Tableaux and plane partitions of truncated shapes

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Abstract

We consider a new kind of straight and shifted plane partitions/Young tableaux whose diagrams are no longer of partition shape, but rather Young diagrams with boxes erased from their upper right ends. We find formulas for the number of standard tableaux in certain cases, namely shifted staircases without a box in their upper right corner and rectangles with a staircase removed from the upper right end. The proofs involve interpretations and formulas for sums of restricted Schur functions and their specializations.

1 Introduction

A central topic in the study of tableaux and plane partition are their enumerative properties. The number of standard tableaux of straight shape is given by the famous hook-length formula of Frame, Robinson and Thrall, similar formula exists for shifted shapes. Not all shapes have such nice enumerative properties though, for example their main generalizations as skew shapes are not counted by any product type formulas.

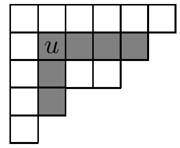
In this paper we find product formulas for special cases of a new type of tableaux and plane partitions, namely ones whose diagrams are not straight or shifted Young diagrams of integer partitions. The diagrams in question are obtained by removing boxes from the north-east corners of a straight or shifted Young diagram and we say that the shape has been truncated by the shape of the boxes removed. We discover formulas for the number of tableaux of specific truncated shapes: a rectangle truncated by a staircase shape and a shifted staircase truncated by one box. The proofs rely on several steps of interpretations. Truncated shapes are interpreted as (tuples of) SSYTs, which translates the problem into specializations of sums of restricted Schur functions. The number of standard tableaux is found as a polytope volume; thus as a certain limit of these specializations whose computations involve complex integration, the Robinson-Schensted-Knuth correspondence, etc.

The consideration of these objects started after R. Adin and Y. Roichman asked for a formula for the number of linear extensions of the poset of triangle-free triangulations, which are equivalent with standard tableaux of shifted staircase shape with upper right corner box removed, [AR]. We find and prove the formula in question, we also prove formulas for the cases of rectangles truncated by a staircase. In [AKR] Adin, King and Roichman have independently found a formula for the case of shifted staircase truncated by one box, they have also considered other case (truncation of staircase and rectangle by a square) by methods fundamentally different from the methods developed here.

2 Definitions

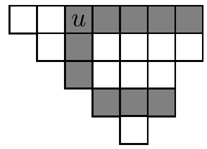
We will refer the reader to [Sta99] and [Mac95] for the basic facts and definitions regarding Young tableaux and symmetric functions, which we will use in this paper. We remind the reader of the hook-length formulas for the number of standard Young tableaux (SYT) of straight shape λ:

$$f_\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h_u}, \qquad \text{hook } h_u:$$



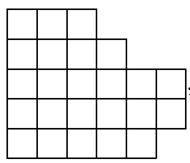
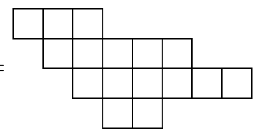
and the number of standard Young tableaux of shifted shape λ

$$g_\lambda = \frac{|\lambda|!}{\prod_u h_u} \qquad \text{hook } h_u:$$



We are now going to define our main objects of study.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be integer partitions, s.t. $\lambda_i \geq \mu_i$. A **straight diagram of truncated shape** $\lambda \setminus \mu$ is a left justified array of boxes, such that row i has $\lambda_i - \mu_i$ boxes. If λ has no equal parts we can define a **shifted diagram of truncated shape** $\lambda \setminus \mu$ as an array of boxes, where row i starts one box to the right of the previous row $i-1$ and has $\lambda_i - \mu_i$ number of boxes. For example

$D_1 =$  , $D_2 =$ 

D_1 is of straight truncated shape $(6,6,6,6,5) \setminus (3,2)$ and D_2 is of shifted truncated shape $(8,7,6,2) \setminus (5,2)$.

We define standard and semi-standard Young tableaux and plane partitions of truncated shape the usual way except this time they are fillings of truncated diagrams. A **standard truncated Young tableaux** of shape $\lambda \setminus \mu$ is a filling of the corresponding truncated diagram with the integers from 1 to $|\lambda| - |\mu|$, such that the entries across rows and down columns are increasing and each number appears exactly once. A **plane partition of truncated shape** $\lambda \setminus \mu$ is a filling of the corresponding truncated diagram with integers such that they weakly increase along rows and down columns. For example

$$T_1 = \begin{array}{ccccc} 1 & 2 & 4 & & \\ & 3 & 5 & 7 & 8 \\ & & 6 & 9 & \end{array}, T_2 = \begin{array}{ccccc} 1 & 1 & 3 & & \\ & 2 & 4 & 4 & 5 \\ & & 5 & 6 & \end{array}, T_3 = \begin{array}{ccccc} 1 & 1 & 3 & & \\ & 1 & 4 & 4 & 6 \\ & & 4 & 5 & \end{array}$$

are respectively a standard Young tableaux (SYT), a semi-standard Young tableaux (SSYT) and a plane partition (PP) of shifted truncated shape $(5,4,2) \setminus (2)$. We also define **reverse** PP, SSYT, SYT by reversing all inequalities in the respective definitions (replacing weakly/strictly increasing with weakly/strictly decreasing). For this paper it will be more convenient to think in terms of the reverse versions of these objects.

In this paper we will consider truncation by staircase shape $\delta_k = (k, k-1, k-2, \ldots, 1)$ of shifted staircases and straight rectangles. We will denote by $T[i,j]$ the entry in the box with coordinate (i,j) in the diagram of T, where i denotes the row number counting from the top and j denotes the $j-$th box in this row counting from the beginning of the row.

For any shape D, let

$$F_D(q) = \sum_{T:\mathrm{sh}(T)=D} q^{\sum T[i,j]}$$

be the generating function for the sum of the entries of all plane partitions of shape D.

3 A bijection with skew SSYT

We will consider a map between truncated plane partitions and skew Semi-Standard Young Tableaux which will enable us to enumerate them using Schur functions.

