The rainbow connectivity of Cayley graphs of Abelian groups*

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Abstract

A path in an edge-colored graph G, where adjacent edges may have the same color, is called a rainbow path if no two edges of the path are colored the same. The rainbow connectivity $rc(G)$ of G is the minimum integer i for which there exists an i-edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. The *strong rainbow connectivity* $src(G)$ of G is the minimum integer i for which there exists an i-edge-coloring of G such that every two distinct vertices u and v of G are connected by a rainbow path of length $d(u,v)$. In this paper, we show that $rc(C(\Gamma,S)) \leq min\{\Sigma_{a\in S^*}\lceil\frac{|a|}{2}\rceil \mid S^* \subseteq S\ is\ a\ minimal\ generating\ set\ of\ \Gamma\}$, where Γ is an Abelian group, and $|a|$ is the order of a. Moreover, If S is a minimal inverse closed generating set of Γ, then $\Sigma_{a\in S^*}\lfloor\frac{|a|}{2}\rfloor \leq rc(C(\Gamma,S)) \leq \Sigma_{a\in S^*}\lceil\frac{|a|}{2}\rceil$. Furthermore, if any element $a \in S$ has even order, then $src(C(\Gamma,S)) = rc(C(\Gamma,S)) = \Sigma_{a\in S^*}\frac{|a|}{2}$.

Keywords: Edge-coloring, Rainbow path, Rainbow connectivity, Abelian group, Cayley graph.

1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book[1] for graph theory notation and terminology not described in this paper. A path in an edge-colored graph G, where adjacent edges may have the same color, is called a *rainbow path* if no two edges of the path are colored the same. A edge-coloring of G is $rainbow\ edge-coloring$ if any two distinct vertices is connected by a rainbow path. Furthermore, the *rainbow connectivity* $rc(G)$ of G is the minimum integer i for which

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there exists a i-edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. If G is disconnected, we say that $rc(G) = 0$ by convention. The *strong rainbow connectivity* $src(G)$ of G is the minimum integer i for which there exists a i-edge-coloring of G such that every two distinct vertices u and v of G are connected by a rainbow path of length $d(u, v)$. It is easy to see that $src(G) \leq rc(G) \leq D(G)$ for any connected graph, where $D(G)$ is the diameter of G.

The concept of rainbow connectivity is of great use in transferring information of high security in communication networks. Reader can see [3] for details.

Let Γ be a group, and let $a \in \Gamma$ be an element. We use $\langle a \rangle$ denote the cyclic subgroup generated by a. The number of elements of $\langle a \rangle$ is called the *order* of a, denoted by $|a|$. A pair of elements a and b in a group commutes if $ab = ba$. A group is *abelian* if every pair of its elements commutes.

The *Cayley Graph* of Γ with respect to S is the graph $C(\Gamma, S)$ with vertex set Γ in which two vertices x and y are adjacent if and only if $xy^{-1} \in S$ (or equivalently $yx^{-1} \in S$), where $S \subseteq \Gamma \setminus 1$ is closed under taking inverse. It is well-known that $C(\Gamma, S)$ is connected if and only if S is a generating set of Γ. We will find the following conception is of great use in our proof. If $xy^{-1} = a \in S$, we call edge xy to be a-edge. It is not difficult to see that an a-edge xy is also an a^{-1}-edge yx since S is closed under taking inverse. Thus we do not distinguish a-edges and a^{-1}-edges in the following arguments.

There are some results on the (strong) rainbow connectivity of graphs, see [2, 4] for examples. We show some results as follows.

Proposition 1. *[2] Let G be a nontrivial connected graph of order n. Then $rc(G) = src(G) = 1$ if and only if $G \cong K_n$.*

Krivelevich et al. show the following upper bound of graph G about order and minimum degree.

