

KINETIC MODELS WITH RANDOMLY PERTURBED BINARY COLLISIONS

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ABSTRACT. We introduce a class of Kac-like kinetic equations on the real line, with general random collisional rules, which include as particular cases models for wealth redistribution in an agent-based market [6], or models for granular gases with a background heat bath [11]. Conditions on these collisional rules which guarantee both the existence and uniqueness of equilibrium profiles and their main properties are found. We show that the characterization of these stationary solutions is of independent interest, since the same profiles are shown to be solutions of different evolution problems, both in the econophysics context [6], and in the kinetic theory of rarefied gases [14, 29].

1. INTRODUCTION

In this paper, we are concerned with the study of the time evolution and the asymptotic behavior of the spatially homogeneous kinetic equation

$$(1) \quad \begin{cases} \partial_t \mu_t + \mu_t = Q^+(\mu_t, \mu_t) \\ \mu(0) = \bar{\mu}_0 \end{cases}$$

which caricatures a Boltzmann-like equation in one spatial dimension. The solution $\mu_t = \mu_t(\cdot)$ is a time-dependent probability measure on \mathbb{R} , describing, in its most common physical applications, the distribution of particle velocity in a homogeneous gas, which is initially distributed according to the probability measure $\bar{\mu}_0$. The gain operator Q^+ models velocity changes due to binary particle collisions. Our fundamental assumption is that Q^+ is a generalized Wild convolution. More precisely, for all bounded and continuous test functions $g \in C_b(\mathbb{R})$, we characterize the probability measure $Q^+(\mu, \mu)$ by

$$(2) \quad \int g(v) Q^+(\mu, \mu)(dv) = \mathbb{E} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} g(v_1 A_1 + v_2 A_2 + A_0) \mu(dv_1) \mu(dv_2) \right],$$

where (A_0, A_1, A_2) is a random vector of \mathbb{R}^3 defined on a probability space (Ω, \mathcal{F}, P) and \mathbb{E} denotes the expectation with respect to P .

The interaction rule generated by the law described in (2) simulates an interaction in which, in addition to the standard binary collision, the post-interaction velocities are randomly modified by the presence of an external background. As we shall see, this modification induces an evolution process for the probability measure which stabilizes in time towards a steady profile heavily dependent of this random collision part. The physical relevance of this generalized collision rule is mainly related to the dissipative Boltzmann equation. Indeed, in a dissipative binary collision process, a particular choice of this random contribution is shown to produce the same steady state of the classical Boltzmann equation with standard dissipative binary collisions, in presence of a thermal bath [14].

A second main example of application of equation (2) is linked to the field of econophysics [6]. In this case, the generalized collision refers to a market based on binary trades between agents, in which part of the traded money is taken away by an external third subject, which redistributes it according to a certain economical random rule.

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For $A_0 = 0$ and for suitable choices of (A_1, A_2) , the one-dimensional kinetic equation (1) reduces to well-known simplified models for a spatially homogeneous gas, in which particles move only in one spatial direction. The basic assumption is that particles change their velocities only because of binary collisions. When two particles collide, then their velocities change from v and w , respectively, to

$$(3) \quad v' = p_1 v + q_1 w \quad w' = p_2 v + q_2 w$$

where (p_1, q_1) and (q_2, p_2) are two identically distributed random vectors (not necessarily independent) with the same law of (A_1, A_2) .

The first model of the type (1)&(2) has been introduced by Kac [22], with the collisional parameters $p_i = \sin \tilde{\theta}$ and $q_i = \cos \tilde{\theta}$, $i = 1, 2$, for a random angle $\tilde{\theta}$, uniformly distributed on $[0, 2\pi)$. The dynamics describes a gas in which the colliding molecules exchange a random fraction of their kinetic energies. This idea has been extended in [25] to gases with inelastically colliding molecules, which lose a random part of their energy in each interaction. The inelastic Kac equation corresponds to (1)&(2) with $p_i = |\sin \tilde{\theta}|^p \sin \tilde{\theta}$ and $q_i = |\cos \tilde{\theta}|^p \cos \tilde{\theta}$, with $p > 0$ being the parameter of inelasticity. Recently, more general versions of (1)&(2) have been considered: their applications range from gases under the influence of a background heat bath [11] to models for the redistribution of wealth in simple market economies [18, 23]. In most of the above mentioned cases, A_1 and A_2 are positive random variables such that $\mathbb{E}[A_1^2 + A_2^2] = 1$ (conservation of energy) [22, 18], or $\mathbb{E}[A_1 + A_2] = 1$ (conservation of momentum) [18, 23].

In the classical Boltzmann equation [12, 13] relaxation to Maxwellian equilibrium (Gaussian density) is shown to be a universal behavior of the solution. Contrary, the corresponding equilibria of model (1), to which the solution is shown to relax, depend heavily on the precise form of the microscopic interactions (3). Furthermore, they are not always explicitly known analytically.

In the case of models of wealth distribution in the society, the comparison of these steady states with realistic data is up to now the only means to evaluate—*a posteriori*—the quality of a proposed model. For instance, it is commonly accepted that the wealth distribution should approach a stationary (or, in general, a self-similar) profile for large times, and that the latter should exhibit a *Pareto tail* [16, 17]. The asymptotic behavior of the solutions of (1), when $A_0 = 0$, has been extensively treated in [2, 3, 9], and it is by now fully understood, in particular if one aims to describe a few analytically accessible properties (e.g. moments and smoothness).

The general situation in which $A_0 \neq 0$, while relevant in various applications which will be dealt with in this paper, has never been touched before. This case corresponds to assume that in a binary interaction the particle velocities change from v and w , respectively, to

$$(4) \quad v' = p_1 v + q_1 w + \eta_1 \quad w' = p_2 v + q_2 w + \eta_2$$

where (p_1, q_1, η_1) and (q_2, p_2, η_2) are two identically distributed random vectors with the same law of (A_1, A_2, A_0) . We will now describe the specific examples we are dealing with.

Kinetic models of a simple market economy with redistribution. In [6] Boltzmann-type kinetic models for wealth redistribution in a simple market economy have been introduced and discussed. The authors focused their attention to models which include taxes to each trading process. Assuming that a percentage of the total wealth involved in the trade is not returned to agents, the goal in [6] was to understand the role of redistribution, there produced by a linear transport-drift type operator.

Here we assume that the economic trades between agents are described by an interaction of type (4). In particular, for a given positive constant $0 < \epsilon < 1$, the post-interaction wealths including redistribution are given by

$$(5) \quad A_1 = (1 - \epsilon)\tilde{A}_1 \quad A_2 = (1 - \epsilon)\tilde{A}_2 \quad A_0 = \epsilon\tilde{A}_0$$

where $\mathbb{E}[\tilde{A}_1 + \tilde{A}_2] = 1$ and $\mathbb{E}[\tilde{A}_0] = m_0 = \int v \bar{\mu}_0(dv)$. Note that within this assumption, the total mean wealth is left unchanged. The mixing parameters $(\tilde{A}_1, \tilde{A}_2)$ can be chosen among the variety of models present in the pertinent literature, see e.g. [23]. The classical model introduced in [15] corresponds to the choice

$$\tilde{A}_1 = \lambda + \tilde{\eta}(1 - \lambda), \quad \tilde{A}_2 = \tilde{\eta}(1 - \lambda)$$

where $\tilde{\eta}$ is a random variable defined on $[0, 1]$ (symmetrically distributed around $1/2$) and $\lambda \in [0, 1]$ is a parameter (the so called saving propensity), while the pure gambling [5] corresponds to fix $\tilde{A}_1 = \tilde{A}_2 = \tilde{\eta}$.

An interesting variant of the previous model is obtained by setting

$$(6) \quad A_1 = (1 - \epsilon\Delta)\tilde{A}_1 \quad A_2 = (1 - \epsilon\Delta)\tilde{A}_2 \quad A_0 = \epsilon\Delta\tilde{A}_0$$

where $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_0)$ and Δ are stochastically independent, and $P\{\Delta = 1\} = 1 - P\{\Delta = 0\} = \delta$. The presence of Δ in (6) simulates a market in which taxation does not act on the totality of trades, but it occurs only with a probability δ .

As we shall see in Section 3.2, one can fix the values of (A_1, A_2, A_0) in such a way that the steady state of the model (1) produces the same steady states as the model considered in [6].

Inelastic Kac models with background. A second interesting application of binary interactions of type (4) is related to the study of a dissipative gas in a thermal bath [11, 14]. In one space-dimension, a dissipative Kac-like model has been introduced and discussed in [25]. As already mentioned, this model corresponds to the choice

$$(7) \quad A_1 = |\sin(\tilde{\theta})|^p \sin(\tilde{\theta}), \quad A_2 = |\cos(\tilde{\theta})|^p \cos(\tilde{\theta})$$

where $\tilde{\theta}$ is uniformly distributed on $[0, 2\pi)$. As shown in [4, 25], in consequence of the dissipation, a solution to the Kac equation corresponding to an initial value with finite second moment converges in time toward the probability mass located in zero. In addition to the physical dissipative interaction (7), let us now assume that particles velocities are subject to random fluctuations η_i , induced by an external background, whose distribution is the same of A_0 , while A_0 and (A_1, A_2) are stochastically independent. In addition let us assume that $A_0 \neq 0$, but $\mathbb{E}[A_0] = 0$.

As extensively discussed in [11], and directly verifiable on the single binary collision, the presence of this random fluctuation of zero mean is such that the post-collision energy is bigger than the corresponding one induced by the dissipative collision without fluctuations, i.e. when $A_0 = 0$. Indeed, since A_0 and (A_1, A_2) are stochastically independent and $\mathbb{E}[A_0] = 0$,

$$\begin{aligned} \mathbb{E}((v')^2 + (w')^2) &= \mathbb{E}(A_1^2 + A_2^2)(v^2 + w^2) + 4\mathbb{E}(A_1 A_2)vw + 2\mathbb{E}(A_0^2) \\ &= \mathbb{E}(A_1 v + A_2 w)^2 + \mathbb{E}(A_1 w + A_2 v)^2 + 2\mathbb{E}(A_0^2) \end{aligned}$$

with $\mathbb{E}(A_0^2) > 0$. The main consequence of this fact is that one can exhibit examples in which the initial value has finite second moment and at the same time the corresponding solution does not converge in time toward a degenerate distribution. The same phenomenon is shown to happen if one adds to the dissipative Boltzmann equation a thermal bath [14].

This allows to establish a direct link between the steady states of the present dissipative collisional models with random fluctuations and the steady states of the dissipative Boltzmann equation in presence of diffusion [14], as well as in presence of friction and/or drift [29]. Indeed, the steady states of the various problems on the dissipative Boltzmann equation quoted above, are steady states of the Boltzmann problem (1), corresponding to suitable choices of the random variables (A_1, A_2, A_0) . We will detail the correspondences between these problems in Section 3.

