CARTIER MODULES ON TORIC VARIETIES

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Abstract. Assume that X is an affine toric variety of characteristic $p > 0$. Let Δ be an effective toric $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p and let $\phi_\Delta : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ be the toric map corresponding to Δ. We identify all ideals I of $\mathcal{O}_X$ with $\phi_\Delta(F^e_* I) = I$ combinatorially and also in terms of a log resolution (giving us a version of these ideals which can be defined in characteristic zero). This is motivated by the fact that in the general, not necessarily toric setting, Blickle and Böckle have recently shown that the set of such ideals is always finite.

1. Introduction

Suppose that R is a ring of characteristic $p > 0$ and $F : R \to R$ is the Frobenius map, we always assume that F is a finite map. If $\phi : R \to R$ is a splitting of Frobenius, then there are finitely many ideals I such that $\phi(I) \subseteq I$, see [KM09] and [Sch09]. These ideals are called *ϕ-compatible* and are an interesting and useful collection of objects to study in their own right (they are closely related to the characteristic zero notion of "log canonical centers").

Much more generally, suppose that R is a reduced ring and $\phi : R \to R$ is an additive map that satisfies the condition $\phi(r^p.x) = r\phi(x)$ for all $r, x \in R$ (for example, a splitting of Frobenius). In [BB09], M. Blickle and G. Böckle generalize the above mentioned finiteness results and show that there are finitely many $I \subseteq R$ such that $\phi(I) = I$ (such ideals we call *ϕ-fixed*).

However, very few examples of the sets of ϕ-fixed ideals are known. In this paper we compute these ideals in the toric setting. In other words, $X = \operatorname{Spec} k[S]$ is an affine toric variety and $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ is a toric map. Here $S = M \cap \sigma^\vee$ for some lattice M and some cone σ with $\sigma^\vee$ in $M_\mathbb{R}$. Because ϕ is a toric map, we can write $\phi(\underline{\ \ }) = \phi_c(x^{-w} \cdot \underline{\ \ })$ for some $w \in M$ where ϕ_c is the canonical splitting[1] of $F^e : k[S] \to k[S]$. Our main result is as follows:

Main Theorem (Theorem 3.4). *Suppose that $\mathscr{F}$ is the set of all faces of $\sigma^\vee$ and for any $\tau \in \mathscr{F}$, set*

$$J_\tau = \langle x^v | v \in \textit{relative interior}(\frac{w}{1-p^e} + \tau) \cap S \rangle.$$

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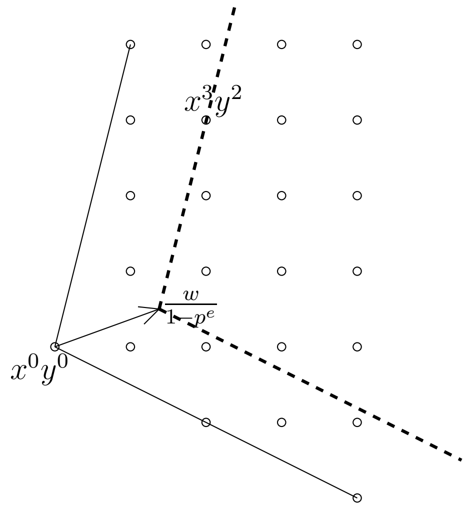
[1]The canonical splitting is the map which sends a monomial x^m to x^{m/p^e} if m/p^e is an integer, and otherwise sends x^m to zero.

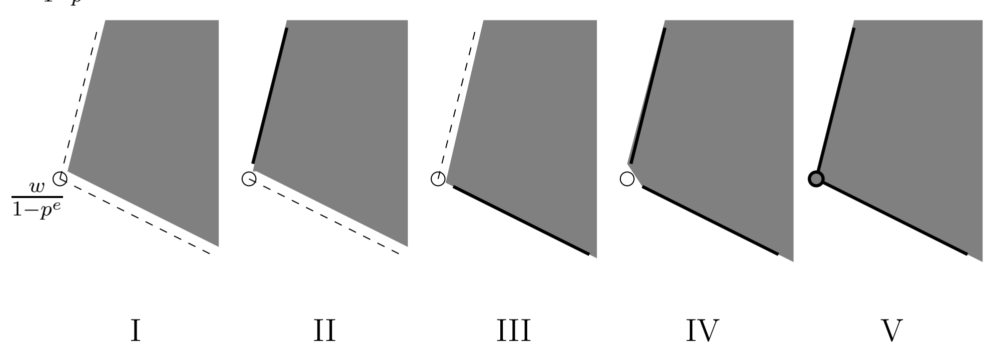
Then a non-zero ideal $I \subseteq S$ is ϕ-fixed if and only if there exists some subset $\mathscr{G} \subseteq \mathscr{F}$ such that

$$I = \sum_{\tau \in \mathscr{G}} \left(\sum_{\tau \subseteq \tau' \text{ in } \mathscr{F}} J_{\tau'} \right)$$

We illustrate this theorem with some figures. In the following diagram, the circles represent the monomials of the semi-group ring $k[S]$ and the solid lines represent the boundaries of $\sigma^{\vee}$. Given ϕ as above, we consider the vector $\frac{w}{1-p^e}$.



The ϕ-fixed ideals will each be generated by monomials contained in the interior or boundary of the above "dotted" region and we can explicitly identify them pictorially. Explicitly, as our main theorem says, each of the ϕ-fixed ideals will be generated by all monomials contained in one of the following shaded regions (in each region, the open circle corresponds to the point $\frac{w}{1-p^e}$).



While each of the ideals associated with these different bodies are potentially different, in many cases (depending on the particular w), they are the same.

As we have already noted, in the case that ϕ is a Frobenius splitting, the ϕ-fixed ideals are closely related to log canonical centers (a notion defined by using a resolution of singularities). It is thus natural to ask if these ideals I such that $\phi(I) = I$ are also related to a notion defined using a resolution of singularities. At least in the toric setting, we identify a class of ideals, defined using a resolution of singularities which coincide with the ϕ-fixed ideals I. Our main result on relating ϕ-fixed ideals and resolutions of singularities is the following.

Theorem (Theorem 5.7)**.** *Let X be an affine toric variety of characteristic $p > 0$ and let Δ be an effective torus-invariant $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ be the toric map corresponding to Δ. Then an ideal $I \subset \mathcal{O}_X$ is ϕ-fixed if and only if there exists a toric log resolution $\pi : X' \to X$ of (X, Δ) and an effective divisor E on X' with $\pi(E) \subset \operatorname{Supp} \Delta \cup \operatorname{Sing} X$ such that*

$$I = \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E \rceil) \textit{ for } 1 \gg \varepsilon > 0.$$

This result perhaps should not be unexpected. Very roughly speaking, ϕ extends to a map of fractional ideals $\phi' : F^e_* \mathscr{M} \to \mathscr{M}$ on X'. The sheaves $\mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E \rceil)$ for the various E correspond to those ϕ'-fixed subsheaves of $\mathscr{M}$ which pushdown to ideals on X. These ideals $\pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E \rceil)$ are also a generalization of the (intermediate) non-LC ideals introduced in [FST10].

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2. Preliminaries for fixed ideals

In this section, R is a normal F-finite domain of characteristic $p > 0$ which admits an additive map $\phi : R \to R$ satisfying the relation that $\phi(r^{p^e} x) = r\phi(x)$ for all $r, x \in R$. Such a ϕ is called a p^{-e}-linear map, which is nothing but an R-linear map $F^e_* R \to R$. Typical examples of such maps are the maps that split the Frobenius map $F : R \to R$.

Under this assumption, R fits into the theory of Cartier modules developed by Blickle and Böckle [BB09]. One of the main theorems in [BB09] guarantees that there are only finitely many ideals I of R satisfying $\phi(I) = I$. We will call such an I a *ϕ-fixed ideal* (according to the terminology of [Bli09], these ideals are also called *F-pure Cartier-submodules of* (R, ϕ)).

In the case where R is Gorenstein and local, there is a canonical p^{-e}-linear map $\phi : R \to R$. The smallest non-zero ideal J such that $\phi(J) = J$ is the (big) test ideal of R, [Sch09]. Furthermore, if ϕ is surjective, the largest such proper ideal is the splitting prime of Aberbach and Enescu, [AE05].

