The Freudenthal product and orbits in the Jordan algebra over the exceptional Lie group of type $F_{4(-20)}$

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Abstract

Let $\mathcal{J}^1$ be the real form of complex simple Jordan algebra with the automorphism group $F_{4(-20)}$. In terms of the Freudenthal product on $\mathcal{J}^1$ and the characteristic polynomial for $X \in \mathcal{J}^1$, a concrete classification of $F_{4(-20)}$-orbits on $\mathcal{J}^1$ is given.

1 Preliminaries.

Let $\mathbb{R}$ be the field of real numbers and $\mathbf{C} := \mathbb{R} \oplus \sqrt{-1}\mathbb{R}$ be the field of complex numbers. For $\mathbf{F} = \mathbb{R}$ or $\mathbf{C}$, let V be a $\mathbf{F}$-linear space, $\mathrm{GL}_{\mathbf{F}}(V)$ the group of $\mathbf{F}$-linear automorphims of V and $\mathrm{End}_{\mathbf{F}}(V)$ the linear space of $\mathbf{F}$-linear endomorphisms on V. For a mapping $f : V \to V$, put $V_f := \{v \in V \mid f(v) = v\}$. For a subgroup G of $\mathrm{GL}_{\mathbf{F}}(V)$, an automorphism ϕ on G and $v \in V$, put $G^{\phi} := \{g \in G \mid \phi g = g\}$, $G_v := \{g \in G \mid gv = v\}$, $\mathcal{O}_G(v) := \{gv \mid g \in G\}$.

For a linear space V over $\mathbb{R}$, its complexification $V^{\mathbf{C}} := V \otimes_{\mathbb{R}} \mathbf{C} = V \oplus \sqrt{-1}V$. For an $f \in \mathrm{End}_{\mathbb{R}}(V)$, its complexification by $f^{\mathbf{C}} \in \mathrm{End}_{\mathbf{C}}(V^{\mathbf{C}})$ is written by the same letter f. The complex conjugation on $V^{\mathbf{C}}$ with respect to V is denoted by $\tau : \tau(u + \sqrt{-1}v) = u - \sqrt{-1}v, \quad u, v \in V$.

Let $\mathbb{O}$ be the $\mathbb{R}$-algebra of octonions[2, 1, 11] with a base 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7 and the multiplications among them are given as follows; 1 is the unit of $\mathbb{R}$; $e_i^2 = -1,\ e_ie_j = -e_je_i$ for $i \neq j$; $e_le_m = e_n$, $e_me_n = e_l$, $e_ne_l = e_m$ for $(l, m, n) \in \{(1, 2, 3), (3, 5, 6), (6, 7, 1), (1, 4, 5), (3, 4, 7), (6, 4, 2), (2, 5, 7)\}$. By convention, $e_0 := 1$. The conjugation is defined as $\overline{\sum_{i=0}^7 x_ie_i} := x_0 - \sum_{i=1}^7 x_ie_i, \quad x_i \in \mathbb{R}$. Put $\mathrm{Re}(x) := \frac{1}{2}(x + \overline{x})$, $\mathrm{Im}(x) := \frac{1}{2}(x - \overline{x})$ and $\mathrm{Im}\mathbb{O} = \{\sum_{i=1}^7 x_ie_i \mid x_i \in \mathbb{R}\}$. And a positive definite inner product and the norm are defined as $(x, y) := \frac{1}{2}(\overline{x}y + \overline{y}x) = \frac{1}{2}(x\overline{y} + y\overline{x}) = \sum_{i=0}^7 x_iy_i$, $|x| := \sqrt{(x, x)}$ for $x = \sum_{i=0}^7 x_ie_i$, $y = \sum_{i=0}^7 y_ie_i \in \mathbb{O}$ with $x_i, y_i \in \mathbb{R}$.

Proposition 1.1. (cf. [2]) *Let* $x, y, a, b \in \mathbb{O}$.

(1) $(ax, ay) = (a, a)(y, y) = (xa, ya)$.

(2) $(ax, y) = (x, \overline{a}y)$, $(xa, y) = (x, y\overline{a})$.

(3) $\overline{\overline{x}} = x$, $\overline{x+y} = \overline{x} + \overline{y}$, $\overline{xy} = \overline{y}\,\overline{x}$.

(4) $a(\overline{a}x) = (a\overline{a})x$, $a(x\overline{a}) = (ax)\overline{a}$, $x(a\overline{a}) = (xa)\overline{a}$, $a(ax) = (aa)x$, $a(xa) = (ax)a$, $x(aa) = (xa)a$.

(5) $\overline{b}(ax) + \overline{a}bx = 2(a, b)x = (xa)\overline{b} + (xb)\overline{a}$.

(6) $(ax)y + x(ya) = a(xy) + (xy)a$, $(xa)y + (xy)a = x(ay) + x(ya)$, $(ax)y + (xa)y = a(xy) + x(ay)$.

(7) $(ax)(ya) = a(xy)a$.

(8) $\mathrm{Re}(xy) = \mathrm{Re}(yx)$, $\mathrm{Re}(x(yz)) = \mathrm{Re}(y(zx)) = \mathrm{Re}(z(xy))$.

Let $\mathbb{O}^{\mathbf{C}}$ be the complexification of $\mathbb{O}$. Similarly $\mathbb{O}^{\mathbf{C}}$ have the product $xy \in \mathbb{O}^{\mathbf{C}}$, the conjugation $\overline{\sum_{i=0}^{7} x_i e_i} := x_0 - \sum_{i=1}^{7} x_i e_i, \quad x_i \in \mathbf{C}$, the inner product $(x, y) := \frac{1}{2}(\overline{x}y + \overline{y}x) = \frac{1}{2}(x\overline{y} + y\overline{x}) \in \mathbf{C}$.

For a linear subspace $\mathbf{K}$ of $\mathbb{O}^{\mathbf{C}}$ over $\mathbb{R}$ or $\mathbf{C}$, let $M(n, \mathbf{K})$ be the set of all $n \times n$- matrices with entries in $\mathbf{K}$. For $A \in M(n, \mathbb{O}^{\mathbf{C}})$, let ${}^tA$ be the transposed matrix of A, $\overline{A}$ the matrix with the conjugate entries of A, and $A^* :={}^t\overline{A}$. Let us complex exceptional Jordan algebra as

$$\mathcal{J}^{\mathbf{C}} := \{X \in M(3, \mathbb{O}^{\mathbf{C}}) \mid X^* = X\}$$

which has the Jordan product $X \circ Y$, the inner product (X, Y) and the identity element E as follows:

$$X \circ Y := \frac{1}{2}(XY + YX),\ (X, Y) := \mathrm{tr}(X \circ Y),\ E := \mathrm{diag}(1, 1, 1).$$

Then an element $X \in \mathcal{J}^{\mathbf{C}}$ has the form

$$X \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbf{C}, x_i \in \mathbb{O}^{\mathbf{C}}.$$

After H. Freudenthal [3], another product $X \times Y$ is defined as

$$X \times Y := \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))E)$$

which is called the *Freudenthal product* and the trilinear form (X, Y, Z), the determinant $\det(X)$ are defined as

$$(X, Y, Z) := (X, Y \times Z), \quad \det(X) := \frac{1}{3}(X, X, X).$$

Proposition 1.2. *Let $X, Y, Z \in \mathcal{J}^{\mathbf{C}}$. By direct calculations, the following formulas hold.*

(1) $X \circ Y = Y \circ X, \quad X \times Y = Y \times X.$

(2) $E \times E = E$, $X \times E = \frac{1}{2}(\mathrm{tr}(X)E - X)$. *In particular,* $(X \times X) \times E = \frac{1}{2}(\mathrm{tr}(X \times X)E - X \times X)$.

(3) $\mathrm{tr}(X \times Y) = \frac{1}{2}(\mathrm{tr}(X)\mathrm{tr}(Y) - (X, Y))$.

(4) $(X, Y, Z) = (Y, Z, X) = (Z, X, Y)$.

(5) $X \circ (X \times X) = \det(X)E, \quad (X \times X) \times (X \times X) = \det(X)X$.

(6) $(X \times X) \times X = \frac{1}{2}(-\mathrm{tr}(X)X \times X - \mathrm{tr}(X \times X)X + (\mathrm{tr}(X \times X)\mathrm{tr}(X) - \det(X))E)$.

A linear Lie group $F_4^{\mathbf{C}}$ is defined as

$$F_4^{\mathbf{C}} := \{\alpha \in \mathrm{GL}_C(\mathcal{J}^{\mathbf{C}}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}.$$

The following result is proved in [10, Lemma 2.1.2, Proposition 2.1.3] after O. Shukuzawa and I. Yokota [7, p.3 Remark.]

Proposition 1.3. *The following formulas hold.*

$$\begin{aligned} F_4^{\mathbf{C}} &= \{\alpha \in F_4^{\mathbf{C}} \mid \mathrm{tr}(\alpha X) = \mathrm{tr}(X)\} \\ &= \{\alpha \in \mathrm{GL}_C(\mathcal{J}^{\mathbf{C}}) \mid \det(\alpha X) = X,\ \alpha E = E\} \\ &= \{\alpha \in \mathrm{GL}_C(\mathcal{J}^{\mathbf{C}}) \mid \det(\alpha X) = X,\ (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \mathrm{GL}_C(\mathcal{J}^{\mathbf{C}}) \mid (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z),\ (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \mathrm{GL}_C(\mathcal{J}^{\mathbf{C}}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}. \end{aligned}$$

Let us define

$$\mathcal{J} := \{ \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \mid \xi_i \in \mathbb{R}, x_i \in \mathbb{O}\}$$

and the complex conjugation τ with respect to $\mathcal{J}$ in $\mathcal{J}^{\mathbf{C}}$. Put $\sigma \in \mathrm{GL}_{\mathbf{C}}(\mathcal{J}^{\mathbf{C}})$:

$$\sigma \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} := \begin{pmatrix} \xi_1 & -x_3 & -\overline{x_2} \\ -\overline{x_3} & \xi_2 & x_1 \\ -x_2 & \overline{x_1} & \xi_3 \end{pmatrix}.$$

Then $\sigma \in F_4^{\mathbf{C}}$ because of $\det(\sigma X) = X$ and $\sigma E = E$. And σ satisfies $\sigma^2 = 1$. Then $\tau\sigma$ induce involutive automorphism $\widetilde{\tau\sigma}$ of $F_4^{\mathbf{C}}$: $\widetilde{\tau\sigma}(\alpha) = \tau\sigma\alpha\sigma\tau, \quad \alpha \in$

$F_4^{\mathbf{C}}$, and so let us define the $\mathbb{R}$-linear space $\mathcal{J}^1$ as $\mathcal{J}^1 := (\mathcal{J}^{\mathbf{C}})_{\tau\sigma}$. Then an element $X \in \mathcal{J}^1$ has a form:

$$X = \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & \sqrt{-1}\overline{x_2} \\ \sqrt{-1}\overline{x_3} & \xi_2 & x_1 \\ \sqrt{-1}x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbb{R}, x_i \in \mathbb{O}.$$

If $X_i \in \mathcal{J}^{\mathbf{C}},\ \ i = 1, 2$ satisfies $\tau\sigma(X_i) = X_i$, then $\tau\sigma(X_1 \circ X_2) = \tau\sigma(X_1) \circ \tau\sigma(X_2) = X_1 \circ X_2$. Thus $\mathcal{J}^1$ is closed under the Jordan product $X \circ Y$ with the identity $E = \mathrm{diag}(1, 1, 1)$ of Jordan product. And for all $X, Y, Z \in \mathcal{J}^1$, the trace $\mathrm{tr}(X) \in \mathbb{R}$, the inner pruduct $(X, Y) \in \mathbb{R}$, the Freudental product $X \times Y \in \mathcal{J}^1$, $(X, Y, Z) := (X, Y \times Z) \in \mathbb{R}$, the determinant $\det(X) = \frac{1}{3}(X, X \times X) \in \mathbb{R}$. Then $\mathcal{J}^1$ satisfies all formulas of Proposition 1.2. Let us define a linear Lie group $F_{4(-20)}$ as

$$F_{4(-20)} := \{\alpha \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\} = (F_4^{\mathbf{C}})^{\tilde{\tau\sigma}}|\mathcal{J}^1.$$

In $\mathcal{J}^1$, we use the following notations:

$$E_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$F_1^1(x) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix},\ F_2^1(x) := \begin{pmatrix} 0 & 0 & \sqrt{-1}\overline{x} \\ 0 & 0 & 0 \\ \sqrt{-1}x & 0 & 0 \end{pmatrix},$$
$$F_3^1(x) := \begin{pmatrix} 0 & \sqrt{-1}x & 0 \\ \sqrt{-1}\overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For all $X \in \mathcal{J}^1$, we can express $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)),\ \ \xi_i \in \mathbb{R}, x_i \in \mathbb{O}$. Put $\epsilon(1) := 1$, $\epsilon(2) := -1$, $\epsilon(3) := -1$. Then the table of the Freudenthal product among them are given as follows:

$$(1.1)\quad \begin{cases} E_i \times E_i = 0, & E_i \times E_{i+1} = \frac{1}{2}E_{i+2}, \\ E_i \times F_i^1(x) = -\frac{1}{2}F_i^1(x), & E_i \times F_j^1(x) = 0 \quad (i \neq j), \\ F_i^1(x) \times F_i^1(y) = -\epsilon(i)(x, y)E_i, & \\ F_{i+1}^1(x) \times F_{i+2}^1(y) = -\epsilon(i)\frac{1}{2}F_i^1(\overline{xy}) & \end{cases}$$

where indices are counted modulo 3. For all $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i))$,

$Y=\sum_{i=1}^3(\eta_i E_i+F_i^1(y_i))\in\mathcal{J}^1$, by direct calculation, we have

$$(1.2)\quad \begin{cases} (X,Y)=\sum_{i=1}^3(\xi_i\eta_i+\epsilon(i)2(x_i,y_i)),\\ \det(X)=\xi_1\xi_2\xi_3+2\mathrm{Re}((x_1x_2)x_3)-\sum_{i=1}^3\epsilon(i)\xi_i(x_i,x_i),\\ X\times X=\sum_{i=1}^3((\xi_{i+1}\xi_{i+2}-\epsilon(i)(x_i,x_i))E_i\\ \qquad\qquad +F_i^1(-\epsilon(i)\overline{x_{i+1}x_{i+2}}-\xi_i x_i)),\end{cases}$$

where indices are counted modulo 3. For all $X=\sum_{i=1}^3(\xi_i E_i+F_i^1(x_i))\in\mathcal{J}^1$, we denote $(X)_{E_i}:=\xi_i$, $(X)_{F_i^1}:=x_i$.

Theorem 1.4. *The following formulas hold.*

$$\begin{aligned} F_{4(-20)} &= \{\alpha\in F_{4(-20)}\mid \mathrm{tr}(\alpha X)=\mathrm{tr}(X)\}\\ &= \{\alpha\in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1)\mid \det(\alpha X)=X,\ \alpha E=E\}\\ &= \{\alpha\in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1)\mid \det(\alpha X)=X,\ (\alpha X,\alpha Y)=(X,Y)\}\\ &= \{\alpha\in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1)|(\alpha X,\alpha Y,\alpha Z)=(X,Y,Z),(\alpha X,\alpha Y)=(X,Y)\}\\ &= \{\alpha\in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1)\mid \alpha(X\times Y)=\alpha X\times\alpha Y\}.\end{aligned}$$

Proof. It follows from $F_{4(-20)}=(F_4^{\mathbf{C}})^{\tilde{\tau\sigma}}|\mathcal{J}^1$ and Proposition 1.3. (cf.[8, p.18].) □

Put $\varphi_X(\lambda):=\lambda E-X$. Let us define the *characteristic polynomial* $\Phi_X(\lambda)$ of $X\in\mathcal{J}^1$ as

$$\Phi_X(\lambda):=\det(\varphi_X(\lambda))=\det(\lambda E-X).$$

And a *characteristic root* of $X\in\mathcal{J}^1$ is a solution of $\Phi_X(\lambda)=0$ in $\mathbf{C}$.

Proposition 1.5. *The following assertions hold.*

(1) $\Phi_X(\lambda)=\lambda^3-\mathrm{tr}(X)\lambda^2+\mathrm{tr}(X\times X)\lambda-\det(X)$. *In particular the polynomial* $\Phi_\lambda(X)$ *is a* $\mathbb{R}$*-coefficient polynomial of* λ *with degree* 3.

(2) *If* $\Phi_X(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$, $\lambda_i\in\mathbf{C}$, *then* $\mathrm{tr}(X)=\lambda_1+\lambda_2+\lambda_3, \mathrm{tr}(X\times X)=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1, \det(X)=\lambda_1\lambda_2\lambda_3$.

(3) *The polynomial* $\Phi_X(\lambda)$ *is invariant by the action of* $F_{4(-20)}$. *In particular, the set of all characteristic roots and thier multiplicities are invariant by the action of* $F_{4(-20)}$.

