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On the last Hilbert-Samuel coefficient of isolated singularities

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1 Introduction

In [18] Lipman presented a proof of the existence of a desingularization for any excellent surface. The strategy of Lipman's proof is based on the finiteness of the number $H(R)$, defined as the supreme of the second Hilbert-Samuel coefficient $e_2(I)$, where I range the set of normal $\mathbf{m}$-primary ideals of a Noetherian complete local ring $(R, \mathbf{m})$. See [18, Theorem* of pag. 158, and Remark B of pag. 160]. On the other hand Huckaba and Huneke proved that if I is a $\mathbf{m}$-primary ideal of a $d-$dimensional, $d \geq 2$, Cohen-Macaulay local ring $(R, \mathbf{m})$ such that I^n is integrally closed for $n \gg 0$, in particular if I is normal, then the associated graded ring of R with respect to I^n has depth at least two for $n \gg 0$, [10, Theorem 3.11].

The problem studied in this paper is the extension of the result of Lipman on $H(R)$ to $\mathbf{m}-$primary ideals I of a d-dimensional Cohen-Macaulay ring R such that $gr_{I^n}(R)$, associated graded ring of R with respect to I^n, is Cohen-Macaulay for $n \gg 0$. We denote by $\mathrm{gCM}(R)$ this set of ideals of R.

We prove the following theorem:

Theorem 3.2 *Let $(R, \mathbf{m})$ be a $d-$dimensional Cohen-Macaulay local ring of dimension $d \geq 1$ essentially of finite type over a characteristic zero field $\mathbf{k} = R/\mathbf{m}$. Assume that the closed point $x \in X = \mathbf{Spec}(R)$ is an isolated singularity. For all $K \in \mathrm{gCM}(R)$ and for all Hironaka ideal $I \in \mathbf{Hir}(R)$ it holds*

$$0 \leq e_d(K) \leq e_d(I) = p_g(X).$$

In particular, if $x \in X$ is a rational singularity then $e_d(K) = 0$ for all $K \in \mathrm{gCM}(R)$.

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An Hironaka ideal of R is an ideal I such that the blow-up of R centered at I is a desingularization of $X = \mathbf{Spec}(R)$. We know from the main result of [9, Main Theorem I, pag. 132] that Hironaka ideals exist. Recall that all Hironaka ideal belongs to gCM(R), this is a consequence of the version of the Grauert-Riemenschneider vanishing theorem due to Sancho de Salas, Proposition 2.2.

An example of Huckaba and Huneke shows that cannot be considered in Theorem 3.2 weaker conditions on the depth of the associated graded ring with respect I^n, and that cannot be extended the result of Lipman to higher dimensions, see Example 3.5. In Proposition 3.8 we relate the ideals such that their last Hilbert-Samuel coefficient equals the geometric genus with the rational singularities. In the last section we study the one-dimension case; in particular we explicitly construct an Hironaka ideal, Example 4.3. We also relate the upper bound of $e_1(I)$ of the main theorem of this paper with the previously obtained in [4] and [29].

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NOTATIONS: Let I be an ideal of R we denote by $\mathcal{R}(I) = \oplus_{n\geq 0} I^n t^n$ the Rees algebra of I, and we denote by $gr_I(R) = \oplus_{n\geq 0} I^n/I^{n+1}$ the associated graded ring of R with respect to I. If I is a $\mathbf{m}-$primary ideal of R we denote by $h_I(n) = length_R(R/I^{n+1})$ the Hilbert-Samuel function of I. Hence there exist integers $e_j(I) \in \mathbb{Z}$ such that

$$p_I(X) = e_0(I)\binom{X+d}{d} - e_1(I)\binom{X+d-1}{d-1} + \cdots + (-1)^d e_d(I)$$

is the Hilbert-Samuel polynomial of I, i.e. $h_I(n) = p_I(n)$ for $n \gg 0$. The integer $e_j(I)$ is the $j-$th normalized Hilbert-Samuel coefficient of I, $j = 0, \cdots, d$. We set $h_R = h_{\mathbf{m}}$ and $p_R = p_{\mathbf{m}}$. We denote by P_I the Poincaré series of an $\mathbf{m}$-primary ideal I, i.e. the power series defined by h_I

$$P_I(Z) = \sum_{n\geq 0} h_I(n)\ Z^n.$$

We know that P_I is a rational function, with $a_i \in \mathbb{Z}$ and $a_s \neq 0$,

$$P_I = \frac{a_0 + a_1 Z + \cdots + a_s Z^s}{(1-Z)^d}.$$

2 On the resolution of singularities

Let $(R, \mathbf{m})$ be a $d-$dimensional reduced Cohen-Macaulay local ring essentially of finite type over a characteristic zero field $\mathbf{k} = R/\mathbf{m}$. We denote by x the closed point of $X = \mathbf{Spec}(R)$. Hironaka proved that there exists an ideal $I \subset R$ such that, [9, Main Theorem I, pag. 132],