Theorem 1. *[4] A connected graph G with n vertices has $rc(G) < 20n/\delta(G)$.*

A *minimal generating set* of a group Γ is generating set X such that on proper subset of X is a generating set of Γ. A *inverse closed minimal generating set* of a group Γ is $X \cup X^{-1}$, where X is a minimal generating set of Γ and $X^{-1} = \{x \mid x \in X\}$. Clearly, A inverse closed minimal generating set of Γ only has one minimal generating set of Γ up to not distinguishing element a and its inverse element a^{-1}.

In the main result, we show that $rc(C(\Gamma, S)) \leq min\{\Sigma_{a \in S^*} \lceil \frac{|a|}{2} \rceil \mid S^* \subseteq S \; is \; a \; minimal$ *generating subset of* $\Gamma\}$, where Γ is an abelian group. Moreover, If S is a minimal inverse closed generating set of Γ, then, $\Sigma_{a \in S^*} \lfloor \frac{|a|}{2} \rfloor \leq rc(C(\Gamma, S)) \leq \Sigma_{a \in S^*} \lceil \frac{|a|}{2} \rceil$, where $S^* \subseteq S$ is a minimal generating set of Γ. Furthermore, if any element $a \in S$ has even order, then $src(C(\Gamma, S)) = rc(C(\Gamma, S)) = \Sigma_{a \in S^*} \frac{|a|}{2}$, where $S^* \subseteq S$ is a minimal generating set of Γ.

2 Main result

In this section, we firstly present the following elementary proposition, secondly show the main result, finally give some corollaries from the main result and one open problem.

Proposition 2. *If H is a spanning subgraph of G, then $rc(h) \leq rc(G)$.*

It is easy to prove above proposition since a rainbow edge-coloring of H induce a rainbow edge-coloring of G.

Theorem 2. *Given an Abelian group Γ and a generating set $S \subseteq \Gamma \setminus \{1\}$ of Γ. We have the following results:*

(i) $rc(C(\Gamma, S)) \leq min\{\Sigma_{a\in S^*}\lceil\frac{|a|}{2}\rceil \mid S^* \subseteq S$ is a minimal generating subset of $\Gamma\}$.

(ii) If S is a minimal inverse closed generating set of Γ, then

$$\Sigma_{a\in S^*}\lfloor\frac{|a|}{2}\rfloor \leq rc(C(\Gamma, S)) \leq \Sigma_{a\in S^*}\lceil\frac{|a|}{2}\rceil,$$

where $S^* \subseteq S$ is a minimal generating set of Γ.

(iii) If S is a minimal inverse closed generating set of Γ, and any element $a \in S$ has even order, then

$$src(C(\Gamma, S)) = rc(C(\Gamma, S)) = \Sigma_{a\in S^*}\frac{|a|}{2},$$

where $S^* \subseteq S$ is a minimal generating set of Γ.

Proof. (i) Note that Cayley graph $C(\Gamma, S)$ is connected if and only if S is a generating set of Γ. Thus $rc(C(\Gamma, S)) = 0$ if and only if S is not a generating subset of Γ. Therefore, (i) holds when S is not a generating set of Γ. Suppose that S is a generating set. We set that $\Gamma = \{v_1, v_2, \cdots, v_n\}$, and we take any one minimal generating set $S^* = \{a_1, a_2, \cdots, a_r\} \subseteq S$ of Γ, then $C(\Gamma, S^{**})$ is a connected spanning subgraph of $C(\Gamma, S)$, where $S^{**} = S^* \cup (S^*)^{-1}$. It suffices to show that $rc(C(\Gamma, S^{**})) \leq \Sigma_{a\in S^*}\lceil\frac{|a|}{2}\rceil$ by Proposition 2. We use $M_i, 1 \leq i \leq r$ denote the edge sets which are a_i-edges. Then, $M_i, 1 \leq i \leq r$ is a partition of $E(C(\Gamma, S^{**}))$.