Proof. (1) It follows from direct calculation using Proposition 1.2.

(2) It follows from (1).

(3) $\det(\lambda E-\alpha X)=\det(\alpha(\lambda E-X))=\det(\lambda E-X)$ for all $\alpha\in F_{4(-20)}$ by Theorem 1.4. □

Let A, B, C be sets. $A = B \coprod C$ means $A = B \cup C$ and $B \cap C = \emptyset$ in this paper. We define hyperbolic planes and null cones.

$$
\begin{aligned}
\mathcal{H} &:= \{X \in \mathcal{J}^1 \mid X \times X = 0, \mathrm{tr}(X) = 1\},\\
\mathcal{H}^+ &:= \{X \in \mathcal{J}^1 \mid X \times X = 0, \mathrm{tr}(X) = 1, (X)_{E_1} \geq 1\},\\
\mathcal{H}^- &:= \{X \in \mathcal{J}^1 \mid X \times X = 0, \mathrm{tr}(X) = 1, (X)_{E_1} \leq 0\},\\
\mathcal{N} &:= \{X \in \mathcal{J}^1 \mid X \neq 0, X \times X = 0, \mathrm{tr}(X) = 0\},\\
\mathcal{N}^+ &:= \{X \in \mathcal{J}^1 \mid X \times X = 0, \mathrm{tr}(X) = 0, (X)_{E_1} > 0\},\\
\mathcal{N}^- &:= \{X \in \mathcal{J}^1 \mid X \times X = 0, \mathrm{tr}(X) = 0, (X)_{E_1} < 0\},\\
\tilde{\mathcal{N}} &:= \mathcal{N} \coprod \{0\} = \{X \in \mathcal{J}^1 \mid X \times X = 0, \mathrm{tr}(X) = 0\}.
\end{aligned}
$$

We denote $N_1 := E_1 - E_2 + F_3^1(1)$ and $N_2 := -E_1 + E_2 + F_3^1(1)$. Using (1.1), $E_1 \in \mathcal{H}^+$, $E_2, E_3 \in \mathcal{H}^-$, $N_1 \in \mathcal{N}^+$ and $N_2 \in \mathcal{N}^-$.

Proposition 1.6. *The following assertions hold.*

(1) $\mathcal{H} = \mathcal{H}^+ \coprod \mathcal{H}^-$.

(2) $\mathcal{N} = \mathcal{N}^+ \coprod \mathcal{N}^-$.

(3) *The group* $F_{4(-20)}$ *acts on* $\mathcal{H}$, $\mathcal{N}$, $\tilde{\mathcal{N}}$.

Proof. (1) $\mathcal{H}^+ \cap \mathcal{H}^- = \emptyset$ is obvious. Take $X = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i)) \in \mathcal{H}$. By (1.2), $0 = (X \times X)_{E_2} = \xi_3\xi_1 + (x_2, x_2)$ and $0 = (X \times X)_{E_3} = \xi_1\xi_2 + (x_3, x_3)$. Therefore $\xi_1(\xi_2 + \xi_3) = -(x_2, x_2) - (x_3, x_3) \leq 0$. Hence $(X)_{E_1} = \xi_1 \leq 0$ or $\xi_2 + \xi_3 \leq 0$. If $\xi_2 + \xi_3 \leq 0$, then $(X)_{E_1} = \xi_1 = 1 - (\xi_2 + \xi_3) \geq 1$ by $\mathrm{tr}(X) = \xi_1 + \xi_2 + \xi_3$. Thus (1) follows.

(2) $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$ is obvious. Suppose that $X \in \mathcal{N}$ satisfies $(X)_{E_1} = 0$. By $\mathrm{tr}(X) = 0$, we can express $X = \xi E_2 - \xi E_3 + \sum_{i=1}^3 F_i^1(x_i)$, $\xi \in \mathbb{R}, x_i \in \mathbb{O}$. Then by (1.2), $0 = (X \times X)_{E_1} = -\xi^2 - (x_1, x_1) \leq 0$ and $0 = (X \times X)_{E_i} = (x_i, x_i) \geq 0$ $(i = 2, 3)$. Therefore $\xi = 0$, $x_i = 0$ $(i = 1, 2, 3)$ so that $X = 0$. It contradicts with $X \neq 0$. Hence (2) follows.

(3) The group $F_{4(-20)}$ acts on $\mathcal{H}$, $\mathcal{N}$, $\tilde{\mathcal{N}}$ by Theorem 1.4. □

For all $X \in \mathcal{J}^1$, put

$$V_X := \{aX \times X + bX + cE \mid a, b, c \in \mathbb{R}\}$$

which is called the *minimal subspace* of X in $\mathcal{J}^1$, because V_X is closed under the Freudenthal product by Proposition 1.2 (2)(5)(6). For $\lambda_1 \in \mathbb{R}$, let us define the following elements in V_X:

$$
\begin{aligned}
E_{X,\lambda_1} &:= \frac{1}{\mathrm{tr}(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))}\varphi_X(\lambda_1)\times\varphi_X(\lambda_1),\\
W_{X,\lambda_1} &:= -\varphi_X(\lambda_1)+\frac{\mathrm{tr}(\varphi_X(\lambda_1))}{2}(E-E_{X,\lambda_1})\\
&= X-(\lambda_1 E_{X,\lambda_1}+\frac{\mathrm{tr}(X)-\lambda_1}{2}(E-E_{X,\lambda_1})).
\end{aligned}
$$

If $\mathrm{tr}(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))\neq 0$, E_{X,λ_1} and W_{X,λ_1} are well-defined.

Proposition 1.7. *Assume that λ_1 is a characteristic root of X in $\mathbb{R}$. Then the following assertions hold.*

(1) $(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))\times(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))=0$.

(2) $\mathrm{tr}(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))=\Phi'_X(\lambda_1)=(\frac{d}{d\lambda}\Phi_X)(\lambda_1)$.

(3) *If λ_1 is a characteristic root of X in $\mathbb{R}$ of multiplicity* 1*, then* $\mathrm{tr}(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))\neq 0$. *Furthermore, $E_{X,\lambda_1}\in\mathcal{H}=\mathcal{H}^+\coprod\mathcal{H}^-$.*

Proof. (1) Since λ_1 is a characteristic root, $\det(\varphi_X(\lambda_1))=0$. By Proposition 1.2(5), $(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))\times(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))=\det(\varphi_X(\lambda_1))\varphi_X(\lambda_1)=0$.

(2) By Proposition 1.2(2), Proposition 1.5(1) and direct calculation, $\mathrm{tr}(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))=3\lambda_1^2-2\mathrm{tr}(X)\lambda_1+\mathrm{tr}(X\times X)=\Phi'_X(\lambda_1)$.

(3) Since λ_1 is a characteristic root of multiplicity 1, $0\neq\Phi'_X(\lambda_1)=\mathrm{tr}(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))$. Using (1), $\mathrm{tr}(\frac{1}{\mathrm{tr}(\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))}\varphi_X(\lambda_1)\times\varphi_X(\lambda_1))=1$ and Proposition 1.6(1), we obtain $E_{X,\lambda_1}\in\mathcal{H}=\mathcal{H}^+\coprod\mathcal{H}^-$. □

Proposition 1.8. *Assume that $X\in\mathcal{J}^1$ has a characteristic root λ_1 of X in $\mathbb{R}$ of multiplicity* 1 *and $\alpha\in F_{4(-20)}$. Then $E_{\alpha X,\lambda_1}$ and $W_{\alpha X,\lambda_1}$ are well-defined. Furthermore,*

$$\alpha E_{X,\lambda_1}=E_{\alpha X,\lambda_1},\ \alpha W_{X,\lambda_1}=W_{\alpha X,\lambda_1}.$$

Proof. By Proposition 1.5(3), $\alpha X\in\mathcal{J}^1$ has a characteristic root $\lambda_1\in\mathbb{R}$ of multiplicity 1. By Proposition 1.7(3), $\mathrm{tr}(\varphi_{\alpha X}(\lambda_1)\times\varphi_{\alpha X}(\lambda_1))\neq 0$, so that $E_{\alpha X,\lambda_1}$ and $W_{\alpha X,\lambda_1}$ are well-defined. Then the last equations follow from Theorem 1.4. □

Let us define the sublinear space $(\mathcal{J}^1)_0:=\{X\in\mathcal{J}^1|\ \mathrm{tr}(X)=0\}$ in $\mathcal{J}^1$ and the $\mathbb{R}$-linear map $p:\mathcal{J}^1\to(\mathcal{J}^1)_0\ ;\ p(X):=X-\frac{1}{3}\mathrm{tr}(X)E$. By Theorem 1.4, $\alpha(p(X))=p(\alpha X)$ for all $\alpha\in F_{4(-20)}$.

Proposition 1.9. *Assume* $X \in \mathcal{J}^1$ *has a characteristic root* λ_1 *in* $\mathbb{R}$ *of multiplicity* 3*. The following assertions hold.*

(1) $\Phi_{p(X)}(\mu) = \mu^3$*. In particular,* $\mathrm{tr}(p(X)) = 0$, $\mathrm{tr}(p(X) \times p(X)) = 0$, $\det(p(X)) = 0$.

(2) *If* $p(X) \times p(X) \neq 0$, *then* $p(X) \times p(X) \in \mathcal{N}$.

Proof. (1) By Proposition 1.5(2), we get $(\lambda - \frac{1}{3}\mathrm{tr}(X))^3 = \Phi_X(\lambda) = \det((\lambda - \frac{1}{3}\mathrm{tr}(X))E - p(X)) = \Phi_{p(X)}(\lambda - \frac{1}{3}\mathrm{tr}(X))$. Hence $\Phi_{p(X)}(\mu) = \mu^3$ and so by Proposition 1.5(1), $0 = \mathrm{tr}(p(X)) = \mathrm{tr}(p(X) \times p(X)) = \det(p(X))$.

(2) By (1), $\det(p(X)) = 0$ and $\mathrm{tr}(p(X) \times p(X)) = 0$. Using Proposition 1.2(5), $(p(X) \times p(X)) \times (p(X) \times p(X)) = \det(p(X))p(X) = 0$. Hence $p(X) \times p(X) \in \mathcal{N}$ follows. □

Main Theorem. $F_{4(-20)}$*-orbits on* $\mathcal{J}^1$ *are classified as follows.*

(I) *Assume that* $X \in \mathcal{J}^1$ *admits the characteristic roots* $\lambda_1 > \lambda_2 > \lambda_3$*. Then there exists the unique* $i \in \{1, 2, 3\}$ *such that* $\mathcal{H}^+ \cap V_X = \{E_{X,\lambda_i}\}$ *and* $\mathcal{H}^- \cap V_X = \{E_{X,\lambda_{i+1}},\ E_{X,\lambda_{i+2}}\}$ *where* i, $i+1$, $i+2$ *are counted modulo* 3*. In this case,* X *can be transformed to one of the following canonical forms by* $F_{4(-20)}$.

Cases	*The canonical forms of* X
1. $E_{X,\lambda_1} \in \mathcal{H}^+$	$\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$
2. $E_{X,\lambda_2} \in \mathcal{H}^+$	$\mathrm{diag}(\lambda_2, \lambda_3, \lambda_1)$
3. $E_{X,\lambda_3} \in \mathcal{H}^+$	$\mathrm{diag}(\lambda_3, \lambda_1, \lambda_2)$

(II) *Assume that* $X \in \mathcal{J}^1$ *admits the characteristic roots* $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ $(q > 0)$*. Then* X *can be transformed to the following canonical form by* $F_{4(-20)}$.

the characteristic roots of X	*The canonical form of* X
4. $\lambda_1 \in \mathbb{R},\ p \pm \sqrt{-1}q\ (q > 0)$	$\begin{pmatrix} p & \sqrt{-1}q & 0 \\ \sqrt{-1}q & p & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$

(III) *Assume that* $X \in \mathcal{J}^1$ *admits the characteristic roots* λ_1 *of multiplicity* 1 *and* λ_2 *of multiplicity* 2*. Then* $W_{X,\lambda_1} \in \tilde{\mathcal{N}}$*. In this case,* X *can be transformed to one of the following canonical forms by* $F_{4(-20)}$.

Cases	*The canonical form of X*
5. $W_{X,\lambda_1} = 0,\ E_{X,\lambda_1} \in \mathcal{H}^+$	$\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)$
6. $W_{X,\lambda_1} = 0,\ E_{X,\lambda_1} \in \mathcal{H}^-$	$\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1)$
7. $W_{X,\lambda_1} \in \mathcal{N}^+$	$\begin{pmatrix} \lambda_2 + 1 & \sqrt{-1} & 0 \\ \sqrt{-1} & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$
8. $W_{X,\lambda_1} \in \mathcal{N}^-$	$\begin{pmatrix} \lambda_2 - 1 & \sqrt{-1} & 0 \\ \sqrt{-1} & \lambda_2 + 1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$

(IV) *Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity 3. Then X can be transformed to one of the following canonical forms by $F_{4(-20)}$.*

Cases	*The canonical form of X*
9. $p(X) = 0$	$\frac{1}{3}\mathrm{tr}(X)E$
10. $p(X) \in \mathcal{N}^+$	$\begin{pmatrix} \frac{1}{3}\mathrm{tr}(X) + 1 & \sqrt{-1} & 0 \\ \sqrt{-1} & \frac{1}{3}\mathrm{tr}(X) - 1 & 0 \\ 0 & 0 & \frac{1}{3}\mathrm{tr}(X) \end{pmatrix}$
11. $p(X) \in \mathcal{N}^-$	$\begin{pmatrix} \frac{1}{3}\mathrm{tr}(X) - 1 & \sqrt{-1} & 0 \\ \sqrt{-1} & \frac{1}{3}\mathrm{tr}(X) + 1 & 0 \\ 0 & 0 & \frac{1}{3}\mathrm{tr}(X) \end{pmatrix}$
12. $p(X) \notin \tilde{\mathcal{N}}$	$\begin{pmatrix} \frac{1}{3}\mathrm{tr}(X) & 0 & \sqrt{-1} \\ 0 & \frac{1}{3}\mathrm{tr}(X) & 1 \\ \sqrt{-1} & 1 & \frac{1}{3}\mathrm{tr}(X) \end{pmatrix}$

(V) *By $F_{4(-20)}$, the above canonical forms cannot be transformed from each other.*

2 Generalized spheres and Hyperbolic planes.

Let $\mathfrak{f}_4^{\mathbf{C}} := \{X \in \mathrm{End}_{\mathbf{C}}(\mathcal{J}^{\mathbf{C}}) \mid \exp(tX) \in F_4^{\mathbf{C}}, t \in \mathbf{C}\}$ be the Lie algebra of $F_4^{\mathbf{C}}$. The following notations are used in this paper:

$$A_1(a) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\overline{a} & 0 \end{pmatrix},\ A_2(a) := \begin{pmatrix} 0 & 0 & -\overline{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix},\ A_3(a) := \begin{pmatrix} 0 & a & 0 \\ -\overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $a \in \mathbb{O}^{\mathbf{C}}$. Then $\tilde{A}_i(a) \in \mathrm{End}_{\mathbf{C}}(\mathcal{J}^{\mathbf{C}})$ is defined as $\tilde{A}_i(a)X := A_i(a)X - XA_i(a),\quad X \in \mathcal{J}^{\mathbf{C}}$. And $\mathfrak{d}_4^{\mathbf{C}} := \{D \in \mathfrak{f}_4^{\mathbf{C}} \mid DE_i = 0,\ i = 1, 2, 3\}$, $\tilde{\mathfrak{u}}_i^{\mathbf{C}} :=$

$\{\tilde{A}_i(a) \mid a \in \mathbb{O}^{\mathbf{C}}\}$. Let $\mathfrak{f}_{4(-20)} := \{X \in \mathrm{End}_{\mathbb{R}}(\mathcal{J}^1) \mid \exp(tX) \in F_{4(-20)}, t \in \mathbb{R}\}$ be the Lie algebra of $F_{4(-20)}$.

Proposition 2.1. *The following assertions hold.*

(1) $\mathfrak{f}_4^{\mathbf{C}} = \mathfrak{d}_4^{\mathbf{C}} \oplus \tilde{\mathfrak{u}}_1^{\mathbf{C}} \oplus \tilde{\mathfrak{u}}_2^{\mathbf{C}} \oplus \tilde{\mathfrak{u}}_3^{\mathbf{C}}$.

(2) $\mathfrak{f}_{4(-20)} = (\mathfrak{f}_4^{\mathbf{C}})_{\tau\sigma} = (\mathfrak{d}_4^{\mathbf{C}})_{\tau\sigma} \oplus (\tilde{\mathfrak{u}}_1^{\mathbf{C}})_{\tau\sigma} \oplus (\tilde{\mathfrak{u}}_2^{\mathbf{C}})_{\tau\sigma} \oplus (\tilde{\mathfrak{u}}_3^{\mathbf{C}})_{\tau\sigma}$. *Here,*$(\mathfrak{d}_4^{\mathbf{C}})_{\tau\sigma} = \{D \in (\mathfrak{f}_4^{\mathbf{C}})_{\tau\sigma} \mid DE_i = 0,\ i = 1,2,3\}$, $(\tilde{\mathfrak{u}}_1^{\mathbf{C}})_{\tau\sigma} = \{\tilde{A}_1(a) \mid a \in \mathbb{O}\}$, $(\tilde{\mathfrak{u}}_i^{\mathbf{C}})_{\tau\sigma} = \{\tilde{A}_i(\sqrt{-1}a) \mid a \in \mathbb{O}\}$ $(i = 2, 3)$.

