ARENS REGULARITY OF TENSOR PRODUCTS AND WEAK AMENABILITY OF BANACH ALGEBRAS

KAZEM HAGHNEJAD AZAR

Abstract. In this note, we study the Arens regularity of projective tensor product $A\hat{\otimes}B$ whenever A and B are Arens regular. We establish some new conditions for showing that the Banach algebras A and B are Arens regular if and only if $A\hat{\otimes}B$ is Arens regular. We also introduce some new concepts as left-weak*-weak convergence property [Lw^*wc-property] and right-weak*-weak convergence property [Rw^*wc-property] and for Banach algebra A, suppose that A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property. Then if A^{**} is weakly amenable, it follows that A is weakly amenable. We also offer some results concerning the relation between these properties with some special derivation $D : A \rightarrow A^*$. We obtain some conclusions in the Arens regularity of Banach algebras.

1. **Preliminaries and Introduction**

Suppose that A and B are Banach algebras. Ülger in [22], has been studied that the Arens regularity of projective tensor product $A\hat{\otimes}B$. He showed that when A and B are Arens regular in general, $A\hat{\otimes}B$ is not Arens regular. He introduced a new concept as biregular mapping and showed that a bounded bilinear mapping $m : A \times B \rightarrow \mathbb{C}$ is biregular if and only if $A\hat{\otimes}B$ is Arens regular. In this paper, we establish some conditions for Banach algebras A and B which follows that $A\hat{\otimes}B$ is Arens regular. Conversely, we investigated if $A\hat{\otimes}B$ is Arens regular, then A or B are Arens regular. In section three, for Banach $A-module\ B$, we introduce new concepts as $left-weak^*-weak$ convergence property [Lw^*wc-property] and $right-weak^*-weak$ convergence property [Rw^*wc-property] with respect to A and we show that if A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property and A^{**} is weakly amenable, then A is weakly amenable. We have also some conclusions regarding Arens regularity of Banach algebras. We introduce some notations and definitions that we used throughout this paper.

Let A be a Banach algebra and let B be a Banach $A-bimodule$. A derivation from A into B is a bounded linear mapping $D : A \rightarrow B$ such that

$$D(xy) = xD(y) + D(x)y \ for\ all\ x,\ y \in A.$$

The space of all continuous derivations from A into B is denoted by $Z^1(A, B)$.
Easy example of derivations are the inner derivations, which are given for each $b \in B$ by

$$\delta_b(a) = ab - ba \ for\ all\ a \in A.$$

2000 *Mathematics Subject Classification.* 46L06; 46L07; 46L10; 47L25.
Key words and phrases. Arens regularity, topological centers, Tensor product, Bilinear form, amenability, weak amenability, derivation, module actions, $left-weak^*-to-weak$ convergence .

The space of inner derivations from A into B is denoted by $N^1(A,B)$. The Banach algebra A is amenable, when for every Banach $A-bimodule$ B, the only inner derivation from A into B^* is zero derivation. It is clear that A is amenable if and only if $H^1(A,B^*)=Z^1(A,B^*)/N^1(A,B^*)=\{0\}$. The concept of amenability for a Banach algebra A, introduced by Johnson in 1972, has proved to be of enormous importance in Banach algebra theory, see [13]. A Banach algebra A is said to be a weakly amenable, if every derivation from A into A^* is inner. Equivalently, A is weakly amenable if and only if $H^1(A,A^*)=Z^1(A,A^*)/N^1(A,A^*)=\{0\}$. The concept of weak amenability was first introduced by Bade, Curtis and Dales in [2] for commutative Banach algebras, and was extended to the noncommutative case by Johnson in [14].

Let A be a Banach algebra and A^*, A^{**}, respectively, be the first and second dual of A. For $a\in A$ and $a'\in A^*$, we denote by $a'a$ and aa' respectively, the functionals in A^* defined by $\langle a'a,b\rangle=\langle a',ab\rangle=a'(ab)$ and $\langle aa',b\rangle=\langle a',ba\rangle=a'(ba)$ for all $b\in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a,a'\rangle$ - $\langle a',a\rangle$ for every $a\in A$ and $a'\in A^*$. We say that a bounded net $(e_\alpha)_{\alpha\in I}$ in A is a left bounded approximate identity ($=LBAI$) [resp. right bounded approximate identity ($=RBAI$)] if, for each $a\in A$, $e_\alpha a\longrightarrow a$ [resp. $ae_\alpha\longrightarrow a$].

Let X,Y,Z be normed spaces and $m:X\times Y\to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**}\times Y^{**}$ into Z^{**} as following

1. $m^*:Z^*\times X\to Y^*$, given by $\langle m^*(z',x),y\rangle=\langle z',m(x,y)\rangle$ where $x\in X$, $y\in Y$, $z'\in Z^*$,
2. $m^{**}:Y^{**}\times Z^*\to X^*$, given by $\langle m^{**}(y'',z'),x\rangle=\langle y'',m^*(z',x)\rangle$ where $x\in X$, $y''\in Y^{**}$, $z'\in Z^*$,
3. $m^{***}:X^{**}\times Y^{**}\to Z^{**}$, given by $\langle m^{***}(x'',y''),z'\rangle=\langle x'',m^{**}(y'',z')\rangle$ where $x''\in X^{**}$, $y''\in Y^{**}$, $z'\in Z^*$.