As a basic setup for this map we first consider truncated shifted plane partition of staircase shape $\delta_n \setminus \delta_k$. Let T be such plane partition. Let $\lambda^j = (T[1,j], T[2,j], \ldots, T[n-j,j])$ - the sequence of numbers in the jth diagonal of T. For example if $T = \begin{array}{cccc} 8 & 6 & 5 & \\ & 6 & 4 & 3 \\ & & 4 & 2 \\ & & & 1 \end{array}$, then $\lambda^1 = (8,6,4,1)$, $\lambda^2 = (6,4,2)$ and $\lambda^3 = (5,3)$.

Let P be a reverse skew tableaux of shape λ^1/λ^{n-k}, such that the entries filling the subshape λ^j/λ^{j+1} are equal to j, i.e. it corresponds to the sequence $\lambda^{n-k} \subset \lambda^{n-k-1} \subset \cdots \subset \lambda^1$. The fact that this is all well defined follows from the inequalities that the $T[i,j]$'s satisfy by virtue of T being a plane partitions. Namely, $\lambda^{j+1} \subset \lambda^j$, because $\lambda^j_i = T[i,j] \geq T[i,j+1] = \lambda^{j+1}_i$. Clearly the rows of P are weakly decreasing. The columns are strictly decreasing, because for each j the entries j in the ith row of P are in positions $T[i,j+1]+1$ to position $T[i,j] < T[i-1,j+1]+1$, so they appear strictly to the left of the js in the row above $(i-1)$.

Define $\phi(T) = P$, ϕ is the map in question. Given a reverse skew tableaux P of shape $\lambda \setminus \mu$ and entries smaller than n we can obtain the inverse shifted truncated plane partition $T = \phi^{-1}(P)$ as $T[i,j] = \max(s | P[i,s] \geq j)$, if no such entry of P exists let $s = 0$.

For example we have that

$$T = \begin{array}{ccccc} 8 & 7 & 6 & 5 & \\ & 7 & 7 & 5 & 4 \\ & & 6 & 5 & 4 \\ & & & 4 & 3 \\ & & & & 2 \end{array} \qquad \xrightarrow{\phi} \qquad P = \begin{array}{cccccccc} & & & & & 3 & 2 & 1 \\ & & & & 3 & 2 & 2 & \\ 3 & 3 & 3 & 3 & 2 & 1 & & \\ 2 & 2 & 2 & 1 & & & & \\ 1 & 1 & & & & & & \end{array} \tag{1}$$

Notice that

$$\sum T[i,j] = \sum P[i,j] + |\lambda^{n-k}|(n-k), \tag{2}$$

where $\mathrm{sh}(P) = \lambda^1/\lambda^{n-k}$.

The map ϕ can be extended to any truncated shape, then the image will be tuples of SSYTs with certain restrictions. For the purposes of this paper we will extend it to truncated plane partitions of shape $(n^m) \setminus \delta_k$ as follows.

Let T be a plane partition of shape ${}^n \setminus \delta_k$ and assume that $n \leq m$ (otherwise we can reflect anti-diagonally). Let $\lambda = (T[1,1], T[2,2], \ldots, T[n,n])$, $\mu = (T[1,n-k], T[2,n-k+1], \ldots, T[k+1,n])$ and let T_1 be the portion of T above and including the main diagonal, hence of shifted truncated shape $\delta_n \setminus \delta_k$, and T_2 the transpose of the lower portion including the main diagonal, a shifted PP of shape $(m, m-1, \ldots, m-n+1)$.

Extend ϕ to T as $\phi(T) = (\phi(T_1), \phi(T_2))$. Here $\phi(T_2)$ is a SSYT of at most n rows (shape λ) and filled with $[1, \ldots, m]$ the same way as in the truncated case.

As an example with $n = 5, m = 6, k = 2$ we have

$$T = \begin{array}{ccccc} 7 & 6 & 4 & & \\ 5 & 6 & 4 & 4 & \\ 4 & 4 & 3 & 3 & 2 \\ 4 & 3 & 2 & 2 & 2 \\ 3 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{array}, \quad T_1 = \begin{array}{ccccc} 7 & 6 & 4 & & \\ & 6 & 4 & 4 & \\ & & 3 & 3 & 2 \\ & & & 2 & 2 \\ & & & & 1 \end{array}, T_2 = \begin{array}{cccccc} 7 & 5 & 4 & 4 & 3 & 2 \\ & 6 & 4 & 3 & 2 & 1 \\ & & 3 & 2 & 2 & 1 \\ & & & 2 & 1 & 1 \\ & & & & 1 & 1 \end{array},$$

$$\phi(T) = \left(\begin{array}{ccccccc} & & & & 2 & 2 & 1 \\ & & & & 1 & 1 & \\ & & 2 & & & & \\ 2 & 2 & & & & & \\ 1 & & & & & & \end{array}, \begin{array}{ccccccc} 6 & 6 & 5 & 4 & 2 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 1 & \\ 4 & 3 & 1 & & & & \\ 3 & 1 & & & & & \\ 2 & & & & & & \end{array} \right)$$

We thus have the following

Proposition 1. The map ϕ is a bijection between shifted truncated plane partitions T of shape $\delta_n \setminus \delta_k$ filled with nonnegative integers and (reverse) skew semi-standard Young tableaux with entries in $[1, \ldots, n-k-1]$ of shape λ/μ with $l(\lambda) \leq n$ and $l(\mu \leq k+1$. Moreover, $\sum_{i,j} T[i,j] = \sum_{i,j} P[i,j] + |\mu|(n-k)$. Similarly ϕ is also a bijection between truncated plane partitions T of shape $n^m \setminus \delta_k$ and pairs of SSYTs (P, Q), s.t. $\mathrm{sh}(P) = \lambda/\mu$, $\mathrm{sh}(Q) = \lambda$ with $l(\lambda) \leq n$, $l(\mu) \leq k+1$ and P is filled with $[1, \ldots, n-k-1]$, Q with $[1, \ldots, m]$. Moreover, $\sum T[i,j] = \sum P[i,j] + \sum Q[i,j] - |\lambda| + |\mu|(n-k)$.