Proof. (1) cf.[2]. (2) It follows from direct calculations. □

For $t \in \mathbb{R}$, let us define elements of $F_{4(-20)}$ as $\beta_1(t) := \exp(t\tilde{A}_1(1))$ and $\beta_3(t) := \exp(t\tilde{A}_3(\sqrt{-1}))$. Put $e^{tA_j(r)} := \sum_{n=0}^{\infty} \frac{1}{n!}(tA_j(r))^n$ where $(j, r) \in \{(1, 1), (3, \sqrt{-1})\}$. By direct calculation,

$$e^{tA_1(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix},$$

$$e^{tA_3(\sqrt{-1})} = \begin{pmatrix} \cosh t & \sqrt{-1}\sinh t & 0 \\ -\sqrt{-1}\sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $X \in \mathcal{J}^1$ and $(j, r) \in \{(1, 1), (3, \sqrt{-1})\}$. By direct calculation, we get $\frac{d}{dt}(e^{tA_j(r)}X)e^{-tA_j(r)} = \tilde{A}_j(r)((e^{tA_j(r)}X)e^{-tA_j(r)})$ and $(e^{0\cdot A_j(r)}X)e^{-0\cdot A_j(r)} = X$. Now $\frac{d}{dt}\beta_j(t)X = \tilde{A}_j(r)\beta_j(t)X$ and $\beta_j(0)X = X$. By the uniqueness of ordinary differential equations, we obtain

$$\beta_j(t) = (e^{tA_j(r)}X)e^{-tA_j(r)}.$$

In particular, put $Y := \beta_1(t)(\sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)))$. Then

$$(2.1)\quad \begin{cases} (Y)_{E_1} &= \xi_1, \\ (Y)_{E_2} &= \frac{\xi_2+\xi_3}{2} + \frac{\xi_2-\xi_3}{2}\cos 2t + \mathrm{Re}(x_1)\sin 2t, \\ (Y)_{E_3} &= \frac{\xi_2+\xi_3}{2} - \frac{\xi_2-\xi_3}{2}\cos 2t - \mathrm{Re}(x_1)\sin 2t, \\ (Y)_{F_1^1} &= \mathrm{Im}(x_1) + \mathrm{Re}(x_1)\cos 2t - \frac{\xi_2-\xi_3}{2}\sin 2t, \\ (Y)_{F_2^1} &= x_2\cos t - \overline{x_3}\sin t, \\ (Y)_{F_3^1} &= x_3\cos t + \overline{x_2}\sin t. \end{cases}$$

Put $Y := \beta_3(t)(\sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)))$. Then

$$
(2.2)\left\{\begin{array}{lcl}
(Y)_{E_1} &=& \frac{\xi_1+\xi_2}{2}+\frac{\xi_1-\xi_2}{2}\cosh 2t-\mathrm{Re}(x_3)\sinh 2t,\\
(Y)_{E_2} &=& \frac{\xi_1+\xi_2}{2}-\frac{\xi_1-\xi_2}{2}\cosh 2t+\mathrm{Re}(x_3)\sinh 2t,\\
(Y)_{E_3} &=& \xi_3,\\
(Y)_{F_1^1} &=& x_1\cosh t+\overline{x_2}\sinh t,\\
(Y)_{F_2^1} &=& x_2\cosh t+\overline{x_1}\sinh t,\\
(Y)_{F_3^1} &=& \mathrm{Im}(x_3)+\mathrm{Re}(x_3)\cosh 2t-\frac{\xi_1-\xi_2}{2}\sinh 2t.
\end{array}\right.
$$

Let us define the subgroup $\tilde{D}_4$ in $\mathrm{SO}(8)^{\times 3}$ and the group homomorphism $p:\tilde{D}_4\to\mathrm{SO}(8)$ as

$$
\begin{array}{rcl}
\tilde{D}_4 &:=& \{(\alpha_1,\alpha_2,\alpha_3)\in\mathrm{SO}(8)^{\times 3}\mid(\alpha_1x)(\alpha_2y)=\overline{\alpha_3(\overline{xy})},\quad x,y\in\mathbb{O}\},\\
&& p(\alpha_1,\alpha_2,\alpha_3)=\alpha_1.
\end{array}
$$

Lemma 2.2. (Y.Mastushima)[5, Lemma 2, Lemma 3]. cf.[9, Lemma 5.3, Lemma 5.4].

Assume that there exists $(\alpha_1,\alpha_2,\alpha_3)\in\mathrm{O}(8)^{\times 3}$ *such that*

$$(\alpha_ix)(\alpha_{i+1}y)=\overline{\alpha_{i+2}(\overline{xy})},\quad x,y\in\mathbb{O}$$

where i, $i+1$, $i+2$ *are counted modulo* 3. *Then the following assertions hold.*

(1) $(\alpha_{i+1}x)(\alpha_{i+2}y)=\overline{\alpha_i(\overline{xy})},\quad x,y\in\mathbb{O}.$

(2) $\alpha_i\in\mathrm{SO}(8)$ *for all* $i\in\{1,2,3\}$.

Let us define the subgroup of $F_{4(-20)}$ as

$$D_4:=\{\alpha\in F_{4(-20)}\mid\alpha_iE_i=E_i,\ i=1,2,3\},$$

and the mapping $\varphi:\tilde{D}_4\to\mathrm{GL}_{\mathbb{R}}(\mathcal{J}^1)$ as

$$\varphi(\alpha_1,\alpha_2,\alpha_3)(\sum_{i=1}^{3}(\xi_iE_i+F_i^1(x_i))):=\sum_{i=1}^{3}(\xi_iE_i+F_i^1(\alpha_ix_i)).$$

Proposition 2.3. *The following assertions hold.*

(1) p *is a group homomorphism* $\tilde{D}_4$ *onto* $\mathrm{SO}(8)$.

(2) φ *is a group isomorphism* $\tilde{D}_4$ *onto* D_4.

Proof. It can be similarly proved as [5, Theorem 1]. cf.[9, Proposition 5.7,5.9]. □

Hereafter, we identify $\tilde{D}_4$ with D_4 via φ in this paper.

Let $\mathcal{E}_i$, $(\mathcal{J}^1)_{2E_i,1}$, $(\mathcal{J}^1)_{2E_i,-1}$ be subspaces of $\mathcal{J}^1$ defined as

$$\begin{array}{rcl} \mathcal{E}_i & := & \{\xi E_i \mid \xi \in \mathbb{R}\}, \\ (\mathcal{J}^1)_{2E_i,1} & := & \{X \in \mathcal{J}^1 \mid 2E_i \times X = X\}, \\ (\mathcal{J}^1)_{2E_i,-1} & := & \{X \in \mathcal{J}^1 \mid 2E_i \times X = -X\}. \end{array}$$

Then by (1.1), $(\mathcal{J}^1)_{2E_i,1} = \{\xi(E_{i+1}+E_{i+2}) \mid \xi \in \mathbb{R}\} = \{\xi(E-E_i) | \ \xi \in \mathbb{R}\}$, $(\mathcal{J}^1)_{2E_i,-1} = \{\xi(E_{i+1}-E_{i+2}) + F_i^1(x) \mid \xi \in \mathbb{R},\ x \in \mathbb{O}\}$ where $i, i+1, i+2$ are counted modulo 3. Next, we define generalized spheres and null cones. For $r > 0$,

$$\begin{array}{rcl} \mathcal{S}^+(E_i;r) & := & \{W \in (\mathcal{J}^1)_{2E_i,-1} \mid (W,W) = r^2\}, \\ \mathcal{S}^-(E_i;r) & := & \{W \in (\mathcal{J}^1)_{2E_i,-1} \mid (W,W) = -r^2\}, \\ \mathcal{N}(E_i) & := & \{W \in (\mathcal{J}^1)_{2E_i,-1} \mid W \neq 0,\ (W,W) = 0\}. \end{array}$$

Since the inner product over $(\mathcal{J}^1)_{2E_1,-1}$ is positive definite, $\mathcal{S}^-(E_1;r) = \mathcal{N}(E_1) = \emptyset$. Moreover, for $i = 2,3$, we put

$$\begin{array}{rcl} \mathcal{S}^+_+(E_i;r) & := & \{W \in \mathcal{S}^+(E_i;r) \mid (W)_{E_1} > 0\}, \\ \mathcal{S}^+_-(E_i;r) & := & \{W \in \mathcal{S}^+(E_i;r) \mid (W)_{E_1} < 0\}, \\ \mathcal{N}_+(E_i) & := & \{W \in \mathcal{N}(E_i) \mid (W)_{E_1} > 0\}, \\ \mathcal{N}_-(E_i) & := & \{W \in \mathcal{N}(E_i) \mid (W)_{E_1} < 0\}. \end{array}$$

Lemma 2.4. *Let X be a set and a group G acts on X. Assume that there are subsets $X_i \subset X$ and elements $v_i \in X$ which satisfy the following conditions* (i)-(iv)*, where I is an index set and $i, j \in I$:*

(i) $X = \cup_{i \in I} X_i$, (ii) $v_i \in X_i$, (iii) $\mathcal{O}_G(v_i) \neq \mathcal{O}_G(v_j) \Leftrightarrow i \neq j$,
(iv) $X_i \subset \mathcal{O}_G(v_i)$.

Then $X_i = \mathcal{O}_G(v_i)$ for all $i \in I$.

Proof. Since G acts on X, $\mathcal{O}_G(v_i) \subset X$. Now take $x \in \mathcal{O}_G(v_i)$. Since $x \in \mathcal{O}_G(v_i) \subset X = \cup_{i \in I} X_i$, there exists $j \in I$ such that $x \in X_j$. By (iv), $x \in X_j \subset \mathcal{O}_G(v_j)$. Then $i = j$ by (iii). Therefore $x \in X_i$, so that $\mathcal{O}_G(v_i) \subset X_i$. Next by (iv), $X_i \subset \mathcal{O}_G(v_i)$. Thus $X_i = \mathcal{O}_G(v_i)$. □

For $Y \in \mathcal{J}^1$, the inner product B_Y on $\mathcal{J}^1$ is defined as $B_Y(X_1, X_2) = (X_1, X_2, Y)$, $\ X_i \in \mathcal{J}^1$.

Proposition 2.5. *For $Y_1, Y_2 \in \mathcal{J}^1$, if B_{Y_i} $(i = 1, 2)$ have different signatures then $\mathcal{O}_{F_{4(-20)}}(Y_1) \neq \mathcal{O}_{F_{4(-20)}}(Y_2)$. In particular:*

(1) $\mathcal{O}_{F_{4(-20)}}(E_1) \neq \mathcal{O}_{F_{4(-20)}}(E_2) = \mathcal{O}_{F_{4(-20)}}(E_3)$:

(2) $\mathcal{O}_{F_{4(-20)}}(E_1 - E_2) \neq \mathcal{O}_{F_{4(-20)}}(-E_1 + E_2)$.

Proof. Suppose that there exists $\alpha \in F_{4(-20)}$ such that $\alpha Y_1 = Y_2$. Using Theorem 1.4, $B_{Y_1}(X_1, X_2) = (X_1, X_2, Y_1) = (\alpha X_1, \alpha X_2, \alpha Y_1) = B_{Y_2}(\alpha X_1, \alpha X_2)$. Therefore inner products B_{Y_1} and B_{Y_2} have the same signatures. This contradicts with the assumption. Thus $\mathcal{O}_{F_{4(-20)}}(Y_1) \neq \mathcal{O}_{F_{4(-20)}}(Y_2)$. Then we have the table of signatures: for $X = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i)) \in \mathcal{J}^1$,

$Y \in \mathcal{J}^1$	$B_Y(X, X)$	types of signature
E_1	$\xi_2\xi_3 - (x_1, x_1)$	$(+ \Rightarrow 1, - \Rightarrow 9)$
E_3	$\xi_1\xi_2 + (x_3, x_3)$	$(+ \Rightarrow 9, - \Rightarrow 1)$
$E_1 - E_2$	$\xi_2\xi_3 - \xi_3\xi_1 - (x_1, x_1) - (x_2, x_2)$	$(+ \Rightarrow 2, - \Rightarrow 18)$
$-E_1 + E_2$	$-\xi_2\xi_3 + \xi_3\xi_1 + (x_1, x_1) + (x_2, x_2)$	$(+ \Rightarrow 18, - \Rightarrow 2)$

Therefore $\mathcal{O}_{F_{4(-20)}}(E_1) \neq \mathcal{O}_{F_{4(-20)}}(E_3)$, $\mathcal{O}_{F_{4(-20)}}(E_1 - E_2) \neq \mathcal{O}_{F_{4(-20)}}(-E_1 + E_2)$. Next, by (2.1), $\beta_1(\frac{\pi}{2})E_3 = E_2$, so that $\mathcal{O}_{F_{4(-20)}}(E_2) = \mathcal{O}_{F_{4(-20)}}(E_3)$. Hence the assertion follows. □

Proposition 2.6. *The following assertions hold.*

(1) *If there exists $X_0 \in (\mathcal{J}^1)_0$ such that $X_0 \times X_0 = N_i$ for some $i \in \{1, 2\}$, then $X_0 \times (X_0 \times X_0) = 0$.*

(2) $\{X_0 \in (\mathcal{J}^1)_0 \mid X_0 \times X_0 = N_1\} = \emptyset$.

(3) $\{X_0 \in (\mathcal{J}^1)_0 \mid X_0 \times X_0 = N_2\}$
$= \{-rN_2 + F_1^1(x) + F_2^1(\overline{x}) | \ r \in \mathbb{R}, x \in \mathbb{O}, |x| = 1\}$.

(4) $\mathcal{O}_{F_{4(-20)}}(N_1) \neq \mathcal{O}_{F_{4(-20)}}(N_2)$.

Proof. (1) Since $X_0 \in (\mathcal{J}^1)_0$, $\mathrm{tr}(X_0 \times X_0) = \mathrm{tr}(N_i) = 0$ and $\det(X_0)X_0 = (X_0 \times X_0) \times (X_0 \times X_0) = N_i \times N_i = 0$ by Proposition 1.2(5), we have $\mathrm{tr}(X_0) = \mathrm{tr}(X_0 \times X_0) = \det(X_0) = 0$. Thus $X_0 \times (X_0 \times X_0) = 0$ follows from Proposition 1.2(6).

(2) Set $P = \{X_0 \in (\mathcal{J}^1)_0 \mid X_0 \times X_0 = N_1\}$. Suppose that there exists $X_0 = \sum_{i=1}^{3}(r_i E_i + F_i^1(x_i)) \in P$. Then by (1) and (1.1), $0 = X_0 \times (X_0 \times X_0) = X_0 \times N_1 = -\frac{r_3}{2}E_1 + \frac{r_3}{2}E_2 + (-\frac{r_1}{2} + \frac{r_2}{2} + (1, x_3))E_3 + \frac{1}{2}F_1^1(-x_1 - \overline{x_2}) - \frac{1}{2}F_2^1(x_2 + \overline{x_1}) - \frac{1}{2}F_3^1(r_3)$. Therefore $r_3 = 0$. However, $1 = (N_1)_{E_1} = (X_0 \times X_0)_{E_1} = r_2 \cdot 0 - (x_1, x_1) = -(x_1, x_1) \leq 0$ by (1.2). It is a contradiction. Hence $\{X_0 \in (\mathcal{J}^1)_0 \mid X_0 \times X_0 = N_1\} = \emptyset$.

(3) Set $P_1 = \{X_0 \in (\mathcal{J}^1)_0 \mid X_0 \times X_0 = N_2\}$ and $P_2 = \{-rN_2 + F_1^1(x) + F_2^1(\overline{x}) \mid r \in \mathbb{R}, x \in \mathbb{O}, |x| = 1\}$. Take $X_0 = \sum_{i=1}^3 (r_i E_i + F_i^1(x_i)) \in P_1$. Then by (1) and (1.1), $0 = X_0 \times (X_0 \times X_0) = X_0 \times N_2 = \frac{r_3}{2}E_1 - \frac{r_3}{2}E_2 + (-\frac{r_2}{2} + \frac{r_1}{2} + (1, x_3))E_3 + \frac{1}{2}F_1^1(x_1 - \overline{x_2}) - \frac{1}{2}F_2^1(-x_2 + \overline{x_1}) - \frac{1}{2}F_3^1(r_3)$. Therefore $r_3 = 0$ and $x_2 = \overline{x_1}$. Next, by (1.2), $N_2 = X_0 \times X_0 = -(x_1, x_1)E_1 + (x_1, x_1)E_2 + (r_1 r_2 + (x_3, x_3))E_3 + F_1^1(-(\overline{x_3} + r_1)x_1) + F_2^1((\overline{x_3} - r_2)\overline{x_1}) + F_3^1(\overline{x_1 \overline{x_1}})$. Since $(N_2)_{E_1} = -1$ and $(N_2)_{F_1^1} = (N_2)_{F_2^1} = 0$, we get $|x_1| = 1$, $x_3 = r_2 = -r_1$. Therefore $X_0 = -r_1 N_2 + F_1^1(x_1) + F_2^1(\overline{x_1})$ where $|x_1| = 1$. Hence $X_0 \in P_2$ and so $P_1 \subset P_2$. Conversely, take $X_0 = -rN_2 + F_1^1(x) + F_2^1(\overline{x}) \in P_2$ where $r \in \mathbb{R}$, $x \in \mathbb{O}$ and $|x| = 1$. By direct calculation, $X_0 \in P_1$, so that $P_2 \subset P_1$. Hence $P_1 = P_2$.