(i) $V(I) = Sing(X)$,

(ii) the natural projection morphism $\pi : \widetilde{X} = \mathbf{Proj}(\mathcal{R}(I)) \longrightarrow X = \mathbf{Spec}(R)$ is a resolution of singularities of X, i.e. $\tilde{X}$ is non-singular and π induces a $\mathbf{k}$-scheme isomorphism

$$\pi : \widetilde{X} \setminus \pi^{-1}(Sing(X)) \longrightarrow X \setminus Sing(X).$$

Definition 2.1. *An ideal I of R is an Hironaka ideal if I satisfies the two above conditions. We denote by $\mathbf{Hir}(R)$ the set of Hironaka ideals of R.*

See Example 4.3 for an explicit computation of an Hironaka ideal.

In the next result we collect three basic properties of Hironaka ideals.

Proposition 2.2. *Let I be an Hironaka ideal, then*

(i) $I^n \in \mathbf{Hir}(R)$, *for* $n \geq 1$,

(ii) $gr_{I^n}(R)$ *is Cohen-Macaulay for all* $n \gg 0$,

(iii) if the closed point x of X is an isolated singularity then Hironaka's ideals are $\mathbf{m}$*-primary.*

Proof. (i) The result follows form the facts $V(I^n) = V(I) = Sing(X)$ and that there exists a natural $\mathbf{k}$-scheme isomorphism

$$\mathbf{Proj}(\mathcal{R}(I)) \cong \mathbf{Proj}(\mathcal{R}(I^n))$$

induced by the degree zero graded morphism $\mathcal{R}(I^n) \subset \mathcal{R}(I)$, $n \geq 1$.
(ii) Follows from the version of the Grauert-Riemenschneider vanishing theorem due to Sancho de Salas, [32]. See [19, Theorem 4.3] for an extension to all Cohen-Macaulay rings.
(iii) If the closed point x of X is an isolated singularity then $V(I) = Sing(X) = \{x\}$, i.e. I is an $\mathbf{m}$-primary ideal. □

Remark 2.3. Recall that if the field $\mathbf{k}$ is of positive characteristic it is an open problem to prove that $\mathbf{Hir}(R)$ is non empty. On the other hand, the fact that Hironaka Ideal I has a Cohen-Macaulay associated graded ring gr_{I^n} for $n \gg 0$ is the key point of the results of this paper. This a consequence of the version of the Grauert-Riemenschneider vanishing theorem due to Sancho de Salas, [32], as we quoted in the proof of Proposition 2.2. Recall that Grauert-Riemenschneider vanishing theorem is true in characteristic zero case but not in the positive characteristic case as Raynaud showed in [28].

Definition 2.4. *If the closed point of X is an isolated singularity, the geometric genus of X is $p_g(X) = \mathrm{Length}_R(\mathbf{R}^{d-1}\pi_*\mathcal{O}_Z)$ for a (all) singularity resolution $\pi : Z \longrightarrow X$. We say that R is a rational singularity if R is normal and for all $i > 0$ $\mathrm{Length}_R(\mathbf{R}^i\pi_*\mathcal{O}_Z) = 0$.*

It is well know that the above definitions do not depend on the resolution Z. In fact from the Grothendieck spectral sequence of the composition of functors and the vanishing result of the higher direct images due to Hironaka we get that $\mathrm{Length}_R(\mathbf{R}^{d-1}\pi_*\mathcal{O}_Z)$ is independent of the resolution of singularities $\pi : Z \longrightarrow X$, see [8, Proposition 2.1]. In the one-dimensional case, $\mathbf{R}^0\pi_*\mathcal{O}_Z \cong \overline{R}/R$ where $\overline{R}$ is the integral closure of R on its full ring of fractions. In the literature the length of $\overline{R}/R$ is also known as the singularity order of $X = \mathbf{Spec}(R)$. In the one-dimensional case, rational means non-singular.

