

The Hodge theory of character varieties

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Abstract

This is a report on joint work with T. Hausel and L. Migliorini, where we prove, for each of the groups $GL_{\mathbb{C}}(2)$, $PGL_{\mathbb{C}}(2)$ and $SL_{\mathbb{C}}(2)$, that the non-Abelian Hodge theorem identifies the weight filtration on the cohomology of the character variety with the perverse Leray filtration on the cohomology of the domain of the Hitchin map. We review the decomposition theorem, Ngô’s support theorem, the geometric description of the perverse filtration and the sub-additivity of the Leray filtration with respect to the cup product.

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1 Introduction

This is an expanded version of notes from my talk at the conference “Classical Algebraic Geometry Today,” M.S.R.I., Berkeley, January 25-29, 2009. The talk consisted of a report on the joint work [9] with T. Hausel at Oxford and L. Migliorini at Bologna.

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Following the recommendation of the editors, I would like this note to be accessible to non specialists and to give a small glimpse into an active area of research. The reader is referred to the introduction of [9] for more details on what follows.

Let C be a nonsingular complex projective curve. We consider the following two moduli spaces associated with C : $\mathcal{M} := \mathcal{M}_{Dolbeault} :=$ the moduli space of stable holomorphic rank two Higgs bundles on C of degree one (see §3) and the character variety $\mathcal{M}' := \mathcal{M}_{Betti}$, i.e. the moduli space of irreducible complex dimension two representations of $\pi_1(C - p)$ subject to the condition that a loop around the chosen point $p \in C$ is sent to $-Id$. There is an analogous picture associated with any complex reductive Lie group G and the above corresponds to the case $G = GL_{\mathbb{C}}(2)$. Our paper [9] deals only with the cases when $G = GL_{\mathbb{C}}(2), PGL_{\mathbb{C}}(2), SL_{\mathbb{C}}(2)$. Both \mathcal{M} and \mathcal{M}' are quasi-projective irreducible and nonsingular of some even dimension $2d$. While \mathcal{M} depends on the complex structure of C , \mathcal{M}' does not. There is a proper flat and surjective map, the Hitchin map, $h : \mathcal{M} \rightarrow \mathbb{C}^d$ with general fibers Abelian varieties of dimension d ; in particular, \mathcal{M} is not affine: it contains complete subvarieties of positive dimension. On the other hand, \mathcal{M}' is easily seen to be affine (it is a GIT quotient of an affine variety).

The non-Abelian Hodge theorem states that the two moduli spaces $\mathcal{M}_{Dolbeault}$ and \mathcal{M}_{Betti} are naturally diffeomorphic, i.e. that there is a natural diffeomorphism $\varphi : \mathcal{M} \simeq \mathcal{M}'$. Since \mathcal{M}' is affine (resp. Stein) and \mathcal{M} is not affine (resp. not Stein), the map φ is not algebraic (resp. not holomorphic). Of course, we can still deduce that φ^* is a natural isomorphism on the singular cohomology groups.

Let us point out that the mixed Hodge structure on the cohomology groups $H^j(\mathcal{M}, \mathbb{Q})$ is in fact pure, i.e. every class has type (p, q) with $p+q = j$, or equivalently, every class has weight j . This follows easily from the fact that, due to the nonsingularity of \mathcal{M} , the weights of $H^j(\mathcal{M}, \mathbb{Q})$ must be $\geq j$. It remains to show that the weights are also $\leq j$: the variety \mathcal{M} admits the fiber $h^{-1}(0)$ of the Hitchin map over the origin $0 \in \mathbb{C}^d$ as a deformation retract; it follows that the restriction map in cohomology, $H^j(\mathcal{M}, \mathbb{Q}) \rightarrow H^j(h^{-1}(0), \mathbb{Q})$ is an isomorphism of mixed Hodge structures; since the central fiber is compact, the weights of $H^j(h^{-1}(0), \mathbb{Q})$ are $\leq j$, and we are done.

On the other hand, the mixed Hodge structure on the cohomology groups $H^j(\mathcal{M}', \mathbb{Q})$ is known to be non pure (cf. [14]), i.e. there are classes of degree j but weight $> j$.

It follows that the isomorphism φ^* is not compatible with the two weight filtrations \mathcal{W} on $H^*(\mathcal{M}, \mathbb{Q})$ and \mathcal{W}' on $H^*(\mathcal{M}', \mathbb{Q})$. This fact raises the following question: *if we transplant the weight filtration \mathcal{W}' onto $H^*(\mathcal{M}, \mathbb{Q})$ via φ^* , can we interpret the resulting filtration on $H^*(\mathcal{M}, \mathbb{Q})$, still called \mathcal{W}' , in terms of the geometry of \mathcal{M} ?*

The main result in [9] is Theorem 5.1 below and it gives a positive answer to the question raised above. In order to state this answer, we need to introduce one more ingredient and to make some trivial renumerations. (In this paper, we only deal with increasing filtrations.) The Hitchin map $h : \mathcal{M} \rightarrow \mathbb{C}^d$ gives rise to the perverse Leray filtration ${}^p\mathcal{L} = {}^p\mathcal{L}_h$ on $H^*(\mathcal{M}, \mathbb{Q})$; this is a suitable variant of the ordinary Leray filtration for h ; for a geometric description of the perverse Leray filtration see Theorem 4.5. We renumerate the filtration ${}^p\mathcal{L}$ so that $1 \in H^0(\mathcal{M}, \mathbb{Q})$ is in place zero (see (10)); the resulting

renumerated filtration on $H^*(\mathcal{M}, \mathbb{Q})$ is denoted by P .

All the actual weights appearing in \mathcal{W}' on $H^*(\mathcal{M}', \mathbb{Q})$ turn out to be multiples of four. We renumerate \mathcal{W}' by setting $W'_k := \mathcal{W}'_{2k}$.

Our answer to the question above is: *the non-Abelian Hodge theorem isomorphism φ^* identifies the weight filtration W' on $H^*(\mathcal{M}', \mathbb{Q})$ with the perverse Leray filtration P on $H^*(\mathcal{M}, \mathbb{Q})$:*

$$P = W'.$$

The nature of these two filtrations being very different, we find this coincidence intriguing, but at present we cannot explain it beyond the fact that we can observe it.

The proof of Theorem 5.1 uses a few ideas from the topology of algebraic maps. Notably, Ngô's support theorem ([18]), the geometric description of the perverse filtration ([8]) and the explicit knowledge of the cohomology ring $H^*(\mathcal{M}_{Betti}, \mathbb{Q})$ ([23]) and of its mixed Hodge structure ([14]).

One of the crucial ingredients we need is Theorem 5.3, which may be of independent interest: it observes that Ngô's support theorem for the Hitchin fibration, i.e. (4) below, can be refined rather sharply, in the rank two cases we consider, as follows: the intersection complexes appearing in (4) are in fact sheaves (up to a dimensional shift).

What follows is a summary of the contents of this paper. §2 is devoted to stating the decomposition theorem for proper maps of algebraic varieties and to defining the associated “supports.” §3 states Ngô's support theorem ([18], §7) and sketches a proof of it in a special case and under a very strong splitting assumption that does not occur in practice; the purpose here is only to explain the main idea behind this beautiful result. §4 is a discussion of the main result of [8], i.e. a description of the perverse filtration in cohomology with coefficients in a complex via the restriction maps in cohomology obtained by taking hyperplane sections. §5 states the main result in [9], Theorem 5.1, and discusses some of the other key ingredients in the proof, notably the use of the sub-additivity of the ordinary Leray filtration with respect to cup products. Since I could not find a reference in the literature for this well-known fact, I have included a proof of it in the more technical §6.

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1.1 Notation

We work with sheaves of either Abelian groups, or of rational vector spaces over complex algebraic varieties. The survey [7] is devoted to the decomposition theorem and contains a more detailed discussion of what follows.

A sheaf F on a variety Y is constructible if there is a finite partition $Y = \coprod T_i$ into nonsingular locally closed irreducible subvarieties that is adapted to F , i.e. such that

each $F|_{T_i}$ is a local system (i.e. a locally constant sheaf) on T_i . A constructible complex K on a variety Y is a bounded complex of sheaves whose cohomology sheaves $\mathcal{H}^i(K)$ are constructible. We denote by D_Y the corresponding full subcategory of the derived category of sheaves on Y . If $K \in D_Y$, then $H^i(Y, K)$ denotes the i -th cohomology group of Y with coefficients in K . Similarly, for $H_c^i(Y, K)$. The complex $K[n]$ has i -th entry K^{n+i} and differential $d_{K[n]}^i = (-1)^n d_K^{n+i}$.

