\infty}$. Let $r_s := \eta(s)$ for all $s \in S$.

Define $\phi : S \to \mathbb{F}$ by $\phi(s) := f(s)$, the crutch function itself. Then, $\operatorname{crh}(\phi) = 1$ so there is a unique bounded $\mathbb{F}$-algebra homomorphism $\hat{\phi} : R \to \mathbb{F}$ such that $\hat{\phi} \circ \eta = \phi$. For all $s \in S$,

$$f(s) = |\phi(s)| \leq \left\|\hat{\phi}\right\|_{\mathcal{B}(R,\mathbb{F})} \|r_s\|_R \leq \left\|\hat{\phi}\right\|_{\mathcal{B}(R,\mathbb{F})} \operatorname{crh}(\eta) f(s)$$

since $\hat{\phi}$ and η are bounded. For $s \notin f^{-1}(0)$, a division yields

$$1 \leq \left\|\hat{\phi}\right\|_{\mathcal{B}(R,\mathbb{F})} \operatorname{crh}(\eta),$$

forcing $\operatorname{crh}(\eta) \neq 0$.

Define $\psi : S \to \mathbb{F}$ by $\psi(s) := 2 \operatorname{crh}(\eta) f(s)$. Notice that $\operatorname{crh}(\psi) = 2 \operatorname{crh}(\eta)$ so there is a unique bounded $\mathbb{F}$-algebra homomorphism $\hat{\psi} : R \to \mathbb{F}$ such that $\hat{\psi} \circ \eta = \psi$. For $n \in \mathbb{N}$,

$$\hat{\psi}\left(r_s^n\right) = \hat{\psi}\left(r_s\right)^n = \psi\left(s\right)^n = 2^n \operatorname{crh}(\eta)^n f(s)^n$$

and

$$\left|\hat{\psi}\left(r_s^n\right)\right| \leq \left\|\hat{\psi}\right\|_{\mathcal{B}(R,\mathbb{F})} \|r_s^n\|_R \leq \left\|\hat{\psi}\right\|_{\mathcal{B}(R,\mathbb{F})} \|r_s\|_R^n \leq \left\|\hat{\psi}\right\|_{\mathcal{B}(R,\mathbb{F})} \operatorname{crh}(\eta)^n f(s)^n.$$

Combining these for $s \notin f^{-1}(0)$, a division yields

$$2^n \leq \left\|\hat{\psi}\right\|_{\mathcal{B}(R,\mathbb{F})},$$

contradicting that $\hat{\psi}$ was bounded.

□

This is initially discouraging, like Propositions 1.1, 2.2.13, and 2.3.12. However, observe that the cause of the failure here was the ability to send a generator to a value potentially larger than its crutch value. This, coupled with the multiplicative structure, forced the norm of the fictional universal map to grow without bound.

This behavior is disallowed in the constrictive case, where the scaled-free construction works perfectly well. To see this, consider the category of $\mathbb{F}$-Banach algebras with contractive $\mathbb{F}$-algebra homomorphisms, $\mathbb{F}\mathbf{BanAlg}_1$. Let $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{BanAlg}_1} : \mathbb{F}\mathbf{BanAlg}_1 \to \mathbf{CSet}_1$ be the restriction of $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$ to $\mathbb{F}\mathbf{BanAlg}_1$.

Now, fix a crutched set (S, f), thought of as a set of generators normed by their values under f. To build a reflection of (S, f) along $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{BanAlg}_1}$, let H_S be the set of all nonempty finite sequences of elements from S, thought of as non-commuting monomials. Under concatenation of lists, H_S is naturally a semigroup, the free semigroup on S.

Next, let B_S be the set of all functions from H_S to $\mathbb{F}$ whose support is finite, thought of as non-commuting polynomials with coefficients from $\mathbb{F}$. Under point-wise addition and scalar multiplication, B_S is naturally a $\mathbb{F}$-vector space with $\mathbb{F}$-linear basis $(\delta_l)_{l\in H_S}$. Vector multiplication is determined by the usual polynomial formula. At this point, B_S is the free $\mathbb{F}$-algebra on S.

To norm B_S, appropriate modifications are applied to Lemma 3.1.1, accounting now for the multiplicative structure.

Lemma 3.2.2. *For each $a \in B_S$, define*

$$\mathscr{T}_a := \left\{ \|\pi(a)\|_{\mathcal{B}} : \begin{array}{c} \mathcal{B} \in \mathrm{Ob}\left(\mathbb{F}\mathbf{BanAlg}_1\right), \\ \pi : B_S \to \mathcal{B} \text{ an } \mathbb{F}\text{-algebra homomorphism}, \\ \|\pi\left(\delta_s\right)\|_{\mathcal{B}} \leq f(s) \forall s \in S \end{array} \right\}.$$

and $\sigma_f : B_S \to [0, \infty)$ by $\sigma_f(a) := \sup \mathscr{T}_a$. Then, σ_f is a sub-multiplicative semi-norm on B_S.

Let $K_f := \{a \in B_S : \sigma_f(a) = 0\}$, a two-sided ideal of B_S. Thus, B_S/K_f is a normed $\mathbb{F}$-algebra, and the completion, $\mathcal{B}_{S,f}$, is an $\mathbb{F}$-Banach algebra. Let $\theta_{S,f} : S \to \mathcal{B}_{S,f}$ by $\theta_{S,f}(s) := [\delta_s]$. Proof like Theorem 3.1.2 yields the reflection result.

Theorem 3.2.3. *The pair $(\mathcal{B}_{S,f}, \theta_{S,f})$ is a reflection of (S, f) along $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{BanAlg}_1}$.*

As in the $\mathbb{F}\mathbf{Ban}_1$ case, several immediate results follow quickly from this key fact.