4 Schur function identities

We will now consider the relevant symmetric function interpretation arising from the map ϕ. Substitute the entries $1, \ldots$ in the skew SSYTs in the image with respective variables $x_1, \ldots$ and $z_1, \ldots$. The idea is to evaluate the resulting expressions at $x = (q, q^2, \ldots)$ and $z = (1, q, q^2, \ldots)$ to obtain generating functions for the sum of entries in the truncated plane partitions which will later allow us to derive enumerative results.

For the case of shifted truncated shape $\delta_n \setminus \delta_k$ we have the corresponding sum

$$S_{n,k}(x;t) = \sum_{\lambda,\mu | l(\lambda) \leq n, l(\mu) \leq k} s_{\lambda/\mu}(x_1, \ldots, x_{n-k-1}) t^{|\mu|}, \tag{3}$$

and for the straight truncated shape $n^m \setminus \delta_k$

$$D_{n,k}(x;z;t) = \sum_{\lambda,\mu | l(\lambda) \leq n, l(\mu) \leq k+1} s_\lambda(z) s_{\lambda/\mu}(x) t^\mu. \tag{4}$$

We need to find formulas when $x_i = 0$ for $i > n-k-1$ and $z_i = 0$ for $i > m$. Keeping the restriction $l(\mu) \leq k+1$ we have that $s_{\lambda/\mu}(x) = 0$ if $l(\lambda) > n$ and this allows us to drop the length restriction on λ in both sums.

From now on the different sums will be treated separately.

Consider another set of variables $y = (y_1, \ldots, y_{k+1})$ which together with $(x_1, \ldots, x_{n-k-1})$ form a set of n variables.

Using Cauchy's identity we have that

$$\begin{aligned}
&\sum_{\lambda | l(\lambda) \leq n} s_\lambda(x_1, \ldots, x_{n-k-1}, ty_1, \ldots, ty_{k+1}) \\
&= \sum_\lambda s_\lambda(x_1, \ldots, x_{n-k-1}, ty_1, \ldots, ty_{k+1}) \\
&= \prod \frac{1}{1-x_i} \prod_{i<j\leq n-k-1} \frac{1}{1-x_i x_j} \prod_{i<j\leq k+1} \frac{1}{1-t^2 y_i y_j} \prod_{i,j} \frac{1}{1-x_i t y_j} \prod \frac{1}{1-ty_i},
\end{aligned}$$

where the length restriction drops because $s_\lambda(u_1, \ldots, u_n) = 0$ when $l(\lambda) > n$. On the other hand we have that

$$\begin{aligned}
&\sum_{\lambda | l(\lambda) \leq n} s_\lambda(x_1, \ldots, x_{n-k-1}, ty_1, \ldots, y_{k+1} t) \\
&= \sum_{\lambda,\mu} s_{\lambda/\mu}(x_1, \ldots, x_{n-k-1}) s_\mu(ty_1, \ldots, ty_{k+1}) \\
&= \sum_{\lambda,\mu | l(\mu) \leq k+1} s_{\lambda/\mu}(x_1, \ldots, x_{n-k-1}) s_\mu(y_1, \ldots, y_{k+1}) t^{|\mu|}
\end{aligned}$$

since again $s_\mu(yt) = 0$ if $l(\mu) > k+1$.

We thus get

$$\prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j} \prod_{i<j\le k+1} \frac{1}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i} \tag{5}$$
$$= \sum_{\lambda,\mu|l(\mu)\le k+1} s_{\lambda/\mu}(x_1,\dots,x_{n-k-1})s_\mu(y_1,\dots,y_{k+1})t^{|mu|}$$

We now need to extract the coefficients of $s_\mu(y)$ from both sides of (5) to obtain a formula for $S_{n,k}(x;t)$. To do so we will use the determinantal formula for the Schur functions, namely that

$$s_\nu(u_1,\dots,u_p) = \frac{a_{\nu+\delta_p}(u)}{a_{\delta_p}(u)} = \frac{\det[u_i^{\nu_j+p-j}]_{i,j=1}^p}{\det[u_i^{p-j}]_{i,j=1}^p}.$$

We also have that $a_{\delta_p}(z_1,\dots,z_p) = \prod_{i<j}(z_i - z_j)$. Substituting $s_\mu(y)$ with $a_{\delta_{k+1}+\mu}(y)/a_{\delta_{k+1}}(y)$ in the right-hand side of (5) and multiplying both sides by $a_{\delta_{k+1}}(y)$ we obtain

$$s_{\lambda/\mu}(x_1,\dots,x_{n-k-1})a_{\mu+\delta_{k+1}}(y_1,\dots,y_{k+1})t^{|mu|} \tag{6}$$
$$= \prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j} \prod_{i<j\le k+1} \frac{y_i-y_j}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i}$$

Observe that $a_\alpha(u_1,\dots,u_p) = \sum_{w\in S_p} \operatorname{sgn}(w)u_{w_1}^{\alpha_1}\cdots u_{w_p}^{\alpha_p}$ with $\alpha_i > \alpha_{i+1}$ has exactly $p!$ different monomials in y, each with a different order of the degrees of u_i (determined by w). Moreover, if $\alpha \neq \beta$ are partitions of distinct parts, then $a_\alpha(u)$ and $a_\beta(u)$ have no monomial in common. Let

$$A(u) = \sum_{\alpha|\alpha_i>\alpha_{i+1}} \sum_{w\in S_p} \operatorname{sgn}(w)u_{w_1}^{\alpha_1}\cdots u_{w_p}^{\alpha_p}.$$

For every β of strictly decreasing parts, every monomial in $a_\beta(u)$ appears exactly once and with the same sign in $A(u)$, so $a_\beta(u)A(u^{-1})$ has coefficient at u^0 equal to $p!$. Therefore we have

$$[y^0]\left(\sum_{\lambda,\mu} s_{\lambda/\mu}(x_1,\dots,x_{n-k-1})a_{\mu+\delta_{k+1}}(y_1,\dots,y_{k+1})t^{|\mu|}A(y^{-1})\right) \tag{7}$$
$$= (k+1)! \sum_{\lambda,\mu|l(\mu)\le k+1} s_{\lambda/\mu}(x_1,\dots,x_{n-k-1})t^{|\mu|} = (k+1)!S_{n,k}(x;t). \tag{8}$$

Using the fact that $a_\alpha(u) = s_\nu(u)a_\delta(u)$ for $\nu = \alpha - \delta_p$ and Cauchy's formula for the sum of the Schur functions we have

$$A(u) = \sum_\nu s_\nu(u)a_{\delta_p}(u) = \prod \frac{1}{1-u_i} \prod_{i<j} \frac{(u_i-u_j)}{1-u_iu_j}.$$

In order for $A(y^{-1})$ and $\sum s_\mu(y)t^{|\mu|}$ to converge we need $1 < |y_i| < |t^{-1}|$ for every i. For $S_{n,k}(x;t)$ to also converge, let $|t| < 1$ and $|x_j| < 1$ for all j.