Set $|a_i| = b_i$. If $b_i = 2$, clearly, $C(\Gamma, S)[M_i]$ is a perfect matching. Then we assign M_i the color $(i, 1)$. If $b_i \geq 3$, firstly picking identity element 1 of Γ. Vertex sequence $(1, a_i 1, a_i^2 1, \cdots, a_i^{b_1} 1 = 1)$ is a cycle, denoted by $C_{i,1}$. Secondly, Picking vertex $u \in \Gamma$ such that $u \notin V(C_{i,1})$, then $(u, a_i u, a_i^2 u, \cdots, a_i^{b_1} u = u)$ is a cycle, denoted by $C_{i,2}$. We could go all the way until on vertex remains. We obtain n/b_i cycles $C_{i,1}, C_{i,2}, \cdots, C_{i,n/b_i}$. For any cycle $C_{i,k}, 1 \leq k \leq n/b_i$, picking vertex $v_k \in V(C_{i,k})$. We color the edge of $C_{i,k} = (v_k, a_i v_k, a_i^2 v_k, \cdots, a_i^{b_i} v_k = v_k)$ as follows.

Case 1. If $b_i \geq 3$ is even. Assign edges $(a_i^j v_k)(a_i^{j+1} v_k), 0 \leq j \leq b_i/2 - 1$ the color $(i, j+1)$, and edges $(a_i^{b_i/2+j} v_k)(a_i^{b_i/2+j+1} v_k), 0 \leq j \leq b_i/2 - 1$ the color $(i, j+1)$.

Case 2. If $b_i \geq 3$ is odd. Assign edges $(a_i^j v_k)(a_i^{j+1} v_k), 0 \leq j \leq (b_i - 1)/2$ the color $(i, j+1)$, and edges $(a_i^{(b_i+1)/2+j} v_k)(a_i^{(b_i+1)/2+j+1} v_k), 0 \leq j \leq (b_i-1)/2$ the color $(i, j+1)$. The edge $(a_i^{(b_i-1)/2} v_k)(a_i^{(b_i+1)/2} v_k)$ can color $(i, (b_i + 1)/2)$.

Note that $C_{i,k}, 1 \leq k \leq n/b_i$ use $\lceil \frac{b_i}{2} \rceil$ colors. Thus, the number of colors that we have used equals $\Sigma_{a\in S^*}\lceil \frac{|a|}{2} \rceil$.

Next we will show that the above edge-coloring is a rainbow edge-coloring, that is, there exists an rainbow path connecting any two distinct vertices x, y of $C(\Gamma, S^{**})$. we can assume that $x = a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}, y = a_1^{j_1}a_2^{j_2}\cdots a_r^{j_r}$ satisfying $0 \leq j_k - i_k \leq \lceil \frac{b_k}{2} \rceil (mod\ b_k), 1 \leq k \leq r$. If some pair satisfies $j_k - i_k > \lceil \frac{b_k}{2} \rceil (mod\ b_k)$, we replace $a_k^{i_k}$ and $a_k^{j_k}$ by $(a_k^{-1})^{b_k - i_k}$ and $(a_k^{-1})^{b_k - j_k}$ respectively since $a_k^{i_k} = (a_k^{-1})^{b_k - i_k}, a_k^{j_k} = (a_k^{-1})^{b_k - j_k}$ and $a_k^{-1} \in S^{**}$. Clearly path $p = (x = a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}, a_1^{i_1+1}a_2^{i_2}\cdots a_r^{i_r}, \cdots, a_1^{j_1}a_2^{i_2}\cdots a_r^{i_r}, \cdots, a_1^{j_1}a_2^{j_2}\cdots a_r^{j_r} = y)$ is a rainbow path. This completes the proof of part (i).

