UNIFORMIZATION OF SIMPLY CONNECTED FINITE TYPE LOG-RIEMANN SURFACES

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Abstract. We consider simply connected log-Riemann surfaces with a finite number of infinite order ramification points. We prove that these surfaces are parabolic with uniformizations given by entire functions of the form $F(z) = \int Q(z)e^{P(z)}\, dz$ where P, Q are polynomials of degrees equal to the number of infinite and finite order ramification points respectively.

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1. Introduction

In [BPM10a] we defined the notion of log-Riemann surface, as a Riemann surface $\mathcal{S}$ equipped with a local diffeomorphism $\pi : \mathcal{S} \to \mathbb{C}$ such that the set of points $\mathcal{R}$ added in the completion $\mathcal{S}^* = \mathcal{S} \sqcup \mathcal{R}$ of $\mathcal{S}$ with respect to the flat metric on $\mathcal{S}$ induced by π is discrete. The mapping π extends to the points $p \in \mathcal{R}$, and is a covering of a punctured neighbourhood of p onto a punctured disk in $\mathbb{C}$; the point p is called a ramification point of $\mathcal{S}$ of order equal to the degree of the covering π near p. The finite order ramification points may be added to $\mathcal{S}$ to give a Riemann surface $\mathcal{S}^{\times}$, called the finite completion of $\mathcal{S}$. In this article we are interested in log-Riemann surfaces of *finite type*, i.e. those with finitely many ramification points and finitely generated fundamental group, in particular simply connected log-Riemann surfaces of finite type. We prove the following:

Theorem 1.1. *Let $\mathcal{S}$ be a log-Riemann surface with $d_1 < +\infty$ infinite order ramification points and $d_2 < +\infty$ finite order ramification points (counted with multiplicity), such that the finite completion $\mathcal{S}^{\times}$ is simply connected. Then $\mathcal{S}$ is biholomorphic to $\mathbb{C}$ and the uniformization $\tilde{F} : \mathbb{C} \to \mathcal{S}^{\times}$ is given by an entire function*

$F = \pi \circ \tilde{F}$ of the form $F(z) = \int Q(z)e^{P(z)}dz$ where P, Q are polynomials of degrees d_1, d_2 respectively.

Conversely we have:

Theorem 1.2. *Let $P, Q \in \mathbb{C}[z]$ be polynomials of degrees d_1, d_2 and F an entire function of the form $F(z) = \int Q(z)e^{P(z)}dz$. Then there exists a log-Riemann surface $\mathcal{S}$ with d_1 infinite order ramification points and d_2 finite order ramification points (counted with multiplicity) such that F lifts to a biholomorphism $\tilde{F} : \mathbb{C} \to \mathcal{S}^{\times}$.*

The entire functions of the above form were first studied by Nevanlinna [Nev32], who essentially proved Theorem 1.1, although his proof is in the classical language. The uniformization theorem was also rediscovered by M. Taniguchi [Tan01] in the form of a representation theorem for a class of entire functions defined by him called "structurally finite entire functions". The techniques we use are very different and adapted to the more general context of log-Riemann surfaces. In a forthcoming article [BPM10b] we use these techniques to generalize the above theorems to a correspondence between higher genus finite type log-Riemann surfaces and holomorphic differentials on punctured Riemann surfaces with isolated singularities of "exponential type" at the punctures (locally of the form $ge^h dz$ where g, h are germs meromorphic at the puncture).

The proof of Theorem 1.1 proceeds in outline as follows: we approximate $\mathcal{S}$ by simply connected log-Riemann surfaces $\mathcal{S}_n^{\times}$ with finitely many ramification points of finite orders such that d_1 ramification points of $\mathcal{S}_n^{\times}$ converge to infinite order ramification points. The surfaces $\mathcal{S}_n$ converge to $\mathcal{S}$ in the sense of Caratheodory (as defined in [BPM10a]) and by the Caratheodory convergence theorem proved in [BPM10a], the uniformizations $\tilde{F}_n$ of $\mathcal{S}_n$ converge to the uniformization $\tilde{F}$ of $\mathcal{S}$. The uniformizations $\tilde{F}_n$ are the lifts of polynomials $F_n = \pi_n \circ \tilde{F}_n$, such that the nonlinearities $G_n = F_n''/F_n'$ are rational functions of uniformly bounded degree with simple poles at the critical points of F_n. As these critical points go to infinity as $n \to \infty$, the nonlinearity of the function $F = \pi \circ \tilde{F}$ is a polynomial, from which it follows that F is of the form $\int Q(z)e^{P(z)}dz$.

To prove Theorem 1.2 we use the converse of Caratheodory convergence theorem: we approximate $F = \int Q(z)e^{P(z)}dz$ by polynomials $F_n = \int Q(z)(1+\frac{P(z)}{n})^n dz$. The polynomials F_n define log-Riemann surfaces $\mathcal{S}_n$ which then converge in the sense of Caratheodory to a log-Riemann surface $\mathcal{S}$ defined by F, and a study of the log-Riemann surfaces $\mathcal{S}_n$ shows that the log-Riemann surface $\mathcal{S}$ has d_1 infinite order ramification points and d_2 finite order ramification points (counted with multiplicity).

We develop the tools necessary for the proofs in the following sections. We first describe a "cell decomposition" for log-Riemann surfaces, which allows one to approximate finite type log-Riemann surfaces by log-Riemann surfaces with finitely many ramification points of finite order. The cell decomposition allows us to read the fundamental group of a log-Riemann surface from an associated graph, and to prove a parabolicity criterion for simply connected log-Riemann surfaces which in particular implies that the log-Riemann surfaces $\mathcal{S}$ and $\mathcal{S}_n$ considered in the proof of Theorem 1.1 are parabolic.