The standard truncation functors are denoted by $\tau_{\leq i}$, the perverse (middle perversity) ones by ${}^p\tau_{\leq i}$. The perverse cohomology sheaves are denoted ${}^p\mathcal{H}^i(K)$, $i \in \mathbb{Z}$. We make some use of these notions in §6. Recall that if $K \in D_Y$, then ${}^p\mathcal{H}^i(K) \neq 0$ for finitely many values of $i \in \mathbb{Z}$. In general, the collection of perverse cohomology sheaves $\{{}^p\mathcal{H}^i(K)\}_{i \in \mathbb{Z}}$ does not determine the isomorphism class of K in D_Y ; e.g. if $j : U \rightarrow X$ is the open immersion of the complement of a point p in a nonsingular surface X , then the sheaves $j_!\mathbb{Q}_U$ and $\mathbb{Q}_X \oplus \mathbb{Q}_p$, viewed as complexes in D_X , yield the same collection ${}^p\mathcal{H}^0(-) = \mathbb{Q}_p$, ${}^p\mathcal{H}^2(-) = \mathbb{Q}_X[2]$. On the other hand, the celebrated decomposition theorem (Theorem 2.4 below) implies that if $f : X \rightarrow Y$ is a proper map of algebraic varieties, with X nonsingular for example, then the direct image complex $Rf_*\mathbb{Q}_X \simeq \bigoplus_i {}^p\mathcal{H}^i(Rf_*\mathbb{Q}_X)[-i]$: this implies that the perverse cohomology sheaves reconstitute, up to an isomorphism, the direct image complex; more is true: each perverse cohomology sheaf splits further into a direct sum of simple intersection complexes (cf. (2)).

We have the following subcategories of D_Y : $D_Y^{\leq 0} := \{K \mid \text{s.t. } \mathcal{H}^i(K) = 0, \forall i > 0\}$ and ${}^pD_Y^{\leq 0} := \{K \mid \dim \text{supp } \mathcal{H}^i(K) \leq -i, \forall i \in \mathbb{Z}\}$. More generally, a perversity p gives rise to truncation functors ${}^p\tau_{\leq i}$, subcategories ${}^pD_Y^{\leq i}$ and cohomology complexes ${}^p\mathcal{H}^i(K)$.

Filtrations on Abelian groups H are assumed to be finite: if the filtration F_\bullet on H is increasing, then $F_i H = 0$ for $i \ll 0$ and $F_i H = H$ for $i \gg 0$; if F^\bullet is decreasing, then it is the other way around. One can switch type by setting $F_i = F^{-i}$. For $i \in \mathbb{Z}$, the i -th graded objects are defined by setting $Gr_i^F H := F_i H / F_{i-1} H$. The increasing standard filtration \mathcal{S} on $H^j(Y, K)$ is defined by setting $\mathcal{S}_i H^j(K) := \text{Im} \{H^j(Y, \tau_{\leq i} K) \rightarrow H^j(Y, K)\}$. Similarly, for ${}^p\mathcal{S}$ and more generally for ${}^p\mathcal{S}$. These filtrations are the abutment of corresponding spectral sequences.

Let $f : X \rightarrow Y$ be a map of varieties. The symbol f_* ($f_!$, resp.) denotes the derived direct image Rf_* (with proper supports $Rf_!$, resp.). Let $C \in D_X$. The direct image sheaves are denoted $R^j f_* C$. We have $H^j(X, C) = H^j(Y, f_* C)$, $H_c^j(X, C) = H_c^j(Y, f_! C)$.

The Leray filtration is defined by setting $\mathcal{L}_i H^j(X, C) := \mathcal{S}_i H^j(Y, f_* C)$ and it is the abutment of the Leray spectral sequence. Similarly, for $H_c^j(X, C)$. Given a perversity p , we have the p -Leray spectral sequence abutting to the p -Leray filtration ${}^p\mathcal{L}$. We reserve the terms perverse Leray spectral sequence and perverse Leray filtration to the case of middle perversity $p = \mathfrak{p}$.

If X is smooth and f is proper, then we let $Y_{reg} \subseteq Y$ be the Zariski open set of regular values of f and we denote by R^i the local system $(R^i f_* \mathbb{Q}_X)|_{Y_{reg}}$ on Y_{reg} .

2 The decomposition theorem

The purpose of this section is to state the decomposition theorem 2.4 and to introduce the related notion of supports.

Let $f : X \rightarrow Y$ be a map of varieties. The Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_* \mathbb{Q}_X) \implies H^{p+q}(X, \mathbb{Q})$$

relates the operation of taking cohomology on Y to the same operation on X . If we have E_2 -degeneration, i.e. $E_2 = E_\infty$, then we have an isomorphism

$$H^j(X, \mathbb{Q}) \simeq \bigoplus_{p+q=j} H^p(Y, R^q f_* \mathbb{Q}). \quad (1)$$

Example 2.1 Let $f : X \rightarrow Y$ be a resolution of the singularities of the projective variety Y . Let us assume, as it is often the case, that the mixed Hodge structure on $H^j(Y, \mathbb{Q})$ is not pure for some j . Then $f^* : H^j(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ is not injective and E_2 -degeneration fails; this is because injectivity would imply the purity of the mixed Hodge structure on $H^j(Y, \mathbb{Q})$.

Example 2.2 Let $f : (\mathbb{C}^2 - \{(0,0)\})/\mathbb{Z} \rightarrow \mathbb{C}P^1$ be a Hopf surface (see [2]) together with its natural holomorphic proper submersion onto the projective line. Since the first Betti number of the Hopf surface is one and the one of a fiber is two, E_2 -degeneration fails.

These examples show that we cannot expect E_2 -degeneration, neither for holomorphic proper submersions of compact complex manifolds, nor for projective maps of complex projective varieties. On the other hand, the following result of P. Deligne [11] shows that E_2 -degeneration is the norm for proper submersions of complex algebraic varieties.

Theorem 2.3 *Let $f : X \rightarrow Y$ be a smooth proper map of complex algebraic varieties. Then the Leray spectral sequence for f is E_2 -degenerate. More precisely, there is an isomorphism in D_Y*

$$f_* \mathbb{Q}_X \simeq \bigoplus_i R^i f_* \mathbb{Q}_X[-i].$$

The decomposition theorem is a far-reaching generalization of Theorem 2.3 that involves intersection cohomology, a notion that we review briefly next. A complex algebraic variety Y of dimension $\dim_{\mathbb{C}} Y = n$ carries intersection cohomology groups $IH^*(Y, \mathbb{Q})$ and $IH_c^*(Y, \mathbb{Q})$ such that

1. Poincaré duality holds: there is a geometrically defined perfect pairing

$$IH^{n+j}(Y) \times IH_c^{n-j}(Y) \longrightarrow \mathbb{Q}.$$

2. There is the intersection complex IC_Y ; it is a constructible complex of sheaves of rational vector spaces on Y such that:

$$IH^j(Y, \mathbb{Q}) = H^{j-n}(Y, IC_Y), \quad IH_c^j(Y, \mathbb{Q}) = H_c^{j-n}(Y, IC_Y).$$

3. If Y is nonsingular, then $IH^*(Y, \mathbb{Q}) = H^*(Y, \mathbb{Q})$ and $IC_Y = \mathbb{Q}_Y[n]$ (complex with the one entry \mathbb{Q}_Y in cohomological degree $-n$).
4. If Y^o is a non-empty open subvariety of the nonsingular locus of Y and L is a local system on Y^o , then we have the twisted intersection complex $IC_Y(L)$ on Y and the intersection cohomology groups $IH^j(Y, L) = H^{j-n}(Y, IC_Y(L))$ of Y with coefficients in L .

Theorem 2.4 (Decomposition theorem) (See [3], Théorème 6.2.5.)

Let $f : X \rightarrow Y$ be a proper map of algebraic varieties. Then

$$f_*IC_X \simeq \bigoplus_{b \in B} IC_{Z_b}(L_b)[d_b] \quad (2)$$

for an uniquely determined finite collection B of triples (Z_b, L_b, d_b) such that $Z_b \subseteq Y$ is a closed irreducible subvariety, $L_b \neq 0$ is a simple local system on some non-empty and nonsingular Zariski open $Z_b^o \subseteq Z_b$ and $d_b \in \mathbb{Z}$.

If, in Theorem 2.4, we replace “simple” with “semisimple,” then we obtain a uniquely determined collection B' by grouping together the terms with the same cohomological shift $[d_b]$ and the same irreducible subvariety Z_b .

Definition 2.5 The varieties $Z_b \subseteq Y$ are called the *supports* of the map $f : X \rightarrow Y$.

The supports Z_b are *among* the closed irreducible subvarieties $Z \subseteq Y$ with the property that

1. $\exists \emptyset \neq Z^o \subseteq Z$ over which all the direct image sheaves $R^i f_* \mathbb{Q}$ are local systems, and
2. Z is maximal with this property.

The following example shows that a support may appear more than once with distinct cohomological shifts. Of course, that happens already in the situation of Theorem 2.3; the point of the example is that this “repeated support” may be smaller than the image $f(X)$.

Example 2.6 Let $f : X \rightarrow Y = \mathbb{C}^3$ be the blowing-up of a point $o \in \mathbb{C}^3$; there is an isomorphism

$$f_* \mathbb{Q}_X[3] \simeq \mathbb{Q}_Y[3] \oplus \mathbb{Q}_o[1] \oplus \mathbb{Q}_o[-1].$$

The next example shows that a variety Z , in this case $Z = v$, that satisfies conditions 1 and 2 above, may fail to be a support.