We mention the following results about the set of ϕ-fixed ideals which we will need.

Proposition 2.1. *Suppose that R is an F-finite domain*[2] *and that $\phi : F^e_* R \to R$ is an R-linear map.*

(i) *The set of ϕ-fixed ideals of R is closed under sum.*
(ii) *If ϕ is surjective, then the set of ϕ-fixed ideals is closed under intersection.*
(iii) *If ϕ is surjective, then any ϕ-fixed ideal is a radical ideal.*
(iv) *If R is a normal domain and ϕ corresponds to a $\mathbb{Q}$-divisor Δ_ϕ as in subsection 2.1 below, then the test ideal*[3] *$\tau(R, \Delta_\phi)$ is the unique smallest non-zero ϕ-fixed ideal of R.*
(v) *There are finitely many ϕ-fixed ideals.*

[2] We make this hypothesis only for simplicity, most of what follows below can be generalized outside of this setting with minimal work.

[3] The reader may take this to be the definition of the test ideal if they are not already familiar with it.

(vi) *For an element $d \in R$, define a new map $\psi(\_\_) = \phi(F^e_* d^{p^e-1} \cdot \_\_)$. Then an ideal $J \subseteq R$ is ϕ-fixed if and only if dJ is ψ-fixed.*

Proof. Part (i) is trivial from the definition. Part (ii) follows from the observation that if ϕ is surjective, then any ideal J satisfying the condition $\phi(F^e_* J) \subseteq J$ is automatically ϕ-fixed, and the set of these ideals is closed under intersection. Part (iii) is again easy. Part (iv) follows from [Sch09] and part (v) is as mentioned, one of the main results of [BB09].

To prove (vi), suppose first that J is ϕ-fixed so that $\phi(F^e_* J) = J$. Then $\psi(F^e_* dJ) = \phi(F^e_* d^{p^e-1} dJ) = d\phi(F^e_* J) = dJ$. The converse statement merely reverses this. $\square$

Remark 2.2. We will see in the toric setting that the set of ϕ-fixed ideals is closed under intersection for any ϕ. It would be interesting to discover if this holds more generally (it also holds if ϕ is surjective).

Remark 2.3. Suppose that R is a normal domain and $\phi : F^e_* R \to R$ is an R-linear map as in Proposition 2.1. One can always extend ϕ to a map $\bar{\phi} : F^e_* K(R) \to K(R)$ where $K(R)$ is the fraction field of R. While it is true that there are only finitely many ϕ-fixed ideals of R, in general there are infinitely many ϕ-fixed fraction ideals of $K(R)$. Remark 3.2 below explicitly provides such an example.

We now give a method for constructing ϕ-fixed ideals. While we will not use it directly, an analog of this result for ideals defined using resolution of singularities is the key observation which allows us to characterize ϕ-fixed ideals via a resolution, compare with Proposition 4.7. We first recall that given $\phi : F^e_* R \to R$, we can compose ϕ with itself to obtain a map $\phi^2 = \phi \circ F^e_* \phi : F^{2e}_* R \to R$ and similarly construct ϕ^n for any positive integer n. While *a priori*, we may have an infinite descending chain of ideals $\phi^n(F^{ne}_* R) \supseteq \phi^{n+1}(F^{(n+1)e}_* R) \supseteq \dots$; it is a theorem of Gabber that this chain eventually stabilizes, see [Gab04] (also see [Bli09] for a generalization and [HS77] for the local dual version in the geometric setting). In particular, this stable ideal which we denote by $\sigma(\phi) \subseteq R$ is automatically ϕ-fixed (and it is by definition, the largest ϕ-fixed ideal).

Proposition 2.4. *Suppose that R is a domain and $d \in R$ is a non-zero element. Then for any $n \gg 0$, if we define a map $\psi_n : F^{ne}_* R \to R$ by the formula $\psi_n(\_\_) = \phi^n(F^{ne}_* d \cdot \_\_)$, we have that $\sigma(\psi_n)$ is ϕ-fixed.*

Proof. We first claim that $\sigma(\psi_n) \subseteq \sigma(\psi_{n+1})$ (compare with [FST10, Proposition 14.11(1)]). To prove this claim, notice that for any $\alpha : F^e_* R \to R$, $\sigma(\alpha) = \sigma(\alpha^m)$. Thus, in order to show the desired containment for ψ, it suffices show that $\psi^{n+1}_n(F^{(n+1)ne}_* R) \subseteq \psi^n_{n+1}(F^{(n+1)ne}_* R)$. However, because

$$1 + p^{ne} + \dots + p^{n^2 e} = \frac{p^{(n+1)ne} - 1}{p^{ne} - 1} \geq \frac{p^{n(n+1)e} - 1}{p^{(n+1)e} - 1} = 1 + p^{(n+1)e} + \dots + p^{(n-1)(n+1)e}$$

we obtain that

$$\begin{aligned} \psi^{n+1}_n(F^{(n+1)ne}_* R) &= \phi^{n(n+1)}(F^{(n+1)ne}_* d^{1+p^{ne}+\dots+p^{n^2 e}} R) \\ &\subseteq \phi^{n(n+1)}(F^{(n+1)ne}_* d^{1+p^{(n+1)e}+\dots+p^{(n-1)(n+1)e}} R) = \psi^n_{n+1}(F^{(n+1)ne}_* R) \end{aligned}$$

which proves the claim.

We choose n which stabilizes this chain $\sigma(\psi_n) \subseteq \sigma(\psi_{n+1})$. Suppose now $m > 0$ is such that $\sigma(\psi_n) = \psi_n^m(F_*^{m(ne)} R)$. Then

$$\begin{aligned}
&\phi(\sigma(\psi_n)) \\
&= \phi(F_*^e \psi_n^{m+1}(F_*^{(m+1)ne} R)) \\
&= \psi_{n+1}\left(F_*^{(n+1)e}(\psi_n^m(F_*^{mne} R))\right) \\
&= \psi_{n+1}\left(F_*^{(n+1)e}\sigma(\psi_n)\right) \\
&= \psi_{n+1}\left(F_*^{(n+1)e}\sigma(\psi_{n+1})\right) \\
&= \sigma(\psi_{n+1}) \\
&= \sigma(\psi_n)
\end{aligned}$$

which proves the proposition. □

2.1. **The relation between ϕ and $\mathbb{Q}$-divisors.** Later, we will relate the ϕ-fixed ideals with the ideals come from a resolution of singularities in characteristic 0 (e.g. multiplier ideals, Fujino's non-LC ideal, and the ideals defining arbitrary unions of log canonical centers). This relation comes from a correspondence between pairs $(X = \operatorname{Spec} R, \Delta)$ and certain p^{-e}-linear maps $\phi : R \to R$ in the theory of F-singularities. The reader is referred to [Sch09] for a detailed account of this correspondence. We only explain a rough idea of it.

Suppose that (X, Δ) is a pair where X is a variety of finite type over an F-finite field k such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by $p > 0$. Further suppose that Δ is an effective $\mathbb{Q}$-divisor such that $(p^e - 1)\Delta$ is integral and $(p^e - 1)(K_X + \Delta)$ is Cartier for some e. Then there is a bijection of sets:

$$\left\{\begin{array}{l}\text{Effective } \mathbb{Q}\text{-divisors } \Delta \text{ on } X \text{ such} \\ \text{that } (p^e-1)(K_X+\Delta) \text{ is Cartier}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Line bundles } \mathscr{L} \text{ and non-zero} \\ \text{elements of } \operatorname{Hom}_{\mathcal{O}_X}(F_*^e\mathscr{L}, \mathcal{O}_X)\end{array}\right\} \Big/ \sim$$

The equivalence relation on the right side identifies two maps $\phi_1 : F_*^e \mathscr{L}_1 \to \mathcal{O}_X$ and $\phi_2 : F_*^e \mathscr{L}_2 \to \mathcal{O}_X$ if there is an isomorphism $\gamma : \mathscr{L}_1 \to \mathscr{L}_2$ and a commutative diagram:

$$\begin{array}{ccc}
F_*^e \mathscr{L}_1 & \xrightarrow{\phi_1} & \mathcal{O}_X \\
{\scriptstyle F_*^e\gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
F_*^e \mathscr{L}_2 & \xrightarrow{\phi_2} & \mathcal{O}_X
\end{array}$$

Given Δ, set $\mathscr{L} = \mathcal{O}_X((1-p^e)(K_X + \Delta))$. Then observe that

$$F_*^e \mathcal{O}_X((p^e-1)\Delta) \cong F_*^e \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{L}, \mathcal{O}_X((1-p^e)K_X)) \cong \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*^e \mathscr{L}, \mathcal{O}_X).$$

The choice of a section $\eta \in \mathcal{O}_X((p^e-1)\Delta)$ corresponding to $(p^e-1)\Delta$ thus gives a map $\phi_\Delta : F_*^e \mathscr{L} \to \mathcal{O}_X$. The choice depends on various isomorphisms selected, but this is harmless for our purposes.