2. Cell decompositions of log-Riemann surfaces

We recall that a log-Riemann surface $(\mathcal{S}, \pi)$ comes equipped with a path metric d induced by the flat metric $|d\pi|$. Any simple arc $(\gamma(t))_{t\in I}$ in $\mathcal{S}$ which is the lift of a straight line segment in $\mathbb{C}$ is a geodesic segment in $\mathcal{S}$; we call such arc *unbroken geodesic segments*. Note that an unbroken geodesic segment is maximal if and only if, as t tends to an endpoint of I not in I, either $\gamma(t)$ tends to infinity, or $\gamma(t) \to p \in \mathcal{R}$.

2.1. **Decomposition into stars.** Let $w_0 \in \mathcal{S}$. Given an angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, for some $0 < \rho(w_0, \theta) \leq +\infty$, there is a unique maximal unbroken geodesic segment $\gamma(w_0, \theta) : [0, \rho(w_0, \theta)) \to \mathcal{S}$ starting at w_0 which is the lift of the line segment $\{\pi(w_0)+te^{i\theta} : 0 \leq t < \rho(w_0, \theta)\}$, such that $\gamma(w_0, \theta)(t) \to w^* \in \mathcal{R}$ if $\rho(w_0, \theta) < +\infty$.

Definition 2.1. *The star of $w_0 \in \mathcal{S}$ is the union of all maximal unbroken geodesics starting at w_0,*

$$V(w_0) := \bigcup_{\theta\in\mathbb{R}/2\pi\mathbb{Z}} \gamma(w_0, \theta)$$

.

Similarly we also define for a ramification point w^* of order $n \leq +\infty$ the star $V(w^*)$ as the union of all maximal unbroken geodesics $\gamma(w^*, \theta)$ starting from w^*, where the angle $\theta \in [-n\pi, n\pi)$:

$$V(w^*) := \{\gamma(w^*, \theta)(t) : 0 \leq t < \rho(w^*, \theta), -n\pi \leq \theta \leq n\pi\}$$

Proposition 2.2. *For $w_0 \in \mathcal{S}$ the star $V(w_0)$ is a simply connected open subset of $\mathcal{S}$. The boundary $\partial V(w_0) \subset \mathcal{S}$ is a disjoint union of maximal unbroken geodesic segments in $\mathcal{S}$.*

Proof: Since $\mathcal{R}$ is closed, the function $\rho(w_0, \theta)$ is upper semi-continuous in θ, from which it follows easily that $V(w_0)$ is open. Moreover π is injective on each $\gamma(w_0, \theta)$, hence is a diffeomorphism from $V(w_0)$ onto its image $\mathbb{C} - F$, where F is the disjoint union of closed line segments $\{\pi(w_0) + te^{i\theta} : \rho(w_0, \theta) < +\infty, t \geq \rho(w_0, \theta)\}$; clearly $\mathbb{C} - F$ is simply connected. By continuity of π, each component C of $\partial V(w_0)$ is contained in $\pi^{-1}(\gamma)$ for some segment γ in F, hence is an unbroken geodesic segment $(\alpha(t))_{t\in I}$. Since C is closed in $\mathcal{S}$, C must be maximal. ⋄

The set of ramification points $\mathcal{R}$ is discrete, hence countable. Let $L \supset \pi(\mathcal{R})$ be the union in $\mathbb{C}$ of all straight lines joining points of $\pi(\mathcal{R})$. Then $\mathbb{C} - L$ is dense in $\mathbb{C}$. By a *generic fiber* we mean a fiber $\pi^{-1}(z_0) = \{w_i\}$ of π such that $z_0 \in \mathbb{C} - L$.

Proposition 2.3. *Let $\{w_i\}$ be a generic fiber. Then:*
(1) The stars $\{V(w_i)\}$ are disjoint.
(2) The connected components of the stars $\partial V(w_i)$ are geodesic rays $\gamma : (0, +\infty) \to \mathcal{S}$ such that $\gamma(t) \to w^* \in \mathcal{R}$ as $t \to 0$, $\gamma(t) \to \infty$ as $t \to \infty$.
(3) The union of the stars is dense in $\mathcal{S}$:

$$\mathcal{S} = \overline{\bigcup_i V(w_i)} = \bigcup_i \overline{V(w_i)}$$

Proof: (1): If $w \in V(w_i) \cap V(w_j)$ then the geodesic segments from w to w_i, w_j are lifts of $[\pi(w), z_0]$, so by uniqueness of lifts (π is a local diffeomorphism) $w_i = w_j$.

(2): By the previous Proposition, each component of $\partial V(w_i)$ is a maximal unbroken geodesic segment $\gamma : (0, r) \to \mathcal{S}$ with $\lim_{t\to 0} \gamma(t) = w^* \in \mathcal{R}$ where w^* is a ramification point such that $\pi(\gamma)$ is a straight line segment contained in the straight line through $\pi(w_i)$ and $\pi(w^*)$. If $r < +\infty$ then $\gamma(t) \to w_1^* \in \mathcal{R}$ as $t \to r$, so $\pi(w_i)$ must lie on the straight line through $\pi(w^*), \pi(w_1^*)$, contradicting the fact that $\{w_i\}$ is a generic fiber. Hence $r = +\infty$.

(3): Given $p \in \mathcal{S}$, if $\pi(p) \neq z_0$, take a path $(p(t))_{0<t<\epsilon} \subset \mathcal{S}$ converging to p as $t \to 0$ such that the line segments $[\pi(p(t)), z_0]$ make distinct angles at z_0, then the discreteness of $\mathcal{R}$ implies that for t small enough these line segments admit lifts; again by discreteness of $\mathcal{R}$ for some i we have $p(t) \in V(w_i)$ for all t small, and $p \in \overline{V(w_i)}$. $\diamond$

It is easy to see that for $w_i \neq w_j$, the components of $\partial V(w_i), \partial V(w_j)$ are either disjoint or equal, and each component can belong to at most two such stars. The above Propositions hence give a cell decomposition of $\mathcal{S}$ into cells $V(w_i)$ glued along boundary arcs $\gamma \subset \partial V(w_i), \partial V(w_j)$.