Corollary 3.2.4. *There is a unique functor $\mathbb{F}\,\mathrm{BanAlg} : \mathbf{CSet}_1 \to \mathbb{F}\mathbf{BanAlg}_1$ such that $\mathbb{F}\,\mathrm{BanAlg}(S, f) = \mathcal{B}_{S,f}$, and $\mathbb{F}\,\mathrm{BanAlg} \dashv F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{BanAlg}_1}$.*

Corollary 3.2.5 (Explicit Universal Property of $\mathbb{F}\,\mathrm{BanAlg} \dashv F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{BanAlg}_1}$)**.** *Let (S, f) be a crutched set and $\mathcal{A}$ be an $\mathbb{F}$-Banach algebra. For any constrictive map $\phi : (S, f) \to F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{BanAlg}_1}\mathcal{A}$, there is a unique contractive $\mathbb{F}$-algebra homomorphism $\hat{\phi} : \mathbb{F}\,\mathrm{BanAlg}(S, f) \to \mathcal{A}$ such that $\hat{\phi} \circ \theta_{S,f} = \phi$.*

Corollary 3.2.6 (Scaled-Free Mapping Property)**.** *Let (S, f) be a crutched set and $\mathcal{A}$ be an $\mathbb{F}$-Banach algebra. For any function $\phi : S \to \mathcal{A}$, there is a unique contractive $\mathbb{F}$-algebra homomorphism $\hat{\phi} : \mathbb{F}\,\mathrm{BanAlg}(S, f) \to \mathcal{A}$ such that for all $s \in S$,*

$$\|\phi(s)\|_{\mathcal{A}} \cdot \hat{\phi}\left([\delta_s]\right) = f(s) \cdot \phi(s).$$

Mimicking Example 3.1.10, for any $\mathbb{F}$-Banach algebra, there is a crutched set (S, f) such that $\mathbb{F}\,\mathrm{BanAlg}(S, f)$ maps surjectively and contractively onto it.

Toward characterization, consider a singleton crutched set $\{(x, \lambda)\}$ for some $\lambda \geq 0$. While λ has some influence on the structure of the resulting algebra, this influence is relatively minor.

Proposition 3.2.7. *Given $\lambda \geq 0$,*

$$\mathbb{F}\,\mathrm{BanAlg}(\{(x,\lambda)\}) \cong_{\mathbb{F}\mathbf{BanAlg}_1} \begin{cases} \mathbb{O}, & \lambda = 0, \\ \mathbb{F}\,\mathrm{BanAlg}(\{(x,1)\}), & \lambda > 0. \end{cases}$$

Proof. For $\lambda = 0$, observe that $\sigma_f(a) = 0$ for all $a \in B_{\{x\}}$, and the result follows immediately.

For $\lambda > 0$, let $\mathcal{A} := \mathbb{F}\,\mathrm{BanAlg}(\{(x,\lambda)\})$ and $\mathcal{B} := \mathbb{F}\,\mathrm{BanAlg}(\{(x,1)\})$. Define $\phi : \{x\} \to \mathcal{B}$ by $\phi(x) := \lambda\,[\delta_x]_{\mathcal{B}}$, a constrictive map. By Theorem 3.2.5, there is a unique contractive $\mathbb{F}$-algebra homomorphism $\hat{\phi} : \mathcal{A} \to \mathcal{B}$ such that $\hat{\phi}\left([\delta_x]_{\mathcal{A}}\right) = \phi(x)$.

Similarly, define $\varphi : \{x\} \to \mathcal{A}$ by $\varphi(x) := \frac{1}{\lambda}\,[\delta_x]_{\mathcal{A}}$, also a constrictive map. By Theorem 3.2.5, there is a unique contractive $\mathbb{F}$-algebra homomorphism $\hat{\varphi} : \mathcal{B} \to \mathcal{A}$ such that $\hat{\varphi}\left([\delta_x]_{\mathcal{B}}\right) = \varphi(x)$.

Note that

$$\left(\hat{\phi} \circ \hat{\varphi}\right)\left([\delta_x]_{\mathcal{B}}\right) = \frac{1}{\lambda}\hat{\phi}\left([\delta_x]_{\mathcal{A}}\right) = \frac{1}{\lambda}\lambda\,[\delta_x]_{\mathcal{B}} = [\delta_x]_{\mathcal{B}}\,.$$

By Theorem 3.2.5, $\hat{\phi} \circ \hat{\varphi} = id_{\mathcal{B}}$. Symmetrically,

$$\left(\hat{\varphi} \circ \hat{\phi}\right)\left([\delta_x]_{\mathcal{A}}\right) = \lambda\hat{\varphi}\left([\delta_x]_{\mathcal{B}}\right) = \lambda\frac{1}{\lambda}\,[\delta_x]_{\mathcal{A}} = [\delta_x]_{\mathcal{A}}\,.$$

By Theorem 3.2.5, $\hat{\varphi} \circ \hat{\phi} = id_{\mathcal{A}}$.

□

Using Proposition 2.2.9 and Corollary 3.2.4, the following canonical form may be taken.

Corollary 3.2.8. *Given a crutched set (S,f),*

$$\mathbb{F}\,\mathrm{BanAlg}(S,f) \cong_{\mathbb{F}\mathbf{BanAlg}_1} \coprod_{s \not\in f^{-1}(0)}^{\mathbb{F}\mathbf{BanAlg}_1} \mathbb{F}\,\mathrm{BanAlg}(\{(0,1)\}).$$

Notice that this coproduct is the free product of the individual factors, equipped with and completed relative to a universal norm like the case of C*-algebras in [21]. While this is a highly abstract characterization, it does mirror the algebraic case, where the free algebra on a set can be seen as the free product of the polynomial algebra in one variable with itself.

3.3. **C*-algebras.** Consider the category of C*-algebras with *- homomorphisms, $\mathbf{C}^*$. Let $F_{\mathbf{C}^*}^{\mathbf{CSet}_1} : \mathbf{C}^* \to \mathbf{CSet}_1$ be the restriction of $F_{\mathbb{C}\mathbf{Ban}_\infty}^{\mathbf{CSet}_\infty}$ to $\mathbf{C}^*$.

Fixing a crutched set (S,f), the construction of a reflection along $F_{\mathbf{C}^*}^{\mathbf{CSet}_1}$ is nearly identical to the case of $\mathbb{C}\mathbf{BanAlg}_1$, though one now must take into consideration the adjoint operation. This version of the construction was done previously in Section 1.3 of [8]. However, this presentation of the material explicitly carried the universal maps of both free *-semigroup and free *-algebra constructions throughout each result. The present work aims to streamline the construction, moving directly from the original crutched set to the constructed algebra.

Let $\hat{S} := S \uplus S := \{0,1\} \times S$, the disjoint union of S with itself. The original set S is identified with $\{0\} \times S$ while elements of $\{1\} \times S$ are denoted s^*, formal adjoints of elements in S. Let $\hat{H}_S$ be the set of all nonempty finite sequences of elements from $\hat{S}$ under concatenation of lists. This structure has a natural involution by reversing order and swapping presence/absence of the *. Hence, $\hat{H}_S$ is a *-semigroup, the free *-semigroup on S.