We also have that for any doubly infinite series $f(y)$, $[y^0]f(y) = \frac{1}{2\pi i}\int_C f(y)y^{-1}dy$, on a circle C in the $\mathbb{C}$ plane centred at 0 and within the region of convergence of f. Hence we have the formula for $S_{n,k}(x;t)$ through a complex integral.

Proposition 2. We have that

$$S_{n,k}(x;t) = (-1)^{\binom{k+1}{2}} \prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j}$$
$$\int_T \prod_{i<j\le k+1} \frac{(y_i-y_j)^2}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i} \prod \frac{1}{y_i-1} \prod_{i<j} \frac{1}{y_iy_j-1} dy_1\cdots dy_{k+1},$$

where $T = C_1 \times C_2 \times \cdots C_p$ and $C_i = \{z \in \mathbb{C} || z| = 1 + \epsilon_i\}$ for $\epsilon_i < |t^{-1}| - 1$.

For the **straight shape case** and $D_{n,k}$ we consider the sum, whose formula we know as Cauchy's identity

$$\sum_{\lambda} s_\lambda(z) s_\lambda(x, ty) = \prod_{i,j} \frac{1}{1 - x_i z_j} \prod_{i,j} \frac{1}{1 - y_i t z_j}. \tag{9}$$

Let $x_i = 0$ if $i > n - k - 1$, $z = (z_1, \ldots, z_m)$ and $y = (y_1, \ldots, y_{k+1})$. Then $s_\lambda(x, ty) = 0$ if $l(\lambda) > n$, so this sum ranges over λ with $l(\lambda) \leq n$. Also, we have that

$$s_\lambda(x, yt) = \sum_{\mu} s_{\lambda/\mu}(x) s_\mu(yt),$$

and since $s_\mu(yt) = 0$ if $l(\mu) > k + 1$, the sum ranges only over μ, s.t. $l(\mu) \leq k + 1$. Thus we have that

$$\prod_{i,j} \frac{1}{1 - x_i z_j} \prod_{i,j} \frac{1}{1 - y_i t z_j} = \sum_{\lambda, \mu | l(\lambda) \leq n, l(\mu) \leq k+1} s_\lambda(z) s_{\lambda/\mu}(x) t^{|\mu|} s_\mu(y)$$

If we expand the left-hand side of the above equation as a linear combination of $s_\mu(y)$, summing all the coefficients on both sides will give us the desired formula for $D_{n,m,k}(x, z; t)$.

We have that

$$\prod_{i,j} \frac{1}{1 - y_i z_j t} = \sum_{\nu} s_\nu(zt) s_\nu(y),$$

and the other factor contains only x and z, a constant over the ring of symmetric polynomials in y. Comparing coefficients we get

Proposition 3. We have that

$$D_{n,m,k}(x, z; t) = \prod_{i,j} \frac{1}{1 - x_i z_j} \left(\sum_{\nu | l(\nu) \leq k+1} s_\nu(zt) \right).$$

For the purpose of enumeration of SYTs we will use this formula as it is. Even though there are formulas, e.g. of Gessel and of King, for the sum of Schur functions of restricted length, they would not give the enumerative answer any easier.

5 A polytope volume as a limit

Plane partitions of specific shape (truncated or not) of size N can be viewed as integer points in a cone in $\mathbb{R}^N$. Let D be the diagram of a plane partition T, the coordinates of $\mathbb{R}^{|D|}$ are indexed by the boxes present in T. Then $C_D = \{(\cdots, x_{i,j}, \cdots) \in \mathbb{R}^N_{\geq 0} : [i,j] \in D, \; x_{i,j} \leq x_{i,j+1}$ if $[i, j+1] \in D, x_{i,j} \leq x_{i+1,j}$ if $[i+1, j] \in D\}$ is the corresponding cone.

Let $P(C)$ be the section of a cone C in $\mathbb{R}^N_{\geq 0}$ with the hyperplane $H = \{x | \sum_{[i,j] \in D} x_{i,j} = 1\}$. Consider the standard tableaux of shape T, these correspond to all linear ordering of the points in C_D and thus also $P_D = P(C_D)$. Considering T as a bijection $D \to [1, \ldots, N]$, P_D is thus subdivided into chambers $\{x : 0 \leq x_{T^{-1}(1)} \leq x_{T^{-1}(2)} \leq \cdots x_{T^{-1}(N)}\} \cap H$ of equal volumes, namely $\frac{1}{N!} Vol(\Delta_N)$, where $\Delta_N = H \cap \mathbb{R}^N$ is the $N-$simplex. Hence the volume of P_D is

$$Vol_{N-1}(P_D) = \frac{\#T : SYT, \mathrm{sh}(T) = D}{N!} Vol(\Delta_N). \tag{10}$$

The following lemma helps determine the volume and thus the number of SYTs of shape D.

Lemma 1. Let P be a $d-1$dimensional rational polytope in $\mathbb{R}^d_{\geq 0}$, s.t. its points satisfy $a_1+\cdots+a_d=1$ for $(a_1,\ldots,a_d)\in P$, and $F_P(q)=\sum_n\sum_{(a_1,\ldots,a_d)\in nP\cap\mathbb{Z}^d} q^{a_1+a_2+\cdots+a_d}$. We have that the $d-1-$dimensional volume of P is $Vol_{d-1}(P)=(\lim_{q\to 1}(1-q)^d F_P(q))Vol(\Delta_d)$, where Δ_d is the $d-1$dimensional simplex.