2.2. **The skeleton and fundamental group.** Let $\pi^{-1}(z_0) = \{w_i\}$ be a generic fiber. The 1-skeleton of the cell decomposition into stars gives an associated graph:

Definition 2.4. *The skeleton $\Gamma(\mathcal{S}, z_0)$ is the graph with vertices given by the stars $V(w_i)$, and an edge between $V(w_i)$ and $V(w_j)$ for each connected component γ of $\partial V(w_i) \cap \partial V(w_j)$. Each edge corresponds to a geodesic ray $\gamma : (0, +\infty) \to \mathcal{S}$ starting at a ramification point. This gives us a map from edges to ramification points, foot : $\gamma \mapsto$ foot$(\gamma) := \lim_{t\to 0} \gamma(t) \in \mathcal{R} \in \overline{V(w_i)} \cap \overline{V(w_j)}$.*

For $w^* \in \mathcal{R}$ we let $C(w^*) = \{\gamma : \text{foot}(\gamma) = w^*\}$.

We omit the proof of the following proposition which is straightforward:

Proposition 2.5. *If w^* is of finite order n then $C(w^*) = (\gamma_i)_{1\leq i\leq n}$ is a cycle of edges in Γ of length n. If w^* is of infinite order then $C(w^*) = (\gamma_i)_{i\in\mathbb{Z}}$ is a bi-infinite path of edges in Γ.*

We can compute the fundamental group of a log-Riemann surface from its skeleton:

Proposition 2.6. *The log-Riemann surface $\mathcal{S}$ deformation retracts onto $\Gamma(\mathcal{S}, z_0)$. In particular $\pi_1(\mathcal{S}) = \pi_1(\Gamma(\mathcal{S}, z_0))$.*

Proof: Let $\partial V(w_i) = \sqcup_{k\in J_i} \gamma_{ik}$ be the decomposition of $\partial V(w_i)$ into its connected components. Choose points $v_{ik} \in \gamma_{ik}$, satisfying $v_{ik} = v_{jl}$ if $\gamma_{ik} = \gamma_{kl}$. Choose simple arcs $\alpha_{ik}, k \in J_i$, joining w_i to v_{ik} within $V(w_i)$, with $\alpha_{ik} \cap \alpha_{ik'} = \{w_i\}$. Then $\overline{V(w_i)}$ deformation retracts onto the union of the arcs α_{ik}; moreover for $i, j \in I$ we can choose the retractions compatibly on arcs $\gamma \subset \partial V(w_i) \cap \partial V(w_j)$, giving a retraction of $\mathcal{S}$ onto the union of all arcs $\alpha_{ik}, i \in I, k \in J_k$, which is homeomorphic to $\Gamma(\mathcal{S}, z_0)$. $\diamond$

The relation of $\Gamma(\mathcal{S}, z_0)$ to the finitely completed log-Riemann surface $\mathcal{S}^\times$ is as follows:

Definition 2.7. *The finitely completed skeleton $\Gamma^\times(\mathcal{S}, z_0)$ is the graph obtained from $\Gamma(\mathcal{S}, z_0)$ as follows: for each finite order ramification point w^*, add a vertex $v = v(w^*)$ to $\Gamma(\mathcal{S}, z_0)$, remove all edges in the cycle $C(w^*)$ and add an edge from v_i to v for each vertex v_i in the cycle $C(w^*)$.*

Then as above we have:

Proposition 2.8. *The finitely completed log-Riemann surface $\mathcal{S}^\times$ deformation retracts onto the finitely completed skeleton $\Gamma^\times(\mathcal{S}, z_0)$.*

Proof: Let w^* be a finite order ramification point. Observe that in the proof of the previous Proposition, for $\gamma = \gamma_{ik}$ an edge in $C(w^*)$, in the finitely completed log-Riemann surface the arc α_{ik} can be be homotoped to an arc $\tilde{\alpha}_{ik}$ from w_i to w^*. Then $\mathcal{S}^\times$ deformation retracts onto the union of the arcs $\alpha_{ik}, \tilde{\alpha}_{ik}$ which is homeomorphic to $\Gamma^\times(\mathcal{S}, z_0)$. $\diamond$

Given a graph Γ satisfying certain compatibility conditions along with the information of the locations of the ramification points, we can also construct an associated log-Riemann surface $\mathcal{S}$ with skeleton Γ:

Proposition 2.9. *Let $\Gamma = (V, E)$ be a connected graph with countable vertex and edge sets and a map $foot : E \to \mathbb{C}$. For each vertex v let E_v be the set of edges with a vertex at v and let $R_v = foot(E_v)$. Assume that the following hold:*
(1) The image $foot(E) \subset \mathbb{C}$ is discrete.
(2) For all vertices v and points $z \in R_v$, the intersection $foot^{-1}(z) \cap E_v$ has exactly two edges, labelled $\{e_z(v, +), e_z(v, -)\}$.
(3) For an edge e between vertices v, v' with $foot(e) = z$, either $e = e_z(v, +) = e_z(v', -)$ or $e = e_z(v, -) = e_z(v', +)$.

Then there exists a log-Riemann surface $\mathcal{S}$ with skeleton $\Gamma(\mathcal{S}, z_0) = \Gamma$ for some $z_0 \in \mathbb{C}$.