Remark 2.5. Recall that if R is Cohen-Macaulay of dimension $d \geq 2$ and the closed point of $X = \mathbf{Spec}(R)$ is an isolated singularity then R is a reduced normal ring.

We assume that $(R, \mathbf{m})$ is a $d-$dimensional Noetherian Cohen-Macaulay local ring of dimension $d \geq 1$.

Definition 2.6. *We denote by* $\mathrm{gCM}(R)$ *(resp.* $\mathrm{agCM}(R)$*) the set of* $\mathbf{m}$*-primary ideals I of R such that $gr_{I^n}(R)$ is Cohen-Macaulay (resp.* $\mathrm{depth}(gr_{I^n}(R)) \geq d-1$*) for $n \gg 0$.*

Notice that $\mathbf{Hir}(R) \subset \mathrm{gCM}(R) \subset \mathrm{agCM}(R)$, these inclusions are in general strict. See [3] and its reference list for a detailed study of the ideals belonging to $\mathrm{gCM}(R)$ or $\mathrm{agCM}(R)$. Moreover, if I is a $\mathbf{m}$-primary ideal it is known that

$$\mathrm{depth}(gr_{I^n}(R)) \geq 1$$

for $n \gg 0$. In particular, if $d \leq 2$ then $\mathrm{agCM}(R)$ agrees with the whole set of $\mathbf{m}$-primary ideals. If $d = 1$ then $\mathrm{agCM}(R) = \mathrm{gCM}(R)$, and this set agrees with the set of $\mathbf{m}$-primary ideals.

The relationship between the Hilbert-Samuel coefficients of I^n and I is the following.

Proposition 2.7. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let I be a* $\mathbf{m}$*-primary ideal. Then it hold:*

(i) $e_d(I^n) = e_d(I)$, $n \geq 1$,

(ii) there exist polynomials f_i^j on n of degree $d-j$, $i = 0, \cdots, d-1$, $j = 0, \cdots, i$, such that

$$e_i(I^n) = \sum_{j=0}^{i} f_i^j(n) e_j(I).$$

Since $e_d(I^n) = e_d(I)$ for all $n \geq 1$, in order to study the last Hilbert-Samuel coefficient of ideals of gCM(R), we may assume that $gr_I(R)$ is Cohen-Macaulay. In the following result we present some inequalities among the Hilbert-Samuel coefficient of $\mathbf{m}$-primary ideals I such that $gr_I(R)$ is Cohen-Macaulay. The same result holds for Hilbert filtrations $\mathcal{F}$ such that its associated graded ring $gr_{\mathcal{F}}(R)$ is Cohen-Macaulay. In [21, Corollary 2] Marley gave several restrictions for the Hilbert-Samuel coefficients of ideals $I \in$ agCM(R).

Proposition 2.8. *Let $(R, \mathbf{m})$ be a $d-$dimensional Cohen-Macaulay local ring of dimension $d \geq 0$. Let I be an $\mathbf{m}$-primary ideal such that $gr_I(R)$ is Cohen-Macaulay with Poincaré series, $a_s \neq 0$,*

$$P_I = \frac{a_0 + a_1 Z + \cdots + a_s Z^s}{(1-Z)^d}.$$

Then the following conditions hold

(i) $s \leq e_0(I) + d + 1 - \text{Length}(I/I^2) \leq e_0(I)$,

(ii) $e_i(I) = 0$, *for* $i = s+1, \cdots, d$,

(iii) $0 \leq (i+1)e_{i+1}(I) \leq (s-i)e_i(I)$, *for* $i = 0, \cdots, s$,

(iv) $0 \leq e_i(I) \leq \binom{s}{i} e_0(I)$, *for* $i = 0, \cdots, d$.