Proof. Let $f_P(n)=\#\{(a_1,\ldots,a_d)\in nP\}$. Then

$$F_P(q)=\sum_n f_P(n)q^n$$

. Moreover, $Vol_{d-1}(P)=\lim_{n\to\infty}\frac{f_P(n)}{n^{d-1}}Vol(\Delta_d)$ since subdividing an embedding of P in $\mathbb{R}^{d-1}$ into $d-1$-hypercubes with side $\frac{1}{n}$ we get $f_P(n)$ cubes of total volume $f_P(n)/n^{d-1}$ which scaled by $Vol(\Delta_d)$ approximate P as $n\to\infty$.

Moreover since $\lim_{n\to\infty}\frac{n^{d-1}}{(n+1)\cdots(n+d-1)}=1$ we also have $\frac{Vol_{d-1}(P)}{Vol(\Delta_d)}=\lim_{n\to\infty}\frac{f_P(n)}{(n+1)\cdots(n+d-1)}$. Let $G(q)=\sum_n\frac{f_P(n)}{(n+1)\cdots(n+d-1)}q^{n+d-1}$. Then

$$\tilde{G}(q)=(1-q)G(q)=\underbrace{(\frac{f_P(n)}{(n+1)\cdots(n+d-1)}-\frac{f_P(n-1)}{(n)\cdots(n+d-2)})}_{b_n}q^{n+d-1}$$

and $\tilde{G}(1)=\sum b_n=\lim_{n\to\infty}b_1+\cdots+b_n=\lim_{n\to\infty}\frac{f_P(n)}{(n+1)\cdots(n+d-1)}=\frac{Vol_d(P)}{Vol(\Delta_d)}$. On the other hand

$$\begin{aligned}\frac{Vol_d(P)}{Vol(\Delta_d)}&=\tilde{G}(1)=\lim_{q\to 1}(1-q)G(q)=\lim_{q\to 1}\frac{G'(q)}{(\frac{1}{1-q})'}\\&=\cdots=\lim_{q\to 1}\frac{G^{(d-1)}(q)}{(\frac{1}{1-q})^{(d-1)}}=\lim_{q\to 1}F_p(q)(1-q)^d\end{aligned}$$

by L'Hopital's rule. ∎

Notice that if $P=P(C_D)$ for some shape D, then

$$\begin{aligned}F_P(q)=&\sum_n\sum_{a\in nP\cap\mathbb{Z}^N}q^n\\=&\sum_{a\in C_D\cap\mathbb{Z}^N}q^{|a|}=\sum_{T:PP,\mathrm{sh}(T)=D}q^{\sum T[i,j]}=F_D(q).\end{aligned}$$

Using (10) and this Lemma we get the key fact to enumerating SYTs of truncated shapes using evaluations of symmetric functions.

Proposition 4. The number of standard tableaux of (truncated) shape D is equal to

$$N!\lim_{q\to 1}(1-q)^N F_D(q).$$

6 Shifted truncated SYTs

We are now going to use Propositions 2 and 4 to find the number of standard shifted tableaux of truncated shape $\delta_n\setminus\delta_1$. Numerical results show that a product formula for the general case of truncation by δ_k does not exist.

First we will evaluate the integral in Proposition 2 by iteration of the Residue theorem.

For simplicity, let $u_0=t$ $u_i=tx_i$, so the integral becomes

$$\int_T\frac{(y_1-y_2)^2}{1-t^2y_1y_2}\frac{1}{y_2-1}\frac{1}{y_1-1}\frac{1}{y_1y_2-1}\prod_{i\geq 0,j=1,2}\frac{1}{1-u_iy_j}dy_1dy_2$$

Integrating by y_1 we have poles at $1, y_2^{-1}t^{-2}$, y_2^{-1} and u_i^{-1}. Only 1 and y_2^{-1} are inside C_1 and the respective residues are

$$\begin{aligned} Res_{y_1=1} &= \prod_{i\geq 0}\frac{1}{1-u_i}\int_{C_2}(-1)\frac{(1-y_2)^2}{1-t^2y_2}\frac{1}{y_2-1}\frac{1}{y_2-1}\prod_{i\geq 0}\frac{1}{1-u_iy_2}dy_2 \\ &= -\prod_{i\geq 0}\frac{1}{1-u_i}\int_{C_2}\frac{1}{(1-t^2y_2)}\prod\frac{1}{1-u_iy_2}dy_2 = 0, \end{aligned}$$

since now the poles for y_2 are all outside C_2.

For the other residue we have that

$$\begin{aligned} Res_{y_1=y_2^{-1}} &= \int_{C_2}\frac{(y_2^{-1}-y_2)^2}{1-t^2}\frac{1}{y_2-1}\frac{1}{y_2^{-1}-1}\frac{1}{y_2}\prod_{i\geq 0}\frac{1}{1-u_iy_2}\prod_{i\geq 0}\frac{1}{1-u_iy_2^{-1}}dy_2 \\ &= \frac{1}{1-t^2}\int_{C_2}\frac{(1+y_2)^2}{y_2^2}\prod_{i\geq 0}\frac{y_2}{y_2-u_i}\prod_{i\geq 0}\frac{1}{1-u_iy_2}dy_2 \\ &= \frac{1}{1-t^2}\int_{C_2}(1+y_2)^2y_2^{n-3}\prod_{i\geq 0}\frac{1}{y_2-u_i}\prod_{i\geq 0}\frac{1}{1-u_iy_2}dy_2 \end{aligned}$$

If $n\geq 3$ the poles inside C_2 are exactly $y_2=u_i$ for all i and so we get a final answer

$$-\sum_{i\geq 0}\frac{1}{1-t^2}(1+u_i)^2u_i^{n-3}\prod_{j\neq i}\frac{1}{u_i-u_j}\prod_{j\geq 0}\frac{1}{1-u_iu_j}$$

and

$$S_{n,1}(x;t) = \frac{(-1)^{\binom{k+1}{2}+1}}{(k+1)!}\prod\frac{1}{1-x_i}\prod_{1\leq i<j\leq n-k-1}\frac{1}{1-x_ix_j}\sum_{i\geq 0}\frac{1}{1-t^2}(1+u_i)^2u_i^{n-3}\prod_{j\neq i}\frac{1}{u_i-u_j}\prod_{j\geq 0}\frac{1}{1-u_iu_j},$$

where $u_i = tx_i$ with $x_0 = 1$.