Proof: Let $L \subset \mathbb{C}$ be the union of all straight lines through pairs of points in foot(E), and let $z_0 \in \mathbb{C} - L$. For each vertex v of Γ, let L_v be the union of the half-lines l_z starting at points $z \in R_v$ with direction $z - z_0$. By assumption (1) this collection of half-lines is locally finite. Let U_v be the domain $\mathbb{C} - L_v$. Equip U_v with the path metric $d(a, b) = \inf_\beta \int_\beta |dz|$ (infimum taken over all rectifiable paths β joining a and b). Then the metric completion U_v^* of U_v is given by adjoining for each $z \in R_v$ two copies of l_z (the two 'sides' of the slit l_z) intersecting at a point z_v, which we denote by

$$U_v^* = U_v \bigsqcup_{z \in R_v} (l_z(v, +) \cup l_z(v, -))$$

where we take $l_z(v, +)$ to be the 'upper side' and $l_z(v, -)$ the 'lower' side (so $z \to l_z(v, +)$ if $z \to l_z$ in U_v with $\arg(z - z_0)$ increasing and $z \to l_z(v, -)$ if $z \to l_z$ in U_v with $\arg(z - z_0)$ decreasing). The inclusion of U_v in $\mathbb{C}$ extends to a local isometry $\pi_v : U_v^* \to \mathbb{C}$ with $\pi_v(l_z(v, +)) = \pi_v(l_z(v, -)) = l_z$.

Let $\mathcal{S}^*$ be

$$\mathcal{S}^* = \bigsqcup_{v\in V} U_v^* / \sim$$

with the following identifications: for each edge e with vertices v, v' and $\text{foot}(\gamma) = z$, if $e = e_z(v,+) = e_z(v',-)$ we paste isometrically the half-lines $l_z(v,+), l_z(v',-)$, otherwise we paste isometrically $l_z(v,-), l_z(v',+)$. The identifications are compatible with the maps π_v, giving a a map $\pi : \mathcal{S}^* \to \mathbb{C}$. We let $\mathcal{R} \subset \mathcal{S}^*$ be the subset corresponding to the points $\{z_v\}$ and $\mathcal{S} = \mathcal{S}^* - \mathcal{R}$.

Since $\pi(\mathcal{R}) = \text{foot}(E)$ is discrete, the set $\mathcal{R}$ is discrete. Moreover π restricted to $\mathcal{S}$ is a local isometry, and the completion of $\mathcal{S}$ with respect to the induced path metric is precisely $\mathcal{S}^*$, hence $\mathcal{S}$ is a log-Riemann surface. The fiber $\pi^{-1}(z_0)$ is generic since $z_0 \in \mathbb{C} - L$. The stars with respect to this fiber are precisely the open subsets $U_v \subset \mathcal{S}$. For any star U_v its closure in $\mathcal{S}^*$ is the image of U_v^* in $\mathcal{S}^*$. For vertices v, v', according to the above identifications between $U_v^*, U_{v'}^*$ in $\mathcal{S}^*$, each component of $\partial U_v \cap \partial U_{v'}$ (if non-empty) is a half-line l arising from an edge e between v_1, v_2, of either the form $l = l_z(v,+) = l_z(v',-)$ or $l = l_z(v,-) = l_z(v',+)$. It follows that $\Gamma(\mathcal{S}, z_0) = \Gamma$. $\diamond$

2.3. **Truncation and approximation by finite sheeted surfaces.** We can use the decomposition into stars to approximate any log-Riemann surface by finite sheeted log-Riemann surfaces by "truncating" infinite order ramification points to finite order ramification points. More precisely we have:

Theorem 2.10. *Let $(\mathcal{S}, p)$ be a pointed log-Riemann surface. Then:*

(1) There exists a sequence of pointed log-Riemann surfaces $(\mathcal{S}_n, p_n)$ converging to $(\mathcal{S}, p)$ in the Caratheodory topology such that each $\mathcal{S}_n$ has only finitely many ramification points all of finite order.

(2) If $\mathcal{S}^\times$ is simply connected then all the surfaces $\mathcal{S}_n^\times$ are simply connected.

We recall the definition of convergence of log-Riemann surfaces in the Caratheodory topology from [BPM10a]: $(\mathcal{S}_n, p_n) \to (\mathcal{S}, p)$ if for any compact $K \subset \mathcal{S}$ containing p there exists $N = N(K) \geq 1$ such that for all $n \geq N$ there is an isometric embedding $\iota_{n,K}$ of K into $\mathcal{S}_n$ mapping p to p_n which is a translation in the charts π, π_n on $\mathcal{S}, \mathcal{S}_n$.

Proof of Theorem 2.10: (1): Since the generic fibers are dense in $\mathcal{S}$ we may assume without loss of generality that $p = w_0$ lies in a generic fiber $\{w_i\} = \pi^{-1}(z_0)$. Let $V_i = V(w_i)$ be the corresponding stars and $\Gamma = \Gamma(\mathcal{S}, z_0)$ the associated skeleton, equipped with the graph metric d_Γ (where each edge has length 1). For any star V_i and $R > 0$, the set $\overline{V_i} \cap \overline{B(w_i, R)}$ is compact, so it contains at most finitely many ramification points. It follows that the collection of edges

$$\mathcal{E}(V_i, R) := \{\gamma : \gamma \text{ is an edge with a vertex at } V_i, \text{foot}(\gamma) \in \overline{B(w_i, R)}\}$$

is finite, and hence so is the corresponding collection of vertices

$$\mathcal{V}(V_i, R) := \{V_j : \gamma \in \mathcal{E}(V_i, R) \text{ is an edge between } V_i, V_j\}.$$

For $n \geq 1$ we define collections of edges and vertices $(\mathcal{E}_{n,k})_{1\leq k\leq n}, (\mathcal{V}_{n,k})_{1\leq k\leq n}$ as follows:

We let $\mathcal{E}_{n,1} = \mathcal{E}(V_0, n), \mathcal{V}_{n,1} = \mathcal{V}(V_0, n)$ and for $1 < k \leq n$,

$$\mathcal{E}_{n,k} := \bigcup_{V_i \in \mathcal{V}_{n,k-1}} \mathcal{E}(V_i, n)$$
$$\mathcal{V}_{n,k} := \bigcup_{V_i \in \mathcal{V}_{n,k-1}} \mathcal{V}(V_i, n)$$