For the converse direction, an element $\phi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*^e \mathscr{L}, \mathcal{O}_X) \cong F_*^e \mathscr{L}^{-1}((1-p^e)K_X)$ determines an $F_*^e \mathcal{O}_X$-linear map

$$F_*^e \mathcal{O}_X \xrightarrow{1 \mapsto \phi} \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*^e \mathscr{L}, \mathcal{O}_X) \xrightarrow{\sim} F_*^e \mathscr{L}^{-1}((1-p^e)K_X)$$

which corresponds to an effective Weil divisor D such that $\mathcal{O}_X(D) \cong \mathscr{L}^{-1}((1-p^e)K_X)$. Set $\Delta_\phi = \frac{1}{p^e-1} D$.

Remark 2.5. Explicitly, suppose $\phi : F^e_* R \to R$ is an R-linear map and $d \in R$. Define a new map $\psi(\_) = \phi(F^e_* d \cdot \_)$. Then $D_\psi = D_\phi + \frac{1}{p^e-1}\operatorname{div}(d)$.

Definition 2.6. Suppose that $\phi : F^e_* R \to R$ corresponds to a divisor Δ. Then we define $\sigma(R, \Delta)$ to be $\sigma(\phi)$ where $\sigma(\phi)$ is defined in the paragraph before Proposition 2.4.

We illustrate the above construction by the case where X is an affine toric variety (which recall has trivial Picard group, so that every line bundle is isomorphic to $\mathcal{O}_X$) Let M be a lattice and σ be a cone. Set $R = k[\sigma^\vee \cap M]$. Suppose Δ is an effective $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$ as above. Then we can write

$$(1 - p^e)(K_X + \Delta) = \operatorname{div}_X(x^w)$$

for some $w \in M$. Then a map ϕ_Δ corresponding to Δ can be expressed as

$$\phi_\Delta(\_) = \phi_c(x^{-w} \cdot \_)$$

where ϕ_c is the canonical splitting on R defined by

$$\phi_c(x^v) = \begin{cases} x^{\frac{v}{p^e}} & \text{if } \frac{v}{p^e} \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Let us explain these claims carefully since this identification is critical for what follows. Our first claim is that the map ϕ_c corresponds to the torus invariant divisor $\Delta_c = -K_X$. It is sufficient to show that ϕ_c fixes every height-one prime torus invariant ideal (which implies that Δ_{ϕ_c} contains each torus invariant divisor as a component) and does not fix any other height-one ideal (which implies that Δ_{ϕ_c} is toric). While both these statements are well-known to experts, we point out that the first statement is simply [Pay09, Proposition 3.2] while the second is a very special case of the proof of Lemma 3.1 below. So now suppose that Δ is a torus invariant divisor such that $(1 - p^e)(K_X + \Delta)$ is Cartier and thus is equal to $\operatorname{div}_X(x^w)$. Therefore

$$(1 - p^e)(K_X + \Delta) = \operatorname{div}_X(x^w) + 0 = \operatorname{div}_X(x^w) + (1 - p^e)(K_X + (-K_X))$$

Dividing through by $(1 - p^e)$ gives us $\Delta = (-K_X) + \frac{1}{p^e-1}\operatorname{div}_X(x^{-w})$. However, it is easy to see that given any map $\beta : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ and any element $d \in F^e_* \operatorname{Frac} R$, the map $\alpha(\_) = \beta(d \cdot \_)$, if it is indeed a map $\alpha : F^e_* \mathcal{O}_X \to \mathcal{O}_X$, has associated divisor $\Delta_\alpha = \Delta_\beta + \frac{1}{p^e-1}\operatorname{div}_X(d)$. In other words, the map ϕ_Δ as described above does indeed correspond to Δ.

Notation. For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^r d_j D_j$ such that D_j's are distinct prime Weil divisors, we define

$$\lceil D \rceil = \sum_{j=1}^r \lceil d_j \rceil D_j \text{ and } \lfloor D \rfloor = \sum_{j=1}^r \lfloor d_j \rfloor D_j,$$

where for each real number x, the round-up (resp. round-down) $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) denotes the integer defined by $x \le \lceil x \rceil < x + 1$ (resp. $x_1 < \lfloor x \rfloor \le x$). We also define

$$D^{\ge k} = \sum_{d_j \ge k} d_j D_j.$$

3. Fixed ideals on toric varieties

Throughout this section, we fix M to be a lattice and σ to be a cone, set $S = \sigma^\vee \cap M$.

Lemma 3.1. *Suppose that $X = \operatorname{Spec} R = \operatorname{Spec} k[S]$ is a toric variety and $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ is a toric p^{-e}-linear map with $\phi(\_\_) = \phi_c(x^{-w}\_\_)$ as before. Then all the ϕ-fixed ideals are generated by monomials. Furthermore, every ϕ-fixed ideal of R is contained inside $\langle x^u | u \in M \cap (\sigma^\vee + \frac{1}{1-p^e} w)\rangle$.*

Proof. Suppose that I is a ϕ-fixed ideal and choose $h \in I$. Write $h = \sum_i c_i x^{m_i}$ where the m_i are multi-indices. Our eventual goal will be to show that $x^{m_i} \in I$. First however, we will bound the m_i. For the convenience of proof, set $w' = \frac{1}{1-p^e} w$ so that $\phi(\_\_) = \phi_c(x^{(p^e-1)w'} \cdot \_\_)$.

Since $h \in I$, there exists an $h' \in I$ such that $\phi(h') = h$. Writing $h' = \sum c'_i x^{m'_i}$, we see that $x^{m_i} = \phi(x^{m'_i})$ for some m'_i. Thus, $m_i = \frac{1}{p^e}(m'_i + (p^e - 1)w')$. But we can perform the same procedure with m'_i in the place of m_i and obtain that $m'_i = \frac{1}{p^e}(m''_i + (p^e - 1)w')$. Plugging this back in we see that

$$m_i = \frac{1}{p^e}(\frac{1}{p^e}(m''_i + (p^e - 1)w') + (p^e - 1)w') = \frac{1}{p^{2e}} m''_i + (1 - \frac{1}{p^{2e}})w'$$

In general, we see that $m_i = \frac{1}{p^{ne}} m_i^{(n)} + (1 - \frac{1}{p^{ne}})w' \in \sigma + (1 - \frac{1}{p^{ne}})w'$. In particular, taking the limit, we see that $m_i \in \sigma + w'$, or in other words that $m_i - w' \in \sigma$.

Notice that

$$\begin{aligned}
&\phi^n(x^{(p^{ne}-1)(m_i - w')} x^{m_j}) \\
=&\phi_c^n(x^{w'(p^{ne}-1)} x^{(p^{ne}-1)(m_i - w')} x^{m_j}) \\
=&\phi_c^n(x^{(p^{ne}-1)m_i + m_j}) \\
=&x^{\frac{1}{p^{ne}}((p^{ne}-1)m_i + m_j)}
\end{aligned}$$

But $\frac{1}{p^{ne}}((p^{ne}-1)m_i + m_j) = m_i + \frac{m_j - m_i}{p^{ne}}$ and this only is an integer for $n \gg 0$ when $m_i = m_j$. In particular, we see that

$$\phi^n\left(x^{(p^{ne}-1)(m_i - w)} h\right) = \phi^n\left(x^{(p^{ne}-1)(m_i - w)} \sum_i c_i x^{m_i}\right) = c_i^{\frac{1}{p^{ne}}} x^{m_i}$$

for $n \gg 0$. Thus $x^{m_i} \in I$ as desired. $\square$

Remark 3.2. Lemma 3.1 only holds for ideals contained in R, it does not hold for fractional ideals. For example, consider the ring $R = k[x]$ with the canonical splitting $\phi_c : F_* R \to R$. Then the fractional ideal generated by $\frac{1}{x+1}$ is ϕ_c-fixed. Indeed, a generating set of this fractional ideal over R^p is $\left\{\frac{x^i}{x+1}\right\}_{0 \leq i \leq p-1}$. Then

$$\phi_c\left(\frac{x^i}{x+1}\right) = \phi_c\left(\frac{(x+1)^{p-1} x^i}{(x+1)^p}\right) = \frac{1}{x+1}\phi_c\left(x^{p-1+i} + \binom{p-1}{1} x^{p-1+i-1} + \cdots + x^i\right) = \frac{x^{\lceil i/p \rceil}}{x+1}$$

which proves the claim. Clearly the fractional ideal generated by $\frac{1}{x+1}$ is not toric. Of course, the same statement holds for the fractional ideal generated by $\frac{1}{x+\lambda}$ for any $\lambda \in k$ as well as for many other ideals.