Proof. We know that, [5],

$$e_i(I) = \sum_{j \geq i} \binom{j}{i} a_j$$

for $i = 0, \cdots, d$, from this we deduce (ii). Since $gr_I(R)$ is Cohen-Macaulay we get that $a_i > 0$ for all $i = 0, \cdots, s$, and $a_1 = \text{Length}(I/I^2) - d$. Then we have $e_i(I) \geq 0$ for all $i = 0, \cdots, d$, and

$$e_0(I) \geq a_0 + a_1 + (a_2 + \cdots + a_s) \geq a_0 + a_1 + s - 1,$$

since $a_0 = \text{Length}(R/I)$, we obtain (i). An easy computation shows that

$$(s-i)\binom{j}{i} \geq (i+1)\binom{j}{i+1}$$

for $j = i, \cdots, s$. From this we get (iii). The inequalities of (iv) follow from the previous ones. □

Example 2.9. Let us assume that $s = 5$. From the last result we get, $e_i = e_i(I)$,

$$10e_0 \geq 2e_1 \geq e_2 \geq e_3 \geq 2e_4 \geq 10e_5 \geq 0.$$

Compare these inequalities with [27, Corollary 3.5].

3 Hilbert-Samuel coefficients of isolated singularities

Let $(R, \mathbf{m})$ be a $d-$dimensional Cohen-Macaulay local ring of dimension $d \geq 2$ essentially of finite type over a characteristic zero field $\mathbf{k} = R/\mathbf{m}$. In this section we always assume that the closed point of $X = \mathbf{Spec}(R)$ is an isolated singularity.

The first part of the next result it is well known, we include it for the sake of completeness.

Proposition 3.1. *(i) Let I be a $\mathbf{m}$-primary ideal of R, and let $\pi : \widetilde{X} = \mathbf{Proj}(\mathcal{R}(I)) \longrightarrow X = \mathbf{Spec}(R)$ be the projection, then*

$$\mathrm{Length}_R(\mathbf{R}^{d-1}\pi_*\mathcal{O}_{\widetilde{X}}) = \mathrm{Length}_R(H^{d-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}})) = \mathrm{Length}_R(H^d_{\mathcal{R}(I)_+}\mathcal{R}(I)_0) < \infty.$$

(ii) For all $I \in \mathrm{agCM}(R)$ and for all $n \geq 1$ it holds

$$e_d(I^n) = \mathrm{Length}_R(\mathbf{R}^{d-1}\pi_*\mathcal{O}_{\widetilde{X}}).$$

(iii) For all Hironaka ideal $I \in \mathbf{Hir}(R)$ and $n \geq 1$ it holds

$$e_d(I^n) = p_g(X).$$

In particular, $e_d(I^n)$ is independent of the ideal $I \in \mathbf{Hir}(R)$ and the integer $n \geq 1$.

Proof. (i) Since X is an affine scheme we get, [7, Chap. III, Proposition 8.5],

$$\mathbf{R}^{d-1}\pi_*\mathcal{O}_{\widetilde{X}} \cong H^{d-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^{\sim}$$

and by [6] we have

$$H^{d-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong H^d_{\mathcal{R}(I)_+}\mathcal{R}(I)_0.$$

Since x is an isolated singularity and I is an $\mathbf{m}$-primary we get that $\mathbf{R}^{d-1}\pi_*\mathcal{O}_{\widetilde{X}}$ is supported in x, so the lengths of (i) are all finite and agree.
(ii) It is well known that for all $n \geq 1$ it holds $e_d(I^n) = e_d(I)$ and there exists a natural $\mathbf{k}$-scheme isomorphism

$$\widetilde{X} = \mathbf{Proj}(\mathcal{R}(I)) \cong \mathbf{Proj}(\mathcal{R}(I^n)).$$

Hence we may assume that $\mathrm{depth}(gr_I(R)) \geq d-1$. From the Grothendieck-Serre formula, see [15, Proposition 3.1], we get

$$\begin{aligned} e_d(I) &= (-1)^d(p_I(-1) - h_I(-1)) = \chi(\mathrm{Length}_R(H^*_{\mathcal{R}(I)_+}\mathcal{R}(I)_0)) \\ &= \mathrm{Length}_R(H^d_{\mathcal{R}(I)_+}\mathcal{R}(I)_0). \end{aligned}$$

From (i) we deduce (ii).

(iii) It is a consequence of (ii) and Proposition 2.2. ∎

In the following result we relate the last Hilbert-Samuel coefficient $e_d(I)$ of a Hironaka ideal I with the last Hilbert-Samuel coefficient $e_d(K)$ of a $\mathbf{m}$-primary ideal $K \in \mathrm{gCM}(R)$. In particular we extend [18, (B) pag.160] to ideals I such that $gr_{I^n}(R)$ is Cohen-Macaulay for $n \gg 0$.