This gives us finite connected subgraphs $\Gamma_n = (\mathcal{V}_{n,n}, \mathcal{E}_{n,n})$ of Γ increasing to Γ. Let

$$\hat{\mathcal{S}}_n = \bigcup_{V \in \mathcal{V}_{n,n}} \overline{V} \subset \mathcal{S}^*$$

be the corresponding union of stars in $\mathcal{S}^*$. It is a Riemann surface with boundary, each boundary component being an edge γ of Γ_n. We paste appropriate boundary components isometrically to obtain a Riemann surface without boundary $\mathcal{S}_n = \hat{\mathcal{S}_n}/\sim$ as follows:

We let $\mathcal{R}_n$ be the set of ramification points $\{\text{foot}(\gamma) : \gamma \in \mathcal{E}_{n,n}\}$. For $w^* \in \mathcal{R}_n$ we let $\Gamma_n(w^*)$ be the subgraph of Γ_n consisting of vertices V_i and edges γ such that $w^* = \text{foot}(\gamma) \in \overline{V_i}$. Two cases arise:

(i) The ramification point w^* is of finite order: Then there are finitely many stars V_i such that $w^* \in \overline{V_i}$. If $\Gamma_n(w^*)$ does not contain all of them, then the union of stars $\overline{V_i}, V_i \in \Gamma_n(w^*)$ has two boundary components, both of which are lifts of a half-line in $\mathbb{C}$ starting at $\pi(w^*)$; in this case we can paste the two components by an isometry which is the identity in charts.

(ii) The ramification point w^* is of infinite order: Then the union of stars $\overline{V_i}, V_i \in \Gamma_n(w^*)$ always has two boundary components, both of which are lifts of a half-line in $\mathbb{C}$ starting at $\pi(w^*)$; we paste the two components by an isometry which is the identity in charts.

Let $q_n : \hat{\mathcal{S}}_n \twoheadrightarrow \hat{\mathcal{S}}_n/\sim$ denote the quotient of $\hat{\mathcal{S}}_n$ under the identifications made in (i), (ii). The subset $\mathcal{S}_n := (\hat{\mathcal{S}}_n/\sim) - q_n(\mathcal{R}_n)$ is a Riemann surface without boundary. Since the identifications are compatible with the map π, π induces a map $\pi_n : \mathcal{S}_n \to \mathbb{C}$ which is a local diffeomorphism. The completion of $\mathcal{S}_n$ with respect to the flat metric induced by π_n is isometric to $\hat{\mathcal{S}}_n/\sim$, so that $\mathcal{S}_n$ is a log-Riemann surface with finite ramification set $q_n(\mathcal{R}_n)$; it is clear from the construction in (i), (ii) above that these ramification points are all of finite order. We let $p_n = q_n(p)$.

Any compact $K \subset \mathcal{S}$ containing p can only intersect finitely many stars V_i and hence $K \subset \hat{\mathcal{S}}_n$ for n large enough. Moreover for n large K does not intersect the boundary of $\hat{\mathcal{S}}_n$ (which is contained in stars going to infinity in Γ as n goes to infinity), hence the quotient map q_n isometrically embeds K in $\mathcal{S}_n$. Thus $(\mathcal{S}_n, p_n)$ converges to $(\mathcal{S}, p)$ as required.

(2): The graph $\Gamma(\mathcal{S}_n, z_0)$ can be obtained by adding edges to the finite graph Γ_n between certain vertices corresponding to edges in the sets $C(w^*), w^* \in \mathcal{R}_n$, to give cycles $C(q_n(w^*))$ in $\Gamma(\mathcal{S}_n, z_0)$. If $\mathcal{S}^\times$ is simply connected then by Proposition 2.8 the graph $\Gamma^\times(\mathcal{S}, z_0)$ is a tree. It follows from the construction of $\Gamma^\times(\mathcal{S}, z_0)$ that $\pi_1(\Gamma(\mathcal{S}, z_0))$ is generated by cycles corresponding to finite order ramification points and hence $\pi_1(\Gamma(\mathcal{S}_n, z_0))$ is generated by the cycles $C(q_n(w^*))$. In constructing $\Gamma^\times(\mathcal{S}_n, z_0)$ from $\Gamma(\mathcal{S}_n, z_0)$ these cycles become trivial so $\pi_1(\Gamma^\times(\mathcal{S}_n, z_0))$ is trivial. ⋄

2.4. **Compactness for uniformly finite type log-Riemann surfaces.** The family of finite type log-Riemann surfaces with a given uniform bound on the number of ramification points is compact, in the following sense:

Theorem 2.11. *Let $(\mathcal{S}_n, p_n)$ be a sequence of pointed log-Riemann surfaces with ramification sets $\mathcal{R}_n$. If for some $M, \epsilon > 0$ we have $\#\mathcal{R}_n \leq M, d(p_n, \mathcal{R}_n) > \epsilon$ for all n then there is a pointed log-Riemann surface $(\mathcal{S}, p)$ with ramification set $\mathcal{R}$ such that $\#\mathcal{R} \leq M$ and $(\mathcal{S}_n, p_n)$ converges to $(\mathcal{S}, p)$ along a subsequence.*

Proof: Composing π_n with a translation if necessary we may assume $\pi_n(p_n) = 0$ for all n. Since $d(p_n, \mathcal{R}_n) > \epsilon$ we can change p_n slightly (within the ball $B(p_n, \epsilon)$) to assume without loss of generality that the fiber $\pi_n^{-1}(0)$ containing p_n is generic. Let Γ_n be the corresponding skeleton and $v_{n,0}$ the vertex containing p_n. Passing to a subsequence we may assume the projections $\pi_n(\mathcal{R}_n)$ converge (in the Hausdorff topology) to a finite set $\{w_1^*, \ldots, w_N^*\} \cup \{\infty\} \subset \hat{\mathbb{C}} - B(0, \epsilon)$ (where $N \leq M$), and for all n lie in small disjoint neighbourhoods $B_1, \ldots, B_N$ and B of the points of $R = \{w_1^*, \ldots, w_N^*\}$ and ∞ respectively.