(ii) Now suppose that S is a minimal inverse closed generating set of Γ. Note that Γ only has one minimal generating set S^* contained in S up to not distinguishing element a and its inverse element a^{-1}, and without loss of generality, set $S^* = \{a_1, a_2, \cdots, a_r\}$ and $|a_i| = b_i$. It suffices to show that $D(C(\Gamma, S)) = \Sigma_{a\in S^*}\lfloor \frac{|a|}{2} \rfloor$. It is well-known that Cayley graphs are vertex-transitive, we only consider the distance from 1 to other vertex x of $C(\Gamma, S)$. Without loss of generality, assume that $x = a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ satisfying $i_k \leq \lfloor \frac{b_k}{2} \rfloor, 1 \leq k \leq r$. Otherwise, we replace $a_k^{i_k}$ by $(a_k^{-1})^{b_k - i_k}$ since $a_k^{i_k} = (a_k^{-1})^{b_k - i_k}$ and $a_k^{-1} \in S$. $p = (1 = a_1^0a_2^0\cdots a_r^0, a_1^1a_2^0\cdots a_r^0, \cdots, a_1^{i_1}a_2^0\cdots a_r^0, \cdots, a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r} = x)$ is a path from 1 to x with length $\Sigma_{1\leq k\leq r} i_k$. Thus, $D(C(\Gamma, S)) \leq \Sigma_{a\in S^*}\lfloor \frac{|a|}{2} \rfloor$. On the other hand, for 1 and $x = a_1^{\lfloor \frac{b_1}{2} \rfloor}a_2^{\lfloor \frac{b_2}{2} \rfloor}\cdots a_r^{\lfloor \frac{b_r}{2} \rfloor}$, if $d(1, x) \leq \Sigma_{a\in S^*}\lfloor \frac{|a|}{2} \rfloor$, then, by pigeonhole principle, there exists integer l such that the number of a_i-edges of the shortest path from 1 to x less than $\lfloor \frac{b_l}{2} \rfloor$, this is impossible. Therefore, $D(C(\Gamma, S)) = \Sigma_{a\in S^*}\lfloor \frac{|a|}{2} \rfloor$. Thus $\Sigma_{a\in S^*}\lfloor \frac{|a|}{2} \rfloor \leq rc(C(\Gamma, S)) \leq \Sigma_{a\in S^*}\lceil \frac{|a|}{2} \rceil$.

(iii) Suppose any element $a \in S$ has even order. We only need show that for any $x, y \in V(C(\Gamma, S))$, there exists a rainbow path with length $d(x, y)$ between x and y. We also can assume that $x = a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}, y = a_1^{j_1}a_2^{j_2}\cdots a_r^{j_r}$ satisfying $0 \leq j_k - i_k \leq \frac{b_k}{2} (mod\ b_k), 1 \leq k \leq r$. By a similar argument of the diameter $D(\Gamma, S)$, we conclude $d(x, y) = \Sigma_{1\leq k\leq r}(j_k - i_k)$. Moreover, path $(x = a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}, a_1^{i_1+1}a_2^{i_2}\cdots a_r^{i_r}, \cdots, a_1^{j_1}a_2^{i_2}\cdots a_r^{i_r}, \cdots, a_1^{j_1}a_2^{j_2}\cdots a_r^{j_r} = y)$ is a rainbow path from x to y with length $d(x, y) = \Sigma_{1\leq k\leq r}(j_k - i_k)$. This completes the proof of this theorem. $\square$

An *n-dimensional hypercube* Q_n has 2^n vertices. Each vertex x can be represented by a sequence of n binary bits $x_1, x_2, \cdots, x_n$ where $x_i \in \{0, 1\}$. Two vertices are adjacent if and only if the binary representations of the two vertices differ in exactly one bit.

The *Cartesian product* of simple graphs G and H is the graph $G\Box H$ whose vertex set is $V(G) \times V(H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$.

Let $\mathbb{Z}_n$ be the additive group of integers modulo n. Note that $\mathbb{Z}_2^n$ is an abelian group. The next corollary follows from Theorem 2 by taking $\Gamma = \mathbb{Z}_2^n$ and $S = \{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)\}$.