Theorem 3.2. *Let $(R, \mathbf{m})$ be a $d-$dimensional Cohen-Macaulay local ring of dimension $d \geq 1$ essentially of finite type over a characteristic zero field $\mathbf{k} = R/\mathbf{m}$. Assume that the closed point $x \in X = \mathbf{Spec}(R)$ is an isolated singularity. For all $K \in \mathrm{gCM}(R)$ and for all Hironaka ideal $I \in \mathbf{Hir}(R)$ it holds*

$$0 \leq e_d(K) \leq e_d(I) = p_g(X).$$

In particular, if $x \in X$ is a rational singularity then $e_d(K) = 0$ for all $K \in \mathrm{gCM}(R)$.

Proof. The one-dimensional case is studied in the last section, so we may assume that $d \geq 2$. We set $\sigma : Z = \mathbf{Proj}(\mathcal{R}(K)) \longrightarrow X$ the blowing-up of X centered in K. Since $e_d(K) = e_d(K^n)$ for all integer $n \geq 1$, we may assume that $gr_K(R)$ is Cohen-Macaulay, Proposition 2.2 (ii).

By the Principalization theorem and the existence of resolution of singularities we get that there exist an Hironaka ideal $I \subset R$ and a commutative diagram, [16, Section 1.9],

$$\begin{array}{ccc} \widetilde{X} = \mathbf{Proj}(\mathcal{R}(I)) & & \\ & \overset{f}{\searrow} & \\ \Big\downarrow \pi & & Z = \mathbf{Proj}(\mathcal{R}(K)) \\ & \overset{\sigma}{\swarrow} & \\ X = \mathbf{Spec}(R) & & \end{array}$$

Let E (resp. $\widetilde{E}$) be the exceptional divisor of σ (resp. π). Since $gr_K(R)$ is Cohen-Macaulay, from [32, Theorem 1.3] we get that

$$H^i_E(Z, \mathcal{O}_Z) = 0$$

for $i < d$. Hence we have the following exact sequence, [6],

$$0 = H^{d-1}_E(Z, \mathcal{O}_Z) \longrightarrow H^{d-1}(Z, \mathcal{O}_Z) \xrightarrow{\rho_Z} H^{d-1}(Z - E, \mathcal{O}_Z),$$

i.e. the restriction morphism ρ_Z is a monomorphism. Since the morphisms f and σ induce isomorphisms

$$\widetilde{X} \setminus \widetilde{E} \overset{f}{\cong} Z \setminus E \overset{\sigma}{\cong} X \setminus \{x\},$$

we have the commutative diagram

$$\begin{array}{ccc} H^{d-1}(Z,\mathcal{O}_Z) & \xrightarrow{\rho_Z} & H^{d-1}(Z-E,\mathcal{O}_Z) \\ \downarrow f & & \downarrow \cong \\ H^{d-1}(\widetilde{X},\mathcal{O}_{\widetilde{X}}) & \xrightarrow{\rho_{\widetilde{X}}} & H^{d-1}(\widetilde{X}-\widetilde{E},\mathcal{O}_{\widetilde{X}}). \end{array}$$

We know that ρ_Z is a monomorphism, so f is also a monomorphism. By Proposition 3.1 we deduce

$$e_d(K) = \mathrm{Length}_R(H^{d-1}(Z,\mathcal{O}_Z)) \leq \mathrm{Length}_R(H^{d-1}(\widetilde{X},\mathcal{O}_{\widetilde{X}})) = e_d(I) = p_g(X).$$

On the other hand, since $gr_K(R)$ is Cohen-Macaulay we have $e_d(K) \geq 0$, Proposition 2.8. □

Remark 3.3. A particular case of the second part of the last result is when R is regular. In this case the result holds for a regular local ring R without any restriction on the field $\mathbf{k}$. Without loss of generality we may assume that k is infinite. This is a well known result and a short proof could be the following. Let $(R,\mathbf{m})$ be a d-dimensional, $d \geq 1$, regular local ring with maximal ideal $\mathbf{m}$ and residue field $\mathbf{k}$. Let I be a $\mathbf{m}$-primary ideal such that $gr_{I^n}(R)$ is Cohen-Macaulay for some $n \geq 1$, we want to prove that $e_d(I) = 0$. If $d = 1$ then R is a DVR and the result is trivial. Since $e_d(I) = e_d(I^n)$ for all integer $n \geq 1$, we may assume that $gr_I(R)$ is Cohen-Macaulay. Let J be a minimal reduction of I. Since R is regular, from Briançon-Skoda theorem we have $I^{d-1} \subset J$, [20] or [13, 13.3.3]. By [11, Proposition 4.6] we get $e_d(I) = 0$.