2. MAIN RESULTS

We start by writing the Boltzmann equation (1) in Fourier variables. By setting $\phi(t, \xi) = \int e^{i\xi v} \mu_t(dv)$, and using Bobylev's identity [7], one obtains that $\phi(t, \xi)$ obeys to the equation

$$(8) \quad \begin{cases} \partial_t \phi(t, \xi) + \phi(t, \xi) = \widehat{Q}^+(\phi(t, \cdot), \phi(t, \cdot))(\xi) & (t > 0, \xi \in \mathbb{R}) \\ \phi(0, \xi) = \phi_0(\xi) \end{cases}$$

where

$$(9) \quad \widehat{Q}^+(\phi(t, \cdot), \phi(t, \cdot))(\xi) := \mathbb{E}[\phi(t, A_1 \xi) \phi(t, A_2 \xi) e^{i\xi A_0}].$$

The initial condition $\phi_0(\xi) = \int e^{i\xi v} \bar{\mu}_0(dv)$ can be seen as the characteristic function of a prescribed real random variable X_0 , i.e. $\phi_0(\xi) = \mathbb{E}[e^{i\xi X_0}]$.

As in the case of the Kac equation, it is easy to see that (8) admits a unique solution ϕ which can be written as a Wild series [31]

$$(10) \quad \phi(t, \xi) = \sum_{n \geq 0} e^{-t} (1 - e^{-t})^n q_n(\xi),$$

where $q_0(\xi) = \phi_0(\xi)$ and, for $n \geq 1$,

$$(11) \quad q_n(\xi) = \frac{1}{n} \sum_{j=0}^{n-1} \hat{Q}^+(q_j, q_{n-1-j})(\xi).$$

Hence, if μ_t is the unique solution of (1) with initial condition $\bar{\mu}_0$, then its Fourier-Stieltjes transform is given by (10).

2.1. Steady states. The stationary equation associated to (8) is

$$(12) \quad \phi_\infty(\xi) = \hat{Q}^+(\phi_\infty, \phi_\infty)(\xi) \quad (\xi \in \mathbb{R}).$$

It can be proven that, under suitable hypotheses, a solution to (12) exist. To show that steady states exist it is enough to recast the problem as a problem of fixed point equation for distributions. In terms of probability distributions, (12) reads

$$(13) \quad Q^+(\mu, \mu) = \mu,$$

where, given any probability distribution μ , by (2), the probability distribution $Q^+(\mu, \mu)$ is the law of the random variable

$$A_0 + Y_1 A_1 + Y_2 A_2,$$

Y_1 and Y_2 having law μ and Y_1, Y_2 and (A_0, A_1, A_2) being stochastically independent.

In what follows, let us set

$$\mathcal{M}_\gamma := \left\{ \mu \text{ probability measure on } \mathcal{B}(\mathbb{R}) : \int_{\mathbb{R}} |x|^\gamma \mu(dx) < +\infty \right\},$$

and, for every m in \mathbb{R} and $\gamma \geq 1$,

$$\mathcal{M}_{\gamma, m} := \left\{ \mu \in \mathcal{M}_\gamma : \int_{\mathbb{R}} x \mu(dx) = m \right\}.$$

Finally, when $\mathbb{E}[A_1 + A_2] \neq 1$, let us define

$$\bar{m} := \frac{\mathbb{E}[A_0]}{1 - \mathbb{E}[A_1 + A_2]}.$$

The convex function $q : [0, \infty) \rightarrow [0, \infty]$ defined by

$$(14) \quad q(\gamma) := \mathbb{E}[|A_1|^\gamma + |A_2|^\gamma],$$

where $0^0 := 0$, will play a very important role in what follows.

First of all, let us collect some known results on the existence of solutions of equation (13).

Proposition 2.1 ([27],[28]). *Assume that there is γ in $(0, 2]$ such that $\mathbb{E}[|A_0|^\gamma] < +\infty$ and $q(\gamma) < 1$.*

- (a) *If $0 < \gamma \leq 1$, then there is a unique solution μ_∞ of (13) in \mathcal{M}_γ . In addition, if $\gamma = 1$, this solution belongs to $\mathcal{M}_{1, \bar{m}}$;*
- (b) *If $1 < \gamma \leq 2$ and $\mathbb{E}[A_1 + A_2] \neq 1$, then there is a unique solution μ_∞ of (13) in \mathcal{M}_γ and this solution belongs to $\mathcal{M}_{\gamma, \bar{m}}$;*
- (c) *If $1 < \gamma \leq 2$, $\mathbb{E}[A_1 + A_2] = 1$ and $\mathbb{E}[A_0] = 0$, then, for every $m_0 \in \mathbb{R}$, there is a unique solution μ_∞ of (13) in $\mathcal{M}_{\gamma, m_0}$.*

Let us notice that in case (a) (cfr. Lemma 5.2), it is possible to describe μ_∞ in terms of a suitable series of random variables.

While it is easy to check when μ_∞ is a degenerate distribution, necessary and sufficient conditions for boundedness of moments up to a certain order are more difficult to obtain. A partial answer to this problem is given in the next proposition.

Proposition 2.2. *Let the same hypotheses of Proposition 2.1 be in force.*

- (i) *In case (a) or (b) of Proposition 2.1 μ_∞ is a degenerate distribution if and only if $m(1 - (A_1 + A_2)) = A_0$ almost surely (a.s.) for some real number m ; in case (c) of Proposition 2.1 μ_∞ is a degenerate distribution if and only if $m_0(1 - (A_1 + A_2)) = A_0$ a.s.;*
- (ii) *If $q(\beta) < 1$ and $\mathbb{E}[|A_0|^\beta] < +\infty$ for some $\beta > 2$, then $q(s) < 1$ for every $\gamma \leq s \leq \beta$ and $\int |x|^\beta \mu_\infty(dx) < +\infty$;*
- (iii) *Let A_0, A_1 and A_2 be positive random variables with $P\{A_0 \neq 0\} > 0$. If, for some $\beta \geq \max\{1, \gamma\}$, $\int |x|^\beta \mu_\infty(dx) < +\infty$ and $\int x \mu_\infty(dx) > 0$, then $\mu_\infty\{[0, +\infty)\} = 1$ and $q(\beta) < 1$.*

2.2. Trend to equilibrium. We recall that the Kantorovich-Wasserstein distance of order $\gamma > 0$ between two probability measures μ and ν is defined by

$$(15) \quad l_\gamma(\mu, \nu) := \inf_{(X', Y')} (\mathbb{E}|X' - Y'|^\gamma)^{1/\max(\gamma, 1)}$$

where the infimum is taken over all pairs (X', Y') of real random variables whose marginal probability distributions are μ and ν , respectively.

If $(\nu_n)_n$ is a sequence of probability measures belonging to \mathcal{M}_γ and $\nu_\infty \in \mathcal{M}_\gamma$, then $l_\gamma(\nu_n, \nu_\infty) \rightarrow 0$ as $n \rightarrow +\infty$ if and only if ν_n converges weakly to ν_∞ and

$$\int |x|^\gamma \nu_n(dx) \rightarrow \int |x|^\gamma \nu_\infty(dx).$$

See, e.g., [26]. Recall that ν_n converges weakly to ν_∞ means that $\int g(x) \nu_n(dx) \rightarrow \int g(x) \nu_\infty(dx)$ for every g in $C_b(\mathbb{R})$.

We are now ready to state our main results concerning the long time behavior of the solutions.

Proposition 2.3. *Let $\gamma \in (0, 1)$. Assume that $\mathbb{E}[|X_0|^\gamma + |A_0|^\gamma] < +\infty$ and $q(\gamma) < 1$. Let μ_∞ be the unique solution in \mathcal{M}_γ to (13). Then, for every $t > 0$*

$$l_\gamma(\mu_t, \mu_\infty) \leq l_\gamma(\mu_0, \mu_\infty) e^{-t(1-q(\gamma))}.$$

In what follows, whenever $\mathbb{E}|X_0| < +\infty$, set $m_0 = \mathbb{E}[X_0]$.

Proposition 2.4. *Assume that $\mathbb{E}[|X_0| + |A_0|] < +\infty$ and that $q(1) < 1$. Let μ_∞ be the unique probability measure in \mathcal{M}_1 which satisfies (13). Then for every $t > 0$*

$$l_1(\mu_t, \mu_\infty) \leq l_1(\mu_0, \mu_\infty) e^{-t(1-q(1))}$$

and $\int v \mu_\infty(dv) = \bar{m}$. Moreover, if $m_0 = \bar{m}$, then $\int v \mu_t(dv) = \bar{m}$ for all $t \geq 0$.

Proposition 2.5. *Assume that, for some $\gamma \in (1, 2]$, $\mathbb{E}[|X_0|^\gamma + |A_0|^\gamma] < +\infty$, $q(\gamma) < 1$, $\mathbb{E}[A_1 + A_2] \neq 1$ and $m_0 = \bar{m}$. Let μ_∞ be the unique solution in \mathcal{M}_γ to (13). Then, for every $t > 0$,*

$$l_\gamma(\mu_t, \mu_\infty) \leq 2^{1/\gamma} l_\gamma(\mu_0, \mu_\infty) e^{-t(1-q(\gamma))/\gamma}$$

and $\int_{\mathbb{R}} v \mu_\infty(dv) = \bar{m}$. Moreover, $\int v \mu_t(dv) = \bar{m}$ for all $t \geq 0$.

Proposition 2.6. *Assume that, for some $\gamma \in (1, 2]$, $\mathbb{E}[|X_0|^\gamma + |A_0|^\gamma] < +\infty$, $q(\gamma) < 1$ and that $\mathbb{E}[A_1 + A_2] = 1$ and $\mathbb{E}[A_0] = 0$. Let μ_∞ be the unique solution in $\mathcal{M}_{\gamma, m_0}$ of (13). Then*

$$l_\gamma(\mu_t, \mu_\infty) \leq 2^{1/\gamma} l_\gamma(\mu_0, \mu_\infty) e^{-t(1-q(\gamma))/\gamma}$$

for every $t > 0$. Moreover, $\int v \mu_t(dv) = m_0$ for all $t \geq 0$.

