Deciding the Continuum Hypothesis with the Inverse Power Set

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Abstract

We introduce the concept of inverse power set by adding two axioms to the Zermelo-Fraenkel set theory. This extends the Zermelo-Fraenkel set theory with a new type of set. We present different ways to extend the definition of cardinality and show that one implies the continuum hypothesis while another disproves the continuum hypothesis.

1 Introduction

Relying on Gödel's results, Cohen proved the independence of Cantor's continuum hypothesis from Zermelo-Fraenkel set theory with the axiom of choice [4, 1, 2]. Since then, many axioms have been considered to extend Zermelo-Fraenkel set theory in a way which would permit a proof or a disproof of the continuum hypothesis. In particular, the axiom of constructibility implies the generalized continuum hypothesis [4] and the Freiling's axiom of symmetry is equivalent to the negation of the continuum hypothesis [3]. More recently, Woodin developed an extensive framework aimed at disproving the continuum hypothesis [7, 8]. Accepting or rejecting such axioms often becomes a matter of taste and 'intuition', but each of those suggestions give rise to new systems of axioms which are worth studying by themselves or in relation with the others. In the following, we give another view which is analogous to numbers.

The negative numbers have been defined to allow us to solve equations such as $2 + y = 1$. Similarly, the complex numbers have been introduced by Euler to consider solutions of equations such as $y^2 = -1$. In the realm of set theory, in particular Zermelo-Fraenkel set theory, we can ask a similar question and try to find a set Y such that the statement $P(Y) = X$ is true for some fixed set X. For example, if we take $X = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, then we find that $Y = \{a, b\}$ satisfies the statement. But if we want a set Y which would satisfy the statement $P(Y) = \{\{a\}, \{a, b\}\}$, we have to define a new type of set in a way that is similar to the introduction of the negative and complex numbers.

The goal of this paper is twofold, first to introduce the concept of the inverse power set, extend the Zermelo-Fraenkel set theory and develop some of its theory. Secondly, in this extended setting, we will show that it is possible to prove or disprove the continuum hypothesis by choosing the suitable extended definition of cardinality.

2 Extending Zermelo-Fraenkel

We extend Zermelo-Fraenkel set theory (*ZF*) by adding the concept of the inverse power set in the form of two axioms and prove a few practical propositions.

2.1 Axioms

We call *EZF* the following extended Zermelo-Fraenkel set theory. We will take ZF along with the new axioms of inverse power set and invertibility. These new axioms apply, in a recursive manner, to sets which arise from EZF. For now, we will only extend the definition of subsets and the axiom of extensionality. We will leave the axiom of power set in the same form but for sets which arise from EZF^p instead of ZF, where EZF^p is a collection of sets which will be defined below. We will keep the other axioms of ZF in the same form and consider that they only apply to sets which arise from ZF.

Axiom 1 (Inverse Power Set)**.**

$$\forall X \exists Y [(P(Y) = X)]$$

We will denote Y as $P^{-1}(X)$ and call it the *inverse power set* of X. Hence, for some fixed X, we introduce a new type of set Y which satisfies the statement $P(Y) = X$. Similarly to what is commonly done with the power of a set, we can consider P^{-1} to be an operator. For example, the set

$Y = \{a, b\}$ satisfies the statement $P(Y) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Also, since $P(\emptyset) = \{\emptyset\}$, we have that $P^{-1}(\{\emptyset\}) = \emptyset$.

We now introduce the second axiom which expresses the notion that P^{-1} is the inverse operator of P.

Axiom 2 (Invertibility)**.**

$$\forall X[P^{-1}(P(X)) = X]$$

If we take the set $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and apply P^{-1} to it, we find

$$P^{-1}(\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}) = P^{-1}(P(\{a, b, c\})) = \{a, b, c\}.$$

We will see that the axiom of invertibility permits us to prove the uniqueness of $P^{-1}(X)$.