The mapping m^{***} is the unique extension of m such that $x''\to m^{***}(x'',y'')$ from X^{**} into Z^{**} is $weak^*-to-weak^*$ continuous for every $y''\in Y^{**}$, but the mapping $y''\to m^{***}(x'',y'')$ is not in general $weak^*-to-weak^*$ continuous from Y^{**} into Z^{**} unless $x''\in X$. Hence the first topological center of m may be defined as following

$$Z_1(m)=\{x''\in X^{**}:\ y''\to m^{***}(x'',y'')\ is\ weak^*-to-weak^*-continuous\}.$$

Let now $m^t:Y\times X\to Z$ be the transpose of m defined by $m^t(y,x)=m(x,y)$ for every $x\in X$ and $y\in Y$. Then m^t is a continuous bilinear map from $Y\times X$ to Z, and so it may be extended as above to $m^{t***}:Y^{**}\times X^{**}\to Z^{**}$. The mapping $m^{t***t}:X^{**}\times Y^{**}\to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***}=m^{t***t}$, then m is called Arens regular. The mapping $y''\to m^{t***t}(x'',y'')$ is $weak^*-to-weak^*$ continuous for every $y''\in Y^{**}$, but the mapping $x''\to m^{t***t}(x'',y'')$ from X^{**} into Z^{**} is not in general $weak^*-to-weak^*$ continuous for every $y''\in Y^{**}$. So we define the second topological center of m as

$$Z_2(m)=\{y''\in Y^{**}:\ x''\to m^{t***t}(x'',y'')\ is\ weak^*-to-weak^*-continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m)=X^{**}$ or $Z_2(m)=Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i\lim_j\langle z',m(x_i,y_j)\rangle=\lim_j\lim_i\langle z',m(x_i,y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [5, 20].
The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}, the second dual of A. By *Goldstin's* Theorem [4, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta \langle a', m(a_\alpha, b_\beta)\rangle = \langle a''b'', a'\rangle$$

and

$$\lim_\beta \lim_\alpha \langle a', m(a_\alpha, b_\beta)\rangle = \langle a''ob'', a'\rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [5, 20].
The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.
Regarding A as a Banach $A-bimodule$, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. In this situation, we shall also simplify our notations. So the first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle.$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a'oa'', a\rangle = \langle a'', aoa'\rangle,$$
$$\langle a''ob'', a'\rangle = \langle b'', a'ob''\rangle.$$

for all $a, b \in A$ and $a' \in A^*$.

2. **Arens regularity of projective tensor product algebras**

The tensor product, $X \otimes Y$, of the vector space X, Y can be constructed as a space of linear functional on $B(X \times Y)$, in the following way:
Let $x \in X$ and $y \in Y$. We denote by $x \otimes y$ the functional given by evaluation at the point (x, y). In other words,

$$\langle x \otimes y, A\rangle = A(x, y),$$

for each bilinear from A on $X \times Y$, so the tensor product $X \otimes Y$ is the subspace of the dual of bounded bilinear forms on $X \otimes Y$, $B(X \times Y)^*$.
We recall that each tensor $u \in X \otimes Y$ acts as a linear functional on the space of bilinear

forms and so we may define a mapping $\tilde{A} : X \otimes Y \to K$ by $u \in X \otimes Y \to \langle A, u\rangle \in K$. In summary, we have

$$B(X \times Y) = (X \otimes Y)^*.$$

Let $X, Y, E \textit{ and } F$ be vector spaces and let $S : X \to E$ and $T : Y \to F$ be linear mappings. Then we may define a bilinear mapping by $(x, y) \in X \times Y \to (Sx) \otimes (Ty) \in E \otimes F$. Linearization gives a linear mapping $(S \otimes T) : X \otimes Y \to E \otimes F$ such that $(S \otimes T)(x \otimes y) = (Sx) \otimes (Ty)$ for every $x \in X$ and $y \in Y$.
By $X\hat{\otimes}Y$ and $X\check{\otimes}Y$ we shall denote, respectively, the projective and injective tensor products of X and Y. That is, $X\hat{\otimes}Y$ is the completion of $X \otimes Y$ for the norm

$$\parallel u \parallel = inf \sum_{i=1}^{n} \parallel x_i \parallel \parallel y_i \parallel,$$

where the infimum is taken over all the representations of u as a finite sum of the form $u = \sum_{i=1}^{n} x_i \otimes y_i$, and $X\check{\otimes}Y$ is the completion of $X \otimes Y$ for the norm

$$\parallel u \parallel = sup\{\mid \sum_{i=1}^{n} \langle x', x_i\rangle\langle y', y_i\rangle \mid : \parallel x' \parallel \leq 1, \ \parallel y' \parallel \leq 1\}.$$