Let $\hat{B}_S$ be the set of all functions from $\hat{H}_S$ to $\mathbb{C}$ whose support is finite under point-wise addition, scalar multiplication, and polynomial multiplication. The involution operation is done on each function by applying the conjugate to the output and the involution of $\hat{H}_S$ to the input. At this point, $\hat{B}_S$ is the free *-algebra over $\mathbb{C}$ on S.

Applying appropriate changes to Lemma 3.2.2 yields a viable semi-norm for $\hat{B}_S$.

Lemma 3.3.1. *For each $a \in \hat{B}_S$, define*

$$\mathscr{U}_a := \left\{ \|\pi(a)\|_{\mathcal{B}} : \begin{array}{l} \mathcal{B} \in \mathrm{Ob}\left(\mathbf{C}^*\right), \\ \pi : \hat{B}_S \to \mathcal{B} \text{ a *-homomorphism}, \\ \left\|\pi\left(\delta_s\right)\right\|_{\mathcal{B}} \leq f(s) \forall s \in S \end{array} \right\}.$$

and $\hat{\sigma}_f : \hat{B}_S \to [0,\infty)$ by $\hat{\sigma}_f(a) := \sup \mathscr{U}_a$. Then, $\hat{\sigma}_f$ is a sub-multiplicative semi-norm on $\hat{B}_S$ satisfying the C-condition.*

Let $\hat{K}_f := \left\{ a \in \hat{B}_S : \hat{\sigma}_f(a) = 0 \right\}$, a two-sided *-ideal of $\hat{B}_S$. Thus, $\hat{B}_S / \hat{K}_f$ is a *-algebra over $\mathbb{C}$ with a C*-norm, and the completion, $\hat{\mathcal{B}}_{S,f}$, is a C*-algebra. Let $\hat{\theta}_{S,f} : S \to \hat{\mathcal{B}}_{S,f}$ by $\hat{\theta}_{S,f}(s) := [\delta_s]$. Similar proof methods yield the reflection result and its consequences.

Theorem 3.3.2. *The pair $\left(\hat{\mathcal{B}}_{S,f}, \hat{\theta}_{S,f}\right)$ is a reflection of (S,f) along $F^{\mathbf{CSet}_1}_{\mathbf{C}^*}$.*

Corollary 3.3.3. *There is a unique functor $\mathrm{C}^*\mathrm{Alg} : \mathbf{CSet}_1 \to \mathbf{C}^*$ such that $\mathrm{C}^*\mathrm{Alg}(S,f) = \hat{\mathcal{B}}_{S,f}$, and $\mathrm{C}^*\mathrm{Alg} \dashv F^{\mathbf{CSet}_1}_{\mathbf{C}^*}$.*

Corollary 3.3.4 (Explicit Universal Property of $\mathrm{C}^*\mathrm{Alg} \dashv F^{\mathbf{CSet}_1}_{\mathbf{C}^*}$)**.** *Let (S,f) be a crutched set and $\mathcal{A}$ be a C*-algebra. For any constrictive map $\phi : (S,f) \to F^{\mathbf{CSet}_1}_{\mathbf{C}^*}\mathcal{A}$, there is a unique *-homomorphism $\hat{\phi} : \mathrm{C}^*\mathrm{Alg}(S,f) \to \mathcal{A}$ such that $\hat{\phi} \circ \hat{\theta}_{S,f} = \phi$.*

Corollary 3.3.5 (Scaled-Free Mapping Property)**.** *Let (S,f) be a crutched set and $\mathcal{A}$ be a C*-algebra. For any function $\phi : S \to \mathcal{A}$, there is a unique *-homomorphism $\hat{\phi} : \mathrm{C}^*\mathrm{Alg}(S,f) \to \mathcal{A}$ such that for all $s \in S$,*

$$\|\phi(s)\|_{\mathcal{A}} \cdot \hat{\phi}\left([\delta_s]\right) = f(s) \cdot \phi(s).$$

Proposition 3.3.6. *Given* $\lambda \geq 0$,

$$\mathrm{C}^*\mathrm{Alg}(\{(x,\lambda)\}) \cong_{\mathbf{C}^*} \begin{cases} \mathbb{O}, & \lambda = 0, \\ \mathrm{C}^*\mathrm{Alg}(\{(x,1)\}), & \lambda > 0. \end{cases}$$

Corollary 3.3.7. *Given a crutched set* (S,f),

$$\mathrm{C}^*\mathrm{Alg}(S,f) \cong_{\mathbf{C}^*} \coprod_{s \notin f^{-1}(0)}^{\mathbf{C}^*} \mathrm{C}^*\mathrm{Alg}(\{(0,1)\}).$$

Here, the coproduct is again the C*-algebra free product of the C*-algebras involved. As in Example 3.1.10, for any C*-algebra, there is a crutched set (S,f) such that $\mathrm{C}^*\mathrm{Alg}(S,f)$ maps surjectively onto it.

A similar construction works in the category of unital C*-algebras and unital *-homomorphisms, $\mathbf{1C}^*$. Little of the construction above actually changes so only the main highlights will be described. Let $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1} : \mathbf{1C}^* \to \mathbf{CSet}_1$ be the restriction of $F_{\mathbb{C}\mathbf{Ban}_\infty}^{\mathbf{CSet}_\infty}$ to $\mathbf{1C}^*$.

Fix a crutched set (S,f), and let M_S free *-monoid on S, $\hat{H}_S$ with the empty list u included. Further, let A_S be all functions from M_S to $\mathbb{C}$ with finite support, equipped with the extensions of the operations on $\hat{B}_S$.

Lemma 3.3.8. *For each* $a \in A_S$, *define*

$$\mathscr{V}_a := \left\{ \|\pi(a)\|_{\mathcal{B}} : \begin{array}{l} \mathcal{B} \in \mathrm{Ob}\left(\mathbf{1C}^*\right), \\ \pi : A_S \to \mathcal{B} \text{ a unital *-homomorphism}, \\ \left\|\pi\left(\delta_s\right)\right\|_{\mathcal{B}} \leq f(s) \forall s \in S \end{array} \right\}.$$

and $\tau_f : A_S \to [0,\infty)$ *by* $\tau_f(a) := \sup \mathscr{V}_a$. *Then,* τ_f *is a sub-multiplicative semi-norm on* A_S *satisfying the C*-condition. Further,* $\tau_f(\delta_u) = 1$.

Let $J_f := \{a \in A_S : \tau_f(a) = 0\}$, a two-sided *-ideal of A_S. Thus, A_S/J_f is a unital *-algebra over $\mathbb{C}$ with a C*-norm, and the completion, $\mathcal{A}_{S,f}$, is a unital C*-algebra. Let $\iota_{S,f} : S \to \mathcal{A}_{S,f}$ by $\iota_{S,f}(s) := [\delta_s]$.