We will say that Y is a *real set* if Y is a set which arise from EZF. We will call a set X which arises from ZF a *Zermelo set*. Sets which arise from EZF but not from ZF will be called *non-Zermelo*. Examples of non-Zermelo sets are $P^{-1}(\{1, 2, 4\})$, $P^{-1}(\{\{1, 2, 4\}\})$, $P^{-1}(\{A\})$, $P^{-1}(\{A, B\})$, $P^{-1}(\{B, \{A, B\}\})$, $P^{-1}(P^{-1}(C))$ and $P^{-1}(P^{-1}(P^{-1}(C)))$ for any sets A, B and C. It is important to note that we will not introduce what are the explicit elements of sets such as $P^{-1}(X)$, but we will explore the concept of cardinality of such sets in the next sections and we will see that the concept of subsets can replace the explicit elements.

We will now introduce the subsets of a non-Zermelo set such as $P^{-1}(X)$. The idea behind the extended definition of subsets is to have

$$P^{-1}(X') \subseteq P^{-1}(X) \text{ if and only if } X' \subseteq X.$$

In analogy with the negative numbers, having defined the negative numbers -1 and -2 by equations such as $2 + y = 1$ and $3 + y = 1$, we need to order them. Since it has proven difficult to introduce explicit elements of a set such as $P^{-1}(X)$, we need to shift our perspective by using subsets ($\subseteq$) instead of membership ($\in$). This is the essence of the modifications which will be made in the axioms of extensionality and of union.

Recall that the classical definition of a subset is

$$A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B).$$

We now give an extended definition of subsets which includes the classical definition and make sure that the new type of sets are ordered via the subset symbol '$\subseteq$'. Note that it is a recursive definition.

Definition 3 (Extended Subsets)**.** *$A \subseteq B$ if and only if*

$$\forall x(x \in A \Rightarrow x \in B) \text{ or } P(A) \subseteq P(B)$$

This reduces to the classical definition if A, B are Zermelo.

Proposition 4. *Let A, B be Zermelo sets, then,*

$$A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B).$$

Proof. ($\Leftarrow$) Immediate by the extended subsets definition.

($\Rightarrow$) If A, B are Zermelo, we have that

$$P(A) \subseteq P(B) \text{ implies } \forall x(x \in A \Rightarrow x \in B),$$

since it is well known that $P(X) \subseteq P(Y) \Leftrightarrow X \subseteq Y$ in ZF (see p.48 of [5]). Hence, we find by the extended subsets definition that

$$A \subseteq B \Rightarrow \forall x(x \in A \Rightarrow x \in B).$$

□

Since we did not define the elements of non-Zermelo sets and since the axiom of extensionality is classically written in terms of membership, we have to write the axiom of extensionality in terms of subsets.

Axiom 5 (Extended Extensionality)**.**

$$\forall S[S \subseteq A \Leftrightarrow S \subseteq B] \Rightarrow A = B$$

We want to show that the axiom of extended extensionality reduces to the axiom of extensionality of ZF in the case where A, B are Zermelo sets. For this purpose, we prove the following.

Proposition 6. *If A, B are Zermelo, then*

$$\forall S(S \subseteq A \Rightarrow S \subseteq B) \Leftrightarrow \forall x(x \in A \Rightarrow x \in B).$$

Proof. ($\Rightarrow$) Take A for S, then we have ($A \subseteq A \Rightarrow A \subseteq B$) and this means that for all $x \in A$ we have that $x \in A \subseteq B$ and hence $x \in B$.

($\Leftarrow$) Take an arbitrary $S \subseteq A$, then for all $x \in S$ we have $x \in S \subseteq A$. By assumption, $x \in A$ implies that $x \in B$. Therefore for every $x \in S$ we must have that $x \in B$, which can be expressed as $S \subseteq B$ by proposition 4. Hence, $\forall S(S \subseteq A \Rightarrow S \subseteq B)$. □

Corollary 7. *Let A, B and all S be Zermelo sets, then*

$$\forall S(S \subseteq A \Leftrightarrow S \subseteq B) \Leftrightarrow \forall x(x \in A \Leftrightarrow x \in B).$$

Proof. Immediate. $\square$

Assuming the axiom of extended extensionality and taking only Zermelo sets, the axiom of extensionality $\forall x[x \subseteq A \Leftrightarrow x \subseteq B] \Rightarrow A = B$ follows immediately from corollary 7.