The dual space of $X\hat{\otimes}Y$ is $B(X \times Y)$, and that of $X\check{\otimes}Y$ is a subspace of $B(X \times Y)$. Although the injective tensor product of two Banach algebra A and B is not always a Banach algebra, their projective tensor product is always a Banach algebra. The natural multiplication of $A\hat{\otimes}B$ is the linear extension of the following multiplication on decomposable tensors $(a \otimes b)(\tilde{a} \otimes \tilde{b}) = a\tilde{a} \otimes b\tilde{b}$. For more information about the tensor product of Banach algebra, see for example [4, 5].
A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact. Pym in [20] showed that this definition to the equivalent following condition
For any two net $(a_i)_i$ and $(b_j)_j$ in $\{a \in A : \parallel a \parallel \leq 1\}$, we have

$$lim_i lim_j \langle a', a_i b_j\rangle = lim_j lim_i \langle a', a_i b_j\rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $\langle a''b'', a'\rangle = \langle a''ob'', a'\rangle$ for every a'', $b'' \in A^{**}$. Thus, it is clear that A is Arens regular if and only if $wap(A) = A^*$. In the following, for Banach algebras A and B, for showing Arens regularity of projective tensor products $A\hat{\otimes}B$, we establish $wap(A\hat{\otimes}B) = (A\hat{\otimes}B)^*$. In all of this section, we regard $A^*\hat{\otimes}B^*$ as a subset of $(A\hat{\otimes}B)^*$.

Theorem 2-1. Suppose that A and B are Banach algebra and for every sequence $(x_i)_i$, $(y_j)_j \subseteq A_1$, $(z_i)_i$, $(w_j)_j \subseteq B_1$ and $f \in \mathbf{B}(A \times B)$, we have

$$\lim_j \lim_i f(x_i z_i, y_j w_j) = \lim_i \lim_j f(x_i z_i, y_j w_j).$$

Then $A\hat{\otimes}B$ is Arens regular.

Proof. Assume that $f \in \mathbf{B}(A \times B)$. Since $\mathbf{B}(A \times B) = (A\hat{\otimes}B)^*$, it is enough to show that $f \in wap(A\hat{\otimes}B)$. Let $(x_i)_i$, $(y_j)_j \subseteq A_1$ and $(z_i)_i$, $(w_j)_j \subseteq B_1$, then we have the following equality

$$\lim_j \lim_i \langle f, (x_i \otimes y_j)(z_i \otimes w_j)\rangle = \lim_j \lim_i \langle f, x_i z_i \otimes y_j w_j\rangle$$

$$= \lim_j \lim_i f(x_i z_i, y_j w_j) = \lim_i \lim_j f(x_i z_i, y_j w_j)$$

$$= \lim_i \lim_j \langle f, (x_i \otimes y_j)(z_i \otimes w_j)\rangle.$$

Consequently by [20], $f \in wap(A\hat{\otimes}B)$. □

Definition 2-2. Assume that B is a Banach $A - bimodule$. We say that B is non-trivial on A, if for every $(a_i)_{i=1}^n \subseteq A_1$ and $(b_j)_{j=1}^n \subseteq B_1$, respectively, basis elements of A and B, we have $\sum_{i=1}^n \alpha_i a_i b_i \neq 0$ where α_i is scaler and every a_i and b_i are distinct for all $1 \leq i \leq n$.
For example, take $B = \mathbb{R} \times \{0\}$ and $A = \mathbb{R}^2$ by the following multiplication

$$(a_1, a_2)(b_1, 0) = (a_1 b_1, 0) \ where \ a_1, a_2, b_1 \in \mathbb{R}.$$

Theorem 2-3. Suppose that A and B are Banach algebras and B is unital. Let B be a Banach $A - bimodule$. Then we have the following assertions:

(1) If $A\hat{\otimes}B$ is Arens regular, then A is Arens regular.

(2) Let B be non-trivial on A and let B be an unital Banach $A - module$. Then A and B are Arens regular if and only if $A\hat{\otimes}B$ is Arens regular.

Proof. (1) Assume that $A\hat{\otimes}B$ is Arens regular and let $u \in B$ be an unit element of B. We show that $wap(A) = A^*$. Assume that $(a_i)_i \subseteq A$, $(c_j)_j \subseteq A$ whenever both iterated limits exist and $a' \in A^*$. Then we define $\phi = a' \otimes b'$ where $b' \in B^*$ and $b'(u) = 1$. Since $A^* \otimes B^* \subseteq (A \otimes B)^*$ and $A\hat{\otimes}B$ is Arens regular, we have $a' \otimes b' \in wap(A \otimes B)$. Hence it follows that

$$\lim_i \lim_j \langle a', a_i c_j\rangle = \lim_i \lim_j \langle a' \otimes b', a_i c_j \otimes u\rangle$$

$$= \lim_i \lim_j \langle a' \otimes b', (a_i \otimes u)(c_j \otimes u)\rangle = \lim_j \lim_i \langle a' \otimes b', (a_i \otimes u)(c_j \otimes u)\rangle$$

$$= \lim_j \lim_i \langle a', a_i c_j\rangle.$$

We conclude that $a' \in wap(A)$, and so A is Arens regular.