(4) Suppose that there exists $\alpha \in F_{4(-20)}$ such that $\alpha N_1 = N_2$. Then $\emptyset = \alpha(\{X_0 \in (\mathcal{J}^1)_0 \mid X_0 \times X_0 = N_1\}) = \{Y_0 \in (\mathcal{J}^1)_0 \mid Y_0 \times Y_0 = N_2\} \neq \emptyset$ by Theorem 1.4, (2) and (3). It is a contradiction, as required. □

Lemma 2.7. *Let* $X = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i)) \in \mathcal{J}^1$ *and* $j \in \{1, 2, 3\}$. *Then there exists* $\varphi(\alpha_1, \alpha_2, \alpha_3) \in D_4$ *such that* $\varphi(\alpha_1, \alpha_2, \alpha_3)X = \sum_{i=1}^3 \xi_i E_i + F_j^1(|x_j|) + F_{j+1}^1(\alpha_{j+1}x_{j+1}) + F_{j+2}^1(\alpha_{j+2}x_{j+2})$ *where* $j, j+1, j+2$ *are counted modulo* 3.

Proof. There exists $\alpha_j \in \mathrm{SO}(8)$ such that $\alpha_j x_j = |x_j|$. By Proposition 2.3(1), there exists $(\alpha_j, \alpha_{j+1}, \alpha_{j+2}) \in \tilde{D}_4$ such that $p(\alpha_j, \alpha_{j+1}, \alpha_{j+2}) = \alpha_j$. Using Lemma 2.2(1), we have $(\alpha_1, \alpha_2, \alpha_3) \in \tilde{D}_4$. Then by Proposition 2.3(2), $\varphi(\alpha_1, \alpha_2, \alpha_3) \in D_4$ satisfies the assertion. □

Proposition 2.8. *The following assertions hold.*

(1) $\mathcal{S}^+(E_1; r) = \mathcal{O}_{(F_{4(-20)})_{E_1}}(\frac{r}{\sqrt{2}}(E_2 - E_3))$.

(2) $\mathcal{S}^+(E_3; r) = \mathcal{S}_+^+(E_3; r) \coprod \mathcal{S}_-^+(E_3; r)$.

(3) $\mathcal{S}_+^+(E_3; r) = \mathcal{O}_{(F_{4(-20)})_{E_3}}(\frac{r}{\sqrt{2}}(E_1 - E_2))$
$\neq \mathcal{O}_{(F_{4(-20)})_{E_3}}(\frac{r}{\sqrt{2}}(-E_1 + E_2)) = \mathcal{S}_-^+(E_3; r)$.

(4) $\mathcal{S}^-(E_3; r) = \mathcal{O}_{(F_{4(-20)})_{E_3}}(F_3^1(\frac{r}{\sqrt{2}}))$.

(5) $\mathcal{N}(E_3) = \mathcal{N}_+(E_3) \coprod \mathcal{N}_-(E_3)$.

(6) $\mathcal{N}_+(E_3) = \mathcal{O}_{(F_{4(-20)})_{E_3}}(N_1) \neq \mathcal{O}_{(F_{4(-20)})_{E_3}}(N_2) = \mathcal{N}_-(E_3)$.

Proof. For all $W \in (\mathcal{J}^1)_{2E_i,-1}$ and $\alpha \in (F_{4(-20)})_{E_i}$, $2E_i \times \alpha W = \alpha(2E_i \times W) = -W$ and $(\alpha W, \alpha W) = (W, W)$ by Theorem 1.4. In particular, $(F_{4(-20)})_{E_1}$ acts on $\mathcal{S}^+(E_1; r)$. And $(F_{4(-20)})_{E_3}$ acts on $\mathcal{S}^+(E_3; r)$, $\mathcal{S}^-(E_3; r)$

and $\mathcal{N}(E_3)$. By (2.1) and (2.2), $\{\beta_1(t)|\ t\in\mathbb{R}\}\subset(F_{4(-20)})_{E_1}$ and $\{\beta_3(t)|\ t\in\mathbb{R}\}\subset(F_{4(-20)})_{E_3}$.

(1) Take $W\in\mathcal{S}^+(E_1;r)$. By Lemma 2.7, there exists $\alpha_0\in D_4\subset(F_{4(-20)})_{E_1}$ such that $\alpha_0 W=\xi(E_2-E_3)+F_1^1(r_0)$ where $r_0\geq 0$. Next by (2.1), $(\beta_1(t)\alpha_0 W)_{E_2}=\xi\cos 2t+r_0\sin 2t=\frac{r}{\sqrt{2}}\cos(2t+t_0)$ for some $t_0\in\mathbb{R}$. Then $\beta_1(-\frac{1}{2}t_0)\alpha_0 W=\frac{r}{\sqrt{2}}(E_2-E_3)$. Hence (1) follows.

(2) $\mathcal{S}^+_+(E_3;r)\cap\mathcal{S}^+_-(E_3;r)=\emptyset$ is obvious. Suppose that there exists $W\in\mathcal{S}^+(E_3;r)$ such that $(W)_{E_1}=0$. Then we can write $W=F_3^1(x)\in\mathcal{S}^+(E_3;r)$, so that $(W,W)=-2(x,x)\leq 0$ by (1.2). It contradicts with $(W,W)=r^2>0$. Thus $(W)_{E_1}\neq 0$ and (2) follows.

(3) We use Lemma 2.4. We know that $(F_{4(-20)})_{E_3}$ acts on $\mathcal{S}^+(E_3;r)$, and so we consider that $X=\mathcal{S}^+(E_3;r)$, $G=(F_{4(-20)})_{E_3}$, $X_1=\mathcal{S}^+_+(E_3;r)$, $X_2=\mathcal{S}^+_-(E_3;r)$, $v_1=\frac{r}{\sqrt{2}}(E_1-E_2)$ and $v_2=\frac{r}{\sqrt{2}}(-E_1+E_2)$ in Lemma 2.4. The first, the condition (i) follows from (2). The second, the condition (ii) follows from direct calculations. The third, we notice $\mathcal{O}_{(F_{4(-20)})_{E_3}}(\epsilon\frac{r}{\sqrt{2}}(E_1-E_2))\subset\mathcal{O}_{F_{4(-20)}}(\epsilon\frac{r}{\sqrt{2}}(E_1-E_2))$ where $\epsilon=\pm 1$. Using Proposition 2.5(2), we have $\mathcal{O}_{F_{4(-20)}}(\frac{r}{\sqrt{2}}(E_1-E_2))\neq\mathcal{O}_{F_{4(-20)}}(\frac{r}{\sqrt{2}}(-E_1+E_2))$. Thus the condition (iii) follows. We will show the condition (iv).

(Case 1) Take $W\in\mathcal{S}^+_+(E_3;r)$. By Lemma 2.7, there exists $\alpha_0\in D_4\subset(F_{4(-20)})_{E_3}$ such that $\alpha_0 W=\xi(E_1-E_2)+F_3^1(r_0)$ where $\xi>0$ and $r_0\geq 0$. Using $\xi>0$ and $2(\xi^2-r_0^2)=r^2$, $\beta_3(\frac{1}{4}\log(\frac{\xi+r_0}{\xi-r_0}))\alpha_0 W=\frac{r}{\sqrt{2}}(E_1-E_2)$. Hence $W\in\mathcal{O}_{(F_{4(-20)})_{E_3}}(\frac{r}{\sqrt{2}}(E_1-E_2))$, and so $\mathcal{S}^+_+(E_3;r)\subset\mathcal{O}_{(F_{4(-20)})_{E_3}}(\frac{r}{\sqrt{2}}(E_1-E_2))$.

(Case 2) Take $W\in\mathcal{S}^+_-(E_3;r)$. By Lemma 2.7, there exists $\alpha_0\in D_4\subset(F_{4(-20)})_{E_3}$ such that $\alpha_0 W=\xi(E_1-E_2)+F_3^1(r_0)$ where $\xi<0$ and $r_0\geq 0$. Using $\xi<0$ and $2(\xi^2-r_0^2)=r^2$, $\beta_3(\frac{1}{4}\log(\frac{\xi+r_0}{\xi-r_0}))\alpha_0 W=\frac{r}{\sqrt{2}}(-E_1+E_2)$. Hence $W\in\mathcal{O}_{(F_{4(-20)})_{E_3}}(\frac{r}{\sqrt{2}}(-E_1+E_2))$. and so $\mathcal{S}^+_-(E_3;r)\subset\mathcal{O}_{(F_{4(-20)})_{E_3}}(\frac{r}{\sqrt{2}}(-E_1+E_2))$.

Hence the condition (iv) follows. Thus (3) follows from Lemma 2.4.

(4) Take $W\in\mathcal{S}^-(E_3;r)$. By Lemma 2.7, there exists $\alpha_0\in D_4\subset(F_{4(-20)})_{E_3}$ such that $\alpha_0 W=\xi(E_1-E_2)+F_3^1(r_0)$ where $r_0>0$ and $(W,W)<0$. Using $2(\xi^2-r_0^2)=-r^2$, $\beta_3(\frac{1}{4}\log(\frac{r_0+\xi}{r_0-\xi}))\alpha_0 W=F_3^1(\frac{r}{\sqrt{2}})$. Hence $\mathcal{S}^-(E_3;r)=\mathcal{O}_{(F_{4(-20)})_{E_3}}(F_3^1(\frac{r}{\sqrt{2}}))$.

(5) $\mathcal{N}_+(E_3)\cap\mathcal{N}_-(E_3)=\emptyset$ is obvious. Suppose that there exists $(0\neq)W\in\mathcal{N}(E_3)$ such that $(W)_{E_1}=0$. Then we can express $W=F_3^1(x)\in\mathcal{N}(E_3)$. Therefore by (1.2), $0=(W,W)=-2(x,x)$ so that $x=0$. It contradicts with $W\neq 0$. Thus $(W)_{E_1}\neq 0$ and (5) follows.

(6) We use Lemma 2.4. We know that $(F_{4(-20)})_{E_3}$ acts on $\mathcal{N}(E_3)$, and so we consider that $X = \mathcal{N}(E_3)$, $G = (F_{4(-20)})_{E_3}$, $X_1 = \mathcal{N}_+(E_3)$, $X_2 = \mathcal{N}_-(E_3)$, $v_1 = N_1$ and $v_2 = N_2$ in Lemma 2.4. The first, the condition (i) follows from (5). The second, the condition (ii) follows from direct calculations. The third, we notice $\mathcal{O}_{(F_{4(-20)})_{E_3}}(N_i) \subset \mathcal{O}_{F_{4(-20)}}(N_i)$ where $i = 1, 2$. Using Proposition 2.6(4), the condition (iii) follows. We will show the condition (iv).

(Case 1) Take $W \in \mathcal{N}_+(E_3)$. By Lemma 2.7, there exists $\alpha_0 \in D_4 \subset (F_{4(-20)})_{E_3}$ such that $\alpha_0 W = \xi(E_1 - E_2) + F_3^1(r_0)$ where $\xi > 0$ and $r_0 > 0$. Using $\xi > 0$ and $\xi^2 - r_0^2 = 0$, $\beta_3(\frac{1}{2}\log\xi)\alpha_0 W = N_1$. Therefore $W \in \mathcal{O}_{(F_{4(-20)})_{E_3}}(N_1)$, so that $\mathcal{N}_+(E_3) \subset \mathcal{O}_{(F_{4(-20)})_{E_3}}(N_1)$.

(Case 2) Take $W \in \mathcal{N}_-(E_3)$. By Lemma 2.7, there exists $\alpha_0 \in D_4 \subset (F_{4(-20)})_{E_3}$ such that $W = \xi(E_1 - E_2) + F_3^1(r_0)$ where $\xi < 0$ and $r_0 > 0$. Using $\xi < 0$ and $\xi^2 - r_0^2 = 0$, $\beta_3(\frac{1}{2}\log|\xi|)\alpha_0 W = N_2$. Therefore $W \in \mathcal{O}_{(F_{4(-20)})_{E_3}}(N_2)$, so that $\mathcal{N}_-(E_3) \subset \mathcal{O}_{(F_{4(-20)})_{E_3}}(N_2)$.

Hence the condition (iv) follows. Thus (6) follows from Lemma 2.4. $\square$

Let us define $\mathcal{J}^1(2; E_3) := \{\xi_1 E_1 + \xi_2 E_2 + F_3^1(x) \in \mathcal{J}^1 \mid \xi_i \in \mathbb{R}, x \in \mathbb{O}\}$.

Lemma 2.9. *The following assertions hold.*

(1) *Assume $X \in \mathcal{J}^1$. Then there exists $\alpha \in F_{4(-20)}$ such that $(\alpha X)_{F_1} = 0$ and $(\alpha X)_{E_1} = (X)_{E_1}$.*

(2) *Assume $X \in \mathcal{J}^1$ satisfies $X \times X = 0$. Then there exists $\alpha \in (F_{4(-20)})_{E_1}$ such that $\alpha X \in \mathcal{J}^1(2; E_3)$ and $(\alpha X)_{E_1} = (X)_{E_1}$.*

(3) *Assume $X \in \mathcal{H}$. Then there exists $\alpha \in F_{4(-20)}$ such that $\alpha X = \frac{1}{2}(E - E_3) + W$ for some $W \in \mathcal{S}^+(E_3; \frac{1}{\sqrt{2}})$), and $(\alpha X)_{E_1} = (X)_{E_1}$.*

(4) *Assume $X \in \mathcal{N}$. Then there exists $\alpha \in F_{4(-20)}$ such that $\alpha X \in \mathcal{N}(E_3)$ and $(\alpha X)_{E_1} = (X)_{E_1}$.*

Proof. (1) Take $X = \sum_{i=1}^3(\xi_i E_i + F_i^1(x_i)) \in \mathcal{J}^1$. Put $(\mathcal{J}^1)_{-\sigma} = \{X \in \mathcal{J}^1 \mid 2E_1 \times X = 0,\ (E_1, X) = 0\}$. By (1.1), $(\mathcal{J}^1)_{-\sigma} = \{F_2^1(x_2) + F_3^1(x_3) \mid x_i \in \mathbb{O}\}$. Then X can be expressed by

$$\begin{aligned} X &= \xi_1 E_1 + \frac{\xi_2 + \xi_3}{2}(E - E_1) + (\frac{\xi_2 - \xi_3}{2}(E_2 - E_3) + F_1^1(x_1)) \\ &\quad + (F_2^1(x_2) + F_3^1(x_3))) \in \mathcal{E}_1 \oplus (\mathcal{J}^1)_{2E_1,1} \oplus (\mathcal{J}^1)_{2E_1,-1} \oplus (\mathcal{J}^1)_{-\sigma}. \end{aligned}$$

For all $\alpha_0 \in (F_{4(-20)})_{E_1}$, by Theorem 1.4, $(\alpha_0 X)_{E_1} = (X)_{E_1}$, $\alpha_0(\frac{\xi_2+\xi_3}{2}(E - E_1)) = \frac{\xi_2+\xi_3}{2}(E - E_1)$, $\alpha_0(\mathcal{J}^1)_{-\sigma} \subset (\mathcal{J}^1)_{-\sigma}$ and $\alpha_0(\mathcal{J}^1)_{2E_1,-1} \subset (\mathcal{J}^1)_{2E_1,-1}$.

By Proposition 2.8(1), there exists $\alpha \in (F_{4(-20)})_{E_1}$ such that $\alpha(\frac{\xi_2-\xi_3}{2}(E_2 - E_3) + F_1^1(x_1)) = r(E_2 - E_3)$ where $r \geq 0$. Then, since $(\mathcal{J}^1)_{-\sigma}$ is invariant over the $(F_{4(-20)})_{E_1}$-action, $\alpha(F_2^1(x_2)+F_3^1(x_3)) = F_2^1(y_2)+F_3^1(y_3)$ for some $y_i \in \mathbb{O}$. Therefore

$$\alpha X = \xi_1 E_1 + (\frac{\xi_2+\xi_3}{2} + r)E_2 + (\frac{\xi_2+\xi_3}{2} - r)E_3 + F_2^1(y_2) + F_3^1(y_3).$$

Hence (1) follows.