2.2 Propositions

We now proceed to prove four propositions concerning the inverse power set. In the following, four propositions $X, Y, Z, ...$ are assumed to be any arbitrary sets from EZF.

Proposition 8. $P(P^{-1}(X)) = X$ *and* $P^{-1}(P(X)) = X$*.*

Proof. By the inverse power set axiom and since we denote Y by $P^{-1}(X)$, we find the first identity $P(P^{-1}(X)) = X$ and by the invertibility axiom we directly find $P^{-1}(P(X)) = X$. $\square$

Proposition 9 (Uniqueness)**.** *There exists a unique set Y satisfying $P(Y) = X$.*

Proof. Suppose there are two sets satisfying $P(Y) = X$, then $P(Y_1) = X = P(Y_2)$. Taking $P(Y_2)$ as X in the inverse power set axiom, we must have that $Y_1 = P^{-1}(P(Y_2))$, but by proposition 8, we find $Y_1 = P^{-1}(P(Y_2)) = Y_2$. $\square$

Proposition 10. $X \subseteq Y$ *if and only if* $P(X) \subseteq P(Y)$*.*

Proof. ($\Leftarrow$) Immediate by definition 3 (extended subsets).

($\Rightarrow$)By the definition of extended subsets, if $X \subseteq Y$, then $\forall x(x \in A \Rightarrow x \in B)$ or $P(A) \subseteq P(B)$. Note that by proposition 4, the case $\forall x(x \in A \Rightarrow x \in B)$ will only happen when A, B are Zermelo. Thus, we only need to show that $\forall x(x \in A \Rightarrow x \in B)$ implies $P(A) \subseteq P(B)$ for A, B Zermelo. By proposition 6, since

$$\forall x(x \in A \Rightarrow x \in B) \Rightarrow \forall S(S \subseteq A \Rightarrow S \subseteq B),$$

we have that each element of $P(A)$ is in $P(B)$ and thus, $P(A) \subseteq P(B)$. $\square$

Proposition 11. *$X \subseteq Y$ if and only if $P^{-1}(X) \subseteq P^{-1}(Y)$.*

Proof. ($\Leftarrow$) By the inverse power set axiom we have that $X = P(P^{-1}(X))$ and $X = P(P^{-1}(Y))$. If $X \subseteq Y$, then we have that $P(P^{-1}(X)) \subseteq P(P^{-1}(Y))$. Thus, by the definition of extended subsets we find $P^{-1}(X) \subseteq P^{-1}(Y)$.

($\Rightarrow$)If $P^{-1}(X) \subseteq P^{-1}(Y)$, then using proposition 10, we find $P(P^{-1}(X)) \subseteq P(P^{-1}(Y))$. By the inverse power set axiom, we find $X \subseteq Y$. $\square$

We now give useful definitions which will help our notation.

Definition 12. *If P^{-1} occurs n times in $P^{-1}(P^{-1}(...P^{-1}(X)...))$, we will denote $P^{-1}(P^{-1}(...P^{-1}(X)...))$ as $P^{-n}(X)$. A set of the form $P^{-n}(X)$ with $n > 0$ and X a Zermelo set such that $X \neq P(X')$ for all Zermelo sets X' will be said to be* powered non-Zermelo set*.*

If P occurs m times in $P(P(...P(Y)...))$, we will denote $P(P(...P(Y)...))$ as $P^n(Y)$. A set of the form $P^m(X)$ with $m > 0$ and X a Zermelo set such that $X \neq P(X')$ for all Zermelo sets X' will be said to be a powered Zermelo set*.*

If a set Z is a powered Zermelo set or a powered non-Zermelo set we will say that it is a powered set*. If Z is neither we will say that it is a* non-powered set*.*

Considering P and P^{-1} as operators, it is important to note that in the previous definition that those operators are only applied a finite number of times.

For the next five propositions, we will formally restrict EZF to EZF^p where EZF^p contains all Zermelo set and all non-Zermelo powered sets. The reason for restricting EZF is to make sure that we explicitly know the form of each set in our domain. In particular, we make sure that we are not considering sets where the operators P and P^{-1} are applied an infinite number of times. The collection EZF^p can also be viewed as ZF augmented with the powered non-Zermelo sets.