(2) Let u be an unit element of B and suppose that B is Arens regular. Then $wap(B) = B^*$. Suppose that $(a_i)_i \subseteq A_1$ and $(b_j)_j \subseteq B_1$ whenever both iterated limits exist. Then $(a_i u)_i \subseteq B_1$, and so for every $b' \in B^*$, we have the following equality

$$\lim_i \lim_j \langle b', (a_i u) b_j\rangle = \lim_j \lim_i \langle b', (a_i u) b_j\rangle.$$

Now let $\phi \in (A\hat{\otimes}B)^*$. We define the mapping $T : A\hat{\otimes}B \to B$ such that $T(\sum_{i=1}^n \alpha_i a_i \otimes b_i) = \sum_{i=1}^n \alpha_i a_i b_i$ where $a_i \in A$, $b_i \in B$ and α_i is a scaler. We show that $\phi o T^{-1} \in B^*$. Since B is not-trivial on A, T^{-1} exist. Now let $e \in A$ be an unit element for B as Banach $A-module$ and let $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \to b$. Then $e \otimes b_\alpha \to e \otimes b$ in $A\hat{\otimes}B$, it follows that

$$\langle \phi o T^{-1}, b_\alpha\rangle = \langle \phi, T^{-1}(b_\alpha)\rangle = \langle \phi, T^{-1}(eb_\alpha)\rangle = \langle \phi, e \otimes b_\alpha\rangle$$
$$\to \langle \phi, e \otimes b\rangle = \langle \phi, T^{-1}(ub)\rangle = \langle \phi o T^{-1}, b\rangle.$$

Consequently $\phi o T^{-1} \in B^*$. Now we have the following equality

$$\lim_i \lim_j \langle \phi, a_i \otimes b_j\rangle = \lim_i \lim_j \langle b' o T, a_i \otimes b_j\rangle$$
$$= \lim_i \lim_j \langle b', T(a_i \otimes b_j)\rangle = \lim_i \lim_j \langle b', a_i b_j\rangle$$
$$= \lim_i \lim_j \langle b', a_i(ub_j)\rangle = \lim_j \lim_i \langle b', (a_i u) b_j\rangle$$
$$= \lim_j \lim_i \langle b', T(a_i \otimes b_j)\rangle = \lim_j \lim_i \langle \phi, a_i \otimes b_j\rangle.$$

It follows that $\phi \in wap(A\hat{\otimes}B)$, and so $A\hat{\otimes}B$ is Arens regular.
The converse by using part (1) hold.

□

Corollary 2-4 . Suppose that A and B are unital Banach algebras and B is an unital Banach as $A-module$. Assume that B is non-trivial on A. Then if A and B are Arens regular, then every bilinear form $m : A \times B \to \mathbb{C}$ is weakly compact.

Proof. By using Theorem 2-3 and [22, Theorem 3.4], proof hold. □

Example 2-5. $(\ell^1 \oplus \mathbb{C})\hat{\otimes}\ell^\infty$ is Arens regular.

Proof. We know that ℓ^∞ is $(\ell^1 \oplus \mathbb{C})-bimodule$ and ℓ^∞ is unital. ℓ^∞ is also non-trivial on $(\ell^1 \oplus \mathbb{C})$. By using [2, Corollary 8] and [5, Example 2.6.22(iii)], respectively, we know that ℓ^∞ and $(\ell^1 \oplus \mathbb{C})$ are Arens regular, and so by Theorem 2-3, $(\ell^1 \oplus \mathbb{C})\hat{\otimes}\ell^\infty$ is Arens regular. □

Let A and B be Banach algebras. A bilinear form $m : A \times B \to \mathbb{C}$ is said to be biregular, if for any two pairs of sequence $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1, we have

$$\lim_i \lim_j m(a_i \tilde{a}_j, b_i \tilde{b}_j) = \lim_j \lim_i m(a_i \tilde{a}_j, b_i \tilde{b}_j)$$

provided that these limits exist.
There are some example of biregular non regular bilinear form that for more information see [22].

Corollary 2-6. Suppose that A and B are Banach algebras. Then we have the following assertions.

(1) By conditions of Theorem 2-1, every bilinear form $m : A \times B \to \mathbb{C}$ is biregular.
(2) By conditions of Theorem 2-3 (2), every bilinear form $m : A \times B \to \mathbb{C}$ is biregular.