The reason that the proof above does not apply to this fractional ideal is that $\frac{1}{x+1}$ cannot be written as a finite sum of monomials.

The next lemma is the key observation which lets us identify the ϕ-fixed ideals of a toric variety.

Lemma 3.3. *Fix a toric map $\phi : F^e_* R \to R$ with $\phi(\_\_) = \phi_c(x^{(p^e-1)w'} \cdot \_\_)$ as above and let I be a torus invariant (possibly fractional) ideal with $\phi(I) \subseteq I$. Suppose that $x^m \in I$ and also that m is in a face F of $\sigma^\vee + w'$. Then I contains every element x^n for $n \in$ relative interior$(F) \cap M$.*

Proof. Choose $n \in$ relative interior$(F) \cap M$. Notice that

$$\phi^l(x^{np^{le}-(p^{le}-1)w'}) = \phi^l_c(x^{(p^{le}-1)w'}x^{np^{le}-(p^{le}-1)w'}) = x^n.$$

We will show that $m+\sigma^\vee$ (which corresponds to the principal ideal generated by x^m) contains $np^{le} - (p^{le}-1)w'$ for l sufficiently large. This will complete the proof.

Notice that $np^{le} - (p^{le}-1)w' = (n-w')(p^{le}-1) + n$. Therefore, we simply have to show that $(n-w')(p^{le}-1) + n - m \in \sigma^\vee$ for l sufficiently large. Let us denote by G the face $F - w'$ of $\sigma^\vee$. Then $(n-w')$ is in the interior of this face and the claim will follow as soon as we see that $n-m$ is contained in the linear subspace $G \otimes \mathbb{R}$ generated by G. But $n - m = (n-w') - (m-w')$ is in $G \otimes \mathbb{R}$ as desired. $\square$

Now, we are ready to describe the ϕ-fixed ideals of a toric variety. Set

$$\mathscr{F} = \{ \text{ the set of all faces of } \sigma^\vee \}.$$

For any $\tau \in \mathscr{F}$, denote $J_\tau = \langle x^v | v \in \text{relative interior}(w'+\tau) \cap S \rangle$ and then set

$$I_\tau = \sum_{\tau \subseteq \tau' \text{ in } \mathscr{F}} J_{\tau'}.$$

Recall that $\phi(\_\_) = \phi_c(x^{-w} \cdot \_\_)$ and $w' = \frac{1}{1-p^e} w$.

Theorem 3.4. *With notation as above, $\phi(F^e_* I) = I$ if and only if $I = \sum_{\tau \in \mathscr{G}} I_\tau$ for some subset $\mathscr{G} \subseteq \mathscr{F}$.*

Proof. The only if part follows immediately from Lemma 3.1 and Lemma 3.3. Conversely, since the ϕ-fixed ideals are closed under sum, it suffices to show that $\phi(I_\tau) = I_\tau$ for any $\tau \in \mathscr{F}$. Let $x^m \in I_\tau$, say $m \in$ relative interior$(w'+\tau')$ for some $\tau' \supseteq \tau$, i.e. $x^m \in J_{\tau'}$. Then

- $\frac{m+(p^e-1)w'}{p^e} = w' + \frac{m-w'}{p^e} \in$ relative interior$(w'+\tau')$, and
- $p^e m - (p^e-1)w' = p^e(m-w') + w' \in$ relative interior$(w'+\tau')$.

Therefore, $\phi(x^m) = x^{\frac{m+(p^e-1)w'}{p^e}} \in J_{\tau'} \subseteq I_\tau$, and $x^m = \phi(x^{p^e m-(p^e-1)w'}) \in \phi(J_{\tau'}) \subseteq \phi(I_\tau)$. $\square$

Corollary 3.5. *In this toric setting above, the set of ideals I such that $\phi(F^e_* I) = I$ are closed under intersection.*

Remark 3.6. It follows from the previous result that in general, the set of ϕ-fixed ideals agrees with the set of $(\phi^2 = \phi \circ F^e_* \phi)$-fixed ideals. This is also true when ϕ is surjective, see for example [Sch09, Proposition 4.1], but it fails in general as the following example shows.

The following example was clarified to us in conversations with Manuel Blickle, also compare with [Die55] and [Bli01, Example 5.28].

Example 3.7. Suppose that $S = \overline{\mathbb{F}_5}[x, y, z]$ and $f = x^4 + y^4 + z^4$. Consider the map $\Phi_S : F_* S \to S$ which sends $x^4 y^4 z^4 \mapsto 1$ and all other monomials $x^i y^j z^k$ to zero (as long as $0 \leq i, j, k \leq 4 = 5 - 1$). Further consider the map $\phi : F_* S \to S$ defined by the rule $\phi(\_\_) = \Phi_S(f^4 \cdot \_\_)$. It is easy to verify that ϕ induces a map on $R = S/(f)$.

Set $\mathfrak{m} = (x, y, z)$. One can verify directly that both $\mathfrak{m}$ and $\mathfrak{m}^2 = (x^2, xy, xz, y^2, yz, z^2)$ are ϕ-fixed (as is (f)). Furthermore, $\phi(x) = 2x$, $\phi(y) = 2y$ and $\phi(z) = 2z$. Therefore, for any elements $a, b, c \in \overline{\mathbb{F}_5}$, $\phi(ax + by + cz) = 2a^{\frac{1}{5}} x + 2b^{\frac{1}{5}} y + 2c^{\frac{1}{5}} z$. Thus the ideal $\mathfrak{m}^2 + (x)$ is also ϕ-fixed, as is $\mathfrak{m}^2 + (ax + by + cz)$ for any elements $a, b, c \in \mathbb{F}_5 \subset \overline{\mathbb{F}_5}$ (this is still a finite set).

However, if one considers $\phi^2 = \phi \circ F_* \phi : F^2_* S \to S$, then one has $\phi^2(ax + by + cz) = 4a^{\frac{1}{25}} x + 4b^{\frac{1}{25}} y + 4c^{\frac{1}{25}} z$ so that $\mathfrak{m}^2 + (ax + by + cz)$ is ϕ^2-fixed for any elements $a, b, c \in \mathbb{F}_{5^2} \subseteq \overline{\mathbb{F}_5}$. Thus we have found ideals that are ϕ^2-fixed but not ϕ-fixed. Even more, continuing in this way, one obtains that the set of ϕ^n-fixed ideals can become arbitrarily large as n increases.

As we saw, toric varieties do not exhibit this phenomena.

4. Intermediate adjoint ideals

In this section, we develop the theory for a potential characteristic zero analog of the ϕ-fixed ideals studied in the previous section. We begin with the following situation.

Suppose that X is a normal variety over a field of characteristic zero and that Z is a closed subset of X such that $X \setminus Z$ is dense in X and that Δ is a (often effective) $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We consider log resolutions $\pi : X' \to X$ of X, Z, Δ. In this context, we can write $K_{X'} - \pi^*(K_X + \Delta) = \sum_i a_i E_i$ (as is standard, for any proper birational map $\pi : \widetilde{X} \to X$, we assume that $\pi_* K_{\widetilde{X}} = K_X$). For each such π, we have the following set of (possibly fractional) ideals:

$$\begin{aligned}\mathscr{I}^{\pi}_Z(X, \Delta) = \{ &\pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + E \rceil) | \\ &E \text{ is a reduced divisor satisfying } \pi(E) \subseteq Z \\ &\text{also such that each component } E_i \text{ of } E \text{ has associated } a_i \in \mathbb{Z}. \}\end{aligned}$$

Definition 4.1. We define

$$\mathscr{I}_Z(X, \Delta) := \bigcup_{\pi} \mathscr{I}^{\pi}_Z(X, \Delta).$$

We call this set the set of *intermediate adjoint ideals with respect to Z*.