Definition 13. *We define the collection EZF^p of real sets from EZF as:*

1. *If U is a Zermelo set, then $U \in$ EZF^p,*
2. *If U is a powered non-Zermelo set, then $U \in$ EZF^p,*

By definition, we have that a set of EZF^p is a Zermelo set or a powered non-Zermelo set.

We now show transitivity in the setting of EZF^p.

Proposition 14. *For A, B, C sets of EZF^p, If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. If A, B, C are Zermelo, by proposition 4, $A \subseteq B$ implies $\forall x(x \in A \Rightarrow x \in B)$. Similarly, $B \subseteq C$ implies $\forall x(x \in B \Rightarrow x \in C)$ and thus, $\forall x(x \in A \Rightarrow x \in C)$ which means that $A \subseteq C$.

If A, B, C are not all Zermelo, let $A = P^{n_a}(X_a)$, $B = P^{n_b}(X_b)$ and $C = P^{n_c}(X_c)$ with X_a, X_b, X_c each a non-powered set and for n_a, n_b, n_c integers different from zero such that at least one is negative. Hence, $P^{n_a}(X_a) \subseteq P^{n_b}(X_b)$ and $P^{n_b}(X_b) \subseteq P^{n_c}(X_c)$. Without loss of generality, take $n_a = \min(n_a, n_b, n_c)$ and note that it is a negative integer. Using proposition 10, we apply the operator P on each side $-n_a$ times and find $P^{-n_a}(P^{n_a}(X_a)) \subseteq P^{-n_a}(P^{n_b}(X_b))$ and $P^{-n_a}(P^{n_b}(X_b)) \subseteq P^{-n_a}(P^{n_c}(X_c))$. By proposition 8, we find $X_a \subseteq P^{n_b-n_a}(X_b)$ and $P^{n_b-n_a}(X_b) \subseteq P^{n_c-n_a}(X_c)$. We now have that $X_a, P^{n_b-n_a}(X_b)$ and $P^{n_c-n_a}(X_c)$ are all Zermelo sets, therefore this implies that $X_a \subseteq P^{n_c-n_a}(X_c)$. Applying P^{n_a} on each side of $X_a \subseteq P^{n_c-n_a}(X_c)$ and by using proposition 8 again, we find $P^{n_a}(X_a) \subseteq P^{n_c}(X_c)$, which can be written as $A \subseteq B$. $\square$

Proposition 15. *$A \subseteq A$.*

Proof. If A is Zermelo, then it is a logical truth that $\forall x(x \in A \Rightarrow x \in A)$, thus by the extended definition of subsets $A \subseteq A$.

If A is not Zermelo, then we can write it in the form $A = P^{-n}(X)$ with $n > 0$ and X a Zermelo set such that $X \neq P(X')$ for all Zermelo sets X'. The set $P^n(P^{-n}(X))$ is Zermelo, thus from the first part of the proof, $P^n(P^{-n}(X)) \subseteq P^n(P^{-n}(X))$. Hence, by proposition 11, we have $P^{-n}(P^n(P^{-n}(X))) \subseteq P^{-n}(P^n(P^{-n}(X)))$, which becomes by the invertibility axiom, $A = P^{-n}(X) \subseteq P^{-n}(X) = A$. $\square$

Proposition 16. *If $A \subseteq B$ and $B \subseteq A$ if and only if $A = B$.*

Proof. ($\Rightarrow$) Let $S \subseteq A$, then by transitivity and since by assumption $A \subseteq B$, we have that $S \subseteq B$ and hence $\forall S(S \subseteq A \Rightarrow S \subseteq B)$. Similarly, from the assumption $B \subseteq A$, we have that $\forall S(S \subseteq B \Rightarrow S \subseteq A)$. Hence by the axiom of extended extensionality, we find $A = B$.