3. EXAMPLES

3.1. Kinetic models of a simple market economy with redistribution. The first application of the results of Section 2 deals with the kinetic model for wealth with redistribution, briefly described in the Introduction. In this leading example, the random variables A_1, A_2, A_0 are given by (6). As already noticed, these assumptions correspond to a kinetic model for wealth distribution in which part of the wealth put into the binary trade is taken away by a third subject, which

at the same time restitutes to agents a certain amount of wealth. This is done in such a way that the mean total amount of wealth into the system is left unchanged. Assuming (6), one has

$$\mathbb{E}[A_1 + A_2] = 1 - \epsilon\delta < 1.$$

Hence, since $\mathbb{E}[\tilde{A}_0] = \int v\bar{\mu}_0(dv) = m_0 < +\infty$, one can invoke Propositions 2.1 (a) and 2.4 to prove both the existence and uniqueness in \mathcal{M}_1 of a steady state and the (exponential) convergence to this steady state of any solution with finite initial moment of order one.

One of the interesting effect of the redistribution is that the steady state can have finite moments of higher order than those of the steady states of the corresponding model without redistribution. This can be easily verified by comparing the steady states corresponding to $\epsilon > 0$, say $\mu_\infty^{(\delta, \epsilon)}$, with the steady states without redistribution, we will denote by $\mu_\infty^{(0,0)}$, obtained by setting $\delta = \epsilon = 0$. Thanks to Theorem 5.3 in [20], it is known that, given $\beta > 1$, $\int v^\beta \mu_\infty^{(0,0)}(dv) < +\infty$ if and only if $\tilde{q}(\beta) := \mathbb{E}[\tilde{A}_1^\beta + \tilde{A}_2^\beta] < 1$. On the other hand, if $\epsilon > 0$ and $\mathbb{E}[A_0^\beta] < +\infty$, by Proposition 2.2 (ii)-(iii), $\int v^\beta \mu_\infty^{(\delta, \epsilon)}(dv) < +\infty$ if and only if $q(\beta) < 1$. Since $q(\beta) = [1 + \delta[(1 - \epsilon)^\beta - 1]]\tilde{q}(\beta)$ and $[1 + \delta[(1 - \epsilon)^\beta - 1]] < 1$, one can easily give examples in which $\int v^\beta \mu_\infty^{(0,0)}(dv) = +\infty$ while $\int v^\beta \mu_\infty^{(\delta, \epsilon)}(dv) < +\infty$.

The previous discussion does not solve another interesting problem connected with wealth taxation and redistribution: the existence of an *optimal* amount of taxation. If one assumes that, given a certain (conserved) amount of money, the optimal redistribution refers to a steady state in which all people in the market ends up with almost the same amount of money, this problem can be solved by looking for the steady state with minimal variance. We leave this point to a further research.

3.2. Connections with other form of redistribution. We show here that the law of (A_1, A_2, A_0) can be fixed in such a way that the steady states of the redistribution model proposed in [6] fit into our framework.

Let us start by briefly outlining the model introduced in [6]. In Fourier variables this model reads

$$(16) \quad \frac{\partial}{\partial t} \phi(t, \xi) + \phi(t, \xi) = \hat{Q}_\epsilon(\phi, \phi)(t, \xi) + \hat{R}_\chi^\epsilon(\phi)(t, \xi)$$

with

$$(17) \quad \hat{Q}_\epsilon(\phi, \phi) := \mathbb{E}[\phi(A_1^* \xi) \phi(A_2^* \xi)],$$

and

$$(18) \quad \hat{R}_\chi^\epsilon(\phi)(\xi) := -\epsilon\chi\xi \frac{\partial}{\partial \xi} \phi(\xi) + i\epsilon(\chi + 1)m_0\xi\phi(\xi) \quad (\chi \geq -1),$$

In (17) (A_1^*, A_2^*) are positive random variables such that $\mathbb{E}[A_1^* + A_2^*] = 1 - \epsilon$, and $\frac{\partial}{\partial \xi} \phi(0, 0) = i \int v\bar{\mu}_0(dv) = im_0$. Note that in (16) the interaction operator consists in a *dissipative* collision operator, given by $\hat{Q}_\epsilon(\phi, \phi)$, and a redistribution (differential) operator $\hat{R}_\chi^\epsilon(\phi)(\xi)$. It is worth recalling that, if ϕ is the Fourier-Stieltjes transform of a (regular) density f , then $\hat{R}_\chi^\epsilon(\phi)$ is the Fourier-Stieltjes transform of

$$R_\chi^\epsilon(f)(v) = \epsilon \frac{\partial}{\partial v} \left[(\chi v - (\chi + 1)m_0) f(v) \right].$$

The possible steady states of (16) must satisfy

$$(19) \quad \phi(\xi) = \hat{Q}_\epsilon(\phi, \phi)(\xi) + \hat{R}_\chi^\epsilon(\phi)(\xi).$$

Existence of a global solution $\phi(\xi, t)$ to (16) has been proved in [6] provided that $\int v\bar{\mu}_0(dv) = m_0$. Anything was proven about the existence (and eventually uniqueness) of a steady state. This problem can be solved in a surprisingly easy way by establishing a connection between the steady states of the model [6] and special cases of our model.

First of all let us fix $\chi = -1$ in (18). In this case the redistribution operator simplifies, and equation (19) reduces to

$$(20) \quad \phi(\xi) = \hat{Q}_\epsilon(\phi, \phi)(\xi) + \epsilon \xi \frac{\partial}{\partial \xi} \phi(\xi).$$

Resorting to the analogous computation in Bobylev, Cercignani and Gamba [8, 9], equality (20) can be equivalently rewritten as

$$(21) \quad \phi(\xi) = \int_0^1 \hat{Q}_\epsilon(\phi, \phi)(\xi u^{-\epsilon}) du.$$

It is immediate to see that equation (21) can be rephrased as

$$(22) \quad \phi(\xi) = \mathbb{E}[\phi(U^{-\epsilon} A_1^* \xi) \phi(U^{-\epsilon} A_2^* \xi)],$$

where U and (A_1^*, A_2^*) are stochastically independent and U is uniformly distributed on $[0, 1]$. Hence, the steady state (20) coincides with the steady state (12) corresponding to $(A_0, A_1, A_2) := (0, U^{-\epsilon} A_1^*, U^{-\epsilon} A_2^*)$. Since in this case $q(1) = \mathbb{E}[A_1 + A_2] = 1$, in order to apply Proposition 2.1 (c) it is necessary that A_1^* and A_2^* satisfy

$$q(\gamma) = \frac{1}{1 - \gamma\epsilon} \mathbb{E}[(A_1^*)^\gamma + (A_2^*)^\gamma] < 1,$$

for some $1 < \gamma \leq \min\{2, 1/\epsilon\}$. If $\mathbb{E}[(A_1^*)^\gamma + (A_2^*)^\gamma] < 1$ this inequality holds true for every $\epsilon < (1 - \mathbb{E}[(A_1^*)^\gamma + (A_2^*)^\gamma])/\gamma$. It should be noticed that in this case, since $A_0 = 0$, and A_1 and A_2 are positive with $\mathbb{E}[A_1 + A_2] = 1$ one can resort also to Theorem 2(a) of [20].

Let us now consider the case in which $\chi = 0$, so that (18) corresponds to a pure transport operator, which produces a uniform redistribution. In this case, equation (19) becomes

$$(23) \quad \phi(\xi) = \hat{Q}_\epsilon(\phi, \phi)(\xi) + i\epsilon m_0 \xi \phi(\xi),$$

or, what is the same,

$$(24) \quad \phi(\xi) = \frac{1}{1 - i\epsilon m_0 \xi} \hat{Q}_\epsilon(\phi, \phi)(\xi).$$

Let us observe that if A_0 is an exponential random variable of mean ϵm_0 , that is with density $h_0(v) = \exp\{-v/(\epsilon m_0)\}/(\epsilon m_0)$ ($v > 0$), then

$$\mathbb{E}[e^{i\xi A_0}] = \int_0^{+\infty} e^{i\xi v} h_0(v) dv = \frac{1}{1 - i\epsilon m_0 \xi}.$$

Under the additional assumption that A_0 and (A_1^*, A_2^*) are stochastically independent, (24) can be equivalently written as

$$\phi(\xi) = \mathbb{E}[e^{i\xi A_0} \phi(A_1^* \xi) \phi(A_2^* \xi)].$$

Hence, it is enough to choose $A_1 = A_1^*$, $A_2 = A_2^*$ and A_0 as above to identify the steady state (23) with the steady state (12). Note that, since in this case $\mathbb{E}[A_1 + A_2] = 1 - \epsilon < 1$, the assumptions of Proposition 2.1 (a) are trivially satisfied for $\gamma = 1$.

Last, let us examine the physically relevant case in which $\chi > -1$ and $\chi \neq 0$. For any given $\epsilon \in (0, 1]$ set $\delta := \epsilon\chi > -\epsilon$. With this choice, (19) becomes

$$\hat{Q}_\epsilon(\phi, \phi)(\xi) = \phi(\xi) + \delta \xi \frac{\partial}{\partial \xi} \phi(\xi) - i(\delta + \epsilon) m_0 \xi \phi(\xi).$$

Multiplying both sides for $e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}-1}$ we get

$$\begin{aligned} \hat{Q}_\epsilon(\phi, \phi)(\xi) e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}-1} &= e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}-1} \left(\phi(\xi) + \delta \xi \frac{\partial}{\partial \xi} \phi(\xi) - i(\delta + \epsilon) m_0 \xi \phi(\xi) \right) \\ &= \frac{\partial}{\partial \xi} \left(\delta e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}} \phi(\xi) \right), \end{aligned}$$

which, integrating over $[0, \xi]$, gives

$$(25) \quad \delta e^{-i\xi m_0 \frac{\delta+\epsilon}{\delta}} \xi^{\frac{1}{\delta}} \phi(\xi) = \int_0^\xi \hat{Q}_\epsilon(\phi, \phi)(\tau) e^{-i\tau m_0 \frac{\delta+\epsilon}{\delta}} \tau^{\frac{1}{\delta}-1} d\tau.$$

By the change of variable $\xi u^\delta = \tau$, we can write the previous equation in the equivalent form

$$\phi(\xi) = \int_0^1 \hat{Q}_\epsilon(\phi, \phi)(\xi u^\delta) e^{i(1-u^\delta)\frac{\delta+\epsilon}{\delta}m_0\xi} du,$$

which can be rephrased as

$$\phi(\xi) = \mathbb{E}[\phi(U^\delta A_1^* \xi) \phi(U^\delta A_2^* \xi) e^{i(1-U^\delta)\frac{\delta+\epsilon}{\delta}m_0\xi}],$$

where (A_1^*, A_2^*) and U are stochastically independent and U is uniformly distributed on $[0, 1]$. Hence (19) is equivalent to

$$\phi(\xi) = \mathbb{E}[\phi(A_1 \xi) \phi(A_2 \xi) e^{iA_0 \xi}]$$

for

$$(26) \quad A_1 = U^\delta A_1^* \quad A_2 = U^\delta A_2^* \quad A_0 = (1 - U^\delta) \frac{\delta + \epsilon}{\delta} m_0.$$

Since $\delta > -\epsilon$, it is immediate to reckon that $\mathbb{E}[A_1 + A_2] = (1 - \epsilon)/(1 + \delta) < 1$. Hence Proposition 2.1 (a) applies. In particular, since $\mathbb{E}[A_0]/(1 - \mathbb{E}[A_1 + A_2]) = m_0$, the solution μ_∞ described in Proposition 2.1 satisfies $\int v \mu_\infty(dv) = m_0$.