We also give an alternative characterization of $\mathscr{I}^{\pi}_Z(X, \Delta)$.

Lemma 4.2. *The set of ideals $\mathscr{I}^{\pi}_Z(X, \Delta)$ is equal to*

$$\begin{aligned}\{ \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E \rceil) | &E \textit{ an effective divisor satisfying } \pi(E) \subseteq Z \\ &\varepsilon \textit{ satisfying } 1 \gg \varepsilon > 0. \}\end{aligned}$$

Proof. It is obvious. ∎

We also recall several related definitions; the *multiplier ideal* and the *maximal non-LC ideal*.

Definition 4.3. [Laz04a], [FST10], [Kaw98] Suppose that X is a normal variety over a field of characteristic zero, that Δ is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf and $t \geq 0$ is a real number. Further suppose that $\pi : X' \to X$ is a log resolution of $(X, \Delta, \mathfrak{a})$ where we set $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-G)$.

- The *multiplier ideal* $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ is defined to be
$$\pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) - tG \rceil)$$
This ideal is independent of the choice of resolution π. If $\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \mathcal{O}_X$, then $(X, \Delta, \mathfrak{a}^t)$ is said to have *Kawamata log terminal (or klt)* singularities.
- If E the reduced divisor on X' with support equal to $\mathrm{Supp}(G) \cup \mathrm{Supp}(\pi_*^{-1}\Delta) \cup \mathrm{exc}(\pi)$, then the *maximal non-LC-ideal* $\mathcal{J}'(X, \Delta, \mathfrak{a}^t)$ is defined to be
$$\pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) - tG + \varepsilon E \rceil)$$
where $\varepsilon > 0$ is arbitrarily small. This ideal is independent of the choice of resolution π assuming that $1 \gg \varepsilon > 0$. If $\mathcal{J}'(X, \Delta, \mathfrak{a}^t) = \mathcal{O}_X$, then $(X, \Delta, \mathfrak{a}^t)$ is said to have *log canonical (or lc)* singularities.
- If $W \subseteq X$ is an irreducible closed subvariety, then W is said to be a *log canonical center* (or *lc center*) of $(X, \Delta, \mathfrak{a}^t)$ if $(X, \Delta, \mathfrak{a}^t)$ is lc at the generic point of W but not klt at the generic point of W.

We prove a number of basic results about the sets $\mathscr{I}_Z^\pi(X, \Delta)$ and $\mathscr{I}_Z(X, \Delta)$.

Lemma 4.4. *Suppose that X, Z, Δ and $\pi : X' \to X$ are as above. Then the following hold:*

(i) *If $Y \subseteq Z$ is closed and π is also a log resolution of Y, then $\mathscr{I}_Y^\pi(X, \Delta) \subseteq \mathscr{I}_Z^\pi(X, \Delta)$.*

(ii) *For any Cartier divisor L such that π is a log resolution of Δ and $\Delta + L$, we have that*
$$\{\mathcal{O}_X(-L) \otimes I | I \in \mathscr{I}_Z^\pi(X, \Delta)\} = \mathscr{I}_Z^\pi(X, \Delta + L)$$

(iii) $\mathscr{I}_Z^\pi(X, \Delta) = \{\pi_* J | J \in \mathscr{I}_{\pi^{-1}(Z)}^{\mathrm{id}}(X', \pi^*(K_X + \Delta) - K_{X'})\}$.

(iv) *Suppose that $\pi' : X'' \to X'$ is a proper birational map such that $\pi \circ \pi' : X'' \to X$ is also a log resolution. Then $\mathscr{I}_Z^{\pi \circ \pi'}(X, \Delta) \supseteq \mathscr{I}_Z^\pi(X, \Delta)$.*

(v) *For any proper birational map $\sigma : Y \to X$ with Y normal, then*
$$\mathscr{I}_Z(X, \Delta) = \{\pi_* I | I \in \mathscr{I}_{\sigma^{-1}(Z)}(Y, -K_Y + \sigma^*(K_X + \Delta))\}.$$

(vi) *If Δ is effective, (X, Δ) is log canonical and Z contains the non-Kawamata-log terminal locus of (X, Δ) and also satisfies $Z \subseteq \mathrm{Sing}(X) \cup \mathrm{Supp}(\Delta)$, then $\mathscr{I}_Z(X, \Delta)$ is the set of ideals defining all unions of log canonical centers of (X, Δ).*

(vii) *If $Z = \mathrm{Sing}\, X \cup \mathrm{Supp}\, \Delta$ and $\Delta \geq 0$, then the unique smallest element of $\mathscr{I}_Z(X, \Delta)$ is the multiplier ideal $\mathcal{J}(X, \Delta)$ and the unique largest element is the maximal non-LC-ideal $\mathcal{J}'(X, \Delta)$ of* [FST10].

(viii) *The set $\mathscr{I}_Z(X, \Delta)$ is finite.*

(ix) *The set $\mathscr{I}_Z(X, \Delta)$ is closed under intersection.*

Proof. (i) is obvious. (ii) is a direct consequence of the projection formula. (iii) is clear from the definition. To prove (iv), notice that for any divisor E on X' such that $\mathrm{Supp}\, E \subseteq \pi^{-1}(Z)$, we have that
$$\mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E \rceil) = \pi'_* \mathcal{O}_{X''}(\lceil K_{X''} - \pi'^* \pi^*(K_X + \Delta) + \varepsilon \pi'^* E) \rceil)$$
by [Laz04b, Lemma 9.2.19]. The result then immediately follows from Lemma 4.2. For (v), simply notice that by (iv), we may restrict ourselves to $\pi : X' \to X$ which factor through σ. (vi) is obvious from the definition. (vii) is the definition of the multiplier ideal and maximal non-LC-ideal respectively.

We now prove (viii). By (v) it is sufficient to show that $\mathscr{I}_{\pi^{-1}(Z)}(X', -K_{X'} + \pi^*(K_X + \Delta))$ is finite. But using (ii), we can reduce to the case of $\mathscr{I}_{\pi^{-1}(Z)}(X', \Delta')$ where $\Delta' \geq 0$ and (X', Δ') is log canonical and furthermore assume that $\pi^{-1}(Z) \subseteq \operatorname{Supp}(\Delta)$. The result then follows from (vi) since the set of log canonical centers of a log canonical pair is finite.

For (ix), suppose that I and J are contained in $\mathscr{I}_Z(X, \Delta)$. By (iv), we may realize both ideals as pushforward from a single log resolution, say $I = \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E \rceil)$ and $J = \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon F \rceil)$ for some sufficiently small positive ε where E and F are reduced divisors with image contained in Z. Let $E \wedge F$ denote the sum of the common components of E and F, i.e., we can write $E = E \wedge F + E'$ and $F = E \wedge F + F'$ where $\operatorname{Supp}(E \wedge F)$, $\operatorname{Supp}(E')$ and $\operatorname{Supp}(F')$ are componentwise disjoint. Note that $E \wedge F$ is also a reduced divisor with $\pi(E \wedge F) \subseteq Z$. We will show that $I \cap J = \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon(E \wedge F) \rceil)$.

For the $\subseteq$ direction, suppose that $g \in I \cap J$, then $\operatorname{div}_{X'}(g) + \lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E \rceil \geq 0$ and $\operatorname{div}_{X'}(g) + \lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon F \rceil \geq 0$. Since $\operatorname{Supp}(E \wedge F)$, $\operatorname{Supp}(E')$ and $\operatorname{Supp}(F')$ are componentwise disjoint, it then immediately follows that

$$\operatorname{div}_{X'}(g) + \lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon(E \wedge F) \rceil \geq 0.$$

Conversely, suppose that $g \in \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon(E \wedge F) \rceil)$. Thus $\operatorname{div}_{X'}(g) + \lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon(E \wedge F) \rceil \geq 0$ which immediately implies that $g \in I$ and $g \in J$ as well since $E \wedge F \leq E, F$. □

Question 4.5. Is the set $\mathscr{I}_Z(X, \Delta)$ is closed under sums as well? In the case that (X, Δ) is log canonical, this follows from the main theorem of [Amb98] as described in [Fuj09, Theorem 3.46].