Corollary 3.4 ([18, (B) pag.160])**.** *Let $X = \mathbf{Spec}(R)$ be a Cohen-Macaulay scheme of dimension $d = 2$. Assume that the closed point of X is an isolated singularity. Let I be an $\mathbf{m}$-primary ideal. If I^n is integrally closed for $n \gg 0$ then*

$$0 \leq e_2(I) \leq p_g(X).$$

Proof. We know that $I \in \mathrm{gCM}(R)$,[10], so the claim follows from the last result. □

Example 3.5. The example of Huckaba and Huneke shows that Theorem 3.2 cannot be extended to ideals of $\mathrm{agCM}(R)$ and that Corollary 3.4 cannot be extended to higher dimensions, [10, Theorem 3.11]. In fact, let us consider $R = \mathbf{k}[X,Y,Z]_{(X,Y,Z)}$ with $\mathbf{k}$ a characteristic zero field. Notice that R is a regular local ring of dimension $d = 3$ defining a rational singularity $X = \mathbf{Spec}(R)$. Let I the ideal of R generated by $(X,Y,Z)^5$ and $X^4, X(Y^3+Z^3), Y(Y^3+Z^3), Z(Y^3+Z^3)$. Huckaba and Huneke proved that I is a normal ideal, $\operatorname{depth}(gr_{I^n}(R)) = 2$ for all $n \geq 1$, i.e. $I \in \operatorname{agCM}(R) \setminus \operatorname{gCM}(R)$, and if $\widetilde{X} = \mathbf{Proj}(\mathcal{R}(I))$ then $H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \neq 0$. From Proposition 3.1 we get that

$$p_g(X) = e_3(R) = 0 < \operatorname{Length}_R(H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})) = e_3(I).$$

Moreover, a standard computation gives us $e_3(I)$:

$$p_I(X) = 76\binom{X+2}{3} - 48\binom{X+1}{2} + 4\binom{X}{1} - 1$$

i.e. $e_3(I) = 1$.

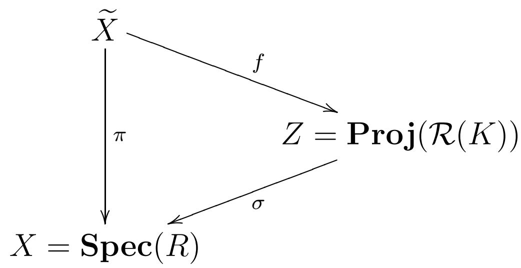
Remark 3.6. Last results can not be extended to the low Hilbert-Samuel coefficients. If $I \in \operatorname{gCM}(R)$ then $I^n \in \operatorname{gCM}(R)$ for all $n \geq 1$, but $e_i(I^n) \to \infty$ when $n \to \infty$ for $i \neq d$, see Section 2.

Remark 3.7. Theorem 3.2 shows that $0 \leq e_d(I) \leq p_g(X)$. The extremal case $e_d(I) = 0$ implies, under some conditions, that the associated graded ring is Cohen-Macaulay. Being R a Cohen-Macaulay ring and I $\mathbf{m}$-primary we have $e_1(I) \geq e_0(I) - \operatorname{Length}_R(R/I) \geq 0$, [26], and $e_2(I) \geq 0$, [31]. Huneke proved that if $e_1(I) = e_0(I) - \operatorname{Length}_R(R/I)$ then $gr_{I^n}(R)$ is Cohen-Macaulay for all $n \geq 1$, [12]. Narita proved that $e_2(I) = 0$ if and only if $gr_{I^n}(R)$ is Cohen-Macaulay for all $n \gg 0$. Marley gave an example of $\mathbf{m}$-primary ideal with $e_2(I) = 0$ and that $gr_I(R)$ is not Cohen-Macaulay. On the other hand, Narita gave an example of an ideal of a Cohen-Macaulay ring with $e_3(I) < 0$, [24]. See [21, example 2] for an ideal of a regular local ring with a negative $e_3(I)$. Itoh proved that if I is normal then $e_3(I) \geq 0$, [14]. Corso, Polini and Rossi proved that if I^n is normal for some $n \gg 0$ then $e_3(I) \geq 0$, and if I^n is integrally closed for all $n \gg 0$ and $e_3(I) = 0$ then $gr_{I^n}(R)$ is Cohen-Macaulay for all $n \gg 0$, [1]. See [2] for the generalization of some of the above results to ideals satisfying the second Vallabrega-Valla condition.