3.3. Inelastic Kac models with background and connection with dissipative models with diffusion. A further application of the results of Section 2, announced in the Introduction, results from the choice

$$(27) \quad A_1 = |\sin(\tilde{\theta})|^p \sin(\tilde{\theta}), \quad A_2 = |\cos(\tilde{\theta})|^p \cos(\tilde{\theta})$$

with $\tilde{\theta}$ uniformly distributed on $[0, 2\pi)$. This assumption leads, when $A_0 = 0$, to the inelastic Kac model [25], which describes the cooling of a one-dimensional spatially homogeneous Maxwell-like gas. In fact, if $A_0 = 0$ and $\int |v|^{2/(1+p)} \mu_0(dv) < +\infty$, μ_t is shown to converge weakly to the probability mass concentrated in 0 (cfr. [4]). As we shall see, the addition of a random fluctuation, described by the random variable $A_0 \neq 0$, is responsible for the formation of non-trivial steady states. If A_1 and A_2 are given by (27), then

$$\mathbb{E}[A_1 + A_2] = 0,$$

and, whenever $\gamma > 2/(1+p)$,

$$q(\gamma) = \mathbb{E}[|A_1|^\gamma + |A_2|^\gamma] = \frac{1}{2\pi} \int_0^{2\pi} [|\sin(\theta)|^{(p+1)\gamma} + |\cos(\theta)|^{(p+1)\gamma}] d\theta < 1.$$

Hence, if for some positive ϵ , with $\epsilon + 2/(1+p) < 2$, $\mathbb{E}[A_0]^{2/(1+p)+\epsilon} < +\infty$ (with $\mathbb{E}[A_0] = m_0 := \int v \mu_0(dv)$ if $p < 1$) and $\int |v|^{2/(1+p)+\epsilon} \bar{\mu}_0(dv) < +\infty$, then Propositions 2.3-2.5 apply. In particular, if $\mathbb{E}[A_0] = 0$ and $P\{A_0 \neq 0\} > 0$, the steady state is a non-degenerate probability distribution with finite moments of all orders.

As a special case let us choose $A_0 = A_{0,a} - A_{0,b}$ with $A_{0,a}$ and $A_{0,b}$ exponentially distributed with density $v \mapsto \exp\{-v/a\}/a$ and $v \mapsto \exp\{-v/b\}/b$ ($v > 0$), and assume that $A_{0,a}, A_{0,b}, A_1, A_2$ are stochastically independent. Since

$$\mathbb{E}[e^{i\xi A_0}] = \mathbb{E}[e^{i\xi A_{0,a}}] \mathbb{E}[e^{-i\xi A_{0,b}}] = \frac{1}{(1 - ia\xi)(1 + ib\xi)} = \frac{1}{1 - i(a-b)\xi + ab\xi^2},$$

if $a := (m_0 + \sqrt{m_0^2 + 4\sigma^2})/2$ and $b := (-m_0 + \sqrt{m_0^2 + 4\sigma^2})/2$ the stationary equation (12) becomes

$$\mathbb{E}[\phi(A_1 \xi) \phi(A_2 \xi)] = \phi(\xi)(1 - im_0\xi + \sigma^2\xi^2)$$

which can be equivalently written, after setting $\hat{Q}_p^+(\phi, \phi) := \mathbb{E}[\phi(A_1 \xi) \phi(A_2 \xi)]$, as

$$(28) \quad \phi(\xi) = \hat{Q}_p^+(\phi, \phi)(\xi) - \sigma^2 \xi^2 \phi(\xi) + im_0 \xi \phi(\xi).$$

Equation (28) describes the steady states of the inelastic Kac equation in presence of a thermal bath and a transport term. Indeed, if ϕ is the Fourier-Stieltjes transform of a density f , then $-\sigma^2 \xi^2 \phi(\xi) + im_0 \xi \phi(\xi)$ is the Fourier-Stieltjes transform of

$$\sigma^2 \frac{\partial^2}{\partial v^2} f(v, t) - m_0 \frac{\partial}{\partial v} f(v, t).$$

In particular, the analysis of Section 2 allows to prove existence of a steady state for the dissipative Kac equation with diffusion. The problem of the solvability of equations of type

$$(29) \quad Q(f, f) + \sigma^2 \Delta f = 0,$$

in terms of nonnegative integrable densities $f \in L^1_+(\mathbb{R}^3)$, and where Q is the Boltzmann collision operator, is a well-known problem in kinetic theory of rarefied gases. When Q is the dissipative collision operator for Maxwellian molecules, existence of non trivial weak solutions has been proved by Cercignani, Illner and Stoica [14].

Also, as clearly discussed by Villani in [29], apart from collisions, other physically relevant problem in kinetic theory of granular gases lead to the addition of various terms which either model external physical forces, or arise from particular situations. One of these situations is described by equation (29). A second one is obtained by subtracting a drift term to the Boltzmann collision operator. This leads to the problem of finding steady states of the equation

$$(30) \quad Q(f, f) - \sigma^2 \nabla \cdot (vf) = 0.$$

Let us remark that in one dimension of the velocity space, equation (30) is a particular case of equation (19) with $\chi = -1$, which has been solved in the previous Sub-section.

4. PROBABILISTIC REPRESENTATION OF THE SOLUTIONS

The core of the proofs of our results is a suitable probabilistic representation of the solution μ_t . The idea to represent the solutions of the Kac equation in a probabilistic way dates back, at least, to the work of McKean [24], but it has been fully formalized and employed in the derivation of analytic results for the Kac equation only in the last decade, starting from [10] and [21].

Our approach here follows the same steps used in [2] and [3] and it is based on the concept of random recursive binary trees. It is worth recalling that a binary tree is a (planar and rooted) tree where each node is either a leaf (that is, it has no successor) or it has 2 successors. We define the size of the binary tree τ , in symbol $|\tau|$, by the number of internal nodes. Hence, any binary tree with $2k + 1$ nodes has size k and possesses $k + 1$ leaves. Any binary tree can be seen as a subset of

$$\mathbb{U} = \{\emptyset\} \cup [\cup_{k \geq 1} \{1, 2\}^k].$$

As usual \emptyset is the root and if $\sigma = (\sigma_1, \dots, \sigma_k)$ ($\sigma_i \in \{1, 2\}$) is a node of a binary tree then the length of σ is $|\sigma| := k$. Moreover $(\sigma, \sigma_{k+1}) := (\sigma_1, \dots, \sigma_k, \sigma_{k+1})$ and for every $1 \leq i \leq k$, $\sigma|i := (\sigma_1, \dots, \sigma_i)$ and $\sigma|0 = \emptyset$.

We now describe a tree evolution process which gives rise to the so called “random binary recursive tree”. The evolution process starts with T_0 , an empty tree, with just an external node (the root). The first step in the growth process is to replace this external node by an internal one with 2 successors that are leaf. In this way one obtains T_1 . Then with probability 1/2 (i.e. one over the number of leaves) one of these 2 leaves is selected and again replaced by an internal node with 2 successors. One continues along the same rules. At every time k , a binary tree T_k with k internal nodes is obtained. For more details on binary recursive trees see, for instance, [19].

In the rest of the paper, given a binary tree τ , we shall denote by $\mathcal{L}(\tau)$ the set of the leaves of τ and by $\mathcal{I}(\tau)$ the set of the internal nodes of τ .

The Wild series expansion (10)-(11) can be translated in a probabilistic representation of the solutions as sums of random variables indexed by binary recursive random trees. On a sufficiently large probability space (Ω, \mathcal{F}, P) let the following be given:

- a family $(X_v)_{v \in \mathbb{U}}$ of independent random variables with common probability distribution $\bar{\mu}_0$;
- a family $(A_0(v), A_1(v), A_2(v))_{v \in \mathbb{U}}$ of independent positive random vectors with the same distribution of (A_0, A_1, A_2) ;
- a sequence of binary recursive random trees $(T_n)_{n \in \mathbb{N}}$;
- a stochastic process $(\nu_t)_{t \geq 0}$ with values in \mathbb{N}_0 such that $P\{\nu_t = k\} = e^{-t}(1 - e^{-t})^k$ for every integer $k \geq 0$.

Write $A(v) = (A_0(v), A_1(v), A_2(v))$ and assume further that

$$(A(v))_{v \in \mathbb{U}}, \quad (T_n)_{n \geq 1}, \quad (X_v)_{v \in \mathbb{U}} \quad \text{and} \quad (\nu_t)_{t > 0}$$

are stochastically independent.

For each node $v = (v_1, \dots, v_k)$ in \mathbb{U} set

$$\varpi(v) := \prod_{i=0}^{|v|-1} A_{v_{i+1}}(v|i)$$

and $\varpi(\emptyset) = 1$. Define

$$W_0 := X_\emptyset \quad \text{and} \quad \Gamma_0 := 0$$

and, for any $n \geq 1$,

$$W_n := \sum_{v \in \mathcal{L}(T_n)} \varpi(v) X_v, \quad \Gamma_n := \sum_{v \in \mathcal{I}(T_n)} \varpi(v) A_0(v), \quad W_n^* := W_n + \Gamma_n.$$

Proposition 4.1. *Equation (8) has a unique solution ϕ , which coincides with the characteristic function of $W_{\nu_t}^*$, i.e.*

$$\phi(t, \xi) = \mathbb{E}[e^{i\xi W_{\nu_t}^*}] = \sum_{n=0}^{\infty} e^{-t} (1 - e^{-t})^n \mathbb{E}[e^{i\xi W_n^*}] \quad (t > 0, \xi \in \mathbb{R}).$$