Remark 4.6. If one instead ignores the restriction on Z and considers the set

$$\begin{aligned} \mathscr{I}^{\pi}(X, \Delta) = \{ \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + E \rceil) | E & \text{ a reduced divisor} \\ & \text{such that each component } E_i \text{ of } E \text{ has associated } a_i \in \mathbb{Z}. \} \end{aligned}$$

then it is fairly easy to see that the set has infinitely many elements (all but finitely many being fractional ideals). Compare with Remarks 2.3 and 3.2 in the characteristic $p > 0$ context.

We now describe a method for producing intermediate adjoint ideals which will be crucial in the next section.

Proposition 4.7. *Suppose that* $(X = \operatorname{Spec} R, \Delta)$ *is a pair, that* $Z = \operatorname{Sing} X \cup \operatorname{Supp} \Delta$ *and* $\Delta \geq 0$. *Then for any* $0 \neq \mathfrak{a} \subseteq R$, $\mathcal{J}'(X, \Delta, \mathfrak{a}^{\delta}) \in \mathscr{I}_Z(X, \Delta)$ *for sufficiently small* $\delta > 0$.

Proof. Choose a log resolution $\pi : X' \to X$ of $(X, \Delta, \mathfrak{a}^{\delta})$ such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-G)$. Set

$$E = \operatorname{Supp}(\pi^*(K_X + \Delta) + \delta G - K_{X'})^{\geq 1}.$$

Note that $E = \operatorname{Supp}(\pi^*(K_X + \Delta) - K_{X'})^{\geq 1}$ for sufficiently small $\delta > 0$, and hence $\pi(E) \subseteq Z$. By definition,

$$\mathcal{J}'(X, \Delta, \mathfrak{a}^{\delta}) = \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) - \delta G + \varepsilon E \rceil)$$

where $0 < \varepsilon \ll 1$ and $\varepsilon \ll \delta$. Set F to be the reduced divisor made up of all components of E that do not appear in $\operatorname{Supp}(G)$ (it might be that $F = 0$). Then we claim that $\pi(F) \subseteq Z$

(which is obvious) and also that

$$\lceil K_{X'} - \pi^*(K_X + \Delta) - \delta G + \varepsilon E \rceil = \lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon F \rceil \tag{1}$$

again for $\delta \gg \varepsilon > 0$ sufficiently small. This will complete the proof. To see this second claim, consider a_i, the E_i-coefficient of $K_{X'} - \pi^*(K_X + \Delta)$, where on E_i is a component of E. There are two cases, if E_i is a component of G, then before rounding up, the E_i-coefficient of the left side of Equation 1 is $a_i - \delta + \varepsilon$ while the E_i-coefficient of the right side is a_i, these have the same round-up since $1 \gg \delta \gg \varepsilon > 0$. On the other hand, if E_i does not appear in G, then along E_i, Equation 1 is already an equality and there is nothing to show. □

Example 4.8. Suppose that $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ and $\Delta = \operatorname{div}_X(xy)$. In this case, (X, Δ) is log canonical and the log canonical centers are $V(x), V(y), V(x, y)$.

5. Intermediate adjoint ideals on toric varieties

In this section we will work only with toric varieties. Throughout this section, we fix M to be a lattice and σ to be a cone, set $S = \sigma^\vee \cap M$. Let X be the affine toric variety $\operatorname{Spec} k[S]$ and suppose that $\Delta \geq 0$ is a torus-invariant $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Set $Z = \operatorname{Supp} \Delta \cup \operatorname{Sing} X$, in this section we describe the set of ideals $\mathscr{I}_Z(X, \Delta)$.

Lemma 5.1. *The ideals of $\mathscr{I}_Z(X, \Delta)$ are torus invariant.*

Proof. First suppose that $\pi : X' \to X$ is a toric log resolution. By Lemma 4.4(v), it is enough to show that $\mathscr{I}_{\pi^{-1}Z}(X', -K_{X'} + \sigma^*(K_X + \Delta))$ is a collection of torus invariant (fractional) ideals since the pushforward of a torus invariant fractional ideal is still torus invariant. However, using the trick of 4.4(ii), it is easy to see that, up to twisting by a line bundle on, $\mathscr{I}_{\pi^{-1}Z}(X', -K_{X'} + \sigma^*(K_X + \Delta))$ is just a finite collection of log canonical centers of a torus invariant pair on X'. This completes the proof. □

Remark 5.2. In the toric setting and any characteristic, we have a sufficiently good theory of resolution of singularities to generalize the notions from Section 4 (alternately, one could define similar notions by considering all proper birational maps instead of log resolutions). Therefore, since in this section we work only with toric varieties, we can now work in any characteristic (either characteristic $p > 0$ or characteristic zero).

We now describe the non-LC ideal sheaf $\mathcal{J}'(X, \Delta, \mathfrak{a}^t)$ in the toric language. First choose $l \in \mathbb{N}$ such that $l(K_X + \Delta) = \operatorname{div}_X(x^m)$ for some $m \in M$.

Proposition 5.3.

$$\mathcal{J}'(X, \Delta, \mathfrak{a}^t) = \langle x^v \mid v - \frac{m}{l} \in t \operatorname{Newt}(\mathfrak{a}) \rangle$$

where $\operatorname{Newt}(\mathfrak{a})$ *is the Newton polygon of* $\mathfrak{a}$*. In particular, the ideal* $\mathcal{J}'(X, \Delta)$ *is generated by the monomials* x^v *such that* $v \in \sigma^\vee$ *and* $v \geq \frac{m}{l}$*.*

Proof. Write $K_{X'} - \pi^*(K_X + \Delta) - tG = \sum a_i E_i$ and set E to be the reduced divisor whose components are made up of E_i such that the corresponding $a_i \leq -1$. It immediately follows that $x^v \in \mathcal{J}'(X, \Delta, \mathfrak{a}^t)$ if and only if

$$\operatorname{div}_{X'}(x^v) + \lceil K_{X'} - \pi^*(K_X + \Delta) - tG + \varepsilon E \rceil \geq 0.$$

We claim that this is true if and only if $\operatorname{div}_{X'}(x^v) - \pi^*(K_X + \Delta) - tG \geq 0$ and we reason as follows.

Let D_i be a torus invariant divisor and let c_i be the coefficient of $\operatorname{div}_{X'}(x^v)$ on D_i. Further set b_i to be the coefficient of $-\pi^*(K_X+\Delta)-tG$ along D_i. It follows that the coefficient of $\lceil K_{X'}-\pi^*(K_X+\Delta)-tG+\varepsilon E\rceil$ along D_i is $\lceil -(1-\varepsilon)+b_i\rceil$. Thus our claim is simply that $c_i \geq -b_i$ if and only if $c_i \geq \lfloor -b_i+(1-\varepsilon)\rfloor$ for $1 \gg \varepsilon > 0$. But this is obvious.

Notice that the integral closure $\bar{\mathfrak{a}} = \pi_*\mathcal{O}_{X'}(-G) = \langle x^v \mid v \in \operatorname{Newt}(\mathfrak{a})\rangle$. If $t>0$ is a rational number, we see that $x^v \in \mathcal{J}'(X,\Delta,\mathfrak{a}^t)$ if and only if $v - \frac{m}{l} \in t\operatorname{Newt}(\mathfrak{a})$ as stated. The general case is achieved by taking a limit. $\square$

We now transition to characteristic $p>0$ and show that $\mathscr{I}_Z(X,\Delta)$ coincides with the ϕ_Δ-fixed ideals. Choose $w \in M$ and $e \in \mathbb{Z}_{\geq 0}$ so that $\operatorname{div}_X(x^w) = (1-p^e)(K_X+\Delta)$.