We can simplify the sum above as follows.

Notice that for any p variables $v = (v_1, \ldots, v_p)$ we have

$$\sum_{i=1}^{p}\frac{v_i^r}{\prod(v_i-v_j)} = \frac{\sum_i(-1)^{i-1}v_i\prod_{s<l;l,s\neq i}(v_s-v_l)}{\prod_{s<l}(v_s-v_l)} = \frac{a_{(r-p+1)+\delta_p}(v)}{a_{\delta_p}(v)} = s_{(r-p+1)}(v) = h_{r-p+1}(v),$$

where $h_s(v) = \sum_{i_1\leq i_2\leq\cdots\leq i_s} v_{i_1}v_{i_2}\cdots v_{i_s}$ is the $s-$th homogeneous symmetric function.

Then we have that

$$\begin{aligned} \sum_{i=0}^{n-k-1}\frac{u_i^s}{\prod(u_i-u_j)}\prod\frac{1}{1-u_iu_j} &= \sum_{i=0}^{n-k-1}\frac{u_i^s}{\prod(u_i-u_j)}\sum_p u_i^p h_p(u) \\ &= \sum_{p\geq 0}\sum_{i=0}^{n-k-1}\frac{u_i^{s+p}}{\prod(u_i-u_j)}h_p(u) \\ &= \sum_{p\geq 0}h_{s-n+k+1+p}(u)h_p(u) = c_{s-n+k+1}(u), \end{aligned} \tag{11}$$

where $c_i = \sum_{n\geq 0} h_nh_{n+i}$.

We then have the new formulas

$$S_{n,1}(x;t) = \prod\frac{1}{1-x_i}\prod_{1\leq i<j\leq n-k-1}\frac{1}{1-x_ix_j}\frac{1}{1-t^2}(c_1(u)+c_0(u)), \tag{12}$$

where $(1+u_i)^2u_i^{n-3} = u_i^{n-3} + 2u_i^{n-2} + u_i^{n-1}$ contributed to $c_{-1}(u) + 2c_0(u) + c_1(u)$ and by its definition $c_{-1} = c_1$. We are now ready to prove the following.

Theorem 1. The number of shifted standard tableaux of shape $\delta_n \setminus \delta_1$ is equal to

$$g_n \frac{C_n C_{n-2}}{2C_{2n-3}},$$

where $g_n = \frac{\binom{n+1}{2}!}{\prod_{0\le i<j\le n}(i+j)}$ is the number of shifted staircase tableaux of shape $(n, n-1, \ldots, 1)$ and $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the $m-$th Catalan number.

Proof. We will use Proposition 4 and the formula (12).

By the properties of ϕ we have that for the shape $D = \delta_n \setminus \delta_k$,

$$F_D(q) = \sum_{T|\,\mathrm{sh}(T)=D} q^{\sum T[i,j]} = \sum_{P=\phi(T)} q^{(n-k)|\mu|+\sum P[i,j]} = S_{n,k}(q, q^2, \ldots, q^{n-k-1}; q^{n-k}).$$

Let now $k = 1$. For the formula in Proposition 4 we have $N = \binom{n+1}{2} - 1$ and plugging $x = (q, \ldots, q^{n-2})$, $t = q^{n-1}$ in (12) we get

$$\lim_{q\to 1}(1-q)^N F_D(q) = \prod \frac{1}{1-q^i} \prod_{1\le i<j\le n-2} \frac{1}{1-q^{i+j}} \frac{1}{1-q^{2(n-1)}}(c_1(u) + c_0(u)),$$

where $u = (q^{n-1}, q^n, \ldots, q^{2n-3})$.

We need to determine $\lim_{q\to 1}(1-q)^{2n-3}c_s(u)$. Let $c_s(x;y) = \sum_l h_l(x)h_{l+s}(y)$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. We have that

$$\begin{aligned} c_s(x;y) &= \sum_l \sum_{i_1\le\cdots\le i_l; j_1\le\cdots\le j_{l+s}} x_{i_1}\cdots x_{i_l} y_{j_1}\cdots y_{j_{s+l}} \\ &= \sum_p h_s(y_1, \ldots, y_p) \sum_{P:(1,p)\to(n,m)} (-1)^{m+n-p-\#P} \sum_l h_l((xy)_P), \end{aligned}$$

where the sum runs over all fully ordered collections of lattice points P in between $(1,p)$ to (n,m) and $(xy)_P = (x_{i_1}y_{j_1}, \ldots)|(i_1, j_1), \ldots \in P$ and the -1s indicate the underlying inclusion-exclusion process. We also have that

$$\sum_l h_l((xy)_P) = \frac{1}{\prod_{(i,j)\in P}(1-x_iy_j)}.$$

The degree of $1-q$ dividing the denominators after substituting $(x_i, y_j) = (u_i, u_j) = (q^{n-2+i}, q^{n-2+j})$ for the evaluation of $c_s(u)$ is equal to the number of points in P. $\#P$ is maximal when the lattice path is from $(1,1)$ to $(n-1, n-1)$ and is saturated, so $\max \#P = 2(n-1) - 1 = 2n-3$. The other summands will contribute 0 when multiplied by the larger power of $(1-q)$ and the limit is taken. For each maximal path we have $\{i+j|(i,j) \in P\} = \{2, \ldots, 2n-2\}$ and the number of these paths is $\binom{2n-4}{n-2}$, so

$$(1-q)^{2n-3}c_s(q^{n-1}, \ldots, q^{2n-3}) = \binom{2n-4}{n-2} \prod_{i=2}^{2n-2} \frac{1-q}{1-q^{2(n-2)+i}} + (1-q)\ldots,$$

where the remaining terms are divisible by $1-q$, hence contribute 0 when the limit is taken.