Proof. We need some preliminary results on recursive binary trees. A very important issue is that any binary tree has a recursive structure. More precisely we can use the following recursive definition of binary trees: a binary tree τ is either just an external node or an internal node with 2 subtrees, $\tau^{(1)}, \tau^{(2)}$, that are again binary trees. For every $k \geq 0$ let \mathbb{T}_k denote the set of all binary trees with size k . By Proposition 3.1 in [2], we know that if $(T_k)_{k \geq 0}$ is a sequence of random binary recursive trees, then for every $k \geq 1$, $j = 0, \dots, k-1$ and every τ in \mathbb{T}_k ,

$$(31) \quad \begin{aligned} P\{T_k^{(1)} = \tau^{(1)}, T_k^{(2)} = \tau^{(2)} \mid |T_k^{(1)}| = j\} \\ = P\{T_j = \tau^{(1)}\} P\{T_{k-j-1} = \tau^{(2)}\} \mathbb{I}\{|\tau^{(1)}| = j\} \end{aligned}$$

and for $k \geq 1$

$$(32) \quad P\{|T_k^{(1)}| = j\} = \frac{1}{k}$$

for every $j = 0, \dots, k-1$. Now observe that, in order to prove the proposition we need only to prove that $q_n(\xi) = \mathbb{E}[e^{i\xi W_n^*}]$, for every $n \geq 0$. This is clearly true for $n = 0$. For $n \geq 1$, write

$$W_n^* = A_0(\emptyset) + \sum_{j=1}^2 A_j(\emptyset) \left\{ \left[\sum_{v \in \mathcal{L}(T_n^{(j)})} \prod_{i=0}^{|v|-1} A_{v_{i+1}}^{(j)}(v|i) X_v^{(j)} \right] + \left[\sum_{v \in \mathcal{I}(T_n^{(j)})} \prod_{i=0}^{|v|-1} A_{v_{i+1}}^{(j)}(v|i) A_0^{(j)}(v) \right] \right\}$$

where $A^{(j)}(v) = A((j, v))$, $X_v^{(j)} = X_{(j,v)}$ and, by convention, if $\mathcal{L}(T_n^{(j)}) = \emptyset$ the terms between square brackets is equal to $X_\emptyset^j = X_j$. Since $(A^{(j)}(v), X_v^{(j)})_{v \in \mathbb{U}}$, $j = 1, 2$, are independent, with the same distribution of $(A(v), X_v)_{v \in \mathbb{U}}$, using (31) and the induction hypothesis one proves that

$$(33) \quad \mathbb{E} \left[e^{i\xi W_n^*} \mid A(\emptyset), |T_n^{(1)}|, |T_n^{(2)}| \right] = \prod_{j=1}^2 q_{|T_n^{(j)}|}(\xi A_j(\emptyset)) e^{i\xi A_0(\emptyset)}.$$

At this stage the conclusion follows easily by using (32); indeed:

$$\mathbb{E}[e^{i\xi W_n^*}] = \mathbb{E} \left[\prod_{j=1}^2 q_{|T_n^{(j)}|}(\xi A_j(\emptyset)) e^{i\xi A_0(\emptyset)} \right] = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E} \left[q_j(\xi A_1) q_{n-j-1}(\xi A_2) e^{i\xi A_0} \right] = q_n(\xi).$$

□

5. PROOFS OF SECTION 2

Proposition 2.1, for $\gamma = 2$, is proved in [27], while for general $\gamma \in (0, 2)$ it can be seen as a special case of a (more general) result contained in [28]. The proof of Proposition 2.1 given in [28] is based on some contraction properties of the Wasserstein metrics. Here we provide a proof for $\gamma \in (0, 2]$ based on a martingale method inspired by [27]. In this way we obtain some additional information on the solution, used to prove Proposition 2.2.

In the following we need to consider a sequence $(T_n^*)_{n \geq 0}$ of (deterministic) binary trees. Any such tree can be seen as a subset of \mathbb{U} : starting from $T_0^* = \emptyset$, for each n denote by T_n^* the binary tree obtained from T_{n-1}^* replacing each leaf by an internal node with 2 successors. Recall that $\mathcal{L}(T_n^*)$ ($\mathcal{I}(T_n^*)$, respectively) denotes the set of the leaves (the internal nodes, respectively) of T_n^* .

Define $K(\mu) := Q^+(\mu, \mu)$ and $K^n(\delta_m)$ as the n -iterate of the transformation K applied to the mass probability concentrated on the real value m . Finally set

$$M_n^* := m \sum_{v \in \mathcal{L}(T_n^*)} \varpi(v) + \sum_{v \in \mathcal{I}(T_n^*)} \varpi(v) A_0(v).$$

In the rest of the paper L^γ will stand for $L^\gamma(\Omega, \mathcal{F}, P)$.

Lemma 5.1. *Let $q(\gamma) < 1$ for some γ in $(1, 2]$ and $\mathbb{E}|A_0|^\gamma < +\infty$. Assume either*

- (i) $\mathbb{E}(A_1 + A_2) \neq 1$ and $m = \mathbb{E}(A_0)/(1 - \mathbb{E}(A_1 + A_2)) = \bar{m}$ or
- (ii) $\mathbb{E}(A_1 + A_2) = 1$, $\mathbb{E}(A_0) = 0$ and m arbitrary.

Then

- (a) $K^n(\delta_m)$ is the law of M_n^* ;
- (b) $(M_n^*)_{n \geq 0}$ is a martingale with respect to $(\mathcal{G}_n^*)_{n \geq 1}$, with $\mathcal{G}_n^* = \sigma(A(v) : v \in T_{n-1}^*)$, such that $\mathbb{E}(M_n^*) = m$ for every n ;
- (c) $\sup_n \mathbb{E}|M_n^*|^\gamma < +\infty$, hence $(M_n^*)_{n \geq 0}$ converges a.s. and in L^1 to a random variable M_∞^* such that $\mathbb{E}(M_\infty^*) = m$ and $\mathbb{E}|M_\infty^*|^\gamma < +\infty$;
- (d) the law μ_∞ of M_∞^* is a solution of (13) in \mathcal{M}_γ ;
- (e) If (ii) holds true then

$$M_\infty^* = mZ_\infty + \sum_{n \geq 0} \sum_{v \in \mathcal{L}(T_n^*)} \varpi(v) A_0(v)$$

where Z_∞ is the almost sure limit of $\sum_{v \in \mathcal{L}(T_n^*)} \varpi(v)$ for $n \rightarrow +\infty$.

Proof. (a) is immediate for $n = 1$. In fact

$$K^1(\delta_m) = \mathcal{D}(A_1(\emptyset)m + A_2(\emptyset)m + A_0(\emptyset)) = \mathcal{D}\left(\sum_{v \in \mathcal{L}(T_1^*)} \varpi(v)m + \sum_{v \in \mathcal{I}(T_1^*)} \varpi(v)A_0(v)\right),$$

where, for every random variable X , $\mathcal{D}(X)$ denotes the law of X . Now, by induction, we obtain

$$\begin{aligned} K^n(\delta_m) &= K(K^{n-1}(\delta_m)) \\ &= K\left(\mathcal{D}\left(\sum_{v \in \mathcal{L}(T_{n-1}^*)} \varpi(v)m + \sum_{v \in \mathcal{I}(T_{n-1}^*)} \varpi(v)A_0(v)\right)\right). \end{aligned}$$

At this stage, denote by T_n^{*1} (T_n^{*2} , respectively) the left (right, respectively) binary subtree of T_n^* . For every v in \mathbb{U} and $i = 1, 2$ set $\varpi^i(v) = \varpi((i, v))/A_i(\emptyset)$ if $A_i(\emptyset) \neq 0$ and $\varpi^i(v) = 0$ if $A_i(\emptyset) = 0$. It is plain to check that

$$(34) \quad \sum_{v \in \mathcal{L}(T_n^{*i})} \varpi^i(v)m + \sum_{v \in \mathcal{I}(T_n^{*i})} \varpi^i(v)A_0((i, v))$$

$i = 1, 2$ are independent random variables with the same law of M_{n-1}^* .

Hence,

$$\begin{aligned}
\mathcal{K}^n(\delta_m) &= \mathcal{D}\left(A_1(\emptyset)\left(\sum_{v \in \mathcal{L}(T_n^{*1})} \varpi^1(v)m + \sum_{v \in \mathcal{I}(T_n^{*1})} \varpi^1(v)A_0((1,v))\right)\right. \\
(35) \quad &+ \left.A_2(\emptyset)\left(\sum_{v \in \mathcal{L}(T_n^{*2})} \varpi^2(v)m + \sum_{v \in \mathcal{I}(T_n^{*2})} \varpi^2(v)A_0((2,v))\right) + A_0(\emptyset)\right) \\
&= \mathcal{D}(M_n^*).
\end{aligned}$$

As for (b) is concerned, clearly M_n^* is integrable and \mathcal{G}_n^* measurable. Moreover,

$$M_n^* = M_{n-1}^* + \sum_{v \in \mathcal{L}(T_{n-1}^*)} \varpi(v)[(A_1(v) + A_2(v) - 1)m + A_0(v)]$$

and hence

$$\mathbb{E}[M_n^* | \mathcal{G}_{n-1}^*] = M_{n-1}^*$$

in both cases (i) and (ii). Furthermore, $\mathbb{E}[M_n^*] = m$ for every n . Since M_n^* is a martingale and $1 < \gamma \leq 2$, we can apply the Topchii-Vatutin inequality, see e.g. [1], to get

$$\begin{aligned}
\mathbb{E}[|M_n^*|^\gamma] &\leq \mathbb{E}[|M_0^*|^\gamma] + 2 \sum_{j=1}^n \mathbb{E}[|M_j^* - M_{j-1}^*|^\gamma] \\
&= m^\gamma + 2 \sum_{j=1}^n \mathbb{E}\left[\left|\sum_{v \in \mathcal{L}(T_{j-1}^*)} \varpi(v)[(A_1(v) + A_2(v) - 1)m + A_0(v)]\right|^\gamma\right].
\end{aligned}$$