Let $\gamma_1, \ldots, \gamma_N$ be generators for the group $G = \pi_1(\mathbb{C} - R)$ where each γ_i is a simple closed curve in $\mathbb{C} - (B \cup_i B_i)$ starting at the origin with winding number one around B_i and zero around $B_j, j \neq i$. There is a natural action of G on the vertices of Γ_n: given a vertex v, let w be the point of the fiber $\pi_n^{-1}(0)$ in v. Then any $g \in G$ has a unique lift $\tilde{g}$ to $\mathcal{S}_n$ starting at w. Let $g \cdot v$ be the vertex of Γ_n containing the endpoint of $\tilde{g}$.

We define a graph $\Gamma_n' = (V_n, E_n)$ as follows: the vertex set V_n is the orbit of $v_{n,0}$ under G. We put an edge e between distinct vertices v, v' of Γ_n' for each generator $\gamma \in \{\gamma_i^\pm, i = 1, \ldots, N\}$ such that $v' = \gamma \cdot v$. We define $\text{foot}_n(e) = w_i^*$ if the edge e corresponds to either of the generators γ_i, γ_i^{-1}. This defines a map $\text{foot}_n : E_n \to R \subset \mathbb{C}$.

For $v \in V_n$ let E_v be the set of edges with a vertex at v and $R_v = \text{foot}_n(E_v) \subset R$. Since $\gamma_i \cdot v = v$ if and only if $\gamma_i^{-1} \cdot v = v$, it follows that for $z = w_i^* \in R_v$, the intersection $\text{foot}_n^{-1}(z) \cap E_v$ consists of precisely the two edges corresponding to the generators γ_i, γ_i^{-1}; we label these edges as $e_z(v, +), e_z(v, -)$.

It is easy to see that the graphs Γ_n' satisfy the hypotheses of Proposition 2.9. Since each vertex has valence at most $2N$, the balls $B(v_{n,0}, k)$ are finite, so we can pass to a subsequence such that the pointed graphs $(\Gamma_n', v_{n,0})$ converge to a limit pointed graph $(\Gamma = (V, E), v_0)$, in the sense that for any $k \geq 1$, for all n large enough there is an isomorphism i_n of the ball $B(v_0, k)$ with $B(v_{n,0}, k)$ taking v_0 to $v_{n,0}$. We may also assume that the isomorphisms i_n for different n are compatible with the mappings foot_n and the labeled edges $e_z(v, +), e_z(v, -)$, thus inducing a corresponding mapping $\text{foot} : E \to R \subset \mathbb{C}$ and a labeling of the edges of Γ. Then the limit graph Γ satisfies the hypotheses of Proposition 2.9 and we obtain a corresponding pointed log-Riemann surface $(\mathcal{S}, p)$ ramified over the points of R such that $\Gamma(\mathcal{S}, 0) = \Gamma$, with p in a generic fiber $\pi^{-1}(0)$, and the star containing p

corresponding to the vertex v_0 of Γ. Moreover $\mathcal{S}$ has at most N ramification points. It is easy to see that any compact $K \subset \mathcal{S}$ containing p embeds isometrically in all the log-Riemann surfaces $\mathcal{S}_n$ via an isometry ι_n such that $\iota_n(p) = p_n, \iota_n'(p) = 1$, hence $(\mathcal{S}_n, p_n)$ converges to $(\mathcal{S}, p)$. $\diamond$

2.5. **Decomposition into Kobayashi-Nevanlinna cells.** Let $\mathcal{S}$ be a log-Riemann surface with $\mathcal{R} \neq \emptyset$. We define a cellular decomposition of $\mathcal{S}$ due to Kobayashi [Kob35] and Nevanlinna ([Nev53] which is useful in determining the type (parabolic or hyperbolic) of simply connected log-Riemann surfaces.

Definition 2.12. *Let $w^* \in \mathcal{R}$. The Kobayashi-Nevanlinna cell of w^* is defined to be the set*

$$W(w^*) := \{w \in \mathcal{S}^* | d(w, w^*) < d(w, \mathcal{R} - \{w^*\})\}$$

Proposition 2.13. *The Kobayashi-Nevanlinna cells satisfy:*
(1) Any $w \in W(w^*)$ lies on an unbroken geodesic $[w^*, w] \subset W(w^*)$. In particular $W(w^*) \subset V(w^*)$ is open and path-connected.
(2) The boundary of $W(w^*)$ is a locally finite union of geodesic segments.
(3) $\mathcal{S} = \overline{\cup_{w^* \in \mathcal{R}} W(w^*)}$

Proof: (1): For any $w \in W(w^*), w \neq w^*$, since $\mathcal{R} \neq \emptyset$ there is a maximal unbroken geodesic $\gamma(w, \theta)$ converging to a point of $\mathcal{R}$ at one end, and since w^* is the point in $\mathcal{R}$ closest to w, there must be such a geodesic $[w, w^*]$ converging to w^*. Moreover for any $w' \in [w, w^*], w_1^* \in \mathcal{R} - \{w^*\}$, we have

$$d(w^*, w') = d(w^*, w) - d(w, w') < d(w_1^*, w) - d(w, w') \leq d(w_1^*, w')$$

hence $[w, w^*] \subset W(w^*)$.

(2): Let $w \in \partial W(w^*)$. By discreteness of $\mathcal{R}$ there are finitely many ramification points $w^* = w_1^*, \dots, w_n^*$ at minimal distance $r > 0$ from w, and $n \geq 2$. The disc $B(w, r)$ is a euclidean disk, with the points w_i^* lying on its boundary; the angular bisectors of the sectors formed by $[w, w_i^*], [w, w_{i+1}^*]$ then are equidistant from w_i^*, w_{i+1}^* and lie in $\partial W(w_i^*) \cap \partial W(w_{i+1}^*)$, while all other points in the disk lie in $W(w_i^*)$ for some i. Hence a neighbourhood of w in $\partial W(w^*)$ is given either by a geodesic segment passing through w (if $n = 2$) or by two geodesic segments meeting at w (if $n > 2$).