Now we can proceed to compute $\lim_{q\to 1}(1-q)^N S_{n,1}(q^1, q^2, \ldots, ; q^{n-1})$. Putting all these together we have that

$$\begin{aligned} &\lim_{q\to 1}(1-q)^{\binom{n+1}{2}-1} S_{n,1}(q^1, \ldots, q^{n-2}; q^{n-1}) \\ &= \lim_{q\to 1} \prod_{1\le i\le n-2} \frac{1-q}{1-q^i} \prod_{1\le i<j\le n-2} \frac{1-q}{1-q^{i+j}} \frac{1-q}{1-q^{2(n-1)}} \left((1-q)^{2n-3}(c_1(u) + c_0(u))\right) \\ &= \prod_{0\le i<j\le n-2} \frac{1}{i+j} \frac{1}{2(n-1)} 2\binom{2n-4}{n-2} \prod_{i=2}^{2n-2} \frac{1}{2n-4+i} = \frac{g_{\delta_{n-2}}}{\binom{n-1}{2}!} \frac{1}{(n-1)} \binom{2n-4}{n-2} \frac{(2n-3)!}{(4n-6)!}, \end{aligned}$$

where $g_{n-2} = \frac{\binom{n-1}{2}!}{\prod_{0\le i<j\le n-2}(i+j)}$ is the number of shifted staircase tableaux of shape $(n-2,\dots,1)$. After algebraic manipulations we arrive at the desired formula. $\square$

7 Straight truncated SYTs

We will compute the number of standard tableaux of straight truncated shape $D = n^m \setminus \delta_k$ using propositions 1, 4 and 3.

By proposition 1 we have that

$$\begin{aligned}F_D(q) &= \sum_{T:\mathrm{sh}(T)=D} q^{\sum T[i,j]} \\ &= \sum_{\lambda,\mu|l(\mu)\le k+1, l(\lambda)\le n} \sum_{P,\mathrm{sh}(P),\lambda/\mu,Q,\mathrm{sh}(Q)=\lambda} q^{\sum P[i,j]+\sum Q[i,j]-|\lambda|+(n-k)|\mu|} \\ &= \sum_{\lambda,\mu|l(\lambda)\le n, l(\mu)\le k+1} s_\lambda(1,q,q^2,\dots,q^{m-1}) s_{\lambda/\mu}(q,q^2,\dots,q^{n-k-1}) q^{(n-k)|\mu|},\end{aligned}$$

which is $D_{n,m,k}(x,z;t)$ for $x=(q,q^2,\dots,q^{n-k-1})$, $z=(1,q,\dots,q^{m-1})$ and $t=q^{n-k}$ and from its simplified formula from proposition 3 and proposition 4 the number of standard tableaux of shape $n^m\setminus\delta_k$ is

$$\begin{aligned}&\lim_{q\to 1}(1-q)^{nm-\binom{k+1}{2}} F_D(q) \qquad (13)\\ &= \lim_{q\to 1}\left(\prod_{i=1,j=0}^{n-k-1,m-1} \frac{1-q}{1-q^{i+j}} (1-q)^{m(k+1)-\binom{k+1}{2}} \big(\sum_{\nu|l(\nu)\le k+1} s_\nu(q^{n-k},q^{n-k+1},\dots,q^{m-1+n-k})\big)\right).\end{aligned}$$

We are thus going to compute the last factor.

Lemma 2. Let $p \ge r$ and $N = rp - \binom{r-1}{2}$. Then for any s we have

$$\lim_{q\to 1}(1-q)^N \sum_{\lambda|l(\lambda)\le r} s_\lambda(q^{1+s},\dots,q^{p+s}) = \frac{g_{(p,p-1,\dots,p-r+1)}}{N!}\frac{E_1(r,p,s)}{E_1(r,p,0)},$$

where

$$E_1(r,p,s) = \prod_{m-1<l<2n-m+1} \frac{1}{(l+2s)^{m/2}} \prod_{l<m} \frac{1}{((l+2s)(2n-l+1+2s))^{\lfloor (l+1)/2\rfloor}}$$

for r even and $E_1(r,p,s) = \frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!}E_1(r-1,p,s)$ when r is odd and g_λ is the number of shifted SYTs of shape λ.

Proof. Consider the Robinson-Schensted-Knuth (RSK) correspondence between SSYTs with no more than r rows filled with $x_1,\dots,x_p$ and symmetric $p\times p$ integer matrices A. The limit on the number of rows translates through Schensted's theorem to the fact that there are no $m+1$ nonzero entries in A with coordinates $(i_1,j_1),\dots,(i_{r+1},j_{r+1})$, s.t. $i_1<\dots<i_{r+1}$ and $j_1>\dots>j_{r+1}$ (i.e. a decreasing subsequence of length $r+1$ in the generalized permutation corresponding to A). Let $\mathcal{A}$ be the set of such matrices. Let $\mathcal{A}\prime \subset \mathcal{A}$ be the set of $0-1$ matrices satisfying this condition, we will refer to them as allowed configurations.

Notice that $A \in \mathcal{A}$ if and only if $B \in \mathcal{B}$, where $B[i,j] = \begin{cases} 1, & \text{if } A[i,j] \neq 0 \\ 0, & \text{if } A[i,j] = 0 \end{cases}$. We thus have that

$$\begin{aligned}
\sum_{\lambda | l(\lambda) \leq r} s_\lambda(x_1, \ldots, x_p) &= \sum_{A \in \mathcal{A}} \prod_i x_i^{A[i,i]} \prod_{i>j} (x_i x_j)^{A[i,j]} \\
&= \sum_{B \in \mathcal{B}} \prod_{i:B[i,i]=1} \left(\sum_{a_{i,i}=1}^{\infty} x_i^{a_{i,i}} \right) \prod_{i>j:B[i,j]=1} \left(\sum_{a_{i,j}=1}^{\infty} (x_i x_j)^{a_{i,j}} \right) \\
&= \sum_{B \in \mathcal{B}} \prod_{i:B[i,i]=1} \frac{x_i}{1-x_i} \prod_{i>j:B[i,j]=1} \frac{x_i x_j}{1 - x_i x_j}.
\end{aligned}$$

Notice that B cannot have more than N nonzero entries on or above the main diagonal. No diagonal $i+j=l$ (i.e. the antidiagonals) can have more than r nonzero entries on it because of the longest decreasing subsequence condition. Also if $l < r$ or $l > 2p-r+1$, the total number of points on such diagonal are l and $2n-l+1$ respectively. Since B is also symmetric the antidiagonals $i+j=l$ will have $r-1$ entries if $l \equiv r (\bmod 2)$ and r is odd. Counting the nonzero entries on each antidiagonal on or above the main diagonal gives always exactly N in each case for the parity of r and p.