Theorem 5.4. *Suppose that (X,Δ) is as above and that $\phi : F^e_*\mathcal{O}_X \to \mathcal{O}_X$ corresponds to Δ. Then every ϕ-fixed ideal appears in the set $\mathscr{I}_Z(X,\Delta)$.*

Proof. Using the notation in Theorem 3.4, we first show that $I_\tau \in \mathscr{I}_Z(X,\Delta)$ for each $\tau \in \mathscr{F}$. Take $a \in$ relative interior$(\tau)\cap S$. By Proposition 4.7, $\mathcal{J}'(X,\Delta+\frac{1}{n}\operatorname{div}_X(x^a)) \in \mathscr{I}_Z(X,\Delta)$ for sufficiently large $n \in \mathbb{N}$. Notice that

$$n(p^e-1)(K_X+\Delta+\frac{1}{n}\operatorname{div}_X(x^a)) = -n\operatorname{div}_X(x^w)+(p^e-1)\operatorname{div}_X(x^a) = \operatorname{div}_X(x^{-nw+(p^e-1)a}).$$

By Proposition 5.3, $\mathcal{J}'(X,\Delta+\frac{1}{n}\operatorname{div}_X(x^a))$ is generated by the monomials x^v such that

$$v \geq \frac{-nw+(p^e-1)a}{(p^e-1)n} = \frac{1}{1-p^e}w + \frac{a}{n}.$$

Therefore, $\mathcal{J}'(X,\Delta+\frac{1}{n}\operatorname{div}_X(x^a))$ coincides with the ideal I_τ.

To complete the proof, we must show that for any subset $\mathscr{G}$ of $\mathscr{F}$, the ideal

$$\sum_{\tau\in\mathscr{G}} I_\tau \in \mathscr{I}_Z(X,\Delta).$$

For each $\tau \in \mathscr{G}$, pick a lattice point a_τ in the relative interior of τ and consider the ideal $\mathfrak{a} = \langle x^{a_\tau} \mid \tau \in \mathscr{G}\rangle$ with integral closure $\bar{\mathfrak{a}}$. We claim that for $0<\delta \ll 1$,

$$\sum_{\tau\in\mathscr{G}} I_\tau = \mathcal{J}'(X,\Delta,\mathfrak{a}^\delta) = \mathcal{J}'(X,\Delta,\bar{\mathfrak{a}}^\delta).$$

Choose a toric log resolution $\pi : X' \to X$ for $X,\Delta,\mathfrak{a}$ so that $\mathfrak{a}\cdot\mathcal{O}_{X'} = \mathcal{O}_{X'}(-G)$ for some Cartier divisor G. Since for each $\tau \in \mathscr{G}$, $\operatorname{div}_{X'}(x^{a_\tau}) \geq G$, we have

$$\begin{aligned} I_\tau &= \mathcal{J}'(X,\Delta+\delta\operatorname{div}_X(x^{a_\tau})) \\ &= \pi_*\mathcal{O}_{X'}(\lceil K_{X'}-\pi^*(K_X+\Delta)-\delta\operatorname{div}_{X'}(x^{a_\tau})+\varepsilon E\rceil) \\ &\subseteq \pi_*\mathcal{O}_{X'}(\lceil K_{X'}-\pi^*(K_X+\Delta)-\delta G+\varepsilon E\rceil) \\ &= \mathcal{J}'(X,\Delta,\mathfrak{a}^\delta). \end{aligned}$$

Given a face γ of σ, set $T_\gamma = \{x^v \mid v \in \text{relative interior}(w'+\gamma)\cap S\}$. It remains to show that if $\mathcal{J}'(X,\Delta,\mathfrak{a}^\delta)\cap T_\gamma \neq \emptyset$ for some face γ of σ then γ contains some face $\tau \in \mathscr{G}$.
Let x^v be a nonzero element in $\mathcal{J}'(X,\Delta,\mathfrak{a}^\delta)\cap T_\gamma$, then $v+\frac{w}{p^e-1} \in \gamma$ (note that $w' = -\frac{1}{p^e-1}w$ in the definition of T_γ) and by Proposition 5.3 $x^{n'v+n'\frac{1}{p^e-1}w} \in \bar{\mathfrak{a}}$ for some $n' \in \mathbb{N}$, Hence $(x^{n'v+n'\frac{1}{p^e-1}w})^{n''} \in \mathfrak{a}$ for some $n'' \in \mathbb{N}$. Since $n''n'(v+\frac{1}{p^e-1}w)$ is also in γ, it follows from the construction of $\mathfrak{a}$ that γ contains a face $\tau \in \mathscr{G}$, as desired. $\square$

Our next goal is to prove that every element of $\mathscr{I}_Z(X, \Delta)$ is ϕ_Δ-fixed. We will need the following in order to do this.

Given an effective $\mathbb{Q}$-divisor Δ such that $(1-p^e)(K_X + \Delta)$ is Cartier, we have the corresponding p^{-e}-linear morphism $\phi_\Delta : F^e_* \mathscr{L} \to \mathcal{O}_X$ for some line bundle $\mathscr{L}$ as described in Section 2. However, even if Δ is not effective, we can still do something similar. Choose a Cartier effective divisor D such that $\Delta + D$ is effective. This produces a map $\phi_{\Delta+D} : F^e_* \mathscr{L} \to \mathcal{O}_X$ where $\mathscr{L} = \mathcal{O}_X((1-p^e)(K_X + \Delta + D))$. Twisting by D and applying the projection formula gives us a map

$$\phi' : F^e_* \mathcal{O}_X((1-p^e)(K_X + \Delta + D) + p^e D) \to \mathcal{O}_X(D)$$

which thus induces

$$\phi_\Delta : F^e_* \mathcal{O}_X((1-p^e)(K_X + \Delta)) \subseteq F^e_* \mathcal{O}_X((1-p^e)(K_X + \Delta) + D) \to \mathcal{O}_X(D) \subseteq K(X),$$

where $K(X)$ is the fraction field of X. This is the map which we declare to be the map corresponding to Δ.

Lemma 5.5. *The composition defining ϕ_Δ above is independent of the choice of D. Explicitly, it is uniquely determined up to the equivalence on maps described in Subsection 2.1.*

Proof. It is sufficient to show this in codimension 1 because the sheaves involved are reflexive, and so we assume that X is the spectrum of a DVR R with uniformizer $r \in R$. We may write $\Delta = \frac{a}{p^e-1} \operatorname{div}(r)$ for some integer a and $D = b \operatorname{div}(r)$ for some integer b where $b + \frac{a}{p^e-1} \geq 0$. Consider the map $\Phi_R : F^e_* R \to R$ which generates $\operatorname{Hom}_R(F^e_* R, R)$ and thus corresponds to the divisor 0. Thus the map $\phi_{\Delta+D}$ sends an element z to $\Phi_R(r^{a+b(p^e-1)} z)$. The map ϕ' above is then defined as by the rule sending z/r^{bp^e} to $\phi_{\Delta+D}(z)/r^b = \Phi_R(r^{a+b(p^e-1)} z)/r^b$. We thus obtain a map ϕ'_Δ, defined at the level of fraction fields, by sending z/r^{p^e} to $\Phi_R(r^{a+b(p^e-1)} z)/r^b$. Making the identification of $\psi : R \cong \langle r^{-p^e+b} \rangle_R$ (corresponding to the inclusion map in the composition defining ϕ_Δ noting that $\mathcal{O}_X((1-p^e)(K_X + \Delta)) \cong \mathcal{O}_X$) gives us a map $\phi_\Delta = \phi'_\Delta \circ \psi : F^e_* R \to K(R)$ which sends $z \in R$ to $z/r^{p^e-b} = zr^b/r^{p^e}$ and then to

$$\Phi_R(r^{a+b(p^e-1)} z r^b)/r^b = \Phi_R(r^{a+bp^e} z)/r^b.$$

Which is clearly independent of the choice of b (and thus of D). $\square$

We now prove a lemma which allows us to identify certain Φ-fixed ideals when Φ corresponds to a simple normal crossings divisor.