(3): Any $w \in \mathcal{S}$ belongs to $\overline{W(w^*)}$ for any ramification point w^* at minimal distance from w. $\diamond$

2.6. **Kobayashi-Nevanlinna parabolicity criterion.** We consider a log-Riemann surface $\mathcal{S}$ such that the finite completion $\mathcal{S}^\times$ is simply connected. We will use the following theorem of Nevanlinna ([Nev53] p. 317):

Theorem 2.14. *Let $F \subset \mathcal{S}^\times$ be a discrete set and $U : \mathcal{S}^\times - F \to [0, +\infty)$ be a continuous function such that:*
(1) U is C^1 except on at most a family of locally finite piecewise smooth curves.
(2) U has isolated critical points.
(3) $U \to +\infty$ as $z \to F$ or as $z \to \infty$.

For $\rho > 0$ let Γ_ρ be the union of the curves where $U = \rho$, and let

$$L(\rho) = \int_{\Gamma_\rho} |grad_z U||dz|.$$

where $|grad_z U||dz|$ is the conformally invariant differential given by $\sqrt{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2}|dz|$ for a local coordinate $z = x + iy$. If the integral

$$\int_0^\infty \frac{d\rho}{L(\rho)}$$

is divergent then the surface $\mathcal{S}^\times$ is parabolic.

We now define a function U on $\mathcal{S}$ as follows:

Let ω be the continuous differential $\omega := |d \arg(w - w^*)|$, where for each $w \in \mathcal{S}$, w^* is a ramification point such that $w \in \overline{W(w^*)}$. Fix a base point $w_0 \in \mathcal{S}$ and define $\tau : \mathcal{S} \to [0, +\infty)$ by

$$\tau(w) := \inf \int_{w_0}^{w} \omega$$

where the infimum is taken over all paths from w_0 to w. We define another non-negative continuous function $\sigma : \mathcal{S} \to [0, +\infty)$ by

$$\sigma(w) := |\log |w - w^*||$$

where as before for each $w \in \mathcal{S}$ the point w^* is a ramification point such that $w \in \overline{W(w^*)}$.

Then the sum $U = \tau + \sigma : \mathcal{S} \to \mathbb{R}$ is a function satisfying the conditions (1)-(3) of the above theorem. The map $t = \sigma + i\tau$ gives a local holomorphic coordinate away from the boundaries of the Kobayashi-Nevanlinna cells, for which we have $|\text{grad}_t U||dt| = \sqrt{2}|dt|$. On a level set $\Gamma_\rho = \{U = \rho\}$ we have $0 \leq \tau \leq \rho, t = (\rho - \tau) + i\tau$, so $|\text{grad}_t U||dt| = \sqrt{2}|dt| = 2|d\tau|$. For a given $\theta > 0$, the connected components of the level set $\{\tau(w) = \theta\}$ are Euclidean line segments which are half-lines or intervals; let $0 \leq n(\theta) \leq \infty$ denote the number of such line segments. Each such segment intersects Γ_ρ in at most one point; hence we obtain

$$L(\rho) = \int_{\Gamma_\rho} |\text{grad}_t U||dt| = 2 \int_{\Gamma_\rho} |d\tau| \leq \int_0^\rho n(\theta) d\theta$$

Using Theorem 2.14 above, we obtain the following:

Theorem 2.15. *Let $\mathcal{S}$ be a log-Riemann surface such that $\mathcal{S}^\times$ is simply connected. For $\theta > 0$ let $0 \leq n(\theta) \leq \infty$ denote the number of connected components of the level set $\{\tau(w) = \theta\}$. If the integral*

$$\int_0^\infty \frac{d\rho}{\int_0^\rho n(\theta) d\theta}$$

is divergent then $\mathcal{S}^\times$ is biholomorphic to $\mathbb{C}$.

This implies:

Corollary 2.16. *Let $\mathcal{S}$ be a log-Riemann surface with a finite number of ramification points such that $\mathcal{S}^\times$ is simply connected. Then $\mathcal{S}$ is biholomorphic to $\mathbb{C}$.*

Proof: In this case the function $n(\theta)$ is bounded above by twice the number of ramification points of $\mathcal{S}$, so $\int_0^\rho n(\theta)d\theta \leq C\rho$ and hence the integral in Theorem 2.15 diverges. $\diamond$

3. Uniformization theorems

We can now prove Theorem 1.1 as follows:

Proof of Theorem 1.1: Let $p \in \mathcal{S}$. Let D_1, D_2 be the numbers of infinite and finite order ramification points respectively of $\mathcal{S}$. By Corollary 2.16 the log-Riemann surface $\mathcal{S}^\times$ is biholomorphic to $\mathbb{C}$. The approximating finitely completed log-Riemann surfaces $\mathcal{S}_n^\times$ given by Theorem 2.10 are also biholomorphic to $\mathbb{C}$ and for n large all have $D_1 + D_2$ ramification points. Let $\tilde{F} : \mathbb{C} \to \mathcal{S}^\times$ and $\tilde{F}_n : \mathbb{C} \to \mathcal{S}_n^\times$ be corresponding normalized uniformizations such that $\tilde{F}(0) = p, \tilde{F}'(0) = 1, \tilde{F}_n(0) = p_n, \tilde{F}_n'(0) = 1$, with inverses $G = \tilde{F}^{-1}, G_n = \tilde{F}_n^{-1}$. By Theorem 1.2 of [BPM10a] the entire functions $F_n = \pi_n \circ \tilde{F}_n$ converge uniformly on compacts to the entire function $F = \pi \circ \tilde{F}$. Since $\pi_n : \mathcal{S}_n^\times \to \mathbb{C}$ is finite to one, the entire function F_n has a pole at ∞ of order equal to the degree of π_n, and is hence a polynomial. The nonlinearities $R_n = F_n''/F_n'$ are rational functions whose poles are simple poles with integer residues at the critical points of F_n, which are images of the ramification points of $\mathcal{S}_n$ under G_n. Thus the rational functions R_n are all of degree $D_1 + D_2$, converging normally to F''/F', so $R = F''/F'$ is a rational function of degree at most D.