Corollary 1. *Let Q_n be an n-dimensional hypercube, then $src(Q_n) = rc(Q_n) = n$.*

Proof. It is easy to see that $Q_n \cong P_2 \Box P_2 \Box P_2 \cong C(\mathbb{Z}_2^n, S)$ by the definitions of n-dimensional hypercube, cartesian product and Cayley graph $C(\mathbb{Z}_2^n, S)$. Note that S is a minimal inverse closed generating set of $\mathbb{Z}_n$, and also is a minimal generating set of $\mathbb{Z}_n$. Thus, $src(Q_n) = rc(Q_n) = n$ follows from Theorem 2 and the fact that $|(\underbrace{0, \cdots, 1}_{k}, \cdots, 0)| = 2$. $\square$

$\mathcal{Z} = \mathbb{Z}_{i_1} \times \mathbb{Z}_{i_2} \times \cdots \times \mathbb{Z}_{i_r}$ is an abelian group, where $i_k \geq 3, 1 \leq k \leq r$, and has a inverse closed minimal generating set $S = \{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1), (i_1 - 1, 0, \cdots, 0), (0, i_2 - 1, \cdots, 0), \cdots, (0, 0, \cdots, i_r - 1)\}$.

Corollary 2. *Let $C_{i_k}, i_k \geq 3, 1 \leq k \leq r$ be cycles, then $\Sigma_{1 \leq k \leq r} \lfloor \frac{i_k}{2} \rfloor \leq rc(C_{i_1} \Box C_{i_2} \Box C_{i_r}) \leq \Sigma_{1 \leq k \leq r} \lceil \frac{i_k}{2} \rceil$. Furthermore, if i_k are even for all $1 \leq k \leq r$, $src(C_{i_1} \Box C_{i_2} \Box C_{i_r}) = rc(C_{i_1} \Box C_{i_2} \Box C_{i_r}) = \Sigma_{1 \leq k \leq r} \frac{i_k}{2}$.*

Proof. We have $C_{i_1} \Box C_{i_2} \Box C_{i_r} \cong C(\mathcal{Z}, S)$ by the definitions of cartesian product and Cayley graph $C(\mathcal{Z}, S)$. Moreover, $\mathcal{Z}$ only has one minimal generating set $S = \{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)\}$ contained in S up to not distinguishing the element a and its inverse element a^{-1}, and $|(\underbrace{0, \cdots, 1}_{k}, \cdots, 0)| = i_k, 1 \leq k \leq r$. Thus the first inequality holds by the second part of Theorem 2. Furthermore, if i_k are even for all $1 \leq k \leq r$, we immediately deduce $src(C_{i_1} \Box C_{i_2} \Box C_{i_r}) = rc(C_{i_1} \Box C_{i_2} \Box C_{i_r}) = \Sigma_{1 \leq k \leq r} \frac{i_k}{2}$ by the third part of Theorem 2. $\square$

A circulant is a Cayley graph $C(\mathbb{Z}_n, S)$ where $S \subseteq \mathbb{Z}_n \backslash \{1\}$ is closed under taking inverse.

Corollary 3. *Let G be a circulant graph with n vertices, then $rc(G) \leq \lceil \frac{n}{2} \rceil$*

For the remainder of this paper we present the following remark, and completes this paper.

Remark: In Theorem 2, we show that, given an Abelian group Γ and a minimal inverse closed generating set $S \subseteq \Gamma \setminus \{1\}$ of Γ, if any element $a \in S$ has even order, then

$$src(C(\Gamma, S)) = rc(C(\Gamma, S)) = \Sigma_{a \in S^*} \frac{|a|}{2},$$

where $S^* \subseteq S$ is a minimal generating set of Γ.

What happens when some element $a \in S$ has odd order? We leave the following open problem: given an Abelian group Γ and a inverse closed minimal generating set $S \subseteq \Gamma \setminus \{1\}$ of Γ, is it true that

$$src(C(\Gamma, S)) = rc(C(\Gamma, S)) = \Sigma_{a \in S^*} \lceil \frac{|a|}{2} \rceil \ ?$$

where $S^* \subseteq S$ is a minimal generating set of Γ.

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