Theorem 3.3.9. *The pair* $(\mathcal{A}_{S,f}, \iota_{S,f})$ *is a reflection of* (S,f) *along* $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}$.

Corollary 3.3.10. *There is a unique functor* $1\mathrm{C}^*\mathrm{Alg} : \mathbf{CSet}_1 \to \mathbf{1C}^*$ *such that* $1\mathrm{C}^*\mathrm{Alg}(S,f) = \mathcal{A}_{S,f}$, *and* $1\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}$.

Corollary 3.3.11 (Explicit Universal Property of $1\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}$)**.** *Let* (S,f) *be a crutched set and* $\mathcal{B}$ *be a unital C*-algebra. For any constrictive map* $\phi : (S,f) \to F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\mathcal{B}$, *there is a unique unital *-homomorphism* $\hat{\phi} : 1\mathrm{C}^*\mathrm{Alg}(S,f) \to \mathcal{B}$ *such that* $\hat{\phi} \circ \iota_{S,f} = \phi$.

Corollary 3.3.12 (Scaled-Free Mapping Property)**.** *Let* (S,f) *be a crutched set and* $\mathcal{B}$ *be a unital C*-algebra. For any function* $\phi : S \to \mathcal{B}$, *there is a unique unital *-homomorphism* $\hat{\phi} : 1\mathrm{C}^*\mathrm{Alg}(S,f) \to \mathcal{B}$ *such that for all* $s \in S$,

$$\|\phi(s)\|_{\mathcal{B}} \cdot \hat{\phi}\left([\delta_s]\right) = f(s) \cdot \phi(s).$$

Proposition 3.3.13. *Given* $\lambda \geq 0$*,*

$$1\mathrm{C}^*\mathrm{Alg}(\{(x,\lambda)\}) \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda = 0, \\ 1\mathrm{C}^*\mathrm{Alg}(\{(x,1)\}), & \lambda > 0. \end{cases}$$

Corollary 3.3.14. *Given a crutched set* (S, f)*,*

$$1\mathrm{C}^*\mathrm{Alg}(S,f) \cong_{\mathbf{1C^*}} \coprod_{s \notin f^{-1}(0)}^{\mathbf{1C^*}} 1\mathrm{C}^*\mathrm{Alg}(\{(0,1)\}).$$

Here, the coproduct is again the C*-algebra free product of the C*-algebras involved, amalgamated along their identities. As in Example 3.1.10, for any unital C*-algebra, there is a crutched set (S, f) such that $1\mathrm{C}^*\mathrm{Alg}(S, f)$ maps surjectively onto it.

If one follows the unital construction the constant function 1, it is precisely the construction of the universal C*-algebra on a set S of contractions, studied in depth within [9] as a *-monoid algebra. In fact, this paper actually terms this algebra the "free C*-algebra" on S. However, the statement is qualified that the algebra is "free" precisely in the sense of Theorem 3.3.11. Also, the scaled-free mapping property of Corollary 3.3.12 substantiates the statement that this algebra is "the closest one gets to free C*-algebras" in [6].

Similarly, [16] and [17] create this same algebra by considering the unit ball functor, as with $\mathbb{F}\mathbf{Ban}_1$. Similarly, the norm has been hardcoded by the choice of the unit ball, forcing any element of S to have norm at most 1. In the construction of the present work, the norms of generators are allowed to vary, and the properties of the unit ball functor are recovered. Further, the functors $F_{\mathbf{C^*}}^{\mathbf{CSet}_1}$ and $F_{\mathbf{1C^*}}^{\mathbf{CSet}_1}$ only remove structure, leaving the underlying set unchanged.

As stated before, Section 1.3 of [8] forms the non-unital algebra of contractions in the way shown above. Section 4.1.2 of [8] holds a comparable analysis of the structure of this object. However, while the initial formulation in Section 1.1 of [8] mentions the forgetful functor and the adjoint situation, the categorical properties of the left adjoint are not used in the work.

In the present work, the properties of the left adjoint have produced the decompositions of Corollaries 3.3.7 and 3.3.14. This functorial relationship is very strong and will be used heavily in future constructions in these categories.

3.4. **Operator Spaces.** Consider the category of operator spaces with completely bounded maps, $\mathbf{O}_\infty$. Here, the objects of $\mathbf{O}_\infty$ will be regarded as *abstract* operator spaces, each a $\mathbb{C}$-vector space equipped with a matrix-norm satisfying Ruan's axioms in [5, p. 20] and complete on the first matrix level. Since there are several norms in play, a choice is made to create the forgetful functor used in the construction. Other choices may yield distinct constructions which may be interesting in their own right.

For $V \in \mathrm{Ob}(\mathbf{O}_\infty)$, V is a set with a nonnegative function $f_V : V \to [0,\infty)$ by $f_V(v) := \|v\|_V$, norm on V itself. Thus, there is a natural forgetful map to $\mathrm{Ob}(\mathbf{CSet}_\infty)$, where one regards V as a crutched set (V, f_V), ignoring all structure except the first norm function. Similarly, given $V, W \in \mathrm{Ob}(\mathbf{O}_\infty)$ and $\phi \in \mathbf{O}_\infty(V,W)$, ϕ is firstly a function from V to W, and

$$\|\phi(v)\|_W \leq \|\phi\|_{\mathcal{CB}(V,W)} \|v\|_V$$

for all $v \in V$ since ϕ is completely bounded. Hence,

$$\phi \in \mathbf{CSet}_\infty\left((V, f_V), (W, f_W)\right).$$

One can quickly check that these two associations define a functor $F_{\mathbf{O}_\infty}^{\mathbf{CSet}_\infty} : \mathbf{O}_\infty \to \mathbf{CSet}_\infty$, where one ignores all data from $\mathbf{O}_\infty$ save the set and the first norm.

Since each operator space is also a Banach space, the construction of a reflection along the above functor will be done by appealing to the construction already established for $\mathbb{C}\mathbf{Ban}_\infty$. Fixing a crutched set (S, f), let $\mathcal{V}_{S,f}$ and $\eta_{S,f} : S \to \mathcal{V}_{S,f}$ be the reflection and inclusion from Theorem 3.1.2. Already, this is a $\mathbb{C}$-Banach space equipped with a constrictive map from (S, f). Now, only the norms on the higher matrix levels must be determined. This is accomplished similarly to Lemma 3.1.1, again using the existing construction for $\mathbb{C}\mathbf{Ban}_1$.