Example 2.7 Let $f : X \rightarrow Y$ be the small resolution of the three-dimensional affine cone $Y \subseteq \mathbb{C}^4$ over a nonsingular quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathcal{Q} \subseteq \mathbb{P}^3$, given by the contraction to the vertex $v \in Y$ of the zero section in the total space X of the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^2$. In this case, we have

$$Rf_* \mathbb{Q}_X[3] = IC_Y.$$

The determination of the supports of a proper map is an important and difficult problem.

3 Ngô's support theorem

B.C. Ngô has proved ([18]) the “fundamental lemma” in the Langlands program. This is a major advance in geometric representation theory, automorphic representation theory and the arithmetic Langlands program. See [17]. One of the crucial ingredients of the proof is the support Theorem 3.1, whose proof applies the decomposition theorem to the Hitchin map associated with a reductive group and a nonsingular projective curve. The support theorem is a rather general result concerning a certain class of fibrations with general fibers Abelian varieties and the Hitchin map is an important example of such a fibration.

In our paper [9] (to which I refer the reader for more context and references), we deal with the Hitchin map in the rank two case, i.e. with the reductive groups $GL_{\mathbb{C}}(2)$, $PGL_{\mathbb{C}}(2)$ and $SL_{\mathbb{C}}(2)$. The simpler geometry allows us to refine the conclusion (4) of the support theorem for the Hitchin map in the form of Fact 5.3, which in turn we use in [9] to prove Theorem 5.1.

In this section, we discuss the support theorem in the case of $GL_{\mathbb{C}}(2)$. This situation is too-simple in the context of the fundamental lemma, but it allows us to concentrate on the main idea underlying the proof of the support theorem, i.e. pursuing the action of Abelian varieties on the fibers of the Hitchin map. In the context of Ngô's work, it is critical to work over finite fields. We ignore this important aspect and, for the sake of exposition, we make the oversimplifying Assumption 3.2 and stick with the situation over \mathbb{C} .

Let C be a compact Riemann surface of genus $g \geq 2$. Let \mathcal{M} be the moduli space of stable rank 2 Higgs bundles on C with determinant of degree one. In this context, a point $m \in \mathcal{M}$ parametrizes a stable pair (E, φ) , where E is a rank two bundle on C with $\deg(\det E) = 1$ and $\varphi : E \rightarrow E \otimes \omega_C$ (where $\omega_C := T_C^*$ denotes the canonical bundle of C) is a map of bundles, i.e. a section of $\text{End}(E) \otimes \omega_C$. Stability is a technical condition on the degrees of the sub-bundles of E preserved by φ . Only the parity of $\deg(\det E)$ counts here: there are only two isomorphism classes of such moduli spaces; the case of even degree yields a singular moduli space and we do not say anything new in that case.

Let $d := 4g - 3$. The variety \mathcal{M} is nonsingular, quasi projective and of dimension $2d$. There is a proper and flat map, called the Hitchin map, onto affine space

$$h : \mathcal{M}^{2d} \longrightarrow \mathbb{A}^d \simeq H^0(C, \omega_C \oplus \omega_C^{\otimes 2}), \quad (3)$$

which is a completely integrable system.

Set-theoretically, the map $h : m = (E, \varphi) \mapsto (\text{trace}(\varphi), \det \varphi)$, where the trace and determinant of the twisted endomorphism φ are viewed as sections of the corresponding powers of ω_C .

A priori, it is far from clear that this map is proper. This fact was first noted and proved by Hitchin. It is a beautiful fact (also due to Hitchin) that each nonsingular fiber $\mathcal{M}_a := h^{-1}(a)$, $a \in \mathbb{A}^d$, is isomorphic to the Jacobian $J(C'_a)$ of what is called the spectral curve C'_a . This curve lives on the surface given by the total space of the line bundle ω_C

and it is given set-theoretically as the double cover of C given by (and this explains the term “spectral”)

$$C \ni \{c\} \longleftrightarrow \{\text{the set of eigenvalues of } \varphi_c\} \in \omega_{C,c}.$$

The genus $g(C'_a) = d$ by Riemann-Roch and by the Hurwitz formula.

The singular fibers of the Hitchin map $h : \mathcal{M} \rightarrow \mathbb{A}^d$ are, and this is an euphemism, difficult to handle.

Let $V \subseteq \mathbb{A}^d$ be the open locus over which the fibers of h are reduced. The sheaf $R_V^{2d} := (R^{2d}f_*\mathbb{Q})|_V$ is the \mathbb{Q} -linearization of the sheaf of finite sets given by the sets of irreducible components of the fibers over V . Let $h_V : \mathcal{M}_V := h^{-1}(V) \rightarrow V$ be the restriction of the Hitchin map over V .

We can now state Ngô’s support theorem in the very special case at hand. Roughly speaking, it states that over V , the highest direct image R_V^{2d} is responsible for all the supports.

Theorem 3.1 (Ngô’s support theorem) *A closed and irreducible subvariety $Z \subseteq V$ appears as a support Z_b in the decomposition theorem (2) for h_V , if and only if there is a dense open subvariety $Z^\circ \subseteq Z$ such that the restriction $(R_V^{2d})|_{Z^\circ}$ is locally constant and Z is maximal with this property.*

If we further restrict to the open set $U \subseteq V$ where the fibers are reduced and irreducible, then the support theorem has the following striking consequence: the only support on U is U itself. The decomposition theorem (2) for h_U takes then the following form (notation as in §1.1)

$$h_{U*}\mathbb{Q}_{\mathcal{M}_U}[2d] \simeq \bigoplus_{i=-d}^d IC_U(R^{i+d})[-i]. \quad (4)$$

The open U is fairly large: its complement has codimension $\geq 2g - 3$.

The remaining part of this section is devoted to discussing the main idea in the proof of the support theorem.

There is a group-variety $\mathcal{P}_V \rightarrow V$ over V acting on the variety $\mathcal{M}_V \rightarrow V$ over V , i.e. a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_V \times \mathcal{M}_V & \xrightarrow{a} & \mathcal{M}_V \\ & \searrow & \swarrow \\ & V & \end{array}$$

satisfying the axioms of an action.

Let us describe this situation over a point $v \in V$. The fiber \mathcal{M}_v is non-canonically isomorphic to a suitable compactification of the identity component \mathcal{P}_v of the Picard group $\text{Pic}(C'_v)$ of the possibly singular spectral curve C'_v . The variety \mathcal{M}_v parametrizes certain torsion free sheaves of rank and degree one on C'_v . The group variety \mathcal{P}_v acts on \mathcal{M}_v via

tensor product. There is an exact sequence (Chevalley devissage) of algebraic groups of the indicated dimensions

$$1 \longrightarrow R_v^{\delta_v} \longrightarrow \mathcal{P}_v^d \longrightarrow A_v^{d-\delta_v} \longrightarrow 1 \quad (5)$$

where A_v is the Abelian variety given by the Picard variety of the normalization of the spectral curve C'_v and R_v is an affine algebraic group. The sequence (5) does not split over the complex numbers, but it splits over a finite field. It turns out that this is enough in order to prove the freeness result on which the proof of the support theorem rests. In order to explain the main idea, let us make the following (over)simplifying assumption.

Assumption 3.2 *There is a splitting of the Chevalley devissage (5).*

A splitting induces an action of A_v on \mathcal{M}_v with finite stabilizers. There is the rational homology algebra $H_*(A_v)$ with product given by the Pontryagin product $H_i(A_v) \otimes H_j(A_v) \rightarrow H_{i+j}(A_v)$ induced by the cross product, followed by push-forward via the multiplication map in A_v . We have the following standard ([18], p.134, Proposition 7.4.5)

Fact 3.3 *Let $A \times T \rightarrow T$ be an action of an Abelian variety A on a variety T such that all stabilizers are finite. Then $H_c^*(T)$ is a free graded $H_*(A)$ -module for the action of the rational homology algebra $H_*(A)$ on $H_c^*(T)$.*

Our assumptions imply that

$$\forall v \in V, \quad H^*(\mathcal{M}_v) \text{ is a free graded } H_*(A_v)\text{-module.}$$

Let Z be a support appearing in the decomposition theorem (2) for h_V . Define a finite set of integers as follows

$$\text{Occ}(Z) := \{n \in \mathbb{Z} \mid \exists b \text{ s.t. } Z_b = Z, d_b = -n\} \subseteq [-d, d].$$

The integers in $\text{Occ}(Z)$ are in one-to-one correspondence with the summands (2) with support Z . By grouping them, we obtain the graded object

$$\mathfrak{I}_Z := \bigoplus_{n \in \text{Occ}(Z)} IC_Z(L^n)[-n].$$

Verdier duality is the generalization of Poincaré duality in the context of complexes. If we apply this duality to (2), we deduce that $\text{Occ}(Z)$ is symmetric about 0.

Every intersection complex $IC_Y(L)$ on an irreducible variety Y restricts to $L[\dim Y]$ on a suitable non-empty open subvariety $Y^o \subseteq Y$. It follows that there is a non-empty open subvariety $V^o \subseteq V$ such that every $IC_Z(L^n)$ restricts to $L^n[\dim Z]$ on $Z^o := Z \cap V^o$. Let us consider the restriction of \mathfrak{I}_Z to Z^o :

$$\mathfrak{L} := \bigoplus_{n \in \text{Occ}(Z)} L^n[\dim Z][-n].$$

If we set $n^+ := \max \text{Occ}(Z)$, then, by the aforementioned symmetry about the origin, the length $l(\mathfrak{L}) = 2n^+$.

The decomposition theorem (2) over V^o implies that

$$\forall n \in \text{Occ}(Z), \quad L^n \text{ is a direct summand of } (R^{2d+n-\dim Z} h_* \mathbb{Q})|_{Z^o}.$$

Since the fibers of h have dimension d , the higher direct images $R^j h_* \mathbb{Q}$ vanish for every $j > 2d$. It follows that the support theorem is equivalent to the following

Claim:

$$L^{n^+} \text{ is a direct summand of } (R^{2d} h_* \mathbb{Q})|_{Z^o}.$$

The Claim is equivalent to having $n^+ - \dim Z = 0$, and, again by the vanishing for the direct images $R^j h_* \mathbb{Q}$, this is equivalent to having $n^+ - \dim Z \geq 0$.

Let $z \in Z^o$ be any point. By adding and subtracting $\dim A_z = d - \delta_z$, we can reformulate the support theorem as follows:

$$[\text{codim } Z - \delta_z] + [n^+ - (d - \delta_z)] \geq 0.$$

It is thus enough to show that each of the two quantities in square brackets is ≥ 0 .

The first inequality $[\text{codim } Z - \delta_z] \geq 0$ follows from the deformation theory of Higgs bundles and Riemann-Roch on the curve C . This point is standard over the complex numbers. At present, in positive characteristic it requires the extra freedom of allowing poles of fixed but arbitrary high order. We do not address this point here.

Since $l(\mathfrak{L}) = 2n^+$, in order to prove the second inequality, we need to show that

$$l(\mathfrak{L}) = l(\mathfrak{L}_z) = 2n^+ \geq 2(d - \delta_z).$$

Since $l(H_*(A_z)) = 2 \dim A_z = 2(d - \delta_z)$, the inequality would follow if we could prove that:

$$\mathfrak{L}_z \text{ is a free graded } H_*(A_z)\text{-module}.$$

By virtue of the decomposition theorem, the graded vector space \mathfrak{L}_z is a graded vector subspace of $H^*(\mathcal{M}_z)$. This is not enough. We need to make sure that it is a free $H_*(A_z)$ -submodule. Once it is known that \mathfrak{L}_z is a submodule, then its freeness is an immediate consequence of standard results from algebra, notably that a projective module over the local graded commutative algebra $H_*(A_z)$ is free. Showing that \mathfrak{L}_z is $H_*(A_z)$ -stable is a delicate point, for a priori the contributions from other supports could enter the picture and spoil it. This problem is solved by means of a delicate specialization argument which we do not discuss here.

4 The perverse filtration and the Lefschetz hyperplane theorem

Let us review the classical construction that relates the Leray filtration on the cohomology of the total space a fiber bundle to the filtration by scheleta on the base.

Let $f : X \rightarrow Y$ be a topological fiber bundle where Y is a cell complex of real dimension n . Let $Y_* := \{Y_0 \subseteq \dots \subseteq Y_k \subseteq \dots \subseteq Y_n = Y\}$ be the filtration by k -scheleta. Let $X_* := \pi^{-1}(Y_*)$ be the corresponding filtration of the total space X .

If \mathcal{L} is the increasing Leray filtration associated with π , then we have (see [21], Ch. 9.4)

$$\mathcal{L}_i H^j(X, \mathbb{Z}) = \text{Ker} \{H^j(X, \mathbb{Z}) \rightarrow H^j(X_{j-i-1}, \mathbb{Z})\}. \quad (6)$$

The key fact that one needs (see [8], [4]) is the π -cellularity of Y_* , i.e. the fact that

$$H^j(Y_p, Y_{p-1}, R^q f_* \mathbb{Z}_X) = 0, \quad \forall j \neq p, \quad \forall q. \quad (7)$$

This condition is verified since, for each fixed p , we are really dealing with bouquets of p -spheres.

This classical result can be viewed as a geometric description of the Leray filtration in the sense that the subspaces of the Leray filtration are exhibited as kernels of restrictions maps to the pre-images of scheleta. The following result of D. Arapura [1] gives a geometric description of the Leray filtration for a projective map of quasi-projective varieties: the important point is that the “scheleta” can be taken to be algebraic subvarieties! For generalizations of Arapura’s result, see [4]. In what follows, for ease of exposition, we concentrate on the case when the target is affine.

Theorem 4.1 (Geometric description of the Leray filtration) *Let $f : X \rightarrow Y$ be a proper map of algebraic varieties with Y affine of dimension n . Then there is a filtration Y_* of Y by closed algebraic subvarieties Y_i of dimension i such that (6) holds.*

Remark 4.2 The flag Y_* is constructed inductively as follows. Choose a closed embedding $Y \subseteq \mathbb{A}^N$. Each Y_i is a complete intersection of Y with $n - i$ sufficiently high degree hypersurfaces in special position. Here “special” refers to the fact that in order to achieve the cellularity condition (7), we need to trace, as p decreases, the Y_{p-1} through the positive-codimension strata of a partition of Y_p adapted to the restricted sheaves $(R^q f_* \mathbb{Z}_X)|_{Y_p}$.

Theorem 4.1 affords a simple proof of the following result of M. Saito ([19]). Recall that the integral singular cohomology of complex algebraic varieties carries a canonical and functorial mixed Hodge structure (mHs).

Corollary 4.3 (The Leray filtration is compatible with mHs) *Let $f : X \rightarrow Y$ be a proper map of algebraic varieties with Y quasi projective. Then the subspaces of the Leray filtration \mathcal{L} on $H^q(X, \mathbb{Z})$ are mixed Hodge substructures.*

Let K be a constructible complex of sheaves on an algebraic variety Y . We have the perverse filtration ${}^p\mathcal{S}_i H^j(Y, K) := \text{Im} \{H^j(Y, {}^p\tau_{\leq i} K) \rightarrow H^j(Y, K)\}$. Let $f : X \rightarrow Y$ be a map of algebraic varieties and let $C \in D_X$. We have the perverse Leray filtration ${}^p\mathcal{L}_i$ on $H^j(X, C)$, i.e the perverse filtration ${}^p\mathcal{S}$ on $H^j(Y, f_* C) = H^j(X, C)$. Similarly, for $H_c^j(X, C)$.

Remark 4.4 In the situation of the decomposition theorem 2.4, if we take X to be nonsingular (if X is singular, then replace cohomology with intersection cohomology in what follows), then the subspace ${}^p\mathcal{L}_i H^j(X, C) \subseteq H^j(X, C)$ is given by the images, via the chosen splitting, of the direct sum of the j -th cohomology groups of the terms with $-d_b \leq i$. The general theory implies that this image is independent of the chosen splitting. However, different splittings yield different embeddings of each of the direct summands into $H^j(X, C)$.

Let $f : X \rightarrow Y$ be a map of varieties where Y is a quasi projective variety. Let $C \in D_X$ and $K \in D_Y$ (integral coefficients). The main result of [8] is a geometric description of the perverse and perverse Leray filtrations. We state a significative special case only.

Theorem 4.5 (Geometric perverse Leray) *Let $f : X \rightarrow Y$ be a map of algebraic varieties with Y affine of dimension n . Then there is a filtration Y_* by closed subvarieties Y_i of dimension i such that if we take $X_* := f^{-1}Y_*$, then*

$${}^p\mathcal{L}_i H^j(X, \mathbb{Z}) := {}^p\mathcal{S}_i H^j(Y, f_*\mathbb{Z}_X) = \text{Ker} \left\{ H^j(X, \mathbb{Z}) \longrightarrow H^j(X_{n+j-i-1}, \mathbb{Z}) \right\}.$$

The main difference with respect to Theorem 4.1 is that Y_* is obtained by choosing general vs. special hypersurfaces (see Remark 4.2). This choice is needed in order to deduce the perverse analogue of the cellularity condition (7), i.e.

$$H^j(Y_p, Y_{p-1}, {}^p\mathcal{H}^q(f_*C) = 0, \quad \forall j \neq 0, \quad \forall q.$$

These vanishing conditions are verified by a systematic use of the Lefschetz hyperplane theorem for perverse sheaves. Unlike [1] and [4], the proof for compactly supported cohomology is completely analogous to the one for cohomology.

A second difference, is that we do not need the map $f : X \rightarrow Y$ to be proper. The choice of general hypersurfaces avoids the usual pitfalls of the failure of the base change theorem (see [4]).

The discrepancy “ $+n$ ” between (6) for Theorem 4.1 and Theorem 4.5 boils down to the fact that for the affine variety Y of dimension n , the cohomology groups $H^j(Y, F)$ with coefficients in a sheaf (perverse sheaf, resp.) F are non-zero only in the interval $[0, n]$ ($[-n, 0]$, resp.).

This geometric description of the perverse filtration in terms of the kernels of restriction maps to subvarieties is amenable to applications to the mixed Hodge theory of algebraic varieties. For example, the analogue of Corollary 4.3 holds, with the same proof. For more applications, see [5].

5 Character varieties and the Hitchin fibration: $P = W'$

In this section, I report on [9], where we prove Theorem 5.1. The main ingredients are the geometric description of the perverse filtration in Theorem 4.5 and the refinement Theorem 5.3 of the support theorem (4) in the case at hand.

We have the Hitchin map (3) for the group $G = GL_{\mathbb{C}}(2)$. There are analogous maps $\check{h} : \check{\mathcal{M}}^{6g-6} \rightarrow \mathbb{A}^{3g-3}$ for $G = SL_{\mathbb{C}}(2)$ and $\hat{h} : \widehat{\mathcal{M}}^{6g-6} \rightarrow \mathbb{A}^{3g-3}$ for $PGL_{\mathbb{C}}(2)$.

Though these three geometries are closely related, this is not the place to detail the toing and froing from one group to another. The main point for this discussion is that we have an explicit description of the cohomology algebra $H^*(\mathcal{M}, \mathbb{Q})$ in view of the canonical isomorphism

$$H^*(\mathcal{M}, \mathbb{Q}) \simeq H^*(\widehat{\mathcal{M}}, \mathbb{Q}) \otimes H^*(\text{Jac}(C), \mathbb{Q}) \quad (8)$$

and of (9) below. In view of (8), the key cohomological considerations towards Theorem 5.1 below can be made in the $PGL_{\mathbb{C}}(2)$ case, for they will imply easily the ones for $GL_{\mathbb{C}}(2)$ and, with some extra considerations which we do not address here, the ones for $SL_{\mathbb{C}}(2)$. For simplicity, ignoring some of the subtle differences between the three groups, let us work with $\hat{h} : \widehat{\mathcal{M}}^{6g-6} \rightarrow \mathbb{A}^{3g-3}$. Though $\widehat{\mathcal{M}}$ is the quotient of a manifold by the action of a finite group, for our purposes we can safely pretend it is a manifold. We set $d := 3g - 3$.

In the context of the non-Abelian Hodge theorem ([20]), the quasi projective variety $\widehat{\mathcal{M}}$ is usually denoted $\widehat{\mathcal{M}}_D$, where D stands for Dolbeault. This is to contrast it with the moduli space $\widehat{\mathcal{M}}_B$ (Betti) of irreducible $PGL_2(\mathbb{C})$ representations of the fundamental group of C ; this is an affine variety.

The non-Abelian Hodge theorem states that there is a natural diffeomorphism $\varphi : \widehat{\mathcal{M}}_B \simeq \widehat{\mathcal{M}}_D$. The two varieties are not isomorphic as complex spaces and, a fortiori, neither as algebraic varieties: the latter contains the fibers of the Hitchin map, i.e. d -dimensional Abelian varieties, while the former is affine.

The diffeomorphism φ induces an isomorphism of cohomology rings $\varphi^* : H^*(\widehat{\mathcal{M}}_D, \mathbb{Q}) \simeq H^*(\widehat{\mathcal{M}}_B, \mathbb{Q})$. This isomorphism is not compatible with the mixed Hodge structures. In fact, the mixed Hodge structure on every $H^j(\widehat{\mathcal{M}}_D, \mathbb{Q})$ is known to be pure (see §1), while the one on $H^j(\widehat{\mathcal{M}}_B, \mathbb{Q})$ is known to be not pure ([14]).

In particular, the weight filtrations do not correspond to each other via φ^* . Our main result in [9] can be stated as follows.

Theorem 5.1 ($P = W'$) *In the cases $G = GL_{\mathbb{C}}(2)$, $PGL_{\mathbb{C}}(2)$, $SL_{\mathbb{C}}(2)$, the non-Abelian Hodge theorem induces an isomorphism in cohomology that identifies the weight filtration for the mixed Hodge structure on the Betti side with the perverse Leray filtration on the Dolbeault side; more precisely, (11) below holds.*

At present, we do not know what happens if the reductive group G has higher rank. Moreover, we do not have a conceptual explanation for the so-far mysterious exchange of structure of Theorem 5.1.

Our paper [10] deals with a related moduli space, i.e. the Hilbert scheme of n points on the cotangent bundle of an elliptic curve, where a similar exchange takes place.

Let us try and describe some of the ideas that play a role in the proof of Theorem 5.1. We refer to [9] for details and attributions.

By the work of several people, the cohomology ring and the mixed Hodge structure of $H^*(\widehat{\mathcal{M}}_B, \mathbb{Q})$ are known. There are tautological classes:

$$\alpha \in H^2, \quad \{\psi_i\}_{i=1}^{2g(C)} \in H^3, \quad \beta \in H^4$$

which generate the cohomology ring. With respect to the mixed Hodge structure, these classes are of weight 4 and pure type (2, 2). Every monomial made of l letters among these tautological classes has weight $4l$ and Hodge type $(2l, 2l)$, i.e. weights are strictly additive for the cup product. In general, weights are only sub-additive. There is a graded \mathbb{Q} -algebra isomorphism

$$H^*(\widehat{\mathcal{M}}_B, \mathbb{Q}) \simeq \frac{\mathbb{Q}[\alpha, \{\psi_i\}, \beta]}{I}, \quad (9)$$

where I is a certain bihomogeneous ideal with respect to weight and cohomological degree. In particular, we have a canonical splitting for the increasing weight filtration \mathcal{W}' on $H^j(\widehat{\mathcal{M}}_B, \mathbb{Q})$ (the trivial weight filtration \mathcal{W} on the pure $H^j(\widehat{\mathcal{M}}_D, \mathbb{Q})$ plays no role here):

$$H^j(\widehat{\mathcal{M}}_B, \mathbb{Q}) = \bigoplus_{w \geq 0} H_w^j, \quad \mathcal{W}'_w H^j = \bigoplus_{w' \leq w} H_{w'}^j.$$

The weights occur in the interval $[0, 4d]$ and they are multiples of four, i.e. $\mathcal{W}'_{4k-i} = \mathcal{W}'_{4k}$ for every $0 \leq i \leq 3$.

By virtue of the decomposition theorem (2) and of the fact that the Hitchin map \widehat{h} is flat of relative dimension d , the increasing perverse Leray filtration ${}^p\mathcal{L}$ has type $[-d, d]$.

In order to compare \mathcal{W}' with ${}^p\mathcal{L}$, we half the weights, i.e. we set $W'_i := \mathcal{W}'_{2i}$, and we translate ${}^p\mathcal{L}$, i.e. we set

$$P := {}^p\mathcal{L}(-d). \quad (10)$$

We still denote these half-weights by w . We have that both W' and P have non-zero graded groups Gr_i only in the interval $i \in [0, 2d]$. The two modified filtrations could still be completely unrelated. After all, they live on the cohomology of different algebraic varieties! The precise formulation of Theorem 5.1 is

$$P = W'. \quad (11)$$

Let us describe our approach to the proof.

We introduce the notion of perversity and, ultimately, we show that the perversity equals the weight. We say that $0 \neq u \in H^j(\widehat{\mathcal{M}}_D, \mathbb{Q})$ has *perversity* $p = p(u)$ if $u \in P_p \setminus P_{p-1}$. By definition, $u = 0$ can be given any perversity. Perversities are in the interval $[0, 2d]$. We write monomials in the tautological classes as $\alpha^r \beta^s \psi^t$, where ψ^t is a short-hand for a product of t classes of type ψ . Then (11) can be re-formulated as follows:

$$p(\alpha^r \beta^s \psi^t) = w(\alpha^r \beta^s \psi^t) = 2(r + s + t). \quad (12)$$

As it turns out, the harder part is to establish the inequality

$$p(\alpha^r \beta^s \psi^t) \leq 2(r + s + t), \quad (13)$$

for once this is done, the reverse inequality is proved by a kind of simple pigeonhole trick. We thus focus on (13).

Recall that $U \subseteq \mathbb{A}^d$ (see (4)) is the dense open set where the fibers of \hat{h} are irreducible. We have the following sharp estimate

$$\text{codim}(\mathbb{A}^d \setminus U) \geq 2g - 3. \quad (14)$$

For every $0 \leq b \leq d$, let $\Lambda^b \subseteq \mathbb{A}^d$ denote a general linear section of dimension b . We have defined the translated perverse Leray filtration P on the cohomology groups of $\widehat{\mathcal{M}}$ for the map \hat{h} that fibers $\widehat{\mathcal{M}}$ over \mathbb{A}^d . We can do so, in a compatible way, over U and over the Λ^b so that the restriction maps respect the resulting P filtrations. All these increasing filtrations start at zero and perversities are in the interval $[0, 2d]$.

The test for perversity Theorem 4.5, now reads

Fact 5.2 *Let $\Lambda^b \subseteq \mathbb{A}^d$ be a general linear subspace of dimension b . Denote by $\widehat{\mathcal{M}}_{\Lambda^b} := \hat{h}^{-1}(\Lambda^b)$. Then*

$$u \in P_{j-b-1}H^j(\widehat{\mathcal{M}}) \iff u|_{\widehat{\mathcal{M}}_{\Lambda^b}} = 0.$$

We need the following strengthening (in the special case we are considering) of the support theorem (4) over U . It is obtained by a study of the local monodromy of the family of spectral curves around the points of U . Let $j : \mathbb{A}_{reg}^d \rightarrow U$ be the open embedding of the set of regular values of \hat{h} .