Example 2-7. Every bilinear form $m : (\ell^1 \oplus \mathbb{C}) \times \ell^\infty \to \mathbb{C}$ is Arens regular.

Proof. By notice to Example 2-5 and [22, Theorem 3.4], proof is hold. □

In the following we give simple proof for biregularity of bilinear form $m : A \times B \to \mathbb{C}$ such that $m(a, b) = \langle u(a), b\rangle$ where $u : A \to B^*$ is continuous linear operator that is introduced in [22 , Theorem 3.4].

Theorem 2-8 [22]. Let A and B be Banach algebras and $u : A \to B^*$ is continuous linear operator. Then the bilinear form $m : A \times B \to \mathbb{C}$ defined by $m(a, b) = \langle u(a), b\rangle$ is biregular.

Proof. Let $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1 be such that the following iterated limits exist

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) \textit{ and } \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j).$$

By [8, p.424], from these sequences we can extract $(a_\alpha)_\alpha$, $(\tilde{a}_\beta)_\beta$ in A and $(b_\alpha)_\alpha$, $(\tilde{b}_\beta)_\beta$ in B such that $a_\alpha \xrightarrow{w^*} a''$ and $\tilde{a}_\beta \xrightarrow{w^*} \tilde{a}''$ in A^{**} and we have also $b_\alpha \xrightarrow{w^*} b''$ and $\tilde{b}_\beta \xrightarrow{w^*} \tilde{b}''$ in B^{**}. Since A and B are Arens regular, we have

$$\lim_\alpha \lim_\beta a_\alpha\tilde{a}_\beta = \lim_\beta \lim_\alpha a_\alpha\tilde{a}_\beta = a''\tilde{a}''$$

$$\textit{and}$$

$$\lim_\alpha \lim_\beta b_\alpha\tilde{b}_\beta = \lim_\beta \lim_\alpha b_\alpha\tilde{b}_\beta = b''\tilde{b}''$$

Then, since u is continuous, we have

$$\lim_\alpha \lim_\beta m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) = \lim_\alpha \lim_\beta \langle u(a_\alpha\tilde{a}_\beta), b_\alpha\tilde{b}_\beta\rangle$$

$$= \langle u''(a''\tilde{a}''), b''\tilde{b}''\rangle.$$

Similarly, we have

$$\lim_\beta \lim_\alpha m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) = \langle u''(a''\tilde{a}''), b''\tilde{b}''\rangle.$$

Consequently we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j).$$

It follows that m is biregular. □

Example 2-9 [22]. Let A be a Banach algebra and $1 < p < \infty$. Then

(1) $\ell^p\hat{\otimes}A$ is Arens regular if and only if A is Arens regular.

(2) Let G be a locally compact group. Then, $L^p(G)\hat{\otimes}A$ is Arens regular if and only if A is Arens regular.

Proof. By using [22, Theorem 3.4] and Theorem 2-8, proof hold. □

3. Weak amenability of Banach algebras

For Banach algebra A, Dales, Rodrigues-Palacios and Velasco in [7] have been studied the weak amenability of A, when its second dual is weakly amenable. Mohamadzadih and Vishki in [19] have given simple solution to this problem with some other results, and Eshaghi Gordji and Filali in [10] have been studied this problem with some new results. In this section, We study this problem in the new way with some new results. Thus, for Banach $A-module$ B, we introduce some new concepts as $left-weak^*-weak$ convergence property [Lw^*wc-property] and $right-weak^*-weak$ convergence property [Rw^*wc-property] with respect to A and we show that if A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property and A^{**} is weakly amenable, then A is weakly amenable. We also show the relations between these properties and weak amenability of A. Now in the following, for left and right Banach $A-module$ B, we define, respectively, Lw^*wc-property and Rw^*wc-property concepts with some examples.

Definition 3-1. Assume that B is a left Banach $A-module$. Let $a''\in A^{**}$ and $(a_\alpha)_\alpha\subset A$ such that $a_\alpha\overset{w^*}{\to}a''$ in A^{**}. We say that $b'\in B^*$ has $left-weak^*-weak$ convergence property Lw^*wc-property with respect to A, if $b'a_\alpha\overset{w}{\to}b'a''$ in B^*.
When every $b'\in B^*$ has Lw^*wc-property with respect to A, we say that B^* has Lw^*wc-property.
The definition of $right-weak^*-weak$ convergence property [$=Rw^*wc-$property] with respect to A is similar and if $b'\in B^*$ has $left-weak^*-weak$ convergence property and $right-weak^*-weak$ convergence property, then we say that $b'\in B^*$ has $weak^*-weak$ convergence property [$=w^*wc-$property].
By using Lemma 3.1 from [17], it is clear that if A^* has Lw^*wc-property, then A is Arens regular.
Assume that B is a left Banach $A-module$. We say that $b'\in B^*$ has $left-weak^*-weak$ convergence property to zero Lw^*wc-property to zero with respect to A, if for every $(a_\alpha)_\alpha\subset A$, $b'a_\alpha\overset{w^*}{\to}0$ in B^* implies that $b'a_\alpha\overset{w}{\to}0$ in B^*.