Now, since $\mathbb{E}[(A_1(v) + A_2(v) - 1)m + A_0(v) | \mathcal{G}_{n-1}^*] = \mathbb{E}[(A_1(v) + A_2(v) - 1)m + A_0(v)] = 0$, by the Bhaahr-Esseen inequality (see [30]) we obtain

$$\mathbb{E}[|M_n^*|^\gamma] \leq m^\gamma + K \sum_{j=1}^n \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_{j-1}^*)} |\varpi(v)|^\gamma\right]$$

where $K = 4\mathbb{E}[|(A_1 + A_2 - 1)m + A_0|^\gamma] < +\infty$ by assumption. Now it is easy to see that, for every $k \geq 0$,

$$\mathbb{E}\left[\sum_{v \in \mathcal{L}(T_k^*)} |\varpi(v)|^\gamma\right] = q(\gamma)^k$$

with $q(\gamma) < 1$. Hence, $\sup_n \mathbb{E}[|M_n^*|^\gamma] < +\infty$ and, from the elementary martingale theory, it follows that $(M_n^*)_{n \geq 0}$ converges a.s. and in L^1 to a random variable M_∞^* such that $\mathbb{E}[M_\infty^*] = m$ and $\mathbb{E}[|M_\infty^*|^\gamma] < +\infty$. The proof of (c) is completed. In order to prove (d) set $\phi_n(\xi) = \mathbb{E}[\exp(i\xi M_n^*)]$. By (34), it is clear that (35) is equivalent to

$$\phi_n(\xi) = \mathbb{E}[\phi_{n-1}(A_1\xi)\phi_{n-1}(\xi A_2)e^{i\xi A_0}].$$

From (c) we know that $\phi_n(\xi)$ converges to $\phi_\infty(\xi) = \mathbb{E}[\exp(i\xi M_\infty^*)]$ as $n \rightarrow +\infty$. Hence, by dominated convergence theorem, we get

$$\phi_\infty(\xi) = \mathbb{E}[\phi_\infty(A_1\xi)\phi_\infty(\xi A_2)e^{i\xi A_0}]$$

and the proof of (d) is completed. Arguing as in the proof of (c) it is easy to see that under (ii), the terms $\sum_{v \in \mathcal{L}(T_n^*)} \varpi(v)m$ and $\sum_{v \in \mathcal{I}(T_n^*)} \varpi(v)A_0(v)$, which form M_n^* , are both uniformly integrable martingales. Hence (e) follows easily. \square

Lemma 5.2. *Let $q(\gamma) < 1$ for some γ in $(0, 1]$ and $\mathbb{E}|A_0|^\gamma < +\infty$. Then, for every m ,*

- (a) $\mathcal{K}^n(\delta_m)$ is the law of M_n^* ;
- (b) $\sum_{v \in \mathcal{L}(T_n^*)} \varpi(v)$ converges to 0 in L^γ ;
- (c) $\Gamma_n^* = \sum_{v \in \mathcal{I}(T_n^*)} \varpi(v)A_0(v)$ is a Cauchy sequence in L^γ and hence it converges in L^γ to the random variable

$$\Gamma_\infty^* = \sum_{n \geq 0} \sum_{v \in \mathcal{L}(T_n^*)} \varpi(v)A_0(v);$$

(d) M_n^* converges to Γ_∞^* in L^γ and the law μ_∞ of Γ_∞^* is a solution of (13) in \mathcal{M}_γ ;

Proof. The proof of (a) is the same of the proof of (a) in Proposition 5.1. Furthermore, since $\gamma \in (0, 1]$,

$$\mathbb{E} \left[\left| \sum_{v \in \mathcal{L}(T_n^*)} \varpi(v) \right|^\gamma \right] \leq \mathbb{E} \left[\sum_{v \in \mathcal{L}(T_n^*)} |\varpi(v)|^\gamma \right] = q(\gamma)^n.$$

Using the fact that $q(\gamma) < 1$, (b) follows. In order to prove (c) observe that, if $n > m$,

$$\mathbb{E}[|\Gamma_n^* - \Gamma_m^*|^\gamma] = \mathbb{E} \left[\left| \sum_{j=m}^{n-1} \sum_{v \in \mathcal{L}(T_n^*)} \varpi(v) A_0(v) \right|^\gamma \right] \leq \mathbb{E}[|A_0|^\gamma] \sum_{j=m+1}^n q(\gamma)^j \rightarrow 0$$

for $n, m \rightarrow +\infty$, i.e. $(\Gamma_n^*)_n$ is a Cauchy sequence in L^γ . As for assertion (d) is concerned, combining (b) and (c) we obtain that $(M_n^*)_n$ converges in L^γ to Γ_∞^* . Finally, arguing as in the proof of (d) of Proposition 5.1 we obtain that Γ_∞^* is a solution of (13) in \mathcal{M}_γ . \square

Proof of Proposition 2.1. The existence of a solution μ_∞ of (13) in \mathcal{M}_γ , as required in (a), (b) and (c), is a consequence of Lemma 5.1 and Lemma 5.2. Let us prove the uniqueness. Let μ_1 and μ_2 be two solutions of (13) in \mathcal{M}_γ . Let $(Y_v^{(1)})_v$ and $(Y_v^{(2)})_v$ two sequences of independent random variables such that, for every $v \in \mathbb{U}$, $Y_v^{(1)}$ ($Y_v^{(2)}$, respectively) has law μ_1 (μ_2 , respectively) and, in addition,

$$(Y_v^{(1)})_v, (Y_v^{(2)})_v, (A(v))_v$$

are stochastically independent. Then, following the same lines of (a) in Lemma 5.1 and Lemma 5.2, it is easy to see that

$$\sum_{v \in \mathcal{L}(T_n^*)} \varpi(v) Y_v^{(i)} + \sum_{v \in \mathcal{I}(T_n^*)} \varpi(v) A_0(v)$$

has law μ_i ($i = 1, 2$). As a consequence, if $\gamma \in (0, 1]$, then

$$\begin{aligned} l_\gamma(\mu_1, \mu_2) &\leq \mathbb{E} \left[\left| \sum_{v \in \mathcal{L}(T_n^*)} \varpi(v) (Y_v^{(1)} - Y_v^{(2)}) \right|^\gamma \right] \\ &\leq \mathbb{E}[|Y_\emptyset^{(1)} - Y_\emptyset^{(2)}|^\gamma] \mathbb{E} \left[\sum_{v \in \mathcal{L}(T_n^*)} |\varpi(v)|^\gamma \right] \\ &= \mathbb{E}[|Y_\emptyset^{(1)} - Y_\emptyset^{(2)}|^\gamma] q(\gamma)^n \end{aligned}$$

and $q(\gamma)^n \rightarrow 0$ for $n \rightarrow +\infty$. Hence $\mu_1 = \mu_2$ and this proves (a). As far as (b) is concerned notice that if $\gamma \in (1, 2]$ and $\mathbb{E}[A_1 + A_2] \neq 1$, it follows that

$$\mathbb{E}[Y_v^{(1)}] = \mathbb{E}[Y_v^{(2)}] = \bar{m}$$

and applying the Bhaaer-Esseen inequality

$$\begin{aligned} l_\gamma^2(\mu_1, \mu_2) &\leq \mathbb{E} \left[\left| \sum_{v \in \mathcal{L}(T_n^*)} \varpi(v) (Y_v^{(1)} - Y_v^{(2)}) \right|^\gamma \right] \\ &\leq 2 \mathbb{E}[|Y_\emptyset^{(1)} - Y_\emptyset^{(2)}|^\gamma] \mathbb{E} \left[\sum_{v \in \mathcal{L}(T_n^*)} |\varpi(v)|^\gamma \right] \\ &= 2 \mathbb{E}[|Y_\emptyset^{(1)} - Y_\emptyset^{(2)}|^\gamma] q(\gamma)^n \end{aligned}$$

and hence $\mu_1 = \mu_2$ again. The case (c) follows in an analogous way since we need to consider only μ_1 and μ_2 in $\mathcal{M}_{\gamma, \bar{m}}$ (i.e. we fix the mean). \square

Proof of Proposition 2.2. The proof of (i) is straightforward.

The proofs of (ii) and (iii) are inspired by the proof of Theorem 5.3 in [20]. Let us first prove (ii). Note that, since q is a convex function, for every λ in $[0, 1]$, $q(\lambda\gamma + (1-\lambda)\beta) \leq \lambda q(\gamma) + (1-\lambda)q(\beta)$,

and hence $q(s) < 1$ for every $\gamma \leq s \leq \beta$. In addition $\mathbb{E}|A_0|^s < +\infty$ since $\mathbb{E}|A_0|^\beta < +\infty$. Now fix $s \leq \beta$, with $1 \leq k < s \leq k+1$, k integer. Then, for $x_i \geq 0$

$$(36) \quad \left(\sum_{i=1}^3 x_i \right)^s = \left(\sum_{i=1}^3 x_i \right)^{\frac{s}{k+1}(k+1)} \leq \sum_{i=1}^3 x_i^s + \sum c_{j_1 j_2 j_3} (x_1^{j_1} x_2^{j_2} x_3^{j_3})^{\frac{s}{k+1}}$$

for suitable constants $c_{j_1 j_2 j_3}$ and j_i are integers such that $j_i \leq k$ and $j_1 + j_2 + j_3 = k+1$. Using (36) it is easy to see that

$$(37) \quad \mathbb{E}[|Y_1 A_1 + Y_2 A_2 + A_0|^s] \leq q(s) \mathbb{E}|Y|^s + c_1 \mathbb{E}[|Y^k|^{\frac{s}{k}}] + c_2$$

if Y, Y_1, Y_2 are independent random variables with the same law ν and (Y, Y_1, Y_2) is independent of (A_1, A_2, A_0) . The constants c_1 and c_2 may depend on β but not on ν . Obviously (37) is equivalent to

$$(38) \quad \int |x|^s (\mathbb{K}\nu)(dx) \leq q(s) \int |x|^s \nu(dx) + c_1 \left[\int |x|^k \nu(dx) \right]^{\frac{s}{k}} + c_2$$

Let us consider first the case in which $\gamma \in (1, 2]$. Choose either $m = \bar{m}$ if $\mathbb{E}[A_1 + A_2] \neq 1$ or $m = m_0 = \mathbb{E}[X_0]$ if $\mathbb{E}[A_1 + A_2] = 1$ and $\mathbb{E}[A_0] = 0$, and take $\nu = \delta_m$. From Lemma 5.1 we know that $\mathbb{K}^n \delta_m$ converges weakly to μ_∞ and that

$$\sup_n \int |x|^\gamma (\mathbb{K}^n \delta_m)(dx) < +\infty.$$

Let us now prove that if for $k \geq 1$ and $k < s \leq k+1$, one has

$$(39) \quad \sup_n \int |x|^k (\mathbb{K}^n \delta_m)(dx) < +\infty \quad \text{and} \quad q(s) < 1$$

then

$$(40) \quad \sup_n \int |x|^s (\mathbb{K}^n \delta_m)(dx) < +\infty \quad \text{and} \quad \int |x|^s \mu_\infty(dx) < +\infty.$$

In fact, applying iteratively (38) starting from $\nu = \delta_m$, since $\int |x|^s \delta_m(dx) = |m|^s < +\infty$, we obtain

$$\int |x|^s (\mathbb{K}^n \delta_m)(dx) \leq |m|^s q(s)^n + C \sum_{j=0}^{n-1} q(s)^j$$

for a suitable constant C . Hence, since $q(s) < 1$, one gets $\sup_n \int |x|^s (\mathbb{K}^n \delta_m)(dx) < +\infty$. Furthermore define $g_M(x) = |x|^s \mathbb{I}_{\{|x| \leq M\}} + M^s \mathbb{I}_{\{|x| > M\}}$ then

$$\begin{aligned} \int |x|^s \mu_\infty(dx) &= \int \liminf_{M \rightarrow +\infty} g_M(x) \mu_\infty(dx) \leq \liminf_{M \rightarrow +\infty} \int g_M(x) \mu_\infty(dx) \\ &\leq \liminf_{M \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int g_M(x) (\mathbb{K}^n \delta_m)(dx) \leq \lim_{n \rightarrow +\infty} \int |x|^s (\mathbb{K}^n \delta_m)(dx) < +\infty. \end{aligned}$$

Now, since $\gamma > 1$ and $\beta > 2$, then (39) is true for $k = 1$ and $s = 2$. As a consequence we obtain (40) for $s = 2$. Let us iterate this procedure for $k < \bar{k}$ with $\bar{k} < \beta \leq \bar{k} + 1$. The last step starts from the validity of (39) for $k = \bar{k}$ and $s = \beta$ which implies $\int |x|^\beta \mu_\infty(dx) < +\infty$.