In the next result we look at the ideals with maximal $e_d(I)$. See [22, Proposition 2.7] for a related result with the the first part of the next result.

Proposition 3.8. *Let $K \in \operatorname{gCM}(R)$ be an ideal such that K^n is integrally closed for $n \gg 0$. Assume that $Z = \mathbf{Proj}(\mathcal{R}(K))$ has only isolated singularities. Then $e_d(K) = p_g(X)$ if and only if Z has only rational singularities.*

Proof. Since $e_d(K) = e_d(K^n)$ for all integer $n \geq 1$ and $K \in \operatorname{gCM}(R)$ we may assume that $gr_K(R)$ is Cohen-Macaulay, Proposition 2.2 (ii). Then $Z = \mathbf{Proj}(\mathcal{R}(K))$ is a normal Cohen-Macaulay scheme. Let $\widetilde{X}$ be a desingularization of Z. It is easy to prove that $\widetilde{X}$ is also a desingularization of X. Hence we have a commutative diagram



Since Z is a normal scheme we have $\mathbf{R}^0 f_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_Z$, [8]. Being Z a Cohen-Macaulay scheme eventually with isolated singularities, $\mathbf{R}^i f_* \mathcal{O}_{\widetilde{X}} = 0$, $i = 1, \cdots, d-2$. Hence from the Grothendieck spectral sequence we get, [30, Theorem 11.3],

$$0 \longrightarrow \mathbf{R}^{d-1}\sigma_* \mathcal{O}_Z \longrightarrow \mathbf{R}^{d-1}\pi_* \mathcal{O}_{\widetilde{X}} \longrightarrow \mathbf{R}^0 \sigma_*(\mathbf{R}^{d-1} f_* \mathcal{O}_{\widetilde{X}}) \longrightarrow \mathbf{R}^d \sigma_* \mathcal{O}_Z \longrightarrow 0.$$

The arithmetical rank of $\mathcal{R}(K)_+$ is $d = \dim(R)$, so

$$\mathbf{R}^d \sigma_* \mathcal{O}_Z \cong H^{d+1}_{\mathcal{R}(K)_+} \mathcal{R}(K)_0 = 0.$$

From the last exact sequence and Proposition 3.1 we get

$$\begin{aligned} \mathrm{Length}_R(\mathbf{R}^0 \sigma_*(\mathbf{R}^{d-1} f_* \mathcal{O}_{\widetilde{X}})) &= \mathrm{Length}_R(\mathbf{R}^{d-1}\pi_* \mathcal{O}_{\widetilde{X}}) - \mathrm{Length}_R(\mathbf{R}^{d-1}\sigma_* \mathcal{O}_Z) \\ &= p_g(X) - e_d(K). \end{aligned}$$

If Z has only rational singularities, then we have $\mathbf{R}^{d-1} f_* \mathcal{O}_{\widetilde{X}} = 0$. From the last equality we get $e_d(K) = p_g(X)$. If $e_d(K) = p_g(X)$ then

$$0 = \mathbf{R}^0 \sigma_*(\mathbf{R}^{d-1} f_* \mathcal{O}_{\widetilde{X}}) \cong H^0(Z, \mathbf{R}^{d-1} f_* \mathcal{O}_{\widetilde{X}})^{\sim}.$$

Since the scheme Z has only isolated singularities we get $\mathbf{R}^{d-1} f_* \mathcal{O}_{\widetilde{X}} = 0$, i.e. Z has only rational singularities. □

See [22] and [23] for results related with the main result of this paper.