Example 3-2 .

(1) Every reflexive Banach $A-module$ has w^*wc-property.
(2) Let Ω be a compact group and suppose that $A=C(\Omega)$ and $B=M(\Omega)$. Let $(a_\alpha)_\alpha\subseteq A$ and $\mu\in B$. Suppose that $\mu a_\alpha\overset{w^*}{\to}0$, then for each $a\in A$, we have

$$\langle\mu a_\alpha,a\rangle=\langle\mu,a_\alpha*a\rangle=\int_\Omega(a_\alpha*a)d\mu\to 0.$$

We set $a = 1_\Omega$. Then $\mu(a_\alpha) \to 0$. Now let $b' \in B^*$. Then

$$\langle b', \mu a_\alpha\rangle = \langle a_\alpha b', \mu\rangle = \int_\Omega a_\alpha b' d\mu \leq \| b' \| \mid \int_\Omega a_\alpha d\mu \mid = \| b' \| \mid \mu(a_\alpha) \mid \to 0.$$

It follows that $\mu a_\alpha \stackrel{w}{\to} 0$, and so that μ has Rw^*wc-property to zero with respect to A.

Let now B be a Banach $A-bimodule$, and let

$$\pi_\ell : \ A \times B \to B \ and \ \pi_r : \ B \times A \to B.$$

be the left and right module actions of A on B, respectively. Then B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{***} : \ A^{**} \times B^{**} \to B^{**} \ and \ \pi_r^{***} : \ B^{**} \times A^{**} \to B^{**}.$$

Similarly, B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{t***t} : \ A^{**} \times B^{**} \to B^{**} \ and \ \pi_r^{t***t} : \ B^{**} \times A^{**} \to B^{**}.$$

For a Banach $A-bimodule$ B, we define the topological centers of the left and right module actions of A on B as follows:

$$\begin{aligned} Z^\ell_{A^{**}}(B^{**}) = Z(\pi_r) = \{ & b'' \in B^{**} : \ the \ map \ a'' \to \pi_r^{***}(b'', a'') \ : \ A^{**} \to B^{**} \\ & is \ weak^* - weak^* \ continuous\} \\ Z^\ell_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{ & a'' \in A^{**} : \ the \ map \ b'' \to \pi_\ell^{***}(a'', b'') \ : \ B^{**} \to B^{**} \\ & is \ weak^* - weak^* \ continuous\} \\ Z^r_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{ & b'' \in B^{**} : \ the \ map \ a'' \to \pi_\ell^{t***}(b'', a'') \ : \ A^{**} \to B^{**} \\ & is \ weak^* - weak^* \ continuous\} \\ Z^r_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{ & a'' \in A^{**} : \ the \ map \ b'' \to \pi_r^{t***}(a'', b'') \ : \ B^{**} \to B^{**} \\ & is \ weak^* - weak^* \ continuous\}. \end{aligned}$$

Theorem 3-3. i) Assume that B is a left Banach $A-module$. If $B^*A^{**} \subseteq B^*$, then B^* has Lw^*wc-property.
ii) Assume that B is a right Banach $A-module$. If $A^{**}B^* \subseteq B^*$ and $Z^r_{A^{**}}(B^{**}) = B^{**}$, then B^* has Rw^*wc-property.

Proof. i) Assume that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \stackrel{w^*}{\to} a''$. Then for every $b'' \in B^{**}$, since $b'a'' \in B^*$, we have

$$< b'', b'a'' >=< a''b'', b' >= \lim_\alpha < a_\alpha b'', b' >= \lim_\alpha < b'', b'a_\alpha > .$$

It follows that $b'a_\alpha \stackrel{w}{\to} b'a''$.
ii) Proof is similar to (i). □

Theorem 3-4. Let A be a Banach algebra and suppose that A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property. If A^{**} is weakly amenable, then A is weakly amenable.

Proof. Assume that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$. Then for each $a' \in A^*$, we have $a_\alpha a' \overset{w^*}{\to} a''a'$ in A^*. Since A^* has Rw^*wc-property, $a_\alpha a' \overset{w}{\to} a''a'$ in A^*. Then for every $x'' \in A^{**}$, we have

$$\langle x''a_\alpha, a'\rangle = \langle x'', a_\alpha a'\rangle \to \langle x'', a''a'\rangle = \langle x''a'', a'\rangle.$$