If $0 < \gamma \leq 1$ then

$$\mathbb{E}|A_0 + A_1 Y_1 + A_2 Y_2| \leq \mathbb{E}|Y|q(1) + \mathbb{E}|A_0|.$$

Since $q(1) < 1$ and $\mathbb{E}|A_0| < +\infty$ we get

$$\int |x| (\mathbb{K}^n \delta_m)(dx) \leq |m|q(1)^n + C \sum_{j=0}^{n-1} q(1)^j$$

and hence, thanks to Lemma 5.2,

$$\sup_n \int |x| (\mathbb{K}^n \delta_m)(dx) < +\infty \quad \text{and} \quad \int |x| \mu_\infty(dx) < +\infty.$$

At this stage (39) is proved for $k = 1$ and $s = k+1 = 2$ and we can go on as in the previous case. This proves (ii).

Let us prove (iii). If $\gamma \leq 1$, from Lemma 5.2 (c) one obtains that $\mu_\infty\{[0, +\infty)\} = 1$. Now assume that $\gamma \in (1, 2]$. Since $P\{A_0 \geq 0\} = 1$ and $P\{A_0 \neq 0\} > 0$, then $\mathbb{E}[A_0] \neq 0$ and only case (b) of Proposition 2.1 has to be considered. By assumption $\bar{m} = \int x\mu_\infty(dx) > 0$ and hence from Lemma 5.1 (c)-(d), since M_n^* is positive a.s. for every $n \geq 1$, again $\mu_\infty\{[0, +\infty)\}$.

As a consequence, using the fact that $\beta \geq 1$ we can write

$$X^\beta =^{\mathcal{D}} (A_0 + A_1X_1 + A_2X_2)^\beta \geq A_0^\beta + A_1^\beta X_1^\beta + A_2^\beta X_2^\beta$$

($=^{\mathcal{D}}$ denotes the identity in distribution) if X, X_1, X_2 are independent random variables with law μ_∞ and (X, X_1, X_2) and (A_0, A_1, A_2) are stochastically independent. Then

$$\mathbb{E}[X^\beta] \geq q(\beta)\mathbb{E}[X_1^\beta] + \mathbb{E}[A_0^\beta] > q(\beta)\mathbb{E}[X_1^\beta]$$

since we are assuming that $P\{A_0 \neq 0\} > 0$. Hence $q(\beta) < 1$. □

Let us state a useful result which is proved, with slightly different notation, in Lemma 2 of [3] (see also Proposition 4.1 in [2]).

Lemma 5.3. *Let $\gamma > 0$ such that $q(\gamma) = \mathbb{E}[|A_1|^\gamma + |A_2|^\gamma] < +\infty$. Then, for every $n \geq 0$,*

$$(41) \quad \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_n)} |\varpi(v)|^\gamma\right] = \frac{\Gamma(q(\gamma) + n)}{\Gamma(n+1)\Gamma(q(\gamma))} =: c_n(\gamma).$$

Proof. Given the sequence $(T_n)_{n \geq 1}$ of random binary recursive trees, one can define a sequence $(V_n)_{n \geq 1}$ of \mathbb{U} -valued random variables such that

$$T_{n+1} = T_n \cup \{(V_n, 1), (V_n, 2)\}$$

for every $n \geq 0$, where $V_0 = \emptyset$ and $V_n \in \mathcal{L}(T_n)$. The random variable V_n corresponds to the random vertex chosen to generate T_{n+1} from T_n . Hence, by construction,

$$P\{V_n = v | T_1, \dots, T_n\} = \mathbb{I}\{v \in \mathcal{L}(T_n)\} \frac{1}{n+1}$$

for every $n \geq 1$. Since $T_0 = \emptyset$ and $\varpi(\emptyset) = 1$, $\mathbb{E}[\sum_{v \in \mathcal{L}(T_0)} |\varpi(v)|^\gamma] = 1$ and hence (41) is true for $n = 0$. For $n \geq 1$,

$$\begin{aligned} \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_n)} |\varpi(v)|^\gamma\right] &= \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_{n-1})} |\varpi(v)|^\gamma \left[(|A_1(v)|^\gamma + |A_2(v)|^\gamma - 1)\mathbb{I}\{V_{n-1} = v\} + 1 \right]\right] \\ &= \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_{n-1})} |\varpi(v)|^\gamma \left(1 + \frac{q(\gamma) - 1}{n}\right)\right] \end{aligned}$$

from the independence assumptions and since $(A_1(v), A_2(v))$ has the same law of (A_1, A_2) for every v . Hence, by induction,

$$\mathbb{E}\left[\sum_{v \in \mathcal{L}(T_n)} |\varpi(v)|^\gamma\right] = \prod_{j=1}^n \left(1 + \frac{q(\gamma) - 1}{j}\right) = \frac{\Gamma(q(\gamma) + n)}{\Gamma(n+1)\Gamma(q(\gamma))}$$

and (41) is proved. □

In the following denote by ζ_n^* the law of W_n^* .

Lemma 5.4. *Assume that, for some γ in $(0, 2]$, $\mathbb{E}[|X_0|^\gamma + |A_0|^\gamma] < +\infty$ and $q(\gamma) < 1$.*

(a) *If $0 < \gamma \leq 1$, then*

$$l_\gamma(\zeta_n^*, \mu_\infty) \leq c_n(\gamma)l_\gamma(\bar{\mu}_0, \mu_\infty)$$

for every $n \geq 0$, where μ_∞ is the unique solution of (13) in \mathcal{M}_γ . Furthermore, if $\gamma = 1$ and $\mathbb{E}(X_0) = \bar{m}$ then $\mathbb{E}(W_n^) = \int v\zeta_n^*(dv) = \bar{m}$ for every $n \geq 1$.*

(b) If $1 < \gamma \leq 2$, $\mathbb{E}(A_1 + A_2) \neq 1$ and $\mathbb{E}(X_0) = \bar{m}$, then

$$(42) \quad l_\gamma^\gamma(\zeta_n^*, \mu_\infty) \leq 2c_n(\gamma)l_\gamma^\gamma(\bar{\mu}_0, \mu_\infty)$$

where μ_∞ is the unique solution of (13) in \mathcal{M}_γ . Furthermore $\mathbb{E}(W_n^*) = \int v\zeta_n^*(dv) = \bar{m}$ for every $n \geq 0$.

(c) If $1 < \gamma \leq 2$, $\mathbb{E}(A_0) = 0$, $\mathbb{E}(A_1 + A_2) = 1$, $\mathbb{E}(X_0) = m_0$ (m_0 arbitrary), and μ_∞ is the unique solution of (13) in $\mathcal{M}_{\gamma, m_0}$, then (42) holds and $\mathbb{E}(W_n^*) = \int v\zeta_n^*(dv) = m_0$ for every $n \geq 0$.

Proof. The existence and uniqueness of μ_∞ in the three cases (a), (b) and (c) is guaranteed by Proposition 2.1. On a sufficiently large probability space (Ω, \mathcal{F}, P) consider a sequence $(Y_v)_{v \in \mathbb{U}}$, such that

- $(A(v))_{v \in \mathbb{U}}, (T_n)_{n \geq 0}$, and $(X_v, Y_v)_{v \in \mathbb{U}}$ are independent;
- (X_v, Y_v) are independent and identically distributed for v in \mathbb{U} , and each (X_v, Y_v) is an optimal transport plan for $l_\gamma(\bar{\mu}_0, \mu_\infty)$, i.e. the law of X_v is $\bar{\mu}_0$, the law of Y_v is μ_∞ and $\mathbb{E}|X_v - Y_v|^\gamma = l_\gamma^{\max(1, \gamma)}(\bar{\mu}_0, \mu_\infty)$.

Let us set $U_n^* = \sum_{v \in \mathcal{L}(T_n)} Y_v \varpi(v) + \Gamma_n$. We now show that, for every n , the law of U_n^* is μ_∞ . In point of fact

$$\begin{aligned} \mathbb{E}[e^{i\xi U_n^*}] &= \mathbb{E}\left[\sum_{\bar{v} \in \mathcal{L}(T_{n-1})} \mathbb{I}\{V_{n-1} = \bar{v}\} \right. \\ &\quad \left. \exp\left\{i\xi\left(\sum_{v \neq \bar{v}} \varpi(v)Y_v + \Gamma_{n-1} + \varpi(\bar{v})\underbrace{(A_1(v)Y_{\bar{v}1} + A_2(\bar{v})Y_{\bar{v}2} + A_0(\bar{v}))}_{=^d Y_{\bar{v}}}\right)\right\}\right] \\ &= \mathbb{E}\left[\sum_{\bar{v} \in \mathcal{L}(T_{n-1})} \mathbb{I}\{V_{n-1} = \bar{v}\} e^{i\xi U_{n-1}^*}\right] \\ &= \mathbb{E}[e^{i\xi U_{n-1}^*}] \end{aligned}$$

where V_n is defined as in the proof of Lemma 5.3. Hence, by induction, U_n^* has the same law of Y_\emptyset , that is μ_∞ . Now denote by \mathcal{G} the σ -field $\sigma(A(v) : v \in \mathbb{U}, (T_n)_{n \geq 1})$ and observe that

$$\begin{aligned} l_\gamma^{\max(1, \gamma)}(\zeta_n^*, \mu_\infty) &\leq \mathbb{E}|W_n^* - U_n^*|^\gamma = \mathbb{E}\left[\mathbb{E}\left[\left|\sum_{v \in \mathcal{L}(T_n)} \varpi(v)(X_v - Y_v)\right|^\gamma \middle| \mathcal{G}\right]\right] \\ &\leq k_\gamma \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_n)} |\varpi(v)|^\gamma \mathbb{E}[|X_v - Y_v|^\gamma | \mathcal{G}]\right]. \end{aligned}$$