4 The one-dimensional case

Let $\widetilde{X}$ be the blowing-up of a finite union of reduced curve singularities $X = \mathbf{Spec}(R)$ centered on one of its singular closed points, i.e. R is a Cohen-Macaulay reduced semilocal ring. Then $\widetilde{X}$ has a finite number of non-singular closed points $P_1, \dots, P_r$, and the germ $(\widetilde{X}, P_i)$ is a curve singularity for $i = 1, \dots, r$. It is known that the set $\{P_1, \dots, P_r\}$ is in correspondence one-to-one with the set of closed points of $\mathbf{Proj}(gr_{\mathbf{m}}(R))$. We can iterate the process we get a sequence of blowing-ups

$$\pi_i : X^{(i+1)} \longrightarrow X^{(i)}$$

$i \geq 0$, with $X^{(0)} = X$, $X^{(i+1)}$ is the blowing-up of X^i centered at a singular point of X^i, and $X^{(i)}$ is non-singular for $i \geq r$. The composition of all maps $\{\pi_i\}_{i=0,\cdots,r}$ is a resolution $\pi : \widetilde{X} \longrightarrow X$. It is well known that $\widetilde{X} \cong \mathbf{Spec}(\overline{R})$, where $\overline{R}$ is the integral closure of R on its ring of fractions. We denote by $\mathbf{Loc}(X)$ the finite set of local rings $\mathcal{O}_{X^{(i)},q}$, q singular point of $X^{(i)}$, appearing in the above resolution process. Recall that the family of Hilbert-Samuel polynomials $p_{\mathcal{O}_C} = e_o(\mathcal{O}_C)(n+1) - e_1(\mathcal{O}_C)$, with $\mathcal{O}_C \in \mathbf{Loc}(X)$ is univocally determined by X.

Definition 4.1. *For a reduced curve singularity $X = \mathbf{Spec}(R)$, the singularity order of X is the finite number*

$$\delta(X) = \text{Length}_R(\overline{R}/R),$$

i.e. the geometric genus $p_g(X)$ of X.

From [25] we have

$$\delta(X) = \sum_{\mathcal{O}_C \in \mathbf{Loc}(X)} e_1(\mathcal{O}_C).$$

Recall that we can decompose $e_1(\mathcal{O}_C)$ as sum of the micro-invariants of the ring extension $\mathcal{O}_C \subset \mathcal{O}_{Bl_{\mathbf{m}}(R)}$, see [2].

Proposition 4.2. *Let $X = \mathbf{Spec}(R)$ be a one-dimensional reduced scheme. Let I be an $\mathbf{m}$-primary ideal, then*

$$0 \leq e_1(I) \leq \delta(X) = p_g(X).$$

If $e_1(I) = \delta(X)$ then I is an Hironaka ideal.

Proof. We denote by $Bl_I(R)$ the semilocal ring of the blow-up of $X = \mathbf{Spec}(R)$ centered at the ideal I. We know that the ring extensions $R \subset Bl_I(R) \subset \overline{R}$ are finite and that, [17],

$$0 \leq e_1(I) = \text{Length}_R(Bl_I(R)/R) \leq \text{Length}_R(\overline{R}/R) = \delta(X).$$

Hence we get the first part of the result. If $e_1(I) = \delta(X)$ then we get $Bl_I(R) = \overline{R}$, i.e. I is an Hironaka ideal. ☐

Example 4.3. Let us consider the one-dimensional Cohen-Macaulay local domain $R_n = \mathbf{k}[x,y]_{(x,y)}/(y^2 - x^n)$, $n \geq 8$; we set $X_n = \mathbf{Spec}(R_n)$. The resolution process consists in $r = [n/2]$ blow-ups, and the singular points appearing in the process are all of multiplicity two. Since the e_1 of a double point is one we get $\delta(X) = r$. Last result shows that for all $\mathbf{m}$-primary ideal $I \subset R_n$

$$0 \leq e_1(I) \leq \delta(X_n) = r.$$

Let us consider the ideal I_n of R_n generated by x^6 and x^2y. The Hilbert-Samuel polynomial of I_n is $p_{I_n}(t) = 12t - 4$, i.e. $e_0(I_n) = 12$, and $e_1(I_n) = 4$. Hence, I_n is an Hironaka ideal of R_n if and only if $n = 8, 9$.

The jacobian ideal $J_n = (y, x^{n-1})$ is not an Hironaka ideal because $e_1(J_n) = 1$. Since $e_0(J_n)-\text{Length}(R_n/J_n) = 1 = e_1(J_n)$, we get that $gr_{J_n^t}(R_n)$ is Cohen-Macaulay for all $t \geq 1$, [12].

On the other hand, in [4] and [29] upper bounds for e_1 are given. In our case we get $e_1(I_n) = 4 < \epsilon(I_n) = 8$, bound of [4], and $e_1(I_n) = 4 < \rho(I_n) = 59$, bound of [29]. See [4, Proposition 2.2] for a comparison between ϵ and ρ.

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