(2) By (1), there exists $\alpha_0 \in (F_{4(-20)})_{E_1}$ such that $\alpha_0 X = \xi_1 E_1 + r_2 E_2 + r_3 E_3 + F_2^1(y_2) + F_3^1(y_3)$ where $r_i \in \mathbb{R}, y_i \in \mathbb{O}$ and $(X)_{E_1} = (\alpha_0 X)_{E_1} = \xi_1$. Now $0 = \alpha_0(X \times X) = \alpha_0 X \times \alpha_0 X$ by Theorem 1.4. Since $0 = (\alpha_0 X \times \alpha_0 X)_{E_1} = r_2 r_3$ by (1.2), we get (i) $r_2 = 0$ or (i) $r_3 = 0$.

In Case (i) $r_2 = 0$, then $0 = (\alpha_0 X \times \alpha_0 X)_{E_3} = (y_3, y_3)$ by (1.2). Therefore $y_3 = 0$. Next step, by (2.1), $\beta_1(\frac{\pi}{2}) \in (F_{4(-20)})_{E_1}$, so that $\beta_1(\frac{\pi}{2})\alpha X = \xi_1 E_1 + r_3 E_2 + F_3^1(\overline{y_2}) \in \mathcal{J}^1(2; E_3)$ and $(\beta_1(\frac{\pi}{2})\alpha X)_{E_1} = \xi_1 = (X)_{E_1}$.

In Case (ii) $r_3 = 0$, then $0 = (\alpha_0 X \times \alpha_0 X)_{E_2} = (y_2, y_2)$ by (1.2). Therefore $y_2 = 0$, so that $\alpha_0 X = \xi_1 E_1 + r_2 E_2 + F_3^1(y_3) \in \mathcal{J}^1(2; E_3)$ and $(\alpha_0 X)_{E_1} = \xi_1 = (X)_{E_1}$.

Hence (2) follows form above Case (i) and Case (ii).

(3) By (2), there exists $\alpha_0 \in (F_{4(-20)})_{E_1}$ such that $\alpha_0 X = \xi_1 E_1 + r E_2 + F_3^1(x) \in \mathcal{J}^1(2; E_3)$. Put $W = \frac{\xi_1 - r}{2}(E_1 - E_2) + F_3^1(x)$. By Theorem 1.4, $1 = \mathrm{tr}(X) = \mathrm{tr}(\alpha X) = \xi_1 + r$, so that $\alpha_0 X = \frac{\xi_1+r}{2}(E_1 + E_2) + W = \frac{1}{2}(E - E_3) + W \in (\mathcal{J}^1)_{2E_3,1} \oplus (\mathcal{J}^1)_{2E_3,-1}$. By Theorem 1.4 and (1.2), $0 = (\alpha(X \times X))_{E_3} = (\alpha X \times \alpha X)_{E_3} = \xi_1 r + (x, x)$. Therefore $(W, W) = \frac{1}{2}(\xi - r)^2 - 2(x, x) = \frac{1}{2}((\xi_1 + r)^2 - 4(\xi_1 r + (x, x))) = \frac{1}{2}$. Hence (3) follows.

(4) By (2), there exists $\alpha_0 \in (F_{4(-20)})_{E_1}$ such that $\alpha_0 X = \xi_1 E_1 + r E_2 + F_3^1(x)$. By Theorem 1.4, $\xi_1 + r = \mathrm{tr}(\alpha_0 X) = \mathrm{tr}(X) = 0$ so that $r = -\xi_1$. Therefore $\alpha_0 X \in (\mathcal{J}^1)_{2E_3,-1}$. By Theorem 1.4 and Proposition 1.2(3), $0 = \mathrm{tr}(X \times X) = \mathrm{tr}(\alpha_0 X \times \alpha_0 X) = \frac{1}{2}(\mathrm{tr}(\alpha_0 X)^2 - (\alpha_0 X, \alpha_0 X))$. By $\mathrm{tr}(\alpha_0 X) = 0$, $(\alpha_0 X, \alpha_0 X) = 0$. Hence (4) follows. □

Proposition 2.10. $F_{4(-20)}$ *acts on* $\mathcal{H}^+$, $\mathcal{H}^-$, $\mathcal{N}^+$, $\mathcal{N}^-$. *Furthermore, the following assertions hold.*

(1) $\mathcal{H}^+ = \mathcal{O}_{F_{4(-20)}}(E_1) \neq \mathcal{O}_{F_{4(-20)}}(E_2) = \mathcal{O}_{F_{4(-20)}}(E_3) = \mathcal{H}^-$.

(2) $\mathcal{N}^+ = \mathcal{O}_{F_{4(-20)}}(N_1) \neq \mathcal{O}_{F_{4(-20)}}(N_2) = \mathcal{N}^-$.

Proof. (1) We use Lemma 2.4. We know that $F_{4(-20)}$ acts on $\mathcal{H}$ by Proposition 1.6(3), and so we consider that $X = \mathcal{H}$, $G = F_{4(-20)}$, $X_1 = \mathcal{H}^+$, $X_2 = \mathcal{H}^-$, $v_1 = E_1$ and $v_2 = E_2$ in Lemma 2.4. The first, the condition (i) follows from Proposition 1.6(1). The second, the condition (ii) follows from direct calculations. The third, the condition (iii) follow from Proposition 2.5(1). We will show the condition (iv).

(Case 1) Take $X \in \mathcal{H}^+$. Then $(X)_{E_1} \geq 1$. By Lemma 2.9(3), there exists $\alpha_0 \in (F_{4(-20)})_{E_1}$ such that $\alpha_0 X = \frac{1}{2}(E - E_3) + W$ where $W \in \mathcal{S}^+(E_3; \frac{1}{\sqrt{2}})$ and $(\alpha_0 X)_{E_1} = (X)_{E_1} \geq 1$. Then $(W)_{E_1} = (\alpha_0 X)_{E_1} - \frac{1}{2} > 0$, so that $W \in \mathcal{S}^+_+(E_3; \frac{1}{\sqrt{2}})$. By Proposition 2.8(3), there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1 W = \frac{1}{2}(E_1 - E_2)$. Therefore by Theorem 1.4, $\alpha_1\alpha_0 X = \frac{1}{2}(E_1 + E_2) + \frac{1}{2}(E_1 - E_2) = E_1$. Hence $X \in \mathcal{O}_{F_{4(-20)}}(E_1)$ and so $\mathcal{H}^+ \subset \mathcal{O}_{F_{4(-20)}}(E_1)$.

(Case 2) Take $X \in \mathcal{H}^-$. Then $(X)_{E_1} \leq 0$. By Lemma 2.9(3), there exists $\alpha_0 \in (F_{4(-20)})_{E_1}$ such that $\alpha_0 X = \frac{1}{2}(E - E_3) + W$ where $W \in \mathcal{S}^+(E_3; \frac{1}{\sqrt{2}})$ and $(\alpha_0 X)_{E_1} = (X)_{E_1} \leq 0$. Then $(W)_{E_1} = (\alpha_0 X)_{E_1} - \frac{1}{2} < 0$ so that $W \in \mathcal{S}^+_-(E_3; \frac{1}{\sqrt{2}})$. By Proposition 2.8(3), there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1 W = \frac{1}{2}(-E_1 + E_2)$. Therefore by Theorem 1.4, $\alpha_1\alpha_0 X = \frac{1}{2}(E_1 + E_2) + \frac{1}{2}(-E_1 + E_2) = E_2$. Hence $X \in \mathcal{O}_{F_{4(-20)}}(E_2)$ and so $\mathcal{H}^- \subset \mathcal{O}_{F_{4(-20)}}(E_2)$.

Therefore the condition (iv) follows. Thus (1) follows from Lemma 2.4.

(2) We use Lemma 2.4. We know that $F_{4(-20)}$ acts on $\mathcal{N}$ by Proposition 1.6(3), and so we consider that $X = \mathcal{N}$, $G = F_{4(-20)}$, $X_1 = \mathcal{N}^+$, $X_2 = \mathcal{N}^-$, $v_1 = N_1$ and $v_2 = N_2$ in Lemma 2.4. The first, the condition (i) follows from Proposition 1.6(2). The second, the condition (ii) follows from direct calculations. The third, the condition (iii) follows from Proposition 2.6(4) . We will show the condition (iv).

(Case 1) Take $X \in \mathcal{N}^+$. Then $(X)_{E_1} > 0$. By Lemma 2.9(4), there exists $\alpha_0 \in (F_{4(-20)})_{E_1}$ such that $\alpha_0 X = W \in \mathcal{N}(E_3)$ and $(\alpha_0 X)_{E_1} = (X)_{E_1} > 0$. Therefore $\alpha_0 X \in \mathcal{N}_+(E_3)$. By Proposition 2.8(6), there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1\alpha_0 X = N_1$. Hence $X \in \mathcal{O}_{F_{4(-20)}}(N_1)$ and so $\mathcal{N}^+ \subset \mathcal{O}_{F_{4(-20)}}(N_1)$.

(Case 2) Take $X \in \mathcal{H}^-$. Then $(X)_{E_1} < 0$. By Lemma 2.9(4), there exists $\alpha_0 \in (F_{4(-20)})_{E_1}$ such that $\alpha_0 X = W \in \mathcal{N}(E_3)$ and $(\alpha_0 X)_{E_1} = (X)_{E_1} < 0$. Therefore $\alpha_0 X \in \mathcal{N}_-(E_3)$. By Proposition 2.8(6), there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1\alpha_0 X = N_2$. Therefore $X \in \mathcal{O}_{F_{4(-20)}}(N_2)$ so that $\mathcal{N}^- \subset \mathcal{O}_{F_{4(-20)}}(N_2)$.

Hence the condition (iv) follows. Thus (2) follows from Lemma 2.4. $\square$

Remark (1) Let us define

$$\mathrm{Herm}'(3, \mathbb{O}) := \{ \begin{pmatrix} r_1 & -\sqrt{-1}\overline{x_1} & -\sqrt{-1}\overline{x_2} \\ \sqrt{-1}x_1 & r_2 & x_3 \\ \sqrt{-1}x_2 & \overline{x_3} & r_3 \end{pmatrix} \mid r_i \in \mathbb{R}, x_i \in \mathbb{O} \}$$

and

$$F_4' := \{\alpha \in \mathrm{GL}_{\mathbb{R}}(\mathrm{Herm}'(3,\mathbb{O})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$$

where $X \circ Y = \frac{1}{2}(XY + YX)$, $(X,Y) = \mathrm{tr}(X \circ Y)$ and $E = \mathrm{diag}(1,1,1)$ for $X, Y \in \mathrm{Herm}'(3,\mathbb{O})$. F.R. Harvey [4, page 296–297] mentions that F_4' is considered to be a simple Lie group of $F_{4(-20)}$ and $F_4'/\mathrm{Spin}(9) \simeq \mathcal{O}_{F_4'}(E_1) = \{A \in \mathrm{Herm}'(3,\mathbb{O}) \mid A \circ A = A, \mathrm{tr}(A) = 1\}$, which is $\{A \in \mathrm{Herm}'(3,\mathbb{O}) \mid A \times A = 0,\ \mathrm{tr}(A) = 1\}$ with $A \times B := \frac{1}{2}(2A \circ B - \mathrm{tr}(A)B - \mathrm{tr}(B)A + (\mathrm{tr}(A)\mathrm{tr}(B) - (A,B))E)$ for $A, B \in \mathrm{Herm}'(3,\mathbb{O})$.

Recall $\mathcal{J} := \left\{ \begin{pmatrix} r_1 & x_3 & \overline{x_2} \\ \overline{x_3} & r_2 & x_1 \\ x_2 & \overline{x_1} & r_3 \end{pmatrix} \mid \xi_i \in \mathbb{R}, x_i \in \mathbb{O} \right\}$ and note that $F_4 := \{\alpha \in \mathrm{GL}_{\mathbb{R}}(\mathcal{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$ is the compact type of $F_{4(-52)}$. Now, there exists a Jordan algebra isomorphism $\Phi : \mathcal{J} \to \mathrm{Herm}'(3,\mathbb{O})$; $\Phi(A) = \mathrm{diag}(-\sqrt{-1},1,1)\ A\ \mathrm{diag}(-\sqrt{-1},1,1)^{-1}$, so that F_4' is the compact type of $F_{4(-52)}$.

(2) By Proposition 2.10(1) and Proposition 1.6(1), we obtain

$$\{X \in \mathcal{J}^1 \mid X \times X = 0,\ \mathrm{tr}(X) = 1\} = \mathcal{O}_{F_{4(-20)}}(E_1) \coprod \mathcal{O}_{F_{4(-20)}}(E_3).$$

3 The characteristic root of multiplicity 1

For $X \in \mathcal{J}^1$, let us define $L_X^{\times} \in \mathrm{End}_{\mathbb{R}}(\mathcal{J}^1)$ as $L_X^{\times} Y := X \times Y$. By Proposition 1.2(1)(4), $L_X^{\times}$ is symmetric; $(L_X^{\times} Y, Z) = (Y, L_X^{\times} Z)$, $Y, Z \in \mathcal{J}^1$. Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity 1. By Proposition 1.7(3), the minimal subspace V_X has the elements $E_{X,\lambda_1} \in \mathcal{H} = \mathcal{H}^+ \coprod \mathcal{H}^-$ and W_{X,λ_1}. Let us define the subspaces of V_X as

$$\begin{aligned} \mathcal{E}_{X,\lambda_1} &:= \{rE_{X,\lambda_1} \mid r \in \mathbb{R}\}, \\ (V_X)_{2E_{X,\lambda_1},1} &:= \{Y \in V_X \mid L^{\times}_{2E_{X,\lambda_1}} Y = 2E_{X,\lambda_1} \times Y = Y\}, \\ (V_X)_{2E_{X,\lambda_1},-1} &:= \{Y \in V_X \mid L^{\times}_{2E_{X,\lambda_1}} Y = 2E_{X,\lambda_1} \times Y = -Y\}, \end{aligned}$$

and denote $\Delta_{X,\lambda_1} := -\frac{1}{2}(3\lambda_1^2 - 2\mathrm{tr}(X)\lambda_1 + \mathrm{tr}(X)^2 - 2(X,X))$.

Proposition 3.1. *Assume that $X \in \mathcal{J}^1$ has a characteristic root λ_1 in $\mathbb{R}$ of multiplicity* 1. *Then the following assertions hold.*

(1) $E - E_{X,\lambda_1} \in (V_X)_{2E_{X,\lambda_1},1}$, $W_{X,\lambda_1} \in (V_X)_{2E_{X,\lambda_1},-1}$.

(2) (i) *If $W_{X,\lambda_1} \neq 0$, then* $\dim V_X = 3$, $V_X = \mathcal{E}_{\lambda_1} \oplus (V_X)_{2E_{X,\lambda_1},1} \oplus (V_X)_{2E_{X,\lambda_1},-1}$ *and*

$$X = \lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X) - \lambda_1}{2}(E - E_{X,\lambda_1}) + W_{X,\lambda_1}.$$

(ii) *If* $W_{X,\lambda_1} = 0$, *then* $\dim V_X = 2$, $V_X = \mathcal{E}_{\lambda_1} \oplus (V_X)_{2E_{X,\lambda_1},1}$ *and*

$$X = \lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X) - \lambda_1}{2}(E - E_{X,\lambda_1}).$$

In particular,

$$(V_X)_{2E_{X,\lambda_1},1} = \{r(E - E_{X,\lambda_1}) \mid r \in \mathbb{R}\},\ (V_X)_{2E_{X,\lambda_1},-1} = \{rW_{X,\lambda_1} \mid r \in \mathbb{R}\}.$$

Furthermore, this decomposition is an orthogonal decomposition and

$$(E_{X,\lambda_1}, E_{X,\lambda_1}) = 1,\ (E - E_{X,\lambda_1}, E - E_{X,\lambda_1}) = 2,\ (W_{X,\lambda_1}, W_{X,\lambda_1}) = \Delta_{X,\lambda_1}.$$

(3) *If* $E_{X,\lambda_1} \in \mathcal{H}^+$, *then there exists* $\alpha \in F_{4(-20)}$ *such that* $\alpha E_{X,\lambda_1} = E_1$ *and*

$$\alpha X = \lambda_1 E_1 + \frac{\mathrm{tr}(X) - \lambda_1}{2}(E - E_1) + \alpha W_{X,\lambda_1}$$

where $\alpha W_{E_{X,\lambda_1}} \in (\mathcal{J}^1)_{2E_1,-1}$ *and* $(\alpha W_{E_{X,\lambda_1}}, \alpha W_{E_{X,\lambda_1}}) = \Delta_{X,\lambda_1} \geq 0$.