Lemma 5.6. *Suppose that X is a regular variety, G is a reduced simple normal crossings divisor on X and that $\phi : F^e_* \mathscr{L} \to K(X)$ is a map such that div_ϕ has support contained in G. Then for any effective divisor E with support also contained in G and any $\varepsilon > 0$ sufficiently small, $\mathcal{O}_X(\lceil -\operatorname{div}_\phi + \varepsilon E \rceil)$ is a ϕ-fixed fractional ideal.*

Proof. The statement is local so we may assume that X is the spectrum of a regular local ring. By the method of Proposition 2.1(vi), it is harmless to assume that the image of ϕ is inside $\mathcal{O}_X$ and thus that div_ϕ is effective. Furthermore, using again Proposition 2.1(vi), we can assume that $\operatorname{div}_\phi = \sum b_i D_i$ is a simple normal crossings divisor with coefficients $0 < b_i \leq 1$ and in particular, the support of div_ϕ is equal to the support of G. Therefore, by Proposition 2.4 and Remark 2.5 it suffices to show that

$$\mathcal{O}_X(\lceil -\operatorname{div}_\phi + \varepsilon E \rceil) = \sigma(X, \operatorname{div}_\phi + \varepsilon(G - E))$$

where ε is chosen sufficiently small and can be written with denominator not divisible by p. Let B be the reduced divisor made up of components of $G - E$ such that the coefficient of $\mathrm{div}_\phi = 1$ (notice, we have just arranged things so that $\mathcal{O}_X(\lceil -\mathrm{div}_\phi + \varepsilon E\rceil) = \mathcal{O}_X(-B)$). Then $\sigma(X, \mathrm{div}_\phi + \varepsilon(G-E)) = \sigma(X, \mathrm{div}_\phi + \varepsilon(G-E) - B + B)$ where we have arranged things such that for each term of B, $\mathrm{div}_\phi + \varepsilon(G-E) - B$ is positive. We then claim that

$$\mathcal{O}_X(-B) \cdot \sigma(X, \mathrm{div}_\phi + \varepsilon(G-E) - B) = \sigma(X, \mathrm{div}_\phi + \varepsilon(G-E) - B + B).$$

To show this, one can either mimic the argument of [FST10, Proposition 14.11(6)] or the argument of Proposition 2.1(vi). However, $(X, \mathrm{div}_\phi + \varepsilon(G-E) - B)$ is F-pure (the divisor is simple normal crossings with coefficients less than or equal to 1). This implies that $\sigma(X, \mathrm{div}_\phi + \varepsilon(G-E) - B) = \mathcal{O}_X$ which completes the proof. $\square$

Theorem 5.7. *Suppose that (X, Δ) is as above and that $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ corresponds to Δ as in Section 2. If $J \in \mathscr{I}_Z(X, \Delta)$, then J is ϕ-fixed.*

Proof. Suppose that $\pi : X' \to X$ is a toric resolution such that $J = \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E\rceil)$. The map ϕ induces a map $\phi' : F^e_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta)\rceil) \to K(X')$, see for example [HW02, Proof #2 of Theorem 3.3] and [Sch10, Theorem 6.7]. It then follows from Lemma 5.6 that $\mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E\rceil)$ is a ϕ-fixed fractional $\mathcal{O}_{X'}$-ideal sheaf (here we extend ϕ to $K(X) = K(X')$ in the natural way). We need to show that $J = \pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E\rceil)$ is also ϕ-fixed. Thus, suppose that $x^m \in J$. It follows that x^m is a section of each affine chart of $\mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E\rceil)$. Since these are ϕ-fixed, there is an element f_j in each such affine chart which is sent to x^m. However, this is a fractional monomial ideal on each affine chart, and so we may assume that each f_j is a monomial. However, there is exactly one monomial in $K(X')$ which is sent to x^m via ϕ, which implies that all the f_j's coincide in a monomial which we will call f. Therefore $f \in J$ as well which completes the proof. $\square$

An alternate proof of Proposition 5.7. Let $\mathcal{J}(E)$ denote $\pi_* \mathcal{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) + \varepsilon E\rceil)$. Then it follows immediately from [Sch10, Theorem 6.7] that $\mathcal{J}(E)$ is ϕ_Δ-stable, i.e. $\phi_\Delta(\mathcal{J}(E)) \subseteq \mathcal{J}(E)$. It remains to prove that $\phi_\Delta|_{\mathcal{J}(E)}$ is surjective.

Assume that $(1-p^e)\pi^*(K_X + \Delta) = \mathrm{div}_{X'}(x^w)$ and hence $\phi_\Delta(x^u) = x^{\frac{u-w}{p^e}}$ if $\frac{u-w}{p^e} \in M$ and 0 otherwise. Given any $x^u \in \mathcal{J}(E)$, it is clear that $\phi_\Delta(x^{p^e u + w}) = x^u$. Therefore, to prove that $\phi_\Delta|_{\mathcal{J}(E)}$ is surjective, it suffices to show that $x^{p^e u + w}$ is contained in $\mathcal{J}(E)$.

write $\frac{1}{1-p^e} \mathrm{div}_{X'}(x^w) = \pi^*(K_X + \Delta) = \sum_i a_i D_i$, where $\{D_i\}$ is the set of toric prime Weil divisors on X' and let v_i be the first lattice point on the ray associated with D_i. Hence

$$\langle w, v_i \rangle = (1-p^e)a_i.$$

Since $x^u \in \mathcal{J}(E)$, we have

$$\mathrm{div}_{X'}(x^u) + \lceil \sum_i (-1 - a_i) D_i + \varepsilon E \rceil \geq 0.$$

Therefore, we have

$$\langle u, v_i \rangle + \lceil -1 - a_i + \varepsilon \rceil \geq 0, \text{ when } D_i \in \mathrm{Supp}(E)$$

and

$$\langle u, v_i \rangle + \lceil -1 - a_i \rceil \geq 0, \text{ when } D_i \notin \mathrm{Supp}(E)$$

It is not hard to see from both inequalities that

$$\langle u, v_i\rangle \geq a_i.$$

Now

$$\begin{aligned}
&\sum_i \langle p^e u + w, v_i\rangle D_i + \lceil \sum_i (-1-a_i)D_i + \varepsilon E\rceil \\
&= \sum_i \langle u, v_i\rangle D_i + \lceil \sum_i (-1-a_i)D_i + \varepsilon E\rceil + (p^e-1)\sum_i \langle u, v_i\rangle D_i + \sum_i \langle w, v_i\rangle D_i \\
&= \operatorname{div}_{X'}(x^u) + \lceil \sum_i (-1-a_i)D_i + \varepsilon E\rceil + (p^e-1)\sum_i (\langle u, v_i\rangle - a_i)D_i \geq 0.
\end{aligned}$$

Therefore, $x^{p^e u + w} \in \mathcal{J}(E)$. This finishes the proof. □

Corollary 5.8. *Suppose that X is an affine toric variety in characteristic zero or characteristic $p > 0$ and that Δ is an effective toric $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier so that $l(K_X + \Delta) = \operatorname{div}(x^m)$. Set $\mathscr{F}$ to be the set of all faces of $\sigma^\vee$ and for any $\tau \in \mathscr{F}$, set*

$$K_\tau = \langle x^v | v \in \textit{relative interior}(\frac{m}{l} + \tau) \cap S\rangle.$$

Then a non-zero ideal $I \subseteq S$ is in $\mathscr{J}_Z(X, \Delta)$ if and only if there exists some subset $\mathscr{G} \subseteq \mathscr{F}$ such that

$$I = \sum_{\tau \in \mathscr{G}} \left(\sum_{\tau \subseteq \tau' \text{ in } \mathscr{F}} K_{\tau'} \right)$$

Proof. The statement in characteristic zero reduces to the characteristic $p > 0$ statement from reduction to characteristic $p \gg 0$. For the characteristic $p > 0$ statement, suppose that $l(K_X + \Delta) = \operatorname{div}(x^m)$ in characteristic $p \gg 0$. Then if $\phi_\Delta(\_\_) = \phi_c(x^{-w} \cdot \_\_)$, we already saw that $\Delta = (-K_X) + \frac{1}{p^e-1}\operatorname{div}_X(x^{-w})$ in Section 2, thus $\operatorname{div}(x^{-w}) = (p^e-1)(K_X + \Delta)$ so that $\frac{w}{1-p^e} = \frac{m}{l}$ and the result follows by Theorems 3.4, 5.4, and 5.7. □

Corollary 5.9. *Suppose that X is an affine toric variety in characteristic zero or characteristic $p > 0$ and that Δ is an effective toric $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then the elements of $\mathscr{J}_Z(X, \Delta)$ are closed under sum.*

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