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ON ASYMPTOTICS OF $\Gamma_q(z)$ AS q APPROACHING 1

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Abstract. In this note we give a derivation of the asymptotic main term for the q-Gamma function as q approaching 1. This formula is valid on all the complex plan except at the poles of the Euler Gamma function.

1. Introduction

Recall that the q-Gamma function is defined as [1, 2, 3]

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(1-q)^{z-1}(q^z;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right), \quad a\in\mathbb{C},\ q\in(0,1).$$

All the standard textbooks on q-series present W. Gosper's heuristic argument for

$$\lim_{q\to 1}\Gamma_q(z) = \Gamma(z),$$

where $\Gamma(z)$ is the Euler Gamma function, without verifying the validity of the term by term limiting process, [1, 2, 3]. An alternative proof by T. Koorwinder is given in [1] using a convexity argument, but all the proofs failed to give an error term. In [5] we give a proof using a q-Beta integral from [1]. In this note we will give yet another proof with error term valid on the whole complex plane except at poles of $\Gamma(z)$.

2. Main Results

Lemma 1. *Let*

$$|z|<1, \quad 0<q<1,$$

then

$$(z;q)_\infty = \exp\left\{-\sum_{k=1}^{\infty}\frac{z^k}{k(1-q^k)}\right\}. \tag{2.1}$$

Proof. From

$$\log(1-z) = -\sum_{k=1}^{\infty}\frac{z^k}{k}, \quad |z|<1$$

we have

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$$\begin{aligned}\log(z,q)_\infty &= \sum_{j=0}^{\infty}\log\left(1-zq^j\right) = -\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\frac{\left(zq^j\right)^k}{k}\\ &= -\sum_{k=1}^{\infty}\frac{z^k}{k}\sum_{j=0}^{\infty}q^{jk} = -\sum_{k=1}^{\infty}\frac{z^k}{k\left(1-q^k\right)}\end{aligned}$$

for $q\in(0,1)$, where all the logarithms are taken as their principle branches. (2.1) follows by taking exponentials. □

Lemma 2. *Let*

$$q=e^{-\pi\tau},\ \tau>0,\quad \Re(w)>0,$$

then,

$$(q^w;q)_\infty = \frac{\sqrt{2\pi}w^{w-1/2}\exp\left(-\frac{\pi}{6\tau}\right)}{\Gamma(w)\left(1-e^{-\tau\pi w}\right)^{w-1/2}}\left\{1+\mathcal{O}(\tau)\right\}, \tag{2.2}$$

as $\tau\to0^+$.

Proof. Take $z=qe^{-\tau\pi w}$ in (2.1) with $\Re(w)>0$ to obtain

$$\left(qe^{-\tau\pi w},q\right)_\infty = \exp\left\{-\sum_{k=1}^{\infty}\frac{q^ke^{-k\tau\pi w}}{k\left(1-q^k\right)}\right\}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{q^ke^{-k\tau\pi w}}{k\left(1-q^k\right)} &= \sum_{k=1}^{\infty}\frac{e^{-k\tau\pi w}}{k}\left\{\frac{q^k}{1-q^k}-\frac{1}{k\pi\tau}+\frac12-\frac{k\pi\tau}{12}\right\}\\ &+\sum_{k=1}^{\infty}\frac{e^{-k\tau\pi w}}{k}\left\{\frac{1}{k\pi\tau}-\frac12+\frac{k\pi\tau}{12}\right\}\\ &= S+\frac{1}{\pi\tau}\sum_{k=1}^{\infty}\frac{e^{-k\tau\pi w}}{k^2}-\frac12\sum_{k=1}^{\infty}\frac{e^{-k\tau\pi w}}{k}+\frac{\pi\tau}{12}\sum_{k=1}^{\infty}e^{-k\tau\pi w}\\ &= S+\frac{1}{\pi\tau}\mathrm{Li}_2\left(\exp(-\pi\tau w)\right)+\frac12\log\left(1-e^{-\tau\pi w}\right)+\frac{\pi\tau}{12\left(\exp(\tau\pi w)-1\right)},\end{aligned}$$

where

$$S=\sum_{k=1}^{\infty}\frac{e^{-k\tau\pi w}}{k}\left\{\frac{1}{e^{k\pi\tau}-1}-\frac{1}{k\pi\tau}+\frac12-\frac{k\pi\tau}{12}\right\}$$

and

$$\mathrm{Li}_2(z)=\sum_{k=1}^{\infty}\frac{z^n}{n^2},\quad |z|\le1.$$

From

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)=\frac{\pi^2}{6}-\log z\cdot\log(1-z)$$