(4) *If* $E_{X,\lambda_1} \in \mathcal{H}^-$, *then there exists* $\alpha \in F_{4(-20)}$ *such that* $\alpha E_{X,\lambda_1} = E_3$ *and*

$$\alpha X = \lambda_1 E_3 + \frac{\mathrm{tr}(X) - \lambda_1}{2}(E - E_3) + \alpha W_{E_{X,\lambda_1}}$$

where $\alpha W_{E_{X,\lambda_1}} \in (\mathcal{J}^1)_{2E_3,-1}$ *and* $(\alpha W_{E_{X,\lambda_1}}, \alpha W_{E_{X,\lambda_1}}) = \Delta_{X,\lambda_1} \in \mathbb{R}$.

Proof. (1) By Proposition 1.7(3), $E_{X,\lambda_1} \in \mathcal{H}$, so that $E_{X,\lambda_1} \times E_{X,\lambda_1} = 0$ and $\mathrm{tr}(E_{X,\lambda_1}) = 1$. By Proposition 1.2(2), $2E_{X,\lambda_1} \times (E - E_{X,\lambda_1}) = E - E_{X,\lambda_1}$. Therefore $E - E_{X,\lambda_1} \in (V_X)_{2E_{X,\lambda_1},1}$. Put $Z = \varphi_X(\lambda_1)$. Using Proposition 1.2(6) and $\det(Z) = \Phi_X(\lambda_1) = 0$, $2E_{X,\lambda_1} \times Z = \frac{2}{\mathrm{tr}(Z \times Z)}(Z \times Z) \times Z = -Z + \mathrm{tr}(Z)(E - E_{X,\lambda_1})$. Therefore $2E_{X,\lambda_1} \times W_{X,\lambda_1} = 2E_{X,\lambda_1} \times (-Z + \frac{\mathrm{tr}(Z)}{2}(E - E_{X,\lambda_1})) = Z - \frac{\mathrm{tr}(Z)}{2}(E - E_{X,\lambda_1}) = -W_{X,\lambda_1}$. Hence $W_{X,\lambda_1} \in (V_X)_{2E_{X,\lambda_1},-1}$.

(2) By Proposition 1.2(3) and $\mathrm{tr}(E_{X,\lambda_1}) = 1$, $0 = \mathrm{tr}(E_{X,\lambda_1} \times E_{X,\lambda_1}) = \frac{1}{2}(\mathrm{tr}(E_{X,\lambda_1})^2 - (E_{X,\lambda_1}, E_{X,\lambda_1}))$. Therefore $(E_{X,\lambda_1}, E_{X,\lambda_1}) = 1$. Using $1 = (E_{X,\lambda_1}, E_{X,\lambda_1}) = \mathrm{tr}(E_{X,\lambda_1}) = (E, E_{X,\lambda_1})$, we get $(E - E_{X,\lambda_1}, E - E_{X,\lambda_1}) = 2$, so that $E - E_{X,\lambda_1} \neq 0$. Note that eigenvectors with different eigenvalues are linearly independent. Therefore if $W_{X,\lambda_1} \neq 0$, then E_{X,λ_1}, $E - E_{X,\lambda_1}$ and W_{X,λ_1} is a basis of V_X because of eigenvectors of $L^{\times}_{2E_{X,\lambda_1}}$ and $\dim V_X \leq 3$. Since $W_{X,\lambda_1} = X - (\lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_{X,\lambda_1}))$, $X = \lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_{X,\lambda_1}) + W_{X,\lambda_1}$. Also, if $W_{X,\lambda_1} = 0$, then $X = \lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_{X,\lambda_1})$. In this case, by direct calculation, $X \times X = \frac{\lambda_1(\mathrm{tr}(X)-\lambda_1)}{2}(E - E_{X,\lambda_1}) + (\frac{\mathrm{tr}(X)-\lambda_1}{2})^2 E_{X,\lambda_1}$ and $E = (E - E_{X,\lambda_1}) + E_{X,\lambda_1}$.

Therefore, since V_X is generated by E, X and $X \times X$, the minimal space V_X is generated by E_{X,λ_1} and $E - E_{X,\lambda_1}$. Hence we obtain the following table:

	$W_{X,\lambda_1} \neq 0$	$W_{X,\lambda_1} = 0$
$\mathbb{R}$-basis	E_{X,λ_1}, $E - E_{X,\lambda_1}$, W_{X,λ_1}	E_{X,λ_1}, $E - E_{X,\lambda_1}$
V_X	$\mathcal{E}_X \oplus (V_X)_{2E_{X,\lambda_1},1} \oplus (V_X)_{2E_{X,\lambda_1},-1}$	$\mathcal{E}_X \oplus (V_X)_{2E_{X,\lambda_1},1}$
dimV_X	3	2

And $(V_X)_{2E_{X,\lambda_1},1} = \{r(E - E_{X,\lambda_1}) \mid r \in \mathbb{R}\}$, $(V_X)_{2E_{X,\lambda_1},-1} = \{rW_{X,\lambda_1} \mid r \in \mathbb{R}\}$. Thus (i) and (ii) follows.

Since $L^{\times}_{2E_{X,\lambda_1}}$ is a symmetric transformation of V_X, the eigenvectors $E_{X,\lambda_1}, E - E_{X,\lambda_1}, W_{X,\lambda_1}$ are orthogonal. And using $W_{X,\lambda_1} = X - (\lambda_1 E_{X,\lambda_1} + \frac{\mathrm{tr}(X)-\lambda_1}{2}(E - E_{X,\lambda_1}))$ and direct calculation, $(W_{X,\lambda_1}, W_{X,\lambda_1}) = -\frac{3}{2}\lambda_1^2 + \mathrm{tr}(X)\lambda_1 - \frac{1}{2}\mathrm{tr}(X)^2 + (X, X) = \Delta_{X,\lambda_1}$. Thus (2) follows.

(3) By Proposition 2.10(1), there exists $\alpha \in F_{4(-20)}$ such that $\alpha E_{X,\lambda_1} = E_1$, so that $\alpha(E - E_{X,\lambda_1}) = E - E_1$. Since $2E_1 \times \alpha W_{X,\lambda_1} = \alpha(2E_{X,\lambda_1} \times W_{X,\lambda_1}) = -\alpha W_{X,\lambda_1}$ by Theorem 1.4, $\alpha W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_1,-1}$. Therefore $\alpha X = \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_1) + \alpha W_{X,\lambda_1}$ where $\alpha W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_1,-1}$. By (2) and Theorem 1.4, $(\alpha W_{X,\lambda_1}, \alpha W_{X,\lambda_1}) = (W_{X,\lambda_1}, W_{X,\lambda_1}) = \Delta_{X,\lambda_1}$. The inner product is positive definite over $(\mathcal{J}^1)_{2E_1,-1}$, so that $\Delta_{X,\lambda_1} = (\alpha W_{X,\lambda_1}, \alpha W_{X,\lambda_1}) \geq 0$.

(4) By Proposition 2.10(1), there exists $\alpha \in F_{4(-20)}$ such that $\alpha E_{X,\lambda_1} = E_3$, so that $\alpha(E - E_{X,\lambda_1}) = E - E_3$. Since $2E_3 \times \alpha W_{X,\lambda_1} = \alpha(2E_{X,\lambda_1} \times W_{X,\lambda_1}) = -\alpha W_{X,\lambda_1}$ by Theorem 1.4, $\alpha W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_3,-1}$. Therefore $\alpha X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + \alpha W_{X,\lambda_1}$ where $\alpha W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_3,-1}$. By (2) and Theorem 1.4, $(\alpha W_{X,\lambda_1}, \alpha W_{X,\lambda_1}) = (W_{X,\lambda_1}, W_{X,\lambda_1}) = \Delta_{X,\lambda_1} \in \mathbb{R}$. □

Proposition 3.2. *Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity* 1. *Then the following assertions hold.*

(1) *If $E_{X,\lambda_1} \in \mathcal{H}^+$ and $\Delta_{X,\lambda_1} > 0$, then*

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_1, \frac{1}{2}(\mathrm{tr}(X) - \lambda_1) + \sqrt{\frac{\Delta_{X,\lambda_1}}{2}}, \frac{1}{2}(\mathrm{tr}(X) - \lambda_1) - \sqrt{\frac{\Delta_{X,\lambda_1}}{2}})).$$

(2) *If $E_{X,\lambda_1} \in \mathcal{H}^+$ and $\Delta_{X,\lambda_1} = 0$, then*

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_1, \frac{1}{2}(\mathrm{tr}(X) - \lambda_1), \frac{1}{2}(\mathrm{tr}(X) - \lambda_1))).$$

(3) *If $E_{X,\lambda_1} \in \mathcal{H}^-$ and $\Delta_{X,\lambda_1} > 0$, then*

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\frac{1}{2}(\mathrm{tr}(X) - \lambda_1) + \sqrt{\frac{\Delta_{X,\lambda_1}}{2}}, \frac{1}{2}(\mathrm{tr}(X) - \lambda_1) - \sqrt{\frac{\Delta_{X,\lambda_1}}{2}}, \lambda_1))$$

or

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\frac{1}{2}(\mathrm{tr}(X)-\lambda_1)-\sqrt{\frac{\Delta_{X,\lambda_1}}{2}}, \frac{1}{2}(\mathrm{tr}(X)-\lambda_1)+\sqrt{\frac{\Delta_{X,\lambda_1}}{2}}, \lambda_1)).$$

(4) *If* $\Delta_{X,\lambda_1} < 0$, *then*

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \lambda_1) + F_3^1(\sqrt{\frac{|\Delta_{X,\lambda_1}|}{2}})).$$

(5) *If* $E_{X,\lambda_1} \in \mathcal{H}^-$ *and* $W_{X,\lambda_1} = 0$, *then*

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \lambda_1)).$$

(6) *If* $\Delta_{X,\lambda_1} = 0$ *and* $W_{X,\lambda_1} \neq 0$, *then* $E_{X,\lambda_1} \in \mathcal{H}^-$ *and* $W_{X,\lambda_1} \in \mathcal{N}$.

(7) *If* $\Delta_{X,\lambda_1} = 0$ *and* $W_{X,\lambda_1} \in \mathcal{N}^+$, *then*

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \lambda_1) + N_1).$$

(8) *If* $\Delta_{X,\lambda_1} = 0$ *and* $W_{X,\lambda_1} \in \mathcal{N}^-$, *then*

$$X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \frac{1}{2}(\mathrm{tr}(X)-\lambda_1), \lambda_1) + N_2).$$

Proof. (1) By Proposition 3.1(3), there exists $\alpha_0 \in F_{4(-20)}$ such that $\alpha_0 X = \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X)-\lambda_1)(E-E_1) + \alpha_0 W_{X,\lambda_1}$ where $\alpha_0 W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_1,-1}$ and $(\alpha_0 W_{X,\lambda_1}, \alpha_0 W_{X,\lambda_1}) = \Delta_{X,\lambda_1} > 0$. Therefore $\alpha_0 W_{X,\lambda_1} \in \mathcal{S}^+(E_1; \sqrt{\Delta_{X,\lambda_1}})$. By Proposition 2.8(1), there exists $\alpha_1 \in (F_{4(-20)})_{E_1}$ such that $\alpha_1\alpha_0 X = \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X)-\lambda_1)(E-E_1) + \sqrt{\frac{\Delta_{X,\lambda_1}}{2}}(E_2 - E_3)$. Hence (1) follows.

(2) By Proposition 3.1(3), there exists $\alpha_0 \in F_{4(-20)}$ such that $\alpha_0 X = \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X)-\lambda_1)(E-E_1) + \alpha_0 W_{X,\lambda_1}$ where $\alpha_0 W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_1,-1}$ and $(\alpha_0 W_{X,\lambda_1}, \alpha_0 W_{X,\lambda_1}) = \Delta_{X,\lambda_1} = 0$. Since the inner product is positive definite over $(\mathcal{J}^1)_{2E_1,-1}$, $\alpha_0 W_{X,\lambda_1} = 0$. Hence $\alpha_0 X = \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E-E_1)$. Thus (2) follows.

(3) By Proposition 3.1(4), there exists $\alpha_0 \in F_{4(-20)}$ such that $\alpha_0 X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X)-\lambda_1)(E-E_3) + \alpha_0 W_{X,\lambda_1}$ where $\alpha_0 W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_3,-1}$ and $(\alpha_0 W_{X,\lambda_1}, \alpha_0 W_{X,\lambda_1}) = \Delta_{X,\lambda_1} > 0$. Therefore $\alpha_0 W_{X,\lambda_1} \in \mathcal{S}^+(E_3; \sqrt{\Delta_{X,\lambda_1}})$. By Proposition 2.8(2), we get (i) $\alpha_0 W_{X,\lambda_1} \in \mathcal{S}^+_+(E_3; \sqrt{\Delta_{X,\lambda_1}})$ or (ii) $W \in \mathcal{S}^+_-(E_3; \sqrt{\Delta_{X,\lambda_1}})$.

In Case (i), by Proposition 2.8(3), there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1\alpha_0 X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X)-\lambda_1)(E-E_3) + \sqrt{\frac{\Delta_{X,\lambda_1}}{2}}(E_1 - E_2)$.

In Case (ii), by Proposition 2.8(3), there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1 \alpha X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + \sqrt{\frac{\Delta_{X,\lambda_1}}{2}}(-E_1 + E_2)$.

Hence (3) follows.

(4) By Proposition 1.7(3), $E_{X,\lambda_1} \in \mathcal{H}^+ \coprod \mathcal{H}^-$. Suppose $E_{X,\lambda_1} \in \mathcal{H}^+$. By Proposition 3.1(3), there exists $\alpha \in F_{4(-20)}$ such that $\alpha X = \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_1) + \alpha W_{X,\lambda_1}$ where $\alpha W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_1,-1}$ and $\Delta_{X,\lambda_1} = (\alpha W_{X,\lambda_1}, \alpha W_{X,\lambda_1}) \geq 0$. It contradicts with $\Delta_{X,\lambda_1} < 0$. Therefore $E_{X,\lambda_1} \in \mathcal{H}^-$. By Proposition 3.1(4), there exists $\alpha_0 \in F_{4(-20)}$ such that $\alpha_0 X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + \alpha_0 W_{X,\lambda_1}$ where $\alpha_0 W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_3,-1}$ and $(\alpha_0 W_{X,\lambda_1}, \alpha_0 W_{X,\lambda_1}) = \Delta_{X,\lambda_1} < 0$. Therefore $\alpha_0 W_{X,\lambda_1} \in \mathcal{S}^-(E_3; \sqrt{|\Delta_{X,\lambda_1}|})$. By Proposition 2.8(4), there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1 \alpha X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + F_3^1(\sqrt{\frac{|\Delta_{X,\lambda_1}|}{2}})$. Hence (4) follows.

(5) By Proposition 3.1(4), there exists $\alpha \in F_{4(-20)}$ such that $\alpha X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3)$. Hence (5) follows.

(6) By Proposition 1.7(3), $E_{X,\lambda_1} \in \mathcal{H}^+ \coprod \mathcal{H}^-$. Suppose $E_{X,\lambda_1} \in \mathcal{H}^+$. Then there exists $\alpha \in F_{4(-20)}$ such that $\alpha X = \lambda_1 E_1 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_1) + \alpha W_{X,\lambda_1}$ where $\alpha W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_1,-1}$. Since $(\alpha W_{X,\lambda_1}, \alpha W_{X,\lambda_1}) = \Delta_{X,\lambda_1} = 0$ and the inner product is positive definite over $(\mathcal{J}^1)_{2E_1,-1}$, $\alpha W_{X,\lambda_1} = 0$ and so $W_{X,\lambda_1} = 0$. It contradicts with $W_{X,\lambda_1} \neq 0$. Therefore $E_{X,\lambda_1} \in \mathcal{H}^-$. By Proposition 3.1(4)(ii), there exists $\alpha_0 \in F_{4(-20)}$ such that $\alpha_0 X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + \alpha_0 W_{X,\lambda_1}$ where $\alpha_0 W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_3,-1}$, $(\alpha_0 W_{X,\lambda_1}, \alpha_0 W_{X,\lambda_1}) = \Delta_{X,\lambda_1} = 0$ and $\alpha_0 W_{X,\lambda_1} \neq 0$. Therefore $\alpha_0 W_{X,\lambda_1} \in \mathcal{N}(E_3)$. By Proposition 2.8(5)(6) and Proposition 2.10(2), $\mathcal{N}(E_3) = \coprod_{i=1}^2 \mathcal{O}_{(F_{4(-20)})_{E_3}}(N_i) \subset \coprod_{i=1}^2 \mathcal{O}_{F_{4(-20)}}(N_i) = \mathcal{N}$. Hence by Proposition 1.6(3), $W_{X,\lambda_1} \in \alpha_0^{-1} \mathcal{N}(E_3) \subset \alpha_0^{-1} \mathcal{N} = \mathcal{N}$. Thus (6) follows.