Lemma 3.4.1. *For each $n \in \mathbb{N}$ and $A \in M_n(\mathcal{V}_{S,f})$, define*

$$\mathscr{W}_{A,n} := \left\{ \frac{\left\|\phi^{(n)}(A)\right\|_{M_n(W)}}{\|\phi\|_{\mathcal{B}(\mathcal{V}_{S,f},W)}} : \begin{array}{l} W \in \mathrm{Ob}(\mathbf{O}_\infty), \\ \phi \in \mathcal{B}(\mathcal{V}_{S,f}, W), \\ \|\phi\|_{\mathcal{B}(\mathcal{V}_{S,f},W)} \neq 0 \end{array} \right\} \cup \{0\}.$$

and $\nu_{f,n} : M_n(\mathcal{V}_{S,f}) \to [0,\infty)$ by $\nu_{f,n}(A) := \sup \mathscr{W}_{A,n}$. Then, $(\nu_{f,n})_{n\in\mathbb{N}}$ is a matrix norm on $\mathcal{V}_{S,f}$. Moreover, $\nu_{f,1}(v) = \|v\|_{\mathcal{V}_{S,f}}$ for all $v \in \mathcal{V}_{S,f}$.

Proof. Fix $n \in \mathbb{N}$ and $A \in M_n(\mathcal{V}_{S,f})$, written as

$$A = \sum_{j,k=1}^{n} a_{i,j} \otimes E_{i,j},$$

where $(E_{i,j})_{i,j=1}^n$ is the usual basis for M_n.

To show $\nu_{f,n}(A)$ finite, let W be an operator space and $\phi : \mathcal{V}_{S,f} \to W$ a bounded $\mathbb{C}$-linear map. Ruan's axioms yield the following computation.

$$\begin{aligned}
\left\|\phi^{(n)}(A)\right\|_{M_n(W)} &\leq \sum_{j,k=1}^{n} \left\|\phi^{(n)}\left(a_{i,j} \otimes E_{i,j}\right)\right\|_{M_n(W)} \\
&= \sum_{j,k=1}^{n} \left\|E_{j,1}\phi^{(n)}\left(a_{i,j} \otimes E_{1,1}\right)E_{1,k}\right\|_{M_n(W)} \\
&\leq \sum_{j,k=1}^{n} \left\|\phi^{(n)}\left(a_{i,j} \otimes E_{1,1}\right)\right\|_{M_n(W)} \\
&= \sum_{j,k=1}^{n} \left\|\phi\left(a_{i,j}\right)\right\|_{W} \\
&\leq \sum_{j,k=1}^{n} \|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)} \|a_{i,j}\|_{\mathcal{V}_{S,f}}
\end{aligned}$$

Therefore,

$$\nu_{f,n}(A) \leq \sum_{j,k=1}^{n} \|a_{i,j}\|_{\mathcal{V}_{S,f}} < \infty.$$

Following the arguments of Lemma 3.1.1, $\nu_{f,n}$ is a semi-norm on $M_n\left(\mathcal{V}_{S,f}\right)$. From the computation above, $\nu_{f,1}(v) \leq \|v\|_{\mathcal{V}_{S,f}}$. Equality comes from considering the standard isometric embedding of $\mathcal{V}_{S,f}$ into continuous functions on the unit ball of its continuous dual space, considered with its canonical operator space structure. This also ensures that $\nu_{f,n}$ is positive definite for all $n \in \mathbb{N}$.

To show satisfaction of Ruan's axioms, let $n, m \in \mathbb{N}$, $A \in M_n\left(\mathcal{V}_{S,f}\right)$, $B \in M_m\left(\mathcal{V}_{S,f}\right)$, and $\alpha, \beta^{\mathrm{T}} \in M_{m,n}$. Observe the following results due to the assumptions on W and ϕ.

$$\begin{aligned}
\frac{\left\|\phi^{(m)}(\alpha A \beta)\right\|_{M_m(W)}}{\|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)}} &= \frac{\left\|\alpha\phi^{(n)}(A)\beta\right\|_{M_m(W)}}{\|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)}} \\
&\leq \|\alpha\|_{M_{m,n}} \frac{\left\|\phi^{(n)}(A)\right\|_{M_n(W)}}{\|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)}} \|\beta\|_{M_{n,m}} \\
&\leq \|\alpha\|_{M_{m,n}} \nu_{f,n}(A) \|\beta\|_{M_{n,m}},
\end{aligned}$$

$$\begin{aligned}
\frac{\left\|\phi^{(n)}(A \oplus B)\right\|_{M_{n+m}(W)}}{\|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)}} &= \frac{\left\|\phi^{(n)}(A) \oplus \phi^{(m)}(B)\right\|_{M_{n+m}(W)}}{\|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)}} \\
&= \max\left\{\frac{\left\|\phi^{(n)}(A)\right\|_{M_n(W)}}{\|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)}}, \frac{\left\|\phi^{(m)}(B)\right\|_{M_m(W)}}{\|\phi\|_{\mathcal{B}\left(\mathcal{V}_{S,f},W\right)}}\right\}
\end{aligned}$$

By taking suprema, $(\nu_{f,n})_{n\in\mathbb{N}}$ is a matricial norm on $\mathcal{V}_{S,f}$.

□

From here forward, $\mathcal{V}_{S,f}$ is considered as an operator space with matrix norm $(\nu_{f,n})_{n\in\mathbb{N}}$. The following theorem is proved analogously to Theorem 3.1.2, as are the consequential results.

Theorem 3.4.2. *The pair* $(\mathcal{V}_{S,f},\eta_{S,f})$ *is a reflection of* (S,f) *along* $F_{\mathbf{O}_\infty}^{\mathbf{CSet}_\infty}$.

Corollary 3.4.3. *There is a unique functor* $\mathrm{OSp}_\infty:\mathbf{CSet}_\infty\to\mathbf{O}_\infty$ *such that* $\mathrm{OSp}_\infty(S,f)=\mathcal{V}_{S,f}$*, and* $\mathrm{OSp}_\infty\dashv F_{\mathbf{O}_\infty}^{\mathbf{CSet}_\infty}$.