($\Leftarrow$) Since $A \subseteq A$ is true and since $A = B$, we can replace the A on the right hand side or the left hand side by B. Hence, we find that $A \subseteq B \wedge B \subseteq A$ is true. ☐

Proposition 17. *$X = Y$ if and only if $P^{-1}(X) = P^{-1}(Y)$.*

Proof. By proposition 16, $X = Y$ if and only if $X \subseteq Y \wedge X \subseteq Y$. Since by proposition 11, we have that $X \subseteq Y \Leftrightarrow P^{-1}(X) \subseteq P^{-1}(Y)$, we find that

$$X \subseteq Y \wedge X \subseteq Y \Leftrightarrow P^{-1}(X) \subseteq P^{-1}(Y) \wedge P^{-1}(X) \subseteq P^{-1}(Y)$$

Thus by proposition 16, $X = Y \Leftrightarrow P^{-1}(X) = P^{-1}(Y)$. ☐

Proposition 18. *$X = Y$ if and only if $P(X) = P(Y)$*

Proof. From proposition 10 and since by proposition 16, $X \subseteq Y \wedge Y \subseteq X \Leftrightarrow X = Y$, we deduce that $X = Y \Leftrightarrow P(X) = P(Y)$. ☐

Let $Y \subseteq P(X)$ such that Y is a Zermelo set and $Y \neq P(Y')$ for all Zermelo sets Y', then by using proposition 11 and axiom 2, we have that $P^{-1}(Y) \subseteq X$ where $P^{-1}(Y)$ is not a Zermelo set. One way to look at this result is to consider that in the EZF^p, we can take 'fractions' of a set X. This is similar to the case of taking a fraction of an integer or like having $\frac{1}{2} \leq 2$. We also find that if $Z \subseteq P(Y)$ such that Z is a Zermelo set and $Z \neq P(Z')$ for all Zermelo sets Z', then by using proposition 11 and axiom 2, we have that $P^{-1}(P^{-1}(Z)) \subseteq P^{-1}(Y)$. Continuing this way, we find ... $\subseteq P^{-1}(P^{-1}(P^{-1}(W))) \subseteq P^{-1}(P^{-1}(Z)) \subseteq P^{-1}(Y) \subseteq X$. Note that this doesn't contradict the axiom of regularity since the axiom of regularity concerns infinite sequences of memberships.

We now have to extend the notion of cardinality. For some sets the usual cardinality definition is adequate, for example $|P^{-1}(P(\mathbb{N})| = \aleph_0$ since $P^{-1}(P(\mathbb{N})) = \mathbb{N}$ by axiom 2. Also, for some finite sets we have a nice inverse, in particular, $|P^{-1}(\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\})| = 3$ since $P^{-1}(\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}) = \{a,b,c\}$. For sets such as $P^{-1}(\{1,2,3,4,5\})$ and $P^{-7}(\mathbb{N})$ it is not clear what is the correct concept of cardinality. This is what we will investigate in the following section.

Note that the first four propositions of this section are all well defined for EZF^p. This is assured because applying P or P^{-1} to a set of EZF^p gives a set of EZF^p.

3 The Continuum Hypothesis

By extending Zermelo-Fraenkel set theory with the concept of the inverse power set, we now have a setting where the truth or falsity of the continuum hypothesis can be decided. This will be done by restricting EZF and by giving two extensions of the definition of cardinality. The extended definitions must apply to non-Zermelo sets such as $P^{-1}(\{1,2,3,4,5\}$ and $P^{-1}(P^{-1}(X))$. As seen in the case of complex numbers, there are many ways to define a norm. We will give three ways to extend the definition of cardinality, in particular two which induce the continuum hypothesis and one which induces its falsity. This can also be seen as giving an explicit model of ZF in which the continuum hypothesis is true and giving a model in which the continuum hypothesis is false.

3.1 Proving the Continuum Hypothesis

We now give an extended definition of cardinality which will induce the continuum hypothesis in the setting of EZF^p. We will use the notation $|X|$ to denote the classical cardinality of a set X.