Theorem 5.3 *The intersection complexes $IC_U(R^i)$ are shifted sheaves and we have*

$$\hat{h}_{U*}\mathbb{Q} \simeq \bigoplus j_*R^i[-i].$$

In particular, the translated perverse Leray filtrations P coincides with the Leray filtration \mathcal{L} on $H^(\widehat{\mathcal{M}}_U, \mathbb{Q})$, and on $H^*(\widehat{\mathcal{M}}_{\Lambda^b}, \mathbb{Q})$ for every $b < 2g - 3$.*

The last statement is a consequence of (14): we can trace Λ^b inside U .

We can now discuss the scheme of proof for (13). We start by establishing the perversities of the multiplicative generators, i.e. by proving that

$$p(\alpha) = p(\beta) = p(\psi_i) = 2.$$

By Fact 5.2, we need to show that α vanishes over the empty set, ψ_i over a point, and β over a line. The first requirement is of course automatic. The second one is a result of M. Thaddeus [22]. He also proved that β vanishes over a point, but we need more.

Fact 5.4 *The class β is zero over a general line $\iota := \Lambda^1 \subseteq \mathbb{A}^{3g-3}$.*

Idea of proof. By (14), we can choose a general line $\mathfrak{l} = \Lambda^1 \subseteq U$. Let $f : \widehat{\mathcal{M}}_{\mathfrak{l}} := \hat{h}^{-1}(\mathfrak{l}) \rightarrow \mathfrak{l}$. In particular, by abuse of notation, we write $\widehat{\mathcal{M}}_{reg} := \hat{h}^{-1}(\mathbb{A}_{reg}^d)$, where $\mathbb{A}_{reg}^d \subseteq \mathbb{A}^d$ is the Zariski open and dense set of regular values of the Hitchin map.

By Theorem 5.3 (as it turns out, since we are working over a curve, here (4) is enough to reach the same conclusion) we have

$$Rf_*\mathbb{Q} \simeq \bigoplus (j_*R^i)_{|\mathfrak{l}}[-i].$$

In particular, there are no skyscraper summands on \mathfrak{l} . A simple spectral sequence argument over the affine curve \mathfrak{l} , implies that the restriction map $H^4(\widehat{\mathcal{M}}_{\mathfrak{l}}) \rightarrow H^4(\widehat{\mathcal{M}}_{reg})$ is injective. (Note that this last conclusion would be clearly false if we had a skyscraper contribution.) It is enough to show that $\beta|_{\widehat{\mathcal{M}}_{reg}}$ is zero. The class β is a multiple of $c_2(\widehat{\mathcal{M}})$. On the other hand, since the Hitchin system is a completely integrable system over the affine space, the tangent bundle can be trivialized, in the C^∞ -sense, over the open set of regular point $\widehat{\mathcal{M}}_{reg}$ using the Hamiltonian vector fields. \square

Having determined the perversity for the multiplicative generators, we turn to (13) which we can re-formulate by saying that perversities are sub-additive under cup product.

In general, I do not know if this is the case: see the discussion following the statement of Theorem 6.1 and also Remark 6.8. On the other hand, the analogous sub-additivity statement for the Leray filtration \mathcal{L} is well-known to hold; see Theorem 6.1.

Let us outline our procedure to prove the sub-additivity of perversity in our case. We want to use the test for perversity Theorem 5.2, for the monomials in (13). First we get rid of α^r : in fact, it is a simple general fact that cupping with a class of degree i , raises the perversity by at most i . It follows that we can concentrate on the case $r = 0$.

Here is the outline of the final analysis.

1. In order to use Theorem 5.3, we need to make sure that we can test the monomials over linear sections Λ^b which can be traced inside U .
2. Theorem 5.3, combined with the sub-additivity of the Leray filtration implies that we have sub-additivity over Λ^b .
3. We deduce that the sub-additivity upper bound on the perversity over Λ^b , when compared with the cohomological degree of the monomial, forces the restricted monomial to be zero, i.e. the monomial passes the test and we are done.

The obstacle in Step 1 is the following: the dimension b of the testing Λ^b increases as $s + t$ increases. On the other hand, by (14) we need $b < 2g - 3$. There are plenty of monomials for which b exceeds this bound. We use the explicit nature of the relations I (9) to find an upper bound for $s + t$. The corresponding upper bound for b is $b \leq 2g - 3$ (sic!) and the only class that needs to be tested on a Λ^{2g-3} is β^{g-1} . This class turns out to require a separate ad-hoc analysis. Step 2 requires no further comment. Step 3 is standard as it is based on the cohomological dimension of affine varieties with respect to perverse sheaves.

6 Appendix: cup product and Leray filtration

We would like to give a (more or less) self-contained proof of Theorem 6.1, i.e. of the fact that the cup product is compatible with the Leray spectral sequence. We have been unable to locate a suitable reference in the literature. As it is clear from our discussion in §5, this fact is used in an essential way in our proof of Theorem 5.1

As it turns out, the same proof shows that the cup product is also compatible with the p -Leray spectral sequence for every non-positive perversity $p \leq 0$, including \mathfrak{p} . However, this statement turns out to be rather weak, unless we are in the standard case when $p \equiv 0$. For example, in the case of middle perversity, it is off the mark by $+d$ with respect to the sub-additivity we need in the proof of Theorem 5.1, as it only implies that $p(\beta^2) \leq 4 + d$, whereas $p(\beta^2) = 4$. Nevertheless, it seems worthwhile to give a unified proof valid for every $p \leq 0$.

The statement involves the cup product operation on the cohomology groups with coefficients in the direct image complex. It is thus natural to state and prove the compatibility result for the p -standard filtration for arbitrary complexes on varieties. The compatibility for the p -Leray filtration is then an immediate consequence. We employ freely the language of derived categories. We work in the context of constructible complexes on algebraic varieties and, just to fix ideas, with integer coefficients. Let us set up the notation necessary to state Theorem 6.1.

Let $p : \mathbb{Z} \rightarrow \mathbb{Z}$ be any function; we call it a *perversity*. Given a partition $X = \coprod S_i$ of a variety X into locally closed nonsingular subvarieties S (strata), we set $p(S) := p(\dim S)$. By considering all possible partitions of X , this data gives rise to a t -structure on D_X (see [3], p. 56). The standard t -structure corresponds to $p(S) \equiv 0$ and the middle perversity t -structure corresponds to the perversity \mathfrak{p} defined by setting $\mathfrak{p}(S) := -\dim S$.

For a given perversity p , the subcategories ${}^p D_X^{\leq i}$ for the corresponding t -structure are defined as follows

$${}^p D_X^{\leq 0} = \left\{ K \in D_X \mid \mathcal{H}^i(K)|_S = 0, \forall i > p(S) \right\}, \quad {}^p D_X^{\leq i} := {}^p D_X^{\leq 0}[-i].$$

If $p = 0$, then ${}^p D_X^{\leq 0} = D_X^{\leq 0}$ is given by the complexes with zero cohomology sheaves in positive degrees. If $p = \mathfrak{p}$ is the middle perversity, then one shows easily that ${}^p D_X^{\leq 0}$ is given by those complexes K such that $\dim \text{supp } \mathcal{H}^i(K) \leq -i$. By using the truncation functors ${}^p \tau_{\leq i}$, we can define (see §1.1) the p -standard ${}^p \mathcal{S}$ and the p -Leray ${}^p \mathcal{L}$ filtrations.

Let $K, L \in D_X$. The tensor product complex $(K \otimes L, d)$ is defined to be

$$(K \otimes L)^i := \bigoplus_{a+b=i} K^a \otimes L^b, \quad d(f_a \otimes g_b) = df \otimes g + (-1)^a f \otimes dg. \quad (15)$$

The left derived tensor product is a bi-functor $\overset{\mathbb{L}}{\otimes} : D_X \times D_X \rightarrow D_X$ defined by first taking a flat resolution $L' \rightarrow L$ and then by setting $K \overset{\mathbb{L}}{\otimes} L := K \otimes L'$. If we use field coefficients, then the left-derived tensor product coincides with the ordinary tensor product: $\overset{\mathbb{L}}{\otimes} = \otimes$.

Let

$$H^a(X, K) \otimes H^b(X, L) \longrightarrow H^{a+b}(X, K \overset{\mathbb{L}}{\otimes} L) \quad (16)$$

be the cup product ([16], p. 134).

The following establishes that the filtration in cohomology associated with a non-positive perversity is compatible with the cup product operation (16).