Corollary 3.4.4 (Explicit Universal Property of $\mathrm{OSp}_\infty\dashv F_{\mathbf{O}_\infty}^{\mathbf{CSet}_\infty}$)**.** *Let* (S,f) *be a crutched set and* W *be an operator space. For any bounded map* $\phi:(S,f)\to F_{\mathbf{O}_\infty}^{\mathbf{CSet}_\infty}W$*, there is a unique completely bounded map* $\hat{\phi}:\mathrm{OSp}_\infty(S,f)\to W$ *such that* $\hat{\phi}\circ\eta_{S,f}=\phi$*. Moreover,*

$$\mathrm{crh}(\phi)=\left\|\hat{\phi}\right\|_{\mathcal{CB}(\mathrm{OSp}_\infty(S,f),W)}.$$

As in the Banach space case, there is a second adjoint relationship. Specifically, consider the category of operator spaces with completely contractive maps, $\mathbf{O}_1$. Let $F_{\mathbf{O}_1}^{\mathbf{CSet}_1}:\mathbf{O}_1\to\mathbf{CSet}_1$ be the restriction of $F_{\mathbf{O}_\infty}^{\mathbf{CSet}_\infty}$ to $\mathbf{O}_1$.

Corollary 3.4.5. *The pair* $(\mathcal{V}_{S,f},\eta_{S,f})$ *is a reflection of* (S,f) *along* $F_{\mathbf{O}_1}^{\mathbf{CSet}_1}$.

Corollary 3.4.6. *There is a unique functor* $\mathrm{OSp}_1:\mathbf{CSet}_1\to\mathbf{O}_1$ *such that* $\mathrm{OSp}_1(S,f)=\mathcal{V}_{S,f}$*, and* $\mathrm{OSp}_1\dashv F_{\mathbf{O}_1}^{\mathbf{CSet}_1}$.

Corollary 3.4.7 (Explicit Universal Property of $\mathrm{OSp}_1\dashv F_{\mathbf{O}_1}^{\mathbf{CSet}_1}$)**.** *Let* (S,f) *be a crutched set and* W *be an operator space. For any constrictive map* $\phi:(S,f)\to F_{\mathbf{O}_1}^{\mathbf{CSet}_1}W$*, there is a unique completely contractive map* $\hat{\phi}:\mathrm{OSp}_1(S,f)\to W$ *such that* $\hat{\phi}\circ\eta_{S,f}=\phi$*. Moreover,*

$$\mathrm{crh}(\phi)=\left\|\hat{\phi}\right\|_{\mathcal{CB}(\mathrm{OSp}_1(S,f),W)}.$$

Corollary 3.4.8 (Scaled-Free Mapping Property)**.** *Let* (S,f) *be a crutched set and* W *be an operator space. For any function* $\phi:S\to W$*, there is a unique completely contractive map* $\hat{\phi}:\mathrm{OSp}_1(S,f)\to W$ *such that for all* $s\in S$*,*

$$\|\phi(s)\|_W\cdot\hat{\phi}\left([\delta_s]\right)=f(s)\cdot\phi(s).$$

As in Example 3.1.10, for any operator space, there is a crutched set (S,f) such that $\mathrm{OSp}_1(S,f)$ maps surjectively and completely contractively onto it.

Moreover, $\mathrm{OSp}_1(S,f)$ is just as familiar as $\mathbb{C}\,\mathrm{BanSp}_1(S,f)$.

Theorem 3.4.9. *Given a crutched set* (S,f)*, let* $T:=S\setminus f^{-1}(0)$*. Then,*

$$\mathrm{OSp}_1(S,f)\cong_{\mathbf{O}_1}\max\left(\ell^1_{\mathbb{C}}(T)\right),$$

the maximal operator space of $\ell^1_{\mathbb{C}}(T)$*.*

Proof. Fix $n \in \mathbb{N}$ and $A \in M_n(\mathcal{V}_{S,f})$. For a $\mathbb{C}$-Hilbert space $\mathcal{H}$ and isometric $\phi : \mathcal{V}_{S,f} \to \mathcal{B}(\mathcal{H})$,

$$\left\|\phi^{(n)}(A)\right\|_{\mathcal{B}(\mathcal{H})} \leq \nu_{f,n}(A).$$

Conversely, let W be an operator space and $\psi : \mathcal{V}_{S,f} \to W$ be bounded and nonzero. Normalizing the map, define

$$\varphi := \frac{1}{\|\psi\|_{\mathcal{B}(\mathcal{V}_{S,f},W)}}\psi$$

and note that

$$\varphi^{(n)} = \frac{1}{\|\psi\|_{\mathcal{B}(\mathcal{V}_{S,f},W)}}\psi^{(n)}.$$

By Ruan's Theorem, there exists a $\mathbb{C}$-Hilbert space $\mathcal{K}$ and a complete isometry $\pi : W \to \mathcal{B}(\mathcal{K})$. Then, $\pi \circ \varphi : \mathcal{V}_{S,f} \to W$ is contractive and gives

$$\begin{aligned}\frac{\left\|\psi^{(n)}(A)\right\|_{M_n(W)}}{\|\psi\|_{\mathcal{B}(\mathcal{V}_{S,f},W)}} &= \left\|\varphi^{(n)}(A)\right\|_{M_n(W)} = \left\|\left(\pi^{(n)} \circ \varphi^{(n)}\right)(A)\right\|_{\mathcal{B}(\mathcal{K})} \\ &= \left\|(\pi \circ \varphi)^{(n)}(A)\right\|_{\mathcal{B}(\mathcal{K})} \leq \|A\|_{M_n(\max(\mathcal{V}_{S,f}))}.\end{aligned}$$

Therefore, $\nu_{f,n} = \|\cdot\|_{M_n(\max(\mathcal{V}_{S,f}))}$ for all $n \in \mathbb{N}$. Already, Theorem 3.1.12 identified $\mathcal{V}_{S,f}$ with $\ell^1_{\mathbb{C}}(T)$, finalizing the result. □

3.5. **Operator Algebras.** Consider the category of operator algebras with completely bounded homomorphisms, $\mathbf{OA}_\infty$. Let $F^{\mathbf{CSet}_\infty}_{\mathbf{OA}_\infty} : \mathbf{OA}_\infty \to \mathbf{CSet}_\infty$ be the restriction of $F^{\mathbf{CSet}_\infty}_{\mathbf{O}_\infty}$ to $\mathbf{OA}_\infty$. Unfortunately, as is the case with $\mathbb{C}\mathbf{BanAlg}_\infty$, only trivial crutched sets have reflections along this functor. The proof is analogous to Proposition 3.2.1

Proposition 3.5.1. *Let (S, f) be a crutched set such that $S \neq f^{-1}(0)$. Then, (S, f) has no reflection along $F^{\mathbf{CSet}_\infty}_{\mathbf{OA}_\infty}$.*

However, all is well in the completely contractive category. Consider the category of operator algebras with completely contractive homomorphisms, $\mathbf{OA}_1$. Let $F^{\mathbf{CSet}_1}_{\mathbf{OA}_1} : \mathbf{OA}_1 \to \mathbf{CSet}_1$ be the restriction of $F^{\mathbf{CSet}_\infty}_{\mathbf{O}_\infty}$ to $\mathbf{OA}_1$.