(7)(8) By (6) and Proposition 3.1(4), there exists $\alpha_0 \in F_{4(-20)}$ such that $\alpha_0 X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + \alpha W_{X,\lambda_1}$ where $\alpha_0 W_{X,\lambda_1} \in (\mathcal{J}^1)_{2E_3,-1}$ and $(\alpha_0 W_{X,\lambda_1}, \alpha_0 W_{X,\lambda_1}) = \Delta_{X,\lambda_1} = 0$. Therefore $\alpha_0 W_{X,\lambda_1} \in \mathcal{N}(E_3)$. Using Proposition 2.10(2), if $W_{X,\lambda_1} \in \mathcal{N}^+$, then $\alpha_0 W_{X,\lambda_1} \in \mathcal{N}^+$, so that $(\alpha_0 W_{X,\lambda_1})_{E_1} > 0$ and $\alpha_0 W_{X,\lambda_1} \in \mathcal{N}_+(E_3)$. Similarly if $W_{X,\lambda_1} \in \mathcal{N}^-$, then $\alpha_0 W_{X,\lambda_1} \in \mathcal{N}^-$, so that $(\alpha_0 W_{X,\lambda_1})_{E_1} < 0$ and $\alpha_0 W_{X,\lambda_1} \in \mathcal{N}_-(E_3)$. By Proposition 2.8(6), if $W_{X,\lambda_1} \in \mathcal{N}^+$, then there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1 \alpha_0 X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + N_1$. And if $W_{X,\lambda_1} \in \mathcal{N}^-$, then there exists $\alpha_1 \in (F_{4(-20)})_{E_3}$ such that $\alpha_1 \alpha_0 X = \lambda_1 E_3 + \frac{1}{2}(\mathrm{tr}(X) - \lambda_1)(E - E_3) + N_2$. Hence (7) and (8) follow. □

Lemma 3.3. *Assume that $X \in \mathcal{J}^1$ has a characteristic root $\lambda_1 \in \mathbb{R}$ of multiplicity* 1. *If $X \in \mathcal{J}^1$ has the characteristic polynomial such that $\Phi(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3), \quad \lambda_2, \lambda_3 \in \mathbf{C}$, then $\Delta_{X,\lambda_1} = \frac{1}{2}(\lambda_2-\lambda_3)^2$.*

Proof. By Proposition 1.2(3) and Proposition 1.5(2), $\Delta_{X,\lambda_1} = -\frac{1}{2}(3\lambda_1^2 - 2\mathrm{tr}(X)\lambda_1 + \mathrm{tr}(X)^2 - 2(X,X)) = -\frac{1}{2}(3\lambda_1^2 - 2\mathrm{tr}(X)\lambda_1 - \mathrm{tr}(X)^2 + 4\mathrm{tr}(X\times X)) = -\frac{1}{2}(3\lambda_1^2 - 2(\lambda_1+\lambda_2+\lambda_3)\lambda_1 - (\lambda_1+\lambda_2+\lambda_3)^2 + 4(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)) = \frac{1}{2}(\lambda_2-\lambda_3)^2$. □

Lemma 3.4. *For all $\alpha \in F_{4(-20)}$, $\alpha(\mathcal{H}^+ \cap V_X) = \mathcal{H}^+ \cap V_{\alpha X}$ and $\alpha(\mathcal{H}^- \cap V_X) = \mathcal{H}^- \cap V_{\alpha X}$.*

Proof. By Theorem 1.4, $\alpha V_X = \alpha(\{aX\times X + bX + cE \mid a,b,c \in \mathbb{R}\}) = \{a(\alpha X\times\alpha X) + b(\alpha X) + cE \mid a,b,c\in\mathbb{R}\} = V_{\alpha X}$. Since α is bijection and Proposition 2.10(1), $\alpha(\mathcal{H}^+\cap V_X) = (\alpha\mathcal{H}^+)\cap(\alpha V_X) = \mathcal{H}^+\cap V_{\alpha X}$ and $\alpha(\mathcal{H}^-\cap V_X) = (\alpha\mathcal{H}^-)\cap(\alpha V_X) = \mathcal{H}^-\cap V_{\alpha X}$. □

Proposition 3.5. *Assume that real numbers r_1, r_2, r_3 are different from each other and $Y = \mathrm{diag}(r_1,r_2,r_3) \in \mathcal{J}^1$. Then :*

(1) *All of characteristic roots of Y are $r_1, \ r_2, \ r_3$.*

(2) *$E_i = E_{Y,r_i} \in V_Y$ for all $i \in \{1,2,3\}$.*

(3) *$\mathcal{H}^+\cap V_Y = \{E_{Y,r_1}\}$ and $\mathcal{H}^-\cap V_Y = \{E_{Y,r_2}, \ E_{Y,r_3}\}$.*

(4) *$\mathcal{H}^+\cap V_{\alpha Y} = \{E_{\alpha Y,r_1}\}$ and $\mathcal{H}^-\cap V_{\alpha Y} = \{E_{\alpha Y,r_2}, \ E_{\alpha Y,r_3}\}$ for all $\alpha\in F_{4(-20)}$.*

(5) *$(F_{4(-20)})_Y = D_4$.*

Proof. (1) It follows from $\Phi_Y(\lambda) = (\lambda-r_1)(\lambda-r_2)(\lambda-r_3)$ by (1.2).

(2) By (1,1), $\varphi_Y(r_i)\times\varphi_Y(r_i) = (r_i-r_{i+1})(r_i-r_{i+2})E_i$, and $\mathrm{tr}(\varphi_Y(r_i)\times\varphi_Y(r_i)) = (r_i-r_{i+1})(r_i-r_{i+2}) \neq 0$. Hence $E_i = \frac{1}{\mathrm{tr}(\varphi_Y(r_i)\times\varphi_Y(r_i))}\varphi_Y(r_i)\times\varphi_Y(r_i) = E_{Y,r_i}\in V_Y$.

(3) By (2), $E_1,E_2,E_3\in V_Y$, so that $\{aE_1+bE_2+cE_3 \mid a,b,c\in\mathbb{R}\}\subset V_Y$. Since $Y\times Y, \ Y, \ E$ are diagonal matrices and $Y\times Y, \ Y, \ E$ are the generators of V_Y, $V_Y\subset\{aE_1+bE_2+cE_3 \mid a,b,c\in\mathbb{R}\}$. Therefore $V_Y = \{aE_1+bE_2+cE_3 \mid a,b,c\in\mathbb{R}\}$. Suppose that $aE_1+bE_2+cE_3\in V_Y\cap\mathcal{H}$. That is $(aE_1+bE_2+cE_3)\times(aE_1+bE_2+cE_3) = bcE_1+caE_2+abE_3 = 0$, and $1 = \mathrm{tr}(aE_1+bE_2+cE_3) = a+b+c$. Hence $bc=ca=ab=0$ and $a+b+c=1$. Therefore $(a,b,c) = (1,0,0), (0,1,0), (0,0,1)$, so that $\mathcal{H}\cap V_Y = \{E_1,E_2,E_3\}$. Then $\mathcal{H}^+\cap V_Y = \{E_1\} = \{E_{Y,r_1}\}$ follows from $(E_i)_{E_1}\geq 1$ iff $i=1$. And $\mathcal{H}^-\cap V_Y = \{E_2,E_3\} = \{E_{Y,r_2},E_{Y,r_3}\}$ follows from $(E_i)_{E_1}\leq 0$ iff $i=2,3$.

(4) By (3), Lemma 3.4 and Proposition 1.8, $\mathcal{H}^+ \cap V_{\alpha Y} = \alpha(\mathcal{H}^+ \cap V_Y) = \{E_{\alpha Y,\lambda_i}\}$ and $\mathcal{H}^- \cap V_{\alpha Y} = \alpha(\mathcal{H}^- \cap V_Y) = \{E_{\alpha Y,\lambda_{i+1}},\ E_{\alpha Y,\lambda_{i+2}}\}$. Hence (4) follows.

(5) If $\alpha \in D_4(\subset F_{4(-20)})$, then $\alpha Y = \alpha(\sum_{i=1}^3 r_i E_i) = \sum_{i=1}^3 r_i E_i = Y$. Therefore $\alpha \in (F_{4(-20)})_Y$, so that $D_4 \subset (F_{4(-20)})_Y$. Conversely, suppose that $\alpha \in F_{4(-20)}$ and $\alpha Y = Y$. By (2) and Proposition 1.8, $\alpha E_i = \alpha E_{Y,r_i} = E_{\alpha Y,r_i} = E_{Y,r_i} = E_i$. Therefore $\alpha \in D_4$, so that $(F_{4(-20)})_Y \subset D_4$. Hence $(F_{4(-20)})_Y = D_4$. □

Proposition 3.6. *Assume that real numbers $\lambda_1,\ \lambda_2,\ \lambda_3$ are different from each other and $Y_i = \mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) \in \mathcal{J}^1$ where $i,\ i+1,\ i+2$ are counted modulo 3. Then orbits $\mathcal{O}_{F_{4(-20)}}(Y_i)$ $(i \in \{1,2,3\})$ are different orbits from each other.*

Proof. We will show that $\mathcal{O}_{F_{4(-20)}}(Y_i) \neq \mathcal{O}_{F_{4(-20)}}(Y_{i+1})$. By Proposition 3.5(3), $\mathcal{H}^+ \cap V_{Y_i} = \{E_{Y_i,\lambda_i}\}$ and $\mathcal{H}^- \cap V_{Y_i} = \{E_{Y_i,\lambda_{i+1}},\ E_{Y_i,\lambda_{i+2}}\}$. Similarly, $\mathcal{H}^+ \cap V_{Y_{i+1}} = \{E_{Y_{i+1},\lambda_{i+1}}\}$ and $\mathcal{H}^- \cap V_{Y_{i+1}} = \{E_{Y_{i+1},\lambda_{i+2}},\ E_{Y_{i+1},\lambda_i}\}$. Suppose there exists $\alpha \in F_{4(-20)}$ such that $\alpha Y_i = Y_{i+1}$. Since $\mathcal{H}^+ \cap V_{Y_i} = \{E_{Y_i,\lambda_i}\}$ and Proposition 3.5(4), $\mathcal{H}^+ \cap V_{\alpha Y_i} = \{E_{\alpha Y_i,\lambda_i}\} = \{E_{Y_{i+1},\lambda_i}\}$. Hence $E_{Y_{i+1},\lambda_i} \in \mathcal{H}^+ \cap \mathcal{H}^-$. It contradicts with $\mathcal{H}^+ \cap \mathcal{H}^- = \emptyset$. Hence $\mathcal{O}_{F_{4(-20)}}(Y_i) \neq \mathcal{O}_{F_{4(-20)}}(Y_{i+1})$. Thus the assertion follows from moving $i \in \{1,2,3\}$. □

Proposition 3.7. *Assume that $X \in \mathcal{J}^1$ has the characteristic polynomial $\Phi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ where $\lambda_i \in \mathbb{R}$ with $\lambda_1 > \lambda_2 > \lambda_3$, and $i,\ i+1,\ i+2$ are counted modulo 3. Then following assertions hold.*

(1) *There exists the unique $i \in \{1,2,3\}$ such that X is transformed to $\mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$ by $F_{4(-20)}$.*

(2) *The following assertions are equivalent:*

(i) $X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2}))$.

(ii) $\mathcal{H}^+ \cap V_X = \{E_{X,\lambda_i}\}$ *and* $\mathcal{H}^- \cap V_X = \{E_{X,\lambda_{i+1}},\ E_{X,\lambda_{i+2}}\}$.

(iii) $E_{X,\lambda_i} \in \mathcal{H}^+$.

(3) *There exists the unique $i \in \{1,2,3\}$ such that $\mathcal{H}^+ \cap V_X = \{E_{X,\lambda_i}\}$ and $\mathcal{H}^- \cap V_X = \{E_{X,\lambda_{i+1}},\ E_{X,\lambda_{i+2}}\}$.*

Proof. (1) By $(0 \neq)\lambda_2 - \lambda_3 \in \mathbb{R}$, Lemma 3.3 and Proposition 1.7(3), $E_{X,\lambda_1} \in \mathcal{H}^+ \coprod \mathcal{H}^-$ and $\Delta_{X,\lambda_1} = \frac{1}{2}(\lambda_2 - \lambda_3)^2 > 0$. By Proposition 3.2(1)(3), X is diagonable over the action of $F_{4(-20)}$. And by Proposition 1.5(3), characteristic roots are invariant by the action of $F_{4(-20)}$. By Proposition 3.5(1),

X can be transformed to one of $\mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$ or $\mathrm{diag}(\lambda_i, \lambda_{i+2}, \lambda_{i+1})$ where $i \in \{1,2,3\}$. Now by (2.1), $\beta_1(\frac{\pi}{2})E_2 = E_3$, $\beta_1(\frac{\pi}{2})E_3 = E_2$ and $\beta_1(\frac{\pi}{2})E_1 = E_1$, so that if X is transformed to $\mathrm{diag}(\lambda_i, \lambda_{i+2}, \lambda_{i+1})$, then $\beta_1(\frac{\pi}{2})\mathrm{diag}(\lambda_i, \lambda_{i+2}, \lambda_{i+1}) = \mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$. Therefore X can be transformed to $\mathrm{diag}(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$ where $i \in \{1,2,3\}$. Now by Proposition 3.6, such i is unique. Hence (1) follows.

(2) (i)$\Rightarrow$(ii) follows from Proposition 3.5(4). (ii)$\Rightarrow$(iii) is obvious. We will show that (iii)$\Rightarrow$(i). By (1), there exists the unique $j \in \{1,2,3\}$ such that $X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_j, \lambda_{j+1}, \lambda_{j+2}))$ where j, $j+1$, $j+2$ are counted modulo 3. Using (i)$\Rightarrow$(ii), $\mathcal{H}^+ \cap V_X = \{E_{X,\lambda_j}\}$ and $\mathcal{H}^- \cap V_X = \{E_{X,\lambda_{j+1}},\ E_{X,\lambda_{j+2}}\}$. Therefore $i = j$. Hence (iii)$\Rightarrow$(i) follows.

(3) It follows from (1) and (2). □

Proposition 3.8. *Assume that $X \in \mathcal{J}^1$ has the characteristic polynomial $\Phi(\lambda) = (\lambda - \lambda_1)(\lambda - (p + \sqrt{-1}q))(\lambda - (p - \sqrt{-1}q))$ where $\lambda_1, p, q \in \mathbb{R}$ and $q > 0$. Then $X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(p,p,\lambda_1) + F_3^1(q))$.*

Proof. λ_1 is a root of $\Phi_X(\lambda)$ of multiplicity 1. Put $\lambda_2 = p + q\sqrt{-1}$, $\lambda_3 = p - q\sqrt{-1}$. By Lemma 3.3, $\Delta_{X,\lambda_1} = \frac{1}{2}(\lambda_2 - \lambda_3)^2 = -2q^2 < 0$. Hence the assertion follows from Proposition 3.2(4). □

Proposition 3.9. *Assume that $X \in \mathcal{J}^1$ have the characteristic polynomial $\Phi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2$ where $\lambda_i \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$. Then $W_{X,\lambda_1} \in \tilde{\mathcal{N}} = \mathcal{N}^+ \coprod \mathcal{N}^- \coprod \{0\}$, $E_{X,\lambda_1} \in \mathcal{H}^+ \coprod \mathcal{H}^-$ and following assertions hold.*

(1) *If $W_{X,\lambda_1} \in \mathcal{N}^+$, then $X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + N_1)$.*

(2) *If $W_{X,\lambda_1} \in \mathcal{N}^-$, then $X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1) + N_2)$.*

(3) *If $W_{X,\lambda_1} = 0$ and $E_{X,\lambda_1} \in \mathcal{H}^+$, then $X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2))$.*

(4) *If $W_{X,\lambda_1} = 0$ and $E_{X,\lambda_1} \in \mathcal{H}^-$, then $X \in \mathcal{O}_{F_{4(-20)}}(\mathrm{diag}(\lambda_2, \lambda_2, \lambda_1))$.*

(5) *By $F_{4(-20)}$, the above canonical forms (1)-(4) cannot be transformed from each other.*

Proof. Since λ_1 is a root of $\Phi_X(\lambda)$ of multiplicity 1, by Proposition 1.7(3), $E_{X,\lambda_1} \in \mathcal{H}^+ \coprod \mathcal{H}^-$. By Lemma 3.3, we have $\Delta_{X,\lambda_1} = \frac{1}{2}(\lambda_2 - \lambda_2)^2 = 0$. Therefore, if $W_{X,\lambda_1} \neq 0$, then by Proposition 3.2(6), $W_{X,\lambda_1} \in \mathcal{N}$. Hence by Proposition 1.6(2), $W_{X,\lambda_1} \in \tilde{\mathcal{N}} = \mathcal{N}^+ \coprod \mathcal{N}^- \coprod \{0\}$. Now (1) and (2) follow from Proposition 3.2(7)(8). And (3) and (4) follow from Proposition 3.2(2)(5).