Theorem 6.1 *Let $p \leq 0$ be a non-positive perversity. The p -standard filtration and, for a map $f : X \rightarrow Y$, the p -Leray filtration are compatible with the cup product:*

$${}^p\mathcal{S}_i H^a(X, K) \otimes {}^p\mathcal{S}_j H^b(X, L) \longrightarrow {}^p\mathcal{S}_{i+j} H^{a+b} \left(X, K \overset{\mathbb{L}}{\otimes} L \right),$$

$${}^p\mathcal{L}_i H^a(X, L) \otimes {}^p\mathcal{L}_j H^b(X, L) \longrightarrow {}^p\mathcal{L}_{i+j} H^{a+b} \left(X, K \overset{\mathbb{L}}{\otimes} L \right).$$

Theorem 6.1 is proved in §6.2. Section 6.3 shows how the cup product and its variants for cohomology with compact supports are related to each other; these variants are listed in (26). Moreover, if we specialize (26) to the case of constant coefficients, and also to the case of the dualizing complex, then we get the usual cup products in cohomology (see the left-hand-side of (27)) and the usual cap products involving homology and Borel-Moore homology (see the right-hand-side of (27)).

Remark 6.2 The obvious variants of the statement of Theorem 6.1 hold also for each of the variants of the cup product mentioned above. The same is true for Theorem 6.7, which is merely a souped-up version of Theorem 6.1. The reader will have no difficulty repeating, for each of these variants, the proof of Theorems 6.1 and 6.7 given in §6.2.

Example 6.3 We consider only the two cases $p \equiv 0$ and $p = \mathfrak{p}$; in the former case we drop the index $p = 0$. Let X be a nonsingular variety of dimension d .

1. Let $K = L = \mathbb{Z}_X$. Then $1 \in \mathcal{S}_0 H^0$ and $1 \cup 1 = 1 \in \mathcal{S}_0 H^0$.
2. Let $K = L = \mathbb{Z}_X[d]$. Then $K \overset{\mathbb{L}}{\otimes} L = \mathbb{Z}_X[2d]$. While $1 = 1 \cup 1 \in {}^p\mathcal{S}_{-d} H^{-2d}(X, \mathbb{Z}_X[2d])$, Theorem 6.1 only predicts $1 \cup 1 \in {}^p\mathcal{S}_0 H^{-2d}(X, \mathbb{Z}_X[2d])$.
3. Let $K = L = \mathbb{Z}_p$, where $p \in X$. We have $1_p = 1_p \cup 1_p \in {}^p\mathcal{S}_0 H^0(X, \mathbb{Z}_p)$ and this agrees with the prediction of Theorem 6.1.
4. Let $f : X = Y \times F \rightarrow Y$ be the projection, and let $K = L = \mathbb{Q}_X$. We have that

$$\mathcal{L}_i H^a(X, \mathbb{Q}) = \bigoplus_{i' \leq i} \left(H^{a-i'}(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} H^{i'}(F, \mathbb{Q}) \right).$$

In this case, Theorem 6.1 is a simple consequence of the compatibility of the Künneth formula with the cup product.

5. Now let us consider ${}^p\mathcal{L}$ for the same projection map $f : Y \times F \rightarrow Y$ as above. We have that $Rf_*\mathbb{Q} \simeq \bigoplus_{i \geq 0} R^i[-i]$, where R^i is the constant local system $R^i f_*\mathbb{Q}$. Let us assume that Y is nonsingular of pure dimension d . Then ${}^p\mathcal{L}_i = \mathcal{L}_{i-d}$, where we use the fact that each $R^i[d]$ is a perverse sheaf due to the nonsingularity of Y (which stems from the one of X). We have that $1 \in {}^p\mathcal{L}_d H^0(X, \mathbb{Q})$. On the other hand, Theorem 6.1 predicts only that $1 = 1 \cup 1 \in {}^p\mathcal{L}_{2d} H^0(X, \mathbb{Q})$.

These examples, which as the reader can verify are not an illusion due to indexing schemes, show that Theorem 6.1 is indeed sharp. However, its conclusions for ${}^p\mathcal{S}$ and ${}^p\mathcal{L}$ are often off the mark. See also Remark 6.8.

Remark 6.4 I do not know an example of a map $f : X \rightarrow Y$, with X and Y nonsingular, f proper and flat of relative dimension d , for which the cup product on $H^*(X, \mathbb{Q})$ does not satisfy

$${}^p\mathcal{L}_i \otimes {}^p\mathcal{L}_j \longrightarrow {}^p\mathcal{L}_{i+j-d} \quad (17)$$

(Theorem 6.1 predicts that the cup product above lands in the bigger ${}^p\mathcal{L}_{i+j}$.) In the paper [9] we need to establish (17) for the Hitchin map. If (17) were true a priori, the proof of the main result of our paper [9] could be somewhat shortened.

Note also that if the shifted perverse Leray filtration $P := {}^p\mathcal{L}(-d)$ (see 10) for the Hitchin map h coincided a priori with the ordinary Leray filtration \mathcal{L} of the map h , then (17) would follow immediately from the case $p = 0$ of Theorem 6.1. At present, we do not know if $P = \mathcal{L}$ for the Hitchin map. In general, i.e. for a map f as above, we have $\mathcal{L} \subseteq P$, but the inclusion can be strict: e.g. the projection to \mathbb{P}^1 of the blowing-up of \mathbb{P}^2 at a point, where the class of the exceptional divisor is in P_1 , but it is not in \mathcal{L}_1 .

6.1 A simple lemma relating tensor product and truncation

The key simple fact behind Theorem 6.1 in the standard case when $p \equiv 0$ is that if two complexes $K, L \in D_{\bar{X}}^{\leq 0}$, i.e. they have non-zero cohomology sheaves in non negative degrees only, then the same is true for their derived tensor product.

Lemma 6.5 shows that the Künneth spectral sequence implies that the analogous fact is true for any non-positive perversity p .

Let us recall the Künneth spectral sequence for the derived tensor product of complexes of sheaves. Define the Tor-sheaves, a collection of bi-functor with variables sheaves A and B , by setting $\mathcal{T}or_i(A, B) := \mathcal{H}^{-i}(A \overset{\mathbb{L}}{\otimes} B)$. We have $\mathcal{T}or_0(A, B) = A \otimes B$ and $\mathcal{T}or_i(A, B) = 0$ for every $i < 0$. Let $K, L \in D_X$. We have the Künneth spectral sequence ([12], III.2., 6.5.4.2, [24], p.7)

$$E_2^{st} = \bigoplus_{a+b=t} \mathcal{T}or_{-s}(\mathcal{H}^a(K), \mathcal{H}^b(L)) \implies \mathcal{H}^{s+t}(K \overset{\mathbb{L}}{\otimes} L). \quad (18)$$

This sequence lives in the II-III quadrants, i.e. where $s \leq 0$. The edge sequence gives a natural map

$$\bigoplus_{a+b=t} \mathcal{H}^a(K) \otimes \mathcal{H}^b(L) \longrightarrow \mathcal{H}^t(K \overset{\mathbb{L}}{\otimes} L). \quad (19)$$

Lemma 6.5 (Tensor product and truncation) *Let $p \leq 0$ be any non-positive perversity. Then*

$$\overset{\mathbb{L}}{\otimes} : {}^p D_X^{\leq i} \times {}^p D_X^{\leq j} \longrightarrow {}^p D_X^{\leq i+j}.$$

Proof. We simplify the notation by dropping the decorations X and p . Since

$$D^{\leq i} \overset{\mathbb{L}}{\otimes} D^{\leq j} = D^{\leq 0}[-i] \overset{\mathbb{L}}{\otimes} D^{\leq 0}[-j] = D^{\leq 0} \overset{\mathbb{L}}{\otimes} D^{\leq 0}[-i-j],$$

it is enough to show that

$$\overset{\mathbb{L}}{\otimes} : D^{\leq 0} \times D^{\leq 0} \longrightarrow D^{\leq 0}.$$

We need to verify that the equality

$$\mathcal{H}^q(K \overset{\mathbb{L}}{\otimes} L)|_S = 0, \quad \forall q > p(S), \quad (20)$$

holds as soon as the same equality is assumed to hold for K and L .

It is enough to prove the analogous equality for the Tor-sheaves on the left-hand-side of (18).

Let us note that $p \leq 0$ implies that if $L \in D^{\leq 0}$, then $\mathcal{H}^b(L) = 0$ for every $b > 0$.

Let $\sigma \geq 0$ and consider

$$\bigoplus_{a+b=q+\sigma} \mathcal{T}or_\sigma \left(\mathcal{H}^a(K)|_S, \mathcal{H}^b(L)|_S \right), \quad \forall q > p(S).$$

If $a > p(S)$, then $\mathcal{H}^a(K)|_S = 0$.

If $a \leq p(S)$, then $b = q - a + \sigma > \sigma \geq 0$, so that $\mathcal{H}^b(L) = 0$. □

6.2 Spectral sequences and multiplicativity

In this section we prove Theorem 6.1 and we also observe that it is the reflection at the level of the abutted filtrations of the more general statement Theorem 6.7 involving spectral sequences.

The most efficient formulation is perhaps the one involving the filtered derived category $D_X F$. We shall quote freely from [15], pp. 285-288. To fix ideas, we deal with the cup product in cohomology. The formulations for the other products in §6.3 are analogous.

Let (K, F_1) and (L, F_2) be two filtered complexes. The filtered derived tensor product $(K \overset{\mathbb{L}}{\otimes} L, F_{12})$ is defined as follows. Let $(L', F'_2) \rightarrow (L, F_2)$ be a left flat filtered resolution.