Each ramification point w^* of $\mathcal{S}$ corresponds to a ramification point w_n^* of $\mathcal{S}_n$ of order converging to that of w^*. We note that for n large any compact $K \subset \mathcal{S}^\times$ containing p embeds into the approximating surfaces $\mathcal{S}_n^\times$. Since the maps G_n converge to G uniformly on compacts of $\mathcal{S}^\times$ by Theorem 1.1 [BPM10a], the images under G_n of ramification points in $\mathcal{S}_n^\times$ corresponding to finite ramification points in $\mathcal{S}$ converge to their images under G, giving in the limit D_2 simple poles of R, with residue at each equal to the order of the corresponding finite ramification point of $\mathcal{S}$ minus one.

On the other hand the infinite order ramification points of $\mathcal{S}$ are not contained in $\mathcal{S}^\times$, so the images of the corresponding ramification points in $\mathcal{S}_n^\times$ under G_n cannot be contained in any compact in $\mathbb{C}$ and hence converge to infinity. The rational functions R_n have a simple zero at infinity, and have D_1 simple poles converging to infinity. Applying the Argument Principle to a small circle around infinity it follows that R has a pole of order $D_1 - 1$ at infinity.

Thus R is of the form

$$\frac{F''}{F'} = \sum_{i=1}^{D_2} \frac{m_i - 1}{z - z_i} + P'(z)$$

where $m_1, \ldots, m_{D_2}$ are the orders of the finite ramification points of $\mathcal{S}$ and P is a polynomial of degree D_1. Integrating the above equation gives

$$F(z) = \pi(p) + \int_0^z (t - z_1)^{m_1 - 1} \ldots (t - z_{D_2})^{m_{D_2} - 1} e^{P(t)} dt$$

as required. ⋄

We can prove the converse using the above Theorem and the compactness Theorem. We need a lemma:

Lemma 3.1. *Let $(\mathcal{S}_n, p_n)$ converge to $(\mathcal{S}, p)$. If all the surfaces $\mathcal{S}_n^\times$ are simply connected then $\mathcal{S}^\times$ is simply connected.*

Proof: We may assume the points p_n, p belong to generic fibers. Let Γ_n, Γ denote the corresponding skeletons. Let γ be a loop in $\mathcal{S}^\times$ based at p. We may homotope γ away from the finite ramification points to assume that $\gamma \subset \mathcal{S}$. By Proposition 2.6, γ corresponds to a path of edges $\alpha = \{e_1, \ldots, e_n\}$. By induction on the number of edges we may assume that α is simple. If $\text{foot}(\alpha) = \{w^*\}$ is a singleton then w^* is a finite ramification point and γ is trivial in $\mathcal{S}^\times$. Otherwise there are distinct ramification points $w_1^*, w_2^* \in \text{foot}(\alpha)$. Considering the isometric embedding of γ in $\mathcal{S}_n$ for n large gives a path γ_n and a corresponding path of edges α_n; for n large, it follows that there are distinct ramification points in $\text{foot}_n(\alpha_n)$, hence γ_n is non-trivial in $\mathcal{S}_n^\times$, a contradiction. ⋄

Proof of Theorem 1.2: Given an entire function F with $F'(z) = Q(z)e^{P(z)}$ we can approximate it by polynomials F_n such that $F_n'(z) = Q(z)(1 + P(z)/n)^n$. Let $Z_n = \{P = -n\} \cup \{Q = 0\} \cup \subset \mathbb{C}$ be the zeroes of F_n'. The pair $(\mathcal{S}_n = \mathbb{C} - Z_n, \pi_n = F_n : \mathbb{C} - Z_n \to \mathbb{C})$ is a log-Riemann surface with finite ramification set $\mathcal{R}_n$ which can be naturally identified with Z_n, the order of a ramification point being the local degree of F_n at the corresponding point of Z_n.

For n large the surfaces $\mathcal{S}_n$ all have the same number of ramification points $D = D_1 + D_2$ where D_1 is the degree of P and D_2 the number of distinct zeroes of Q. Moreover since F_n' converge uniformly on compacts, choosing a point z_0 such that $Q(z_0) \neq 0$, for all n large $|F_n'|$ is uniformly bounded away from 0 on a fixed neighbourhood of z_0, so $d(z_0, \mathcal{R}_n)$ is uniformly bounded away from 0. It follows from Theorem 2.11 that $(\mathcal{S}_n, p_n = z_0)$ converge along a subsequence to a limit log-Riemann surface $(\mathcal{S}, p)$ with finitely many ramification points such that $\pi(p) = z_0$. Since $\mathcal{S}_n^\times$ is simply connected for all n, by the previous Lemma $\mathcal{S}^\times$ is simply connected. By Theorem 2.16, $\mathcal{S}^\times$ is biholomorphic to $\mathbb{C}$. Let $\tilde{F} : \mathbb{C} \to \mathcal{S}^\times$ be a normalized uniformization such that $\tilde{F}(z_0) = p, \tilde{F}'(z_0) = F'(z_0)$. It follows from Theorem 1.2 of [BPM10a] that the maps F_n converge normally to $\pi \circ \tilde{F}$, so $F = \pi \circ \tilde{F}$. Thus F defines the uniformization of a simply connected log-Riemann surface with finitely many ramification points. The degrees of Q, P relate to the numbers of finite poles and poles at infinity respectively of the nonlinearity F''/F'; the relations between the degrees of Q, P and the numbers of finite and infinite order ramification points of $\mathcal{S}$ then follow from the previous Theorem. ⋄

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