It follows that $x''a_\alpha \overset{w^*}{\to} x''a''$. Since A^{**} has Lw^*wc-property with respect to A, $x''a_\alpha \overset{w}{\to} x''a''$. If $D : A \to A^*$ is a bounded derivation, we extend it to a bounded linear mapping D'' from A^{**} into A^{***}. Suppose that $a'', b'' \in A^{**}$ and $(a_\alpha)_\alpha, (b_\beta)_\beta \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$ and $b_\beta \overset{w^*}{\to} b''$. Since $x''a_\alpha \overset{w}{\to} x''a''$ for every $x'' \in A^{**}$, we have

$$\lim_\alpha \langle D''(b''), x''a_\alpha\rangle = \langle D''(b''), x''a''\rangle.$$

In the following we take limit on the *weak** topologies. Thus we have

$$\lim_\alpha \lim_\beta D(a_\alpha)b_\beta = D''(a'')b''.$$

Consequently, we have

$$D''(a''b'') = \lim_\alpha \lim_\beta D(a_\alpha b_\beta) = \lim_\alpha \lim_\beta D(a_\alpha)b_\beta + \lim_\alpha \lim_\beta a_\alpha D(b_\beta)$$
$$= D''(a'')b'' + a''D''(b'').$$

Since A^{**} is weakly amenable, there is $a''' \in A^{***}$ such that $D'' = \delta_{a'''}$. We conclude that $D = D'' \mid_A = \delta_{a'''} \mid_A$. Hence for each $x' \in A^*$, we have $D = x'a''' \mid_A - a''' \mid_A x'$. Take $a' = a''' \mid_A$. It follows that $H^1(A, A^*) = 0$. □

Theorem 3-5. Let A be a Banach algebra and suppose that $D : A \to A^*$ is a surjective derivation. If D'' is a derivation, then we have the following assertions.

(1) A^* and A^{**}, respectively, have w^*wc-property and Lw^*wc-property with respect to A.
(2) For every $a'' \in A^{**}$, the mapping $x'' \to a''x''$ from A^{**} into A^{**} is *weak*$^*-$ *weak* continuous.
(3) A is Arens regular.
(4) If A has $LBAI$, then A is reflexive.

Proof. (1) Since D is surjective, D'' is surjective, and so by using [19, Theorem 2.2], we have $A^{***}A^{**} \subseteq D''(A^{**})A^{**} \subseteq A^*$. Suppose that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$. Then for each $x' \in A^*$, we have $x'a_\alpha \overset{w^*}{\to} x'a''$. Since $A^{***}A^{**} \subseteq A^*$, $x'a'' \in A^*$. Then for every $x'' \in A^{**}$, we have

$$\langle x'', x'a_\alpha\rangle = \langle x''x', a_\alpha\rangle \to \langle a'', x''x'\rangle = \langle x'a'', x''\rangle = \langle x'', x'a''\rangle.$$

It follows that $x'a_\alpha \overset{w}{\to} x'a''$ in A^*. Thus x' has Lw^*wc-property with respect to A. The proof that x' has Rw^*wc-property with respect to A is similar, and so A^* has w^*wc-property.

Suppose that $x''' \in A^{***}$. Since $A^{***}A^{**} \subseteq A^*$, $x''a_\alpha \overset{w^*}{\to} x''a''$ for each $x'' \in A^{**}$. Then

$$\langle x''', x''a_\alpha\rangle = \langle x'''x'', a_\alpha\rangle \to \langle x'''x'', a''\rangle = \langle x''', x''a''\rangle.$$

It follows that $x''a_\alpha \overset{w}{\to} x''a''$. Thus x'' has Lw^*wc-property with respect to A.

(2) Suppose that $(a''_\alpha)_\alpha \subseteq A^{**}$ and $a''_\alpha \overset{w^*}{\to} a''$. Let $x'' \in A^{**}$. Then for every $x''' \in A^{***}$, since $A^{***}A^{**} \subseteq A^*$, we have

$$\langle x''', x''a''_\alpha\rangle = \langle x'''x'', a''_\alpha\rangle \to \langle x'''x'', a''\rangle = \langle x''', x''a''\rangle.$$

(3) It follows from (2).

(4) Let $(e_\alpha)_\alpha \subseteq A$ be a $BLAI$ for A. Then without loss generality, let e'' be a left unit for A^{**} such that $e_\alpha \overset{w^*}{\to} e''$. Suppose that $(a''_\alpha)_\alpha \subseteq A^{**}$ and $a''_\alpha \overset{w^*}{\to} a''$. Then for every $a''' \in A^{***}$, since $A^{***}A^{**} \subseteq A^*$, we have

$$\langle a''', a''_\alpha\rangle = \langle a''', e''a''_\alpha\rangle = \langle a'''e'', a''_\alpha\rangle \to \langle a'''e'', a''\rangle = \langle a''', a''\rangle.$$

It follows that $a''_\alpha \overset{w}{\to} a''$. Consequently A is reflexive.

□

Corollary 3-6. Let A be a Banach algebra and suppose that $D : A \to A^*$ is a surjective derivation. Then the following statements are equivalent.

(1) A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property.

(2) For every $a'' \in A^{**}$, the mapping $x'' \to a''x''$ from A^{**} into A^{**} is $weak^* - weak$ continuous.