(5) By Proposition 1.8 and Proposition 2.10(1), $E_{\alpha X,\lambda_1} = \alpha E_{X,\lambda_1} \in \alpha\mathcal{H}^+ = \mathcal{H}^+$. Therefore, if $E_{X,\lambda_1} \in \mathcal{H}^+$, then $E_{\alpha X,\lambda_1} \in \mathcal{H}^+$ for all $\alpha \in F_{4(-20)}$. Similarly by Proposition 1.8 and Proposition 2.10(1), if $E_{X,\lambda_1} \in \mathcal{H}^-$ is $E_{\alpha X,\lambda_1} \in \mathcal{H}^-$ for all $\alpha \in F_{4(-20)}$. Next by Proposition 1.8 and Proposition 2.10(2), $W_{\alpha X,\lambda_1} = \alpha W_{X,\lambda_1} \in \alpha\mathcal{N}^+ = \mathcal{N}^+$ for all $\alpha \in F_{4(-20)}$. Therefore, if $W_{X,\lambda_1} \in \mathcal{N}^+$ then $W_{\alpha X,\lambda_1} \in \mathcal{N}^+$. Similarly by Proposition 1.8 and Proposition 2.10(2), if $W_{X,\lambda_1} \in \mathcal{N}^-$, then $W_{\alpha X,\lambda_1} \in \mathcal{N}^-$ for all $\alpha \in F_{4(-20)}$. And by Proposition 1.8, if $W_{X,\lambda_1} = 0$, then $W_{\alpha X,\lambda_1} = 0$ for all $\alpha \in F_{4(-20)}$. Hence canonical forms (1)-(4) cannot be transformed from each other. $\square$

4 The characteristic root of multiplicity 3

By Proposition 2.1(2), $\tilde{A}_1(-1) + \tilde{A}_2(\sqrt{-1}) \in \mathfrak{f}_{4(-20)}$, so that let us define

$$\beta_{1,2}(t) := \exp(t(\tilde{A}_1(-1) + \tilde{A}_2(\sqrt{-1}))) \in F_{4(-20)}, \quad t \in \mathbb{R}.$$

Put $M := A_1(-1) + A_2(\sqrt{-1})$ and $e^{tM} := \sum_{n=0}^{\infty} \frac{1}{n!}(tM)^n$. Then by direct calculation, $e^{tM} = \begin{pmatrix} 1+\frac{t^2}{2} & -\sqrt{-1}\frac{t^2}{2} & -\sqrt{-1}t \\ -\sqrt{-1}\frac{t^2}{2} & 1-\frac{t^2}{2} & -t \\ \sqrt{-1}t & t & 1 \end{pmatrix}$. Set $X \in \mathcal{J}^1$. By direct calculation, $\frac{d}{dt}(e^{tM}X)e^{-tM} = (\tilde{A}_1(-1) + \tilde{A}_2(\sqrt{-1}))((e^{tM}X)e^{-tM})$ and $(e^{0\cdot M}X)e^{-0\cdot M} = X$. Now, $\frac{d}{dt}\beta_{1,2}(t)X = (\tilde{A}_1(-1) + \tilde{A}_2(\sqrt{-1}))\beta_{1,2}(t)X$ and $\beta_{1,2}(0)X = X$. Since the uniqueness of ordinary differential equations, we have $\beta_{1,2}(t)X = (e^{tM}X)e^{-tM}$. In particular, take $X_0 = -rN_2 + F_1^1(x) + F_2^1(\overline{x})$ where $r \in \mathbb{R}$ and $x \in \mathbb{O}$. Then by direct calculation,

$$(4.1) \quad \beta_{1,2}(t)X_0 = (-r - 2t\mathrm{Re}(x))N_2 + F_1^1(x) + F_2^1(\overline{x}).$$

If $(r, x) = (-1, 0)$ in (4.1), then $\beta_{1,2}(t)N_2 = N_2$, and so $\beta_{1,2}(t) \in (F_{4(-20)})_{N_2}$. In this section, we use the exceptional Lie group G_2 (cf.[4, 11]):

$$G_2 := \mathrm{Aut}(\mathbb{O}) = \{\alpha \in \mathrm{GL}_{\mathbb{R}}(8) \mid \alpha(xy) = (\alpha x)(\alpha y), \quad x, y \in \mathbb{O})\}.$$

Note that $\alpha 1 = 1$, $\alpha\overline{x} = \overline{\alpha x}$ and $(\alpha x, \alpha y) = (x, y)$ for all $\alpha \in G_2$. Using $\alpha\overline{x} = \overline{\alpha x}$, $(\alpha x)(\alpha y) = \alpha(xy) = \overline{\alpha(\overline{xy})}$. It implies $G_2 \subset \tilde{D}_4 \cong D_4 \subset F_{4(-20)}$. Therefore we consider that G_2 is the subgroup of $F_{4(-20)}$ as

$$\alpha(\sum_{i=1}^{3}(\xi_i E_i + F_i^1(x_i))) = \sum_{i=1}^{3}(\xi_i E_i + F_i^1(\alpha x_i)), \quad \alpha \in G_2.$$

Since $\alpha 1 = 1$ for all $\alpha \in G_2$, G_2 is the subgroup of $(F_{4(-20)})_{N_2}$. Set

$$S^6 := \{a \in \mathrm{Im}\mathbb{O} \mid |x| = 1\}.$$

Note that the group G_2 acts on S^6 transitively (cf.[11, Theorem 1.9.2]). That is, if $x \in S^6$, then there exists $\alpha \in G_2$ such that $\alpha x = e_1$.

Lemma 4.1. *For all $a, b \in S^6$, let $\beta_i \in \mathrm{End}_{\mathbb{R}}(\mathbb{O})$ such that*

$$\beta_1 x = b(ax),\ \ \beta_2 x = (xa)b,\ \ \beta_3 x = b(axa)b, \quad x \in \mathbb{O}.$$

Then $\varphi(\beta_1, \beta_2, \beta_3) \in (F_{4(-20)})_{N_2}$.

Proof. Let $p \in S^6$. By Proposition 1.1(1), $(px, py) = (p, p)(x, y) = (x, y)$ and $(xp, yp) = (x, y)(p, p) = (x, y)$, $\ x, y \in \mathbb{O}$. Therefore $\beta_i \in \mathrm{O}(8)$ $(i = 1, 2, 3)$. Now by Proposition 1.1(7)(3) and $a, b \in S^6 \subset \mathrm{Im}\mathbb{O}$,

$$(\beta_1 x)(\beta_2 y) = (b(ax))((ya)b) = b(axa)b = \overline{b(a(\overline{xy})a)b} = \overline{\beta_3(\overline{xy})}.$$

Therefore, by Lemma 2.2(2) and Proposition 2.3(2), we get $\beta_i \in \mathrm{SO}(8)$ and $\varphi(\beta_1, \beta_2, \beta_3) \in D_4 \subset F_{4(-20)}$. Next, by direct calculation, $\varphi(\beta_1, \beta_2, \beta_3) N_2 = N_2$. Hence the assertion follows. ☐

By Lemma 4.1, for $a, b \in S^6$, we can define $\varphi_{a,b} \in (F_{4(-20)})_{N_2}$ as $\varphi_{a,b} := \varphi(\beta_1, \beta_2, \beta_3)$ where

$$\beta_1 x := b(ax),\ \ \beta_2 x := (xa)b,\ \ \beta_3 x := b(axa)b, \quad x \in \mathbb{O}.$$

Proposition 4.2. *Assume that $X_0 \in (\mathcal{J}^1)_0$ satisfies $X_0 \times X_0 = N_2$. Then there exists $\alpha \in (F_{4(-20)})_{N_2}$ such that $\alpha X_0 = F_1^1(1) + F_2^1(1)$.*

Proof. By Proposition 2.6(3), we can write $X_0 = -rN_2 + F_1^1(x) + F_2^1(\overline{x})$ where $r \in \mathbb{R}$, $x \in \mathbb{O}$ and $|x| = 1$.

(Step 1) We will show the following assertion: If $x \in \mathrm{Im}\mathbb{O}$, then there exists $\beta_0 \in (F_{4(-20)})_{N_2}$ such that $\beta X_0 = -rN_2 + F_1^1(x') + F_2^1(\overline{x'})$ where $x' \in \mathbb{O}$, $|x'| = 1$ and $\mathrm{Re}(x') \neq 0$. Suppose that $x \in \mathrm{Im}\mathbb{O}$. Then $x \in S^6$. Since G_2 acts on S^6 transitively, there exists $\beta_1 \in G_2 \subset (F_{4(-20)})_{N_2}$ such that $\beta_1 X_0 = -rN_2 + F_1^1(e_1) + F_2^1(-e_1)$. Now $e_2, e_3 \in S^6$, so that $\varphi_{e_2,e_3} \in (F_{4(-20)})_{N_2}$. And $\varphi_{e_2,e_3}\beta_1 X_0 = -rN_2 + F_1^1(1) + F_2^1(1)$. Hence (Step 1) follows.

(Step 2) We may assume $X_0 = -rN_2 + F_1^1(x) + F_2^1(\overline{x})$ where $r \in \mathbb{R}$, $x \in \mathbb{O}$, $|x| = 1$ and $\mathrm{Re}(x) \neq 0$ by (Step 1). Then $\beta_{1,2}(t) \in (F_{4(-20)})_{N_2}$, and by (4.1), $\beta_{1,2}(-\frac{r}{2\mathrm{Re}(x)}) X_0 = F_1^1(x) + F_2^1(\overline{x})$.

(Step 3) We may assume $X_0 = F_1^1(x) + F_2^1(\overline{x})$ where $x \in \mathbb{O}$ and $|x| = 1$ by (Step 2). Then $x = \cos\theta + a\sin\theta$ for some $a \in \mathrm{Im}\mathbb{O}$ and $\theta \in \mathbb{R}$. Since G_2 acts on S^6 transitively, there exists $\alpha_1 \in G_2 \subset (F_{4(-20)})_{N_2}$ such that $\alpha_1 X_0 = F_1^1(\alpha_1 x) + F_2^1(\overline{\alpha_1 x}) = F_1^1(\cos\theta + e_1\sin\theta) + F_2^1(\cos\theta - e_1\sin\theta)$. Next $e_2, e_1 \in S^6$, so that $\varphi_{e_2,e_1} \in (F_{4(-20)})_{N_2}$. And $\varphi_{e_2,e_1}\alpha_1 X_0 = F_1^1(e_3\cos\theta + e_2\sin\theta) + F_2^1(-e_3\cos\theta - e_2\sin\theta)$. Since G_2 acts on S^6 transitively, there exists $\alpha_3 \in G_2 \subset (F_{4(-20)})_{N_2}$ such that $\alpha_3\varphi_{e_2,e_1}\alpha_1 X_0 = F_1^1(e_1) + F_2^1(-e_1)$. And last, $e_2, e_3 \in S^6$, so that $\varphi_{e_2,e_1} \in (F_{4(-20)})_{N_2}$. And $\varphi_{e_2,e_3}\alpha_3\varphi_{e_2,e_1}\alpha_1 X_0 = F_1^1(1) + F_2^1(1)$. Hence the assertion follows. □

Proposition 4.3. *Assume that $X \in \mathcal{J}^1$ satisfies $\Phi_X(\lambda) = (\lambda - \lambda_1)^3$. Then following assertions hold.*

(1) *If $p(X) = 0$, then $X = \frac{1}{3}\mathrm{tr}(X)E$.*

(2) *If $p(X) \in \mathcal{N}^+$, then $X \in \mathcal{O}_{F_{4(-20)}}(\frac{1}{3}\mathrm{tr}(X)E + N_1)$.*

(3) *If $p(X) \in \mathcal{N}^-$, then $X \in \mathcal{O}_{F_{4(-20)}}(\frac{1}{3}\mathrm{tr}(X)E + N_2)$.*

(4) *If $p(X) \notin \tilde{\mathcal{N}}$, then $X \in \mathcal{O}_{F_{4(-20)}}(\frac{1}{3}\mathrm{tr}(X)E + F_1^1(1) + F_2^1(1))$.*

(5) *By $F_{4(-20)}$, the above canonical forms* (1)-(4) *cannot be transformed from each other.*

Proof. (1) By $0 = p(X) = X - \frac{1}{3}\mathrm{tr}(X)E$, (1) follows.

(2) By Proposition 2.10(2), there exists $\alpha \in F_{4(-20)}$ such that $\alpha p(X) = N_1$. Hence $\alpha X = \alpha(\frac{1}{3}\mathrm{tr}(X)E + p(X)) = \frac{1}{3}\mathrm{tr}(X)E + N_1$ by Theorem 1.4.

(3) By Proposition 2.10(2), there exists $\alpha \in F_{4(-20)}$ such that $\alpha p(X) = N_2$. Hence $\alpha X = \alpha(\frac{1}{3}\mathrm{tr}(X)E + p(X)) = \frac{1}{3}\mathrm{tr}(X)E + N_2$ by Theorem 1.4.

(4) Suppose that $p(X) \times p(X) = 0$. By $\mathrm{tr}(p(X)) = 0$, $p(X) \in \tilde{\mathcal{N}}$. It contradicts with $p(X) \notin \tilde{\mathcal{N}}$. Therefore $p(X) \times p(X) \neq 0$. Put $Y = p(X) \times p(X)$. By Proposition 1.9(2) and Proposition 1.6(2), $Y \in \mathcal{N}^+$ or $Y \in \mathcal{N}^-$. Suppose that $Y \in \mathcal{N}^+$. By Proposition 2.10(2), there exists $\alpha \in F_{4(-20)}$ such that $\alpha Y = N_1$. Then $N_1 = \alpha(p(X) \times p(X)) = p(\alpha X) \times p(\alpha X)$ and $\mathrm{tr}(p(\alpha X)) = \mathrm{tr}(\alpha(p(X))) = \mathrm{tr}(p(X)) = 0$ by Theorem 1.4. Using Proposition 2.6(1), this can not be happened. Therefore $Y \in \mathcal{N}^-$, so that by Proposition 2.10(2), there exists $\alpha_1 \in F_{4(-20)}$ such that $\alpha_1 Y = N_2$. Then by Theorem 1.4, $N_2 = p(\alpha_1 X) \times p(\alpha_1 X)$ and $p(\alpha_1 X) \in (\mathcal{J}^1)_0$. Now by Proposition 4.2, there exists $\alpha_2 \in (F_{4(-20)})_{N_2}$ such that $\alpha_2\alpha_1(p(X)) = \alpha_2(p(\alpha_1 X)) = F_1^1(1) + F_2^1(1)$. Hence $\alpha_2\alpha_1 X = \alpha_2\alpha_1(\frac{1}{3}\mathrm{tr}(X)E + p(X)) = \frac{1}{3}\mathrm{tr}(X)E + F_1^1(1) + F_2^1(1)$ by Theorem 1.4.

(5) By Theorem 1.4 and Proposition 2.10(2), $p(\alpha X) = \alpha p(X) \in \alpha\mathcal{N}^+ = \mathcal{N}^+$. Therefore, if $p(X) \in \mathcal{N}^+$, then $p(\alpha X) \in \mathcal{N}^+$ for all $\alpha \in F_{4(-20)}$.

Similarly if $p(X) \in \mathcal{N}^-$, then $p(\alpha X) \in \mathcal{N}^-$ for all $\alpha \in F_{4(-20)}$. And by Proposition 1.6(3), if $p(X) \notin \tilde{\mathcal{N}}$, then $p(\alpha X) \notin \tilde{\mathcal{N}}$ for all $\alpha \in F_{4(-20)}$. Thus canonical forms (1)-(4) cannot be transformed from each other. □

Proof of Main Theorem. For $X \in \mathcal{J}^1$, by Proposition 1.5(1), $\Phi_\lambda(X)$ is a $\mathbb{R}$-coefficient polynomial of λ with degree 3, so that we have the following Cases (I)-(IV) by the set of characteristic roots and their multiplicities.

(I) $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_i \in \mathbb{R}$ and $\lambda_1 > \lambda_2 > \lambda_3$.

(II) $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ $(q > 0)$.

(III) $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_i \in \mathbb{R}$ and λ_1 of multiplicity 1 and λ_2 of multiplicity 2.

(IV) $X \in \mathcal{J}^1$ admits the characteristic real root of multiplicity 3.

By Proposition 1.5(3), the set of characteristic roots and their multiplicities are invariant by the action of $F_{4(-20)}$, the difference of the set of characteristic roots and their multiplicities induce the difference of orbits in $\mathcal{J}^1$ over the $F_{4(-20)}$-actions. Therefore Cases (I)-(IV) are different cases of orbits from each other.

In Case (I), by Proposition 3.7(2)(3), we obtain the canonicals form of X. And by Proposition 3.6, the cases are different orbits from each other.

In Case (II), by Proposition 3.8, we obtain the canonical form of X.

In Case (III), by Proposition 3.9(1)-(4), we obtain the canonicals form of X. And by Proposition 3.9(5), the cases are different orbits from each other.

In Case (IV), by Proposition 4.3(1)-(4), we obtain the canonical form of X. And by Proposition 4.3(5), the cases are different orbits from each other.

Thus we obtain a concrete classification of $F_{4(-20)}$-orbits on $\mathcal{J}^1$. □

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