If B has less than N nonzero entries, then

$$\lim_{q \to 1} (1-q)^N \prod_{i:B[i,i]=1} \frac{q^{i+1}}{1-q^{i+s}} \prod_{i>j:B[i,j]=1} \frac{q^{i+j+2s}}{1-q^{i+j+2s}} =$$

$$\lim_{q \to 1} (1-q)^{N-|B|>0} \prod_{i:B[i,i]=1} \frac{q^{i+1}(1-q)}{1-q^{i+s}} \prod_{i>j:B[i,j]=1} \frac{q^{i+j+2s}(1-q)}{1-q^{i+j+2s}} = 0,$$

so such Bs won't contribute to the final answer.

Consider now only Bs with maximal possible number of nonzero entries (i.e. N), which forces them to have exactly r (or $r-1$) nonzero entries on every diagonal $i+j=l$ for $r<l<2p-r$ if and all entries in $i+j<r$ and $i+j>2p-r$.

If r is even, then there are no entries on the main diagonal when $r<l<2p-r$ and so there are $r/2$ terms on each diagonal $i+j=l$. Thus every such B contributes the same factor when evaluated at $x = (q^{1+s}, \ldots)$:

$$E_q(r,p,s) := \prod_{r-1<l<2p-r+1} \frac{q^{(l+2s)r/2}}{(1-q^{l+2s})^{r/2}} \prod_{l<r} \frac{q^{(l+4s+2p-l+1)\lfloor (l+1)/2 \rfloor}}{((1-q^{l+2s})(1-q^{2p-l+1+2s}))^{\lfloor (l+1)/2 \rfloor}}.$$

If r is odd, then the entries on the main diagonal will all be present with the rest being as in the even case with $r-1$, so the contribution is

$$E_q(r,p,s) := \prod_{\frac{r+1}{2} \leq i \leq p - \frac{r+1}{2}+1} \frac{q^{i+s}}{1-q^{i+s}} E_q(r-1,p,s).$$

Let now M be the number of such maximal Bs in $\mathcal{A}_0$. The final answer after taking the limit is $ME_1(r,p,s)$, where $E_1(r,p,s) = \lim_{q \to 1} (1-q)^N E_q(r,p,s)$, so if r is even

$$E_1(r,p,s) = \prod_{m-1<l<2n-m+1} \frac{1}{(l+2s)^{m/2}} \prod_{l<m} \frac{1}{((l+2s)(2n-l+1+2s))^{\lfloor (l+1)/2 \rfloor}}$$

and if r is odd $E_1(r,p,s) = \frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!} E_1(r-1,p,s)$.

In order to find M observe that the case of $s=0$ gives

$$\lim_{q \to 1} (1-q)^N \sum_{\lambda | l(\lambda) \leq r} s_\lambda(q^1, \ldots, q^p) = ME_1(r,p,0), \tag{14}$$

on one hand. On the other hand via the bijection ϕ we have that

$$\sum_{\lambda|l(\lambda\leq m} s_\lambda(q^1,\ldots,q^n) = \sum_T q^{\sum T[i,j]},$$

where the sum on the right goes over all shifted plane partitions T of shape $(p, p-1, \ldots, p-r+1)$. Multiplying by $(1-q)^N$ and taking the limit on the right hand side gives us by the inverse of proposition 4 $\frac{1}{N!}$ times the number of standard shifted tableaux of that shape. This number is well known and is $g_{(p,p-1,\ldots,p-r+1)} = \frac{N!}{\prod_u h_u}$, where the product runs over the hook lengths of all boxes on or above the main diagonal of (p^r, r^{p-r}). Putting all this together gives

$$ME_1(r,p,0) = \frac{g_{(n,n-1,\ldots,n-m+1)}}{N!}.$$

Solving for M we obtain the final answer as

$$\frac{g_{(n,n-1,\ldots,n-m+1)}}{N!}\frac{E_1(r,p,s)}{E_1(r,p,0)}.$$

□

We can now put all of this together and state

Theorem 2. The number of truncated straight tableaux of shape $(\underbrace{n-k, n-k+1, \ldots, n, \cdots, n}_{m})$ $(m > n)$is equal to

$$\left(mn - \binom{k+1}{2}\right)! \times \frac{f_{(n-k)^m}}{((n-k)m)!} \times \frac{g_{(n,n-1,\ldots,n-k)}}{((k+1)n - \binom{k}{2})!}$$
$$\times \prod_{k<l<2n-k}\left(\frac{l}{l+2s}\right)^{(k+1)/2} \prod_{l<k+1}\left(\frac{l(2n-l+1)}{(l+2s)(2n-l+1+2s)}\right)^{\lfloor (l+1)/2 \rfloor},$$

if k is odd and

$$\left(mn - \binom{k+1}{2}\right)! \times \frac{f_{(m-k)^n}}{((m-k)n)!} \times \frac{g_{(m,m-1,\ldots,m-k)}}{((k+1)m - \binom{k}{2})!}$$
$$\times \frac{(k/2+s)!}{(m-k/2+s)!}\prod_{k-1<l<2m-k+1}\left(\frac{l}{l+2s}\right)^{k/2} \prod_{l<k}\left(\frac{l(2m-l+1)}{(l+2s)(2m-l+1+2s)}\right)^{\lfloor (l+1)/2 \rfloor},$$

when k is even, where $s = n-k$.

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