Problems.

1. Let G be a locally compact group. What can say for the following sets?

(1) $Z^\ell_{L^1(G)^{**}}((L^1(G)\hat{\otimes}L^1(G))^{**}) =?$, $Z^r_{L^1(G)^{**}}((L^1(G)\hat{\otimes}L^1(G))^{**}) =?$

(2) $Z^\ell_{(L^1(G)\hat{\otimes}L^1(G))^{**}}(L^1(G)^{**}) =?$, $Z^r_{(L^1(G)\hat{\otimes}L^1(G))^{**}}(L^1(G)^{**}) =?$

(3) $Z^\ell_{L^1(G)^{**}}(L^1(G)^{**}\hat{\otimes}L^1(G)^{**}) =?$, $Z^r_{L^1(G)^{**}}(L^1(G)^{**}\hat{\otimes}L^1(G)^{**}) =?$

2. Suppose that S is a compact semigroup. Dose $L^1(S)^*$ and $M(S)^*$ have Lw^*wc-property or Rw^*wc-property?

References

1. R. E. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839-848.
2. W. G. Bade, P.C. Curtis and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebra*, Proc. Lodon Math. Soc. **50** (1987) 359-377.
3. J. Baker, A. T. Lau, J.S. Pym *Module homomorphism and topological centers associated with weakly sequentially compact Banach algebras*, Journal of Functional Analysis. **158** (1998), 186-208.
4. F. F. Bonsall, J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin 1973.
5. H. G. Dales, *Banach algebra and automatic continuity*, Oxford 2000.

6. H. G. Dales, F. Ghahramani, N. Grønbæk *Derivation into iterated duals of Banach algebras* Studia Math. **128** 1 (1998), 19-53.
7. H. G. Dales, A. Rodrigues-Palacios, M.V. Velasco, *The second transpose of a derivation*, J. London. Math. Soc. **2** 64 (2001) 707-721.
8. N. Dunford, J. T. Schwartz, *Linear operators.I*, Wiley, New york 1958.
9. K. Haghnejad Azar, A. Riazi *Topological centers of the $n-th$ dual of module actions*, To apper in Bulletin of Iranian Math. Soc.
10. M. Eshaghi Gordji, M. Filali, *Weak amenability of the second dual of a Banach algebra*, Studia Math. **182** 3 (2007), 205-213.
11. F. Gourdeau, *Amenability of Lipschits algebra*, Math. Proc. Cambridge. Philos. Soc. **112** (1992), 581-588.
12. E. Hewitt, K. A. Ross, *Abstract harmonic analysis*, Springer, Berlin, Vol I 1963.
13. B.E. Johoson, *Cohomology in Banach algebra*, Mem. Amer. Math. Soc. **127**, 1972.
14. B. E. Johoson, *Derivation from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$*, Harmonic analysis. Luxembourg 1987, 191-198 Lecture Note in Math., **1359**, Springer, Berlin, 1988. MR **90a**:46122.
15. B. E. Johoson, *Weak amenability of group algebra*, Bull. London. Math. Soc. **23**(1991), 281-284.
16. A. T. Lau, V. Losert, *On the second Conjugate Algebra of locally compact groups*, J. London Math. Soc. **37** (2)(1988), 464-480.
17. A. T. Lau, A. Ülger, *Topological center of certain dual algebras*, Trans. Amer. Math. Soc. **348** (1996), 1191-1212.
18. V. Losert, *The derivation problem for group algebra*, Annals of Mathematics, **168** (2008), 221-246.
19. S. Mohamadzadih, H. R. E. Vishki, *Arens regularity of module actions and the second adjoint of a derivation*, Bulletin of the Australian Mathematical Society **77** (2008), 465-476.
20. J. S. Pym, *The convolution of functionals on spaces of bounded functions*, Proc. London Math Soc. **15** (1965), 84-104.
21. V. Runde, *Lectures on the amenability*, springer-verlag Berlin Heideberg NewYork.
22. A. Ülger, *Arens regularity of the algebra $A\hat{\otimes}B$*, Trans. Amer. Math. Soc. **305** (2) (1988) 623-639.
23. A. Ülger, *Some stability properties of Arens regular bilinear operators*, Proc. Amer. Math. Soc. (1991) **34**, 443-454.
24. A. Ülger, *Arens regularity of weakly sequentialy compact Banach algebras*, Proc. Amer. Math. Soc. **127** (11) (1999), 3221-3227.
25. P. K. Wong, *The second conjugate algebras of Banach algebras*, J. Math. Sci. **17** (1) (1994), 15-18.
26. Y. Zhing, *Weak amenability of module extentions of Banach algebras*, Trans. Amer. Math. Soc. **354** (10) (2002), 4131-4151.
27. Y. Zhing, *Weak amenability of a class of Banach algebra*, Cand. Math. Bull. **44** (4) (2001) 504-508.