Fixing a crutched set (S, f), let B_S be the free $\mathbb{C}$-algebra on S as in the $\mathbb{C}$-Banach algebra case. First, B_S is normed in much the same way as before, but with the category of interest changed. The proof is analogous to Lemma 3.2.2.

Lemma 3.5.2. *For each $a \in B_S$, define*

$$\mathscr{X}_a := \left\{ \|\pi(a)\|_{\mathcal{B}} : \begin{array}{l} \mathcal{B} \in \operatorname{Ob}(\mathbf{OA}_1), \\ \pi : B_S \to \mathcal{B} \textit{ a } \mathbb{C}\textit{-algebra homomorphism}, \\ \|\pi(\delta_s)\|_{\mathcal{B}} \leq f(s) \forall s \in S \end{array} \right\}.$$

and $\kappa_f : B_S \to [0,\infty)$ *by* $\kappa_f(a) := \sup \mathscr{X}_a$. *Then,* κ_f *is a sub-multiplicative semi-norm on* B_S.

Let $L_f := \{a \in B_S : \kappa_f(a) = 0\}$, a two-sided ideal of B_S. Thus, B_S/L_f is a normed $\mathbb{C}$-algebra, and the completion, $\tilde{\mathcal{B}}_{S,f}$, is an $\mathbb{C}$-Banach algebra. Let $\tilde{\theta}_{S,f} : S \to \tilde{\mathcal{B}}_{S,f}$ by $\tilde{\theta}_{S,f}(s) := [\delta_s]$.

For the matrix norm structure, analogous arguments to those of Lemma 3.4.1 accomplish the following.

Lemma 3.5.3. *For each* $n \in \mathbb{N}$ *and* $A \in M_n\left(\tilde{\mathcal{B}}_{S,f}\right)$, *define*

$$\mathscr{Y}_{A,n} := \left\{ \left\|\phi^{(n)}(A)\right\|_{M_n(\mathcal{C})} : \begin{array}{l} \mathcal{C} \in \mathrm{Ob}\left(\mathbf{OA}_1\right), \\ \phi : \tilde{\mathcal{B}}_{S,f} \to \mathcal{C} \text{ a contractive homomorphism} \end{array} \right\}.$$

and $\mu_{f,n} : M_n\left(\tilde{\mathcal{B}}_{S,f}\right) \to [0,\infty)$ *by* $\mu_{f,n}(A) := \sup \mathscr{Y}_{A,n}$. *Then,* $\left(\mu_{f,n}\right)_{n\in\mathbb{N}}$ *is a matrix norm on* $\tilde{\mathcal{B}}_{S,f}$, *which makes the multiplication on* $\tilde{\mathcal{B}}_{S,f}$ *completely contractive. Moreover,* $\mu_{f,1}(a) = \|a\|_{\tilde{\mathcal{B}}_{S,f}}$ *for all* $a \in \tilde{\mathcal{B}}_{S,f}$.

From here forward, $\tilde{\mathcal{B}}_{S,f}$ is considered as an operator algebra with matrix norm $\left(\mu_{f,n}\right)_{n\in\mathbb{N}}$. As in the previous constructions, the consequential results follow by appropriately modified proofs.

Theorem 3.5.4. *The pair* $\left(\tilde{\mathcal{B}}_{S,f}, \tilde{\theta}_{S,f}\right)$ *is a reflection of* (S,f) *along* $F_{\mathbf{OA}_1}^{\mathbf{CSet}_1}$.

Corollary 3.5.5. *There is a unique functor* $\mathrm{OAlg} : \mathbf{CSet}_1 \to \mathbf{OA}_1$ *such that* $\mathrm{OAlg}(S,f) = \tilde{\mathcal{B}}_{S,f}$, *and* $\mathrm{OAlg} \dashv F_{\mathbf{OA}_1}^{\mathbf{CSet}_1}$.

Corollary 3.5.6 (Explicit Universal Property of $\mathrm{OAlg} \dashv F_{\mathbf{OA}_1}^{\mathbf{CSet}_1}$)**.** *Let* (S,f) *be a crutched set and* $\mathcal{B}$ *be an operator algebra. For any constrictive map* $\phi : (S,f) \to F_{\mathbf{OA}_1}^{\mathbf{CSet}_1}\mathcal{B}$, *there is a unique completely contractive homomorphism* $\hat{\phi} : \mathrm{OAlg}(S,f) \to \mathcal{B}$ *such that* $\hat{\phi} \circ \tilde{\theta}_{S,f} = \phi$.

Corollary 3.5.7 (Scaled-Free Mapping Property)**.** *Let* (S,f) *be a crutched set and* $\mathcal{B}$ *be an operator algebra. For any function* $\phi : S \to \mathcal{B}$, *there is a unique completely contractive homomorphism* $\hat{\phi} : \mathrm{OAlg}(S,f) \to \mathcal{B}$ *such that for all* $s \in S$,

$$\|\phi(s)\|_{\mathcal{B}} \cdot \hat{\phi}\left([\delta_s]\right) = f(s) \cdot \phi(s).$$

As in Example 3.1.10, for any operator algebra, there is a crutched set (S,f) such that $\mathrm{OAlg}(S,f)$ maps surjectively and completely contractively onto it.

4. Failure of Hilbert Spaces

The previous section showed that several categories have objects satisfying the scaled-free mapping property. However, the cases of $\mathbb{F}\mathbf{BanAlg}_\infty$ and $\mathbf{OA}_\infty$ illustrated that the fundamental adjoint construction may not be achievable with certain classes of maps. This section considers another such failure result for Hilbert spaces.

Specifically, consider the category of $\mathbb{F}$-Hilbert spaces and $\mathbb{F}$-linear contractions, $\mathbb{F}\mathbf{Hilb}_1$. Let $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Hilb}_1} : \mathbb{F}\mathbf{Hilb}_1 \to \mathbf{CSet}_1$ be the restriction of $F^{\mathbf{CSet}_\infty}_{\mathbb{F}\mathbf{Ban}_\infty}$ to $\mathbb{F}\mathbf{Hilb}_1$. As in the previous failure cases, most interesting crutched sets cannot have a reflection along this functor.