Define $K \overset{\mathbb{L}}{\otimes} L := K \otimes L'$ and define F_{12} to be the product filtration of F_1 and F_2' . We have natural isomorphisms

$$\bigoplus_{s+s'=\sigma} \left(Gr_{F_1}^s K \overset{\mathbb{L}}{\otimes} Gr_{F_2}^{s'} L \right) \xrightarrow{\simeq} Gr_{F_{12}}^\sigma \left(K \overset{\mathbb{L}}{\otimes} L \right). \quad (21)$$

We have the filtered version of [16], p.134, i.e. a map in $D_{pt}F$

$$(R\Gamma(X, K), F_1) \overset{\mathbb{L}}{\otimes} (R\Gamma(X, L), F_2) \longrightarrow \left(R\Gamma(X, K \overset{\mathbb{L}}{\otimes} L), F_{12} \right) \quad (22)$$

inducing (cf. [16], Ex.I.24.a) the filtered cup product map

$$(H^a(X, K), F_1) \otimes (H^b(X, L), F_2) \longrightarrow \left(H^{a+b} \left(X, K \overset{\mathbb{L}}{\otimes} L \right), F_{12} \right). \quad (23)$$

In view of Theorem 6.7, by first recalling the notion of bilinear pairing of spectral sequences ([21], p. 491), we have a bilinear pairing of spectral sequences

$$E_1^{st}(K, F_1) \otimes E_1^{s't'}(L, F_2) \longrightarrow E_1^{s+s', t+t'} \left(K \overset{\mathbb{L}}{\otimes} L, F_{12} \right) \quad (24)$$

that on the E_1 -term coincides with the cup product map (16) induced by (21), and on the E_∞ -term is the graded cup product associated with the filtered cup product (23).

Given $(M, F) \in D_X F$, we have the spectral sequence

$$E_1^{st} = E_1^{st}(M, F) = H^{s+t}(X, Gr_F^s M) \implies H^{s+t}(X, M), \quad E_\infty^{st} = Gr_F^s H^{s+t}(X, M),$$

with abutment the filtration induced by (M, F) on the cohomology groups $H^*(X, M)$. Clearly, we can always compose with the map of spectral sequences induced by any filtered map $(K \overset{\mathbb{L}}{\otimes} L, F_{12}) \rightarrow (M, F)$.

We apply the machinery above to the case when the filtrations F_i are the p -standard decreasing filtrations ${}^p\mathcal{S}$ induced by the t -structure associated with a non-positive perversity $p \leq 0$. The construction of ${}^p\mathcal{S}$ is performed via the use of injective resolutions ([8], §3.1). The product filtration F_{12} on the derived tensor product is *not* the p -standard filtration, not even up to isomorphism in the filtered derived category $D_X F$; see Remark 6.8 below.

The upshot of this discussion is that Lemma 6.5 implies the following

Lemma 6.6 *There is a canonical lift*

$$u : \left(K \overset{\mathbb{L}}{\otimes} L, F_{12} \right) \longrightarrow \left(K \overset{\mathbb{L}}{\otimes} L, {}^p\mathcal{S} \right)$$

of the identity on $K \overset{\mathbb{L}}{\otimes} L$ to $D_X F$.

Proof. Let N denote the derived tensor product of K with L . It is enough to show that $F_{12}^\sigma N \in {}^pD_{\bar{X}}^{\leq -\sigma}$, for every $\sigma \in \mathbb{Z}$. We prove this by decreasing induction on σ . The statement is clearly true for $\sigma \gg 0$. We have the short exact sequence

$$0 \longrightarrow F_{12}^{\sigma+1}N \rightarrow F_{12}^\sigma N \rightarrow Gr_{F_{12}}^\sigma N \longrightarrow 0.$$

Lemma 6.5 implies that $Gr_{F_{12}}^\sigma N \in {}^pD_{\bar{X}}^{\leq -\sigma}$ and the inductive hypothesis gives $F_{12}^{\sigma+1}N \in {}^pD_{\bar{X}}^{\leq -\sigma-1} \subseteq {}^pD_{\bar{X}}^{\leq -\sigma}$. We have the following simple fact: if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle and $A, C \in {}^pD_{\bar{X}}^{\leq i}$, then $B \in {}^pD_{\bar{X}}^{\leq i}$. We conclude the proof by applying this fact to the distinguished triangle associated with the short exact sequence above. \square

We are now ready for the

Proof of Theorem 6.1:

Apply the construction (23) to the case $F_i = {}^pS$. Compose the resulting filtered cup product map with the canonical lift u of Lemma 6.6 and obtain the filtered cup product map of Theorem 6.1. \square

As mentioned earlier, Theorem 6.1 is the abutted reflection of the following statement concerning spectral sequences:

Theorem 6.7 *There is a natural bilinear pairing of spectral sequences*

$$E_1^{st}(K, {}^pS) \otimes E_1^{s't'}(L, {}^pS) \longrightarrow E_1^{s+s', t+t'}\left(K \overset{\mathbb{L}}{\otimes} L, {}^pS\right)$$

such that:

1. on the E_1 -term it coincides with the cup product map induced by (21), which in this case reads

$${}^p\mathcal{H}^{-s}(K)[s] \overset{\mathbb{L}}{\otimes} {}^p\mathcal{H}^{-s'}(K)[s'] \longrightarrow {}^p\mathcal{H}^{-s-s'}\left(K \overset{\mathbb{L}}{\otimes} L\right)[s+s'], \quad (25)$$

2. on the E_∞ -term it is the graded cup product associated with the filtered cup product (23).

Proof: Compose (24) with the map of spectral sequences induced by the canonical map u of Lemma 6.6. \square

Remark 6.8 Unless we are in the case $p \equiv 0$, the product filtration F_{12} of the p -standard filtrations is often strictly smaller than the p -standard filtration. As a result, the graded pairing is often trivial. One can see this on the E_1 -page in terms of the map (25). Here is an example. Let X be nonsingular of pure dimension d , take middle perversity and perverse complexes $K = L = \mathbb{Q}_X[d]$. The pairing in question is $\mathbb{Q}_X[d] \otimes \mathbb{Q}_X[d] \longrightarrow {}^p\mathcal{H}^0(\mathbb{Q}_X)[2d] = 0$.

6.3 Cup and cap

The methods employed in the previous sections are of course susceptible of being applied to the other usual constructions, such as the cup product in cohomology with compact supports and cap products in homology and in Borel-Moore homology.

By taking various flavors of (22) with compact supports, we obtain the commutative diagram of cup product maps

$$\begin{array}{ccc}
 H_c^i(X, K) \otimes H_c^j(X, L) & \longrightarrow & H_c^{i+j}(X, K \overset{\mathbb{L}}{\otimes} L) \\
 \downarrow & \searrow & \downarrow = \\
 H^i(X, K) \otimes H_c^j(X, L) & \longrightarrow & H_c^{i+j}(X, K \overset{\mathbb{L}}{\otimes} L) \\
 \downarrow & \searrow & \downarrow \\
 H^i(X, K) \otimes H^j(X, L) & \longrightarrow & H^{i+j}(X, K \overset{\mathbb{L}}{\otimes} L).
 \end{array} \tag{26}$$

Theorem 6.7 applies to each row, each vertical arrow is a filtered map for the product filtrations and, as a result, the conclusion of Theorem 6.7 apply to the diagonal products as well.

We have the following important special cases. Take K and L to be either \mathbb{Z}_X and/or ω_X (the Verdier dualizing complex of X). We have $\mathbb{Z}_X \overset{\mathbb{L}}{\otimes} \omega_X = \omega_X$ as well as the following equalities (decorations omitted)

$$H^i(X, \mathbb{Z}) = H^i(X, \mathbb{Z}_X) = H^i \mathbb{Z}_X = H^i, \quad H_c^i = H_c^i \mathbb{Z}_X, \quad H_i = H_c^{-i} \omega_X, \quad H_i^{BM} = H^{-i} \omega_X.$$

Then we have the following commutative diagrams expressing the well-known compatibilities of the cup and cap products:

$$\begin{array}{ccc}
 H_c^i \otimes H_c^j & \longrightarrow & H_c^{i+j} \\
 \downarrow & \searrow & \downarrow = \\
 H^i \otimes H_c^j & \longrightarrow & H_c^{i+j} \\
 \downarrow & \searrow & \downarrow \\
 H^i \otimes H^j & \longrightarrow & H^{i+j}
 \end{array}
 \qquad
 \begin{array}{ccc}
 H_c^i \otimes H_j & \longrightarrow & H_{j-i} \\
 \downarrow & \searrow & \downarrow = \\
 H^i \otimes H_j & \longrightarrow & H_{j-i} \\
 \downarrow & \searrow & \downarrow \\
 H^i \otimes H_j^{BM} & \longrightarrow & H_{j-i}^{BM}.
 \end{array} \tag{27}$$

One also has the variants in relative cohomology and in relative cohomology with compact supports, the variants with supports on locally closed subvarieties, as well as the variants involving a map $f : X \rightarrow Y$ (e.g. $H^*(X)$ as a $H^*(Y)$ -module etc). The reader can sort these variants out.

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