The last inequality is immediate for $\gamma \leq 1$ with $k_\gamma = 1$ while, if $1 < \gamma \leq 2$, it follows with $k_\gamma = 2$ from Bhaaer-Esseen inequality, since $\mathbb{E}[X_v] = \mathbb{E}[Y_v]$ which implies $\mathbb{E}\left[\sum_{v \in \mathcal{L}(T_n)} \varpi(v)(X_v - Y_v) \middle| \mathcal{G}\right] = 0$. Hence, using (41),

$$l_\gamma^{\max(1, \gamma)}(\zeta_n^*, \mu_\infty) \leq k_\gamma \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_n)} |\varpi(v)|^\gamma\right] l_\gamma^{\max(1, \gamma)}(\mu_0, \mu_\infty) = k_\gamma c_n(\gamma) l_\gamma^{\max(1, \gamma)}(\mu_0, \mu_\infty).$$

In order to conclude the proof, let us study $\mathbb{E}[W_n^*]$ when γ belongs to $[1, 2]$. Observe that

$$\mathbb{E}[W_n^*] = \mathbb{E}[W_{n-1}^*] + \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_{n-1})} \varpi(v)[A_0(v) + A_1(v)X_{v1} + A_2(v)X_{v2} - X_v] \mathbb{I}\{V_n = v\}\right].$$

If $\mathcal{G}_{n-1} = \sigma(T_1, \dots, T_n; \varpi(v) : v \in \mathcal{L}(T_{n-1}))$, then

$$\begin{aligned} &\mathbb{E}\left[\sum_{v \in \mathcal{L}(T_{n-1})} \varpi(v)[A_0(v) + A_1(v)X_{v1} + A_2(v)X_{v2} - X_v] \mathbb{I}\{V_n = v\}\right] \\ &= \mathbb{E}\left[\sum_{v \in \mathcal{L}(T_{n-1})} \varpi(v) \mathbb{I}\{V_n = v\} \mathbb{E}[A_0(v) + A_1(v)X_{v1} + A_2(v)X_{v2} - X_v | \mathcal{G}_{n-1}]\right] \end{aligned}$$

and

$$\mathbb{E}[A_0(v) + A_1(v)X_{v1} + A_2(v)X_{v2} - X_v | \mathcal{G}_{n-1}] = \mathbb{E}[A_1 + A_2 - 1]\mathbb{E}[X_0] + \mathbb{E}[A_0] = 0$$

either in case (b) and (c) or in case (a) when $\gamma = 1$ and $\mathbb{E}[X_0] = \bar{m}$. Hence, $\mathbb{E}[W_n^*] = \mathbb{E}[W_0^*] = \mathbb{E}[X_0]$, which completes the proof. \square

Proof of Proposition 2.3. Using Proposition 4.1, to the convexity of the Wasserstein distance l_γ ($\gamma < 1$) and Lemma 5.4 (a) one gets

$$\begin{aligned} l_\gamma(\mu_t, \mu_\infty) &\leq \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n l_\gamma(\zeta_n^*, \mu_\infty) \\ &\leq \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n c_n(\gamma) l_\gamma(\bar{\mu}_0, \mu_\infty) = e^{-t(1-q(\gamma))} l_\gamma(\bar{\mu}_0, \mu_\infty). \end{aligned}$$

\square

Proof of Proposition 2.4. Since $q(1) < 1$, in this case $\mathbb{E}[A_1 + A_2] \neq 1$. Hence, thanks to Proposition 4.1, to (41) and to the convexity of the Wasserstein distance

$$\begin{aligned} l_1(\mu_t, \mu_\infty) &\leq \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n l_1(\zeta_n^*, \mu_\infty) \\ &\leq \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n c_n(1) l_1(\mu_0, \mu_\infty) = e^{-t(1-q(1))} l_1(\mu_0, \mu_\infty). \end{aligned}$$

Furthermore, if $\mathbb{E}[X_0] = \bar{m}$, then

$$\int v \mu_t(dv) = \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n \mathbb{E}[W_n^*] = \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n \bar{m} = \bar{m},$$

since $\mathbb{E}[W_n^*] = \bar{m}$ as stated in (b) of Lemma 5.4. \square

Proof of Proposition 2.5. Using Proposition 4.1, to the convexity of l_γ^γ ($\gamma \geq 1$) and Lemma 5.4 (b) one gets

$$\begin{aligned} l_\gamma^\gamma(\mu_t, \mu_\infty) &\leq \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n l_\gamma^\gamma(\zeta_n^*, \mu_\infty) \\ &\leq \sum_{n \geq 0} e^{-t}(1 - e^{-t})^n c_n(\gamma) l_\gamma^\gamma(\bar{\mu}_0, \mu_\infty) = e^{-t(1-q(\gamma))} l_\gamma^\gamma(\bar{\mu}_0, \mu_\infty). \end{aligned}$$

Arguing exactly in as in the previous proof using Lemma 5.4 (b) in place of Lemma 5.4 (a) one proves $\int v \mu_t(dv) = \bar{m}$. \square

Proof of Proposition 2.6. Analogous to the proof of Proposition 2.5, using Lemma 5.4 (c) in place of Lemma 5.4 (b). \square

REFERENCES

- [1] G. ALSMEYER and U. RÖSLER (2003). The best constant in the Topchii-Vatutin inequality for martingales. *Statist. Probab. Lett.* **65** 199–206.
- [2] F. BASSETTI and L. LADELLI (2010) Self similar solutions in one-dimensional kinetic models: a probabilistic view. arXiv:1003.5527
- [3] F. BASSETTI, L. LADELLI and D. MATTHES (2010). Central limit theorem for a class of one-dimensional kinetic equations. *Probab. Theory Related Fields* DOI:10.1007/s00440-010-0269-8 (Published on line) .
- [4] F. BASSETTI, L. LADELLI and E. REGAZZINI (2008) Probabilistic study of the speed of approach to equilibrium for an inelastic Kac model. *J. Stat. Phys.* **133** 683–710.
- [5] F. BASSETTI and G. TOSCANI (2010) Explicit equilibria in a kinetic model of gambling. *Phys. Rev. E* **81**, 066115.
- [6] M. BISI, G. SPIGA and G. TOSCANI (2009). Kinetic models of conservative economies with wealth redistribution. *Commun. Math. Sci.* **7** 901–916.
- [7] A.V. BOBYLEV (1988). The theory of the spatially Uniform Boltzmann equation for Maxwell molecules. *Sov. Sci. Review C* **7**, 112–229.
- [8] A.V. BOBYLEV and C. CERCIGNANI (2003). Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions. *J. Statist. Phys.* **110** 333–375.

- [9] A. V. BOBYLEV, C. CERCIGNANI and I. M. GAMBA (2009). On the self-similar asymptotics for generalized nonlinear kinetic maxwell models. *Comm. Math. Phys.* **291** 599–644.
- [10] E.A. CARLEN, M.C. CARVALHO and E. GABETTA (2000). Central limit theorem for Maxwellian molecules and truncation of the Wild expansion. *Comm. Pure Appl. Math.* **53** 370–397 .
- [11] J.A. CARRILLO, S. CORDIER and G. TOSCANI (2009) Over-populated tails for conservative-in-the-mean inelastic Maxwell models. *Discrete Contin. Dyn. Syst.* **24** no. 1, 59–81.
- [12] C. CERCIGNANI (1988). *The Boltzmann equation and its applications*, Springer Series in Applied Mathematical Sciences, Vol. **67** Springer–Verlag 1988.
- [13] C. CERCIGNANI, R. ILLNER and M. PULVIRENTI (1994). The mathematical theory of dilute gases, *Springer Series in Applied Mathematical Sciences*, Vol. **106** Springer–Verlag 1994.
- [14] C. CERCIGNANI, R. ILLNER and C. STOICA (2001) On diffusive equilibria in generalized kinetic theory. *J. Stat. Phys.* **105** 337–352.
- [15] A. CHAKRABORTI and B.K. CHAKRABARTI (2000) Statistical Mechanics of Money: Effects of Saving Propensity, *Eur. Phys. J. B* **17**, 167–170.
- [16] A. CHATTERJEE and B.K. CHAKRABARTI (2007). Kinetic Exchange Models for Income and Wealth Distributions. *Eur. Phys. J. B* **60** , 135–149.
- [17] A. CHATTERJEE, Y. SUDHAKAR and B.K. CHAKRABARTI (2005). *Econophysics of Wealth Distributions* New Economic Windows Series, Springer, Milan, 2005.
- [18] S. CORDIER, L. PARESCHI and G. TOSCANI (2005) On a kinetic model for a simple market economy. *J. Stat. Phys.* **120** no. 1-2, 253–277.
- [19] M. DRMOTA (2009). *An interplay between combinatorics and probability. Random trees*. Springer Wien NewYork, Vienna.
- [20] R. DURRETT and T. LIGGETT (1983). Fixed points of the smoothing transformation. *Z. Wahrsch. Verw. Gebiete* **64** 275–301.
- [21] GABETTA E. and REGAZZINI E. (2008). Central limit theorem for the solution of the Kac equation. *Ann. Appl. Probab.* **18**, 2320-2336.
- [22] M. KAC (1959) *Probability and related topics in physical sciences* (Lectures in Applied Mathematics. Proceedings of the Summer Seminar, Boulder, Colo., 1957, Vol. I Interscience Publishers, London-New York 1959.)
- [23] MATTHES, D. and TOSCANI, G. (2008). On steady distributions of kinetic models of conservative economies. *J. Statist. Phys.* **130** 1087-1117.
- [24] H. P. MCKEAN JR (1966). Speed of approach to equilibrium for Kac’s caricature of a Maxwellian gas. *Arch. Rational Mech. Anal.* **21** 343–367.
- [25] PULVIRENTI, A. and TOSCANI, G. (2004). Asymptotic properties of the inelastic Kac model. *J. Statist. Phys.* **114** 1453–1480.
- [26] RACHEV, S. T. and RÜSCHENDORF, L. (1995). Probability metrics and recursive algorithms. *Adv. Appl. Prob.* **27**, 770–799.
- [27] U. RÖSLER (1992). A fixed point theorem for distributions. *Stochastic Process. Appl.* **42** 195–214.
- [28] L. RÜSCHENDORF (2006). On stochastic recursive equations of sum and max type. *J. Appl. Probab.* **43** 687–703.
- [29] C. VILLANI (2006) Mathematics of granular materials. *J. Stat. Phys.* **124** 781–822.
- [30] B. VON BAHR and C.G. ESSEEN (1965). Inequalities for the r th absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Statist* **36** 299–303.
- [31] E. WILD (1951). On Boltzmann’s equation in the kinetic theory of gases. *Proc. Cambridge Philos. Soc.* **47** 602–609.