Proposition 4.1. *Let (S, f) be a crutched set such that* $\operatorname{card}\left(S \setminus f^{-1}(0)\right) \geq 2$. *Then, (S, f) has no reflection along $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Hilb}_1}$.*

Proof. Consider first when $\mathbb{F} = \mathbb{C}$. For purposes of contradiction, assume that (S, f) has a reflection (R, η) along $F^{\mathbf{CSet}_1}_{\mathbb{F}\mathbf{Hilb}_1}$. Let $v_s := \eta(s)$ for all $s \in S$.

First, the norms of each generator are determined. Since η is constrictive, $\|v_s\|_R \leq f(s)$ for all $s \in S$. Consider the function $\psi : S \to \mathbb{C}$ by $\phi(s) := f(s)$, the crutch function itself. Then, $\operatorname{crh}(\psi) = 1$ so there is a unique $\mathbb{C}$-linear contraction $\hat{\psi} : R \to \mathbb{C}$ such that $\hat{\psi} \circ \eta = \psi$. Therefore, for all $s \in S$,

$$\|v_s\|_R \geq \left|\hat{\psi}\left(v_s\right)\right| = f(s),$$

which forces equality.

Next, consider the inner product of two generators via the polarization identity. For $n \in \mathbb{N}$ and $s \neq t$,

$$\|v_s + \imath^n v_t\|_R \leq \|v_s\|_R + \|v_t\|_R \leq f(s) + f(t).$$

Define $\phi_{s,t,n} : S \to \mathbb{C}$ by

$$\phi_{s,t,n}(u) := \begin{cases} f(s), & u = s, \\ \imath^{-n} f(t), & u = t, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\operatorname{crh}(\phi_{s,t,n}) = 1$ so there is a unique $\mathbb{C}$-linear contraction $\hat{\phi}_{s,t,n} : R \to \mathbb{C}$ such that $\hat{\phi} \circ \eta = \phi$. Hence,

$$\begin{aligned} \|v_s + \imath^n v_t\|_R &\geq \left|\hat{\phi}_{s,t,n}\left(v_s + \imath^n v_t\right)\right| \\ &= \left|\hat{\phi}_{s,t,n}\left(v_s\right) + \imath^n \hat{\phi}_{s,t,n}\left(v_t\right)\right| \\ &= \left|f(s) + \imath^n \imath^{-n} f(t)\right| \\ &= f(s) + f(t), \end{aligned}$$

forcing equality. Using the polarization identity,

$$\langle v_s, v_t \rangle_R = \frac{1}{4} \sum_{n=0}^{3} \imath^n \|v_s + \imath^n v_t\|_R = \frac{1}{4} \sum_{n=0}^{3} \imath^n \left(f(s) + f(t)\right) = 0$$

Thus, $(v_s)_{s \in S}$ is an orthogonal set in R.

Using Parseval's identity, for $s \neq t$,

$$\begin{aligned}\|v_s + v_t\|_R^2 &= \|v_s\|_R^2 + \|v_t\|_R^2 \\ &= f(s)^2 + f(t)^2,\end{aligned}$$

but

$$\begin{aligned}\|v_s + v_t\|_R^2 &= (f(s) + f(t))^2 \\ &= f(s)^2 + 2f(s)f(t) + f(t)^2.\end{aligned}$$

Together, these imply that if $s \neq t$, $f(s)f(t) = 0$. However, for distinct $s, t \notin f^{-1}(0)$, this is impossible. Therefore, this reflection can never have existed.

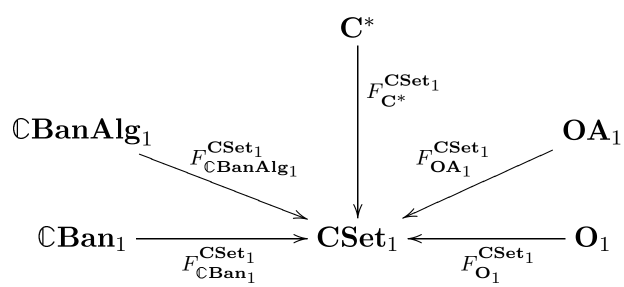
The case for $\mathbb{F} = \mathbb{R}$ follows by considering the real version of the polarization identity.

□

From the proof, the issue here was due to the incompatibility of the universal property with Parseval's identity. The universal property imposes that the norm on the reflection be an ℓ^1-norm, like the case of $\mathbb{F}$-Banach spaces, but this cannot happen in a $\mathbb{F}$-Hilbert space other than $\mathbb{F}$ or $\mathbb{O}$.

5. Universal Algebra for Normed Objects

After performing all the constructions of Section 3, consider the following diagram of categories and functors.



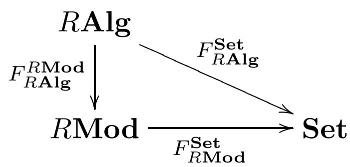
While not drawn, there are natural forgetful functors for the categories placed on the exterior of this diagram, and they will automatically make this large diagram commute. For example, there is a natural forgetful functor $F_{\mathbf{C}^*}^{\mathbb{C}\mathbf{BanAlg}_1} : \mathbf{C}^* \to \mathbb{C}\mathbf{BanAlg}_1$ by ignoring the involution and its properties. A quick check shows that

$$F_{\mathbb{C}\mathbf{BanAlg}_1}^{\mathbf{CSet}_1} F_{\mathbf{C}^*}^{\mathbb{C}\mathbf{BanAlg}_1} = F_{\mathbf{C}^*}^{\mathbf{CSet}_1}.$$

Section 3 has shown that each of the functors to $\mathbf{CSet}_1$ has a corresponding left adjoint functor, given by the appropriate scaled-free construction. If the undrawn forgetful functors have left adjoints as well, general category theory has that the composition of left adjoints is again a left adjoint.

That is, the diagram given by the left adjoints would also commute, up to a natural transformation.

This would give, in a sense, a coherent means of understanding normed objects. This would be comparable to the understanding found in pure algebra. In summary, consider the following commutative diagram of categories and functors for a fixed ring R. Here, the functors are likewise forgetful functors, stripping away structure.



For algebraic categories, **Set** plays a central role, allowing consideration of objects with minimal structure. From the constructions of Section 3 and the properties found in Section 2, $\mathbf{CSet}_1$ may play a similar role for normed objects, a category of objects with structure similar to normed objects, but minimal.

Plans are to investigate these relationships in subsequent papers, relying heavily on the foundation laid in this work.

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