to get

$$\begin{aligned}\mathrm{Li}_2\left(\exp(-\pi\tau w)\right) &= -\mathrm{Li}_2\left(1-\exp(-\pi\tau w)\right) + \frac{\pi^2}{6} + \pi\tau w\log\left(1-\exp(-\pi\tau w)\right)\\ &= -\pi\tau w + \frac{\pi^2}{6} + \pi\tau w\log\left(1-\exp(-\pi\tau w)\right) + \mathcal{O}\left(\tau^2\right),\end{aligned}$$

hence

$$\sum_{k=1}^{\infty}\frac{q^k e^{-k\tau\pi w}}{k\left(1-q^k\right)} = S - w + \frac{\pi}{6\tau} + \left(w+\frac{1}{2}\right)\log\left(1-\exp(-\pi\tau w)\right) + \frac{\pi\tau}{12\left(\exp(\tau\pi w)-1\right)} + \mathcal{O}\left(\tau\right),$$

as $\tau\to 0^+$. From

$$\log\Gamma(w) = \left(w-\frac{1}{2}\right)\log w - w + \frac{\log(2\pi)}{2} + \int_0^{\infty}\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt$$

to obtain

$$\begin{aligned}&\int_0^{\infty}\left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt\\ &= \log\Gamma(w) - \left(w-\frac{1}{2}\right)\log w + w - \frac{\log(2\pi)}{2} - \frac{1}{12w}\end{aligned}$$

for $\Re(w)>0$. Write

$$\begin{aligned}I &= \int_0^{\infty}\left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt\\ &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau}\left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t}dt\end{aligned}$$

and

$$f(t) = \left(\frac{1}{2}-\frac{1}{t}-\frac{t}{12}+\frac{1}{e^t-1}\right)\frac{e^{-tw}}{t},$$

then

$$f'(t) = \mathcal{O}\left(t\right)$$

for $t\to 0^+$ and

$$f'(t) = \mathcal{O}\left(\exp\left(-t\Re(w)\right)\right)$$

for $t\to+\infty$. Hence,

$$\begin{aligned}S - I &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau}dt\int_t^{k\pi\tau}f'(y)dy\\ &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau}f'(y)\int_{(k-1)\pi\tau}^{y}dtdy\\ &= \sum_{k=1}^{\infty}\int_{(k-1)\pi\tau}^{k\pi\tau}f'(y)\left(y-(k-1)\pi\tau\right)dy,\end{aligned}$$

thus,

$$|S-I| \le \pi\tau\int_0^{\infty}\left|f'(y)\right|dy$$

and

$$S - I = \mathcal{O}\left(\pi\tau\right)$$

as $\tau \to 0^+$. Thus,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{q^k e^{-k\tau\pi w}}{k\left(1-q^k\right)} &= \log\Gamma(w) - \left(w-\frac{1}{2}\right)\log w - \frac{\log(2\pi)}{2} \\ &+ \frac{\pi\tau}{12}\left(\frac{1}{\exp(\tau\pi w)-1} - \frac{1}{\pi\tau w}\right) + \frac{\pi}{6\tau} \\ &+ \left(w+\frac{1}{2}\right)\log\left(1-\exp(-\pi\tau w)\right) + \mathcal{O}(\tau) \\ &= \log\Gamma(w) - \left(w-\frac{1}{2}\right)\log w - \frac{\log(2\pi)}{2} \\ &+ \frac{\pi}{6\tau} + \left(w+\frac{1}{2}\right)\log\left(1-\exp(-\pi\tau w)\right) + \mathcal{O}(\tau),\end{aligned}$$

as $\tau \to 0^+$. Hence, for $\Re(w) > 0$ we have

$$\left(q^{w+1};q\right)_\infty = \frac{\sqrt{2\pi}w^{w-1/2}\exp\left(-\frac{\pi}{6\tau}\right)}{\Gamma(w)\left(1-e^{-\tau\pi w}\right)^{w+1/2}}\left\{1+\mathcal{O}(\tau)\right\}$$

and

$$\left(q^w;q\right)_\infty = \left(1-e^{-\tau\pi w}\right)\left(qe^{-\tau\pi w},q\right)_\infty = \frac{\sqrt{2\pi}w^{w-1/2}\exp\left(-\frac{\pi}{6\tau}\right)}{\Gamma(w)\left(1-e^{-\tau\pi w}\right)^{w-1/2}}\left\{1+\mathcal{O}(\tau)\right\}$$

as $\tau \to 0^+$. □

Theorem 3. *Let $q = \exp(-\pi\tau)$ with $\tau > 0$, then*

$$\Gamma_q(w) = \Gamma(w)\left\{1+\mathcal{O}(\tau)\right\} \tag{2.3}$$

as $\tau \to 0^+$for $-z \notin \mathbb{N}\cup\{0\}$.

Proof. From (2.2) to get

$$(q;q)_\infty = \frac{\sqrt{2\pi}\exp\left(-\frac{\pi}{6\tau}\right)}{\left(1-e^{-\tau\pi}\right)^{1/2}}\left\{1+\mathcal{O}(\tau)\right\}$$

as $\tau \to 0^+$. Hence, for $\Re(w) > 0$ we have

$$\Gamma_q(w) = \frac{(q;q)_\infty}{(1-q)^{w-1}(q^w;q)_\infty} = \Gamma(w)\left\{\frac{1-e^{-\pi\tau w}}{w\left(1-e^{-\pi\tau}\right)}\right\}^{w-\frac{1}{2}}\left\{1+\mathcal{O}(\tau)\right\}$$

as $\tau \to 0^+$. The above equation and

$$\left\{\frac{1-e^{-\pi\tau w}}{w\left(1-e^{-\pi\tau}\right)}\right\}^{w-\frac{1}{2}} = 1+\mathcal{O}(\tau)$$

as $\tau \to 0^+$imply (2.3) for $\Re(w) > 0$.

From [4]

$$\theta_1(v|t) = 2\sum_{k=0}^{\infty}(-1)^k p^{(k+1/2)^2}\sin(2k+1)\pi v,$$

$$\theta_1(v|t) = 2p^{1/4}\sin\pi v(p^2;p^2)_\infty(p^2e^{2\pi iv};p^2)_\infty(p^2e^{-2\pi iv};p^2)_\infty$$

and

$$\theta_1\left(\frac{v}{t}\mid -\frac{1}{t}\right) = -i\sqrt{\frac{t}{i}}e^{\pi i v^2/t}\theta_1\left(v\mid t\right),$$

where $p = e^{\pi i t}$, $\quad \Im(t) > 0$, to get

$$\left(q, q^{1+w}, q^{1-w}; q\right)_\infty = \frac{\exp\left(\frac{\pi\tau}{8} + \frac{\pi\tau w^2}{2}\right)\theta_1\left(w|\frac{2i}{\tau}\right)}{\sqrt{2\tau}\sinh\frac{\pi\tau w}{2}}$$

and

$$(q;q)_\infty^3 = \frac{\sqrt{2}\exp\left(\frac{\pi\tau}{8}\right)\theta_1'\left(0|\frac{2i}{\tau}\right)}{\pi\tau^{3/2}}$$

for $q = \exp(-\pi\tau)$ and $\tau > 0$. Hence

$$\Gamma_q\left(1+w\right)\Gamma_q\left(1-w\right) = \frac{(q;q)_\infty^3}{\left(q, q^{1+w}, q^{1-w}; q\right)_\infty} = \frac{2\sinh\frac{\pi\tau w}{2}\theta_1'\left(0|\frac{2i}{\tau}\right)}{\pi\tau\exp\left(\frac{\pi\tau w^2}{2}\right)\theta_1\left(w|\frac{2i}{\tau}\right)}$$

and

$$\Gamma_q\left(w\right)\Gamma_q(1-w) = \frac{1-q}{1-q^w}\Gamma_q(1+w) = \frac{\left(e^{\pi\tau}-1\right)\theta_1'\left(0|\frac{2i}{\tau}\right)}{\pi\tau\exp\left(\frac{\pi\tau(w^2+w+2)}{2}\right)\theta_1\left(w|\frac{2i}{\tau}\right)}$$

for $w \notin \mathbb{Z}$. From

$$\frac{e^{\pi\tau}-1}{\pi\tau} = 1 + \mathcal{O}\left(\tau\right),$$

$$\theta_1'\left(0|\frac{2i}{\tau}\right) = 2\pi\exp\left(-\frac{\pi}{2\tau}\right)\left\{1 + \mathcal{O}\left(\tau\right)\right\},$$

$$\theta_1\left(w|\frac{2i}{\tau}\right) = 2\sin\pi w\exp\left(-\frac{\pi}{2\tau}\right)\left\{1 + \mathcal{O}\left(\tau\right)\right\}$$

and

$$\Gamma\left(w\right)\Gamma\left(1-w\right) = \frac{\pi}{\sin\pi w}, \quad w \notin \mathbb{Z}$$

to obtain

$$\Gamma_q\left(w\right) = \frac{\pi}{\sin\pi w}\frac{1}{\Gamma\left(1-w\right)}\left\{1 + \mathcal{O}\left(\tau\right)\right\} = \Gamma\left(w\right)\left\{1 + \mathcal{O}\left(\tau\right)\right\}$$

for $\Re(w) < 1$ and $w \notin \mathbb{Z}$ as $\tau \to 0^+$. The theorem follows by combining the $\Re(w) > 0$ and $\Re(w) < 1$ cases. □

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