Definition 19 (CH-cardinality)**.** *Let U be a set of EZF^p. If U is a Zermelo set then*

$$|U|_{ch} = |U|$$

Otherwise, if U is a powered non-Zermelo set, let U be written as $P^{-n}(X)$ with $n \geq 1$ and X a Zermelo set which is not a powered Zermelo set (i.e. X cannot be written as $P^k(X)$ for some $k \geq 1$). Then,

$$|U|_{ch} = |X|.$$

Definition 20. *Let U, U' be sets of EZF^p. Let $|U|_{ch} = |Y|$ and $|U|_{ch} = |Y'|$. Then,*

$$|U|_{ch} \leq |U'|_{ch} \text{ if and only if } |Y| \leq |Y'|.$$

Note that if U is a Zermelo set, the definitions reduce to the usual definition of cardinality. We will refer to this extension of cardinality as the *CH-cardinality.*

We now prove the extension of the Shroeder-Bernstein theorem.

Proposition 21. *If $|U|_{ch} \leq |U'|_{ch}$ and $|U'|_{ch} \leq |U|_{ch}$ then $|U|_{ch} = |U'|_{ch}$.*

Proof. Let $|U|_{ch} = |Y|$ and $|U|_{ch} = |Y'|$, then by definition 20 we have that $|Y| \leq |Y'|$ and $|Y'| \leq |Y|$ which implies by the Shroeder-Bernstein theorem that $|Y| = |Y'|$. Hence, we have $|U|_{ch} = |U'|_{ch}$. □

Proposition 22. *If $|U|_{ch} \leq |V|_{ch}$ and $|V|_{ch} \leq |W|_{ch}$ then $|U|_{ch} \leq |W|_{ch}$.*

Proof. Let $|U|_{ch} = |Y|$, $|V|_{ch} = |Y'|$ and $|W|_{ch} = |Y''|$, then by assumption and by the transitivity of cardinal numbers, we have $|Y| \leq |Y''|$ and thus, $|U|_{ch} \leq |W|_{ch}$. □

Following the approach given in [5], we extend a few useful definitions.

Definition 23. *α is a CH-cardinal number if and only if there is a set X of EZF^p such that $|X|_{ch} = \alpha$.*

In [5], there is a similar definition, but for cardinal number in ZF. The validity of that definition is assured by the axiom of cardinality, for more details, see p.111 of [5]. In EZF^p, the validity of definition 23 is guaranteed by the axiom of cardinality of [5] and since, by the CH-cardinality definition, we have that $|U|_{ch} = |V|$ for a particular Zermelo set V.

Proposition 24. *If α is a cardinal number, then α is a CH-cardinal number.*

Proof. If α is a cardinal number, by the usual definition of cardinality there is a set of ZF such that $|X| = \alpha$. By the CH-cardinality definition 19, we have $|X|_{ch} = |X| = \alpha$ since X is Zermelo, thus by definition 23, α is a CH-cardinal number. □

In [5], Suppes give the two following definitions regarding the context of ZF.

Definition 25. *Let A, B be sets of ZF, then $A \preceq B$ if and only if there exists a set C of ZF such that $|C| = |A|$ and $C \subseteq B$.*

Definition 26. *$\alpha \leq \alpha'$ if and only if there are sets A and B of ZF such that $|A| = \alpha$, $|B| = \alpha'$ and $A \preceq B$.*

We extend those definitions to EZF^p in the following way.

Definition 27. *Let A, B be sets of EZF^p, then $A \preceq B$ if and only if there is a set C of EZF^p such that $|C|_{ch} = |A|_{ch}$ and $C \subseteq B$.*

Definition 28. *$\alpha \leq \alpha'$ if and only if there are sets A and B of EZF^p such that $|A|_{ch} = \alpha$, $|B|_{ch} = \alpha'$ and $A \preceq B$.*

Note that if we take A, B to be Zermelo sets, the definitions 27 and 28 reduce to 25 and 26, since by the CH-cardinality definition $|A|_{ch} = |A|$ and $|B|_{ch} = |B|$.

Lemma 29. *Let A be a set of EZF^p and let B be a Zermelo set, then $|A|_{ch} \leq |B|_{ch}$ if and only if there is a Zermelo set C'' such that $|C''|_{ch} = |A|_{ch}$ and $C'' \subseteq B$.*

Proof. ($\Rightarrow$)Using the definition 28 and 27, there is a set C of EZF^p such that $|C|_{ch} = |A|_{ch}$ and $C \subseteq B$. By the definition of CH-cardinality, $|C|_{ch} = |C'|$ for some Zermelo set C' and thus $|C|_{ch} = |C'|_{ch}$. Since B is Zermelo we have $|B|_{ch} = |B|$ and since $|C|_{ch} = |A|_{ch}$ and $|C|_{ch} = |C'|_{ch} = |C'|$, we have that $|A|_{ch} = |C'|$. By assumption, $|A|_{ch} \leq |B|_{ch}$, hence by replacing $|A|_{ch}$ and $|B|_{ch}$, we find $|C'| \leq |B|$. By definition 26 and 25, there exists a set C'' of ZF such that $|C''| = |C'|$ and $C'' \subseteq B$. This is what we were looking for since $|C''|_{ch} = |C''| = |C'| = |C|_{ch} = |A|_{ch}$.

($\Leftarrow$)by definition 27 and 28 and since Zermelo sets are sets of EZF^p we find that $|A|_{ch} \leq |B|_{ch}$. ∎

Definition 30. *$\alpha < \alpha'$ if and only if $\alpha \leq \alpha'$ and not $\alpha' \leq \alpha$.*

Definition 31. *Let U be a set of EZF^p and let $|U|_{ch} = |Y|$, then U is finite if and only if $|Y|$ is finite. A set of EZF^p is infinite if it is not finite.*

We will now evaluate the CH-cardinality of $P^{-1}(\mathbb{N})$. It is important to note that by definition, $\mathbb{N}$ is a Zermelo set, but it is not clear if it is a powered Zermelot set or not.

Theorem 32. *Let A be a Zermelo set such that $|A|_{ch} = |\mathbb{N}|_{ch}$ then $|P^{-1}(A)|_{ch} = |\mathbb{N}|_{ch}$.*

Proof. If A is not a powered Zermelo set, by definition of CH-cardinality, we have $|P^{-1}(A)|_{ch} = |A|$. Since $|\mathbb{N}| = |\mathbb{N}|_{ch}$ and by assumption $|A|_{ch} = |\mathbb{N}|_{ch}$, we find that $|P^{-1}(A)|_{ch} = |\mathbb{N}|_{ch}$. Thus, if we show that A cannot be a powered Zermelo set, we are finished.

Assume that A is a powered Zermelo set, then there is a Zermelo set A' such that $P(A') = A$.

Suppose that $|P^{-1}(A)|_{ch} \geq |\mathbb{N}|_{ch}$, then replacing A we find $|P^{-1}(P(A'))|_{ch} \geq |\mathbb{N}|_{ch}$ which becomes $|A'|_{ch} \geq |\mathbb{N}|_{ch}$ by the invertibility axiom. Since $|A'| = |A'|_{ch} \geq |\mathbb{N}|_{ch} = |\mathbb{N}|$, we find $|P(A')| \geq |P(\mathbb{N})|$ by using a known result $|X| \geq |Y| \Rightarrow |P(X)| \geq |P(Y)|$ (see the lemma on p.95 of [6]). But this means that $|\mathbb{N}| = |A| = |P(A')|_{ch} \leq |P(\mathbb{N})|$, a contradiction to Cantor's theorem.

Now, suppose that $|P^{-1}(A)|_{ch} < |\mathbb{N}|_{ch}$. Since A is a powered Zermelo set, then there is a Zermelo set A' such that $P(A') = A$. Replacing on the left hand side of $|P^{-1}(A)|_{ch} < |\mathbb{N}|_{ch}$ we find $|P^{-1}(P(A'))|_{ch} < |\mathbb{N}|_{ch}$ and thus by the invertibility axiom and because A' and $\mathbb{N}$ are Zermelo sets, $|A'| = |A'|_{ch} < |\mathbb{N}|_{ch} = |\mathbb{N}|$. Using the definition of [6] p.74, which says that a subset S is at most countable if $|S| \leq |\mathbb{N}|$, this mean that A' is at most countable. By the corollary of p.74 of [6], a set is at most countable if and only if it is finite or countable. If A' is finite, we find that $P(A') = A$ is finite, a contradiction with $|A| = |\mathbb{N}|$. Thus, we must have that A' is countable. This means that $|A'| = \aleph_0$. Thus we have that $\aleph_0 = |A'| = |A'|_{ch} < |\mathbb{N}|_{ch} = |\mathbb{N}| = \aleph_0$, an impossibility. $\square$

We are now in a context where we can prove the continuum hypothesis.

Theorem 33. *In EZF^p, there exists no CH-cardinal number β such that $\aleph_0 < \beta < \mathfrak{c}$ where $\aleph_0$ is the CH-cardinality of a countable set and $\mathfrak{c}$ is the CH-cardinality of the continuum.*

Proof. Assume that there is such a CH-cardinal β. Since $\aleph_0 = |\mathbb{N}|_{ch} = |\mathbb{N}|$ and $\mathfrak{c} = |P(\mathbb{N})|_{ch} = |P(\mathbb{N})|$ we have that by definition 23, there exists set B of EZF^p such that $|\mathbb{N}|_{ch} < |B|_{ch} < |P(\mathbb{N})|_{ch}$.

Since $|B|_{ch} < |P(\mathbb{N})|_{ch}$, by lemma 29, there exists a Zermelo set B' such that $B' \subset P(\mathbb{N})$, $|B'|_{ch} = |B|_{ch}$ and $|B'|_{ch} \neq |P(\mathbb{N})|_{ch}$. Also, since $|\mathbb{N}|_{ch} < |B|_{ch} = |B'|_{ch}$, by lemma 29, there exists a Zermelo set N' such that $N' \subset B'$, $|N'|_{ch} = |\mathbb{N}|_{ch}$ and $|N'|_{ch} \neq |B'|_{ch}$.

Hence, by the proposition 14 of transitivity, we have $N' \subset B' \subset P(\mathbb{N})$. By proposition 11, we find $P^{-1}(N') \subset P^{-1}(B') \subset P^{-1}(P(\mathbb{N}))$ and by proposition 8, we get $P^{-1}(N') \subset P^{-1}(B') \subset \mathbb{N}$. By definition 27 and 28 this means that $|P^{-1}(N')|_{ch} \leq |P^{-1}(B')|_{ch} \leq |\mathbb{N}|_{ch}$. By theorem 32, we have $|\mathbb{N}|_{ch} = |P^{-1}(N')|_{ch} \leq |P^{-1}(B')|_{ch} \leq |\mathbb{N}|_{ch}$ and by proposition 21, we find that $|P^{-1}(B')|_{ch} = |\mathbb{N}|_{ch}$.

We now have two cases to consider: B' is a powered Zermelo set or B' is not a powered Zermelo set.

If B' is a powered Zermelo set, then we can write B' as $P^n(X')$ with X' a Zermelo set. Since the power set of a Zermelo set is Zermelo, we can write B' as $P(X)$ with X being a Zermelo set. Replacing $B' = P(X)$ in $|\mathbb{N}|_{ch} = |P^{-1}(B')|_{ch}$, we find $|\mathbb{N}|_{ch} = |P^{-1}(P(X))|_{ch}$ which becomes by proposition 8, $|\mathbb{N}|_{ch} = |X|_{ch}$. Since $\mathbb{N}$ and X are Zermelo we have, by the classical result $|X| = |Y| \Rightarrow |P(X)| = |P(Y)|$ (see p.95 lemma of [6]), that $|P(\mathbb{N})|_{ch} = |P(X)|_{ch}$. But since $B' = P(X)$ and $|B'|_{ch} = |B|_{ch}$ we have $|P(\mathbb{N})|_{ch} = |B|_{ch}$ which is a contradiction with our assumption.

If B' is not a powered Zermelo set, then by definition of B' it must be a non-powered Zermelo set. The CH-cardinality definition tells us that $|B'|_{ch} = |P^{-1}(B')|_{ch}$. Thus, since we have found that $|P^{-1}(B')|_{ch} = |\mathbb{N}|_{ch}$ and since $|B'|_{ch} = |B|_{ch}$, we have that $|B|_{ch} = |B'|_{ch} = |P^{-1}(B')|_{ch} = |\mathbb{N}|_{ch}$, a contradiction with the assumption that $|\mathbb{N}|_{ch} < |B|_{ch}$.

□

Corollary 34. *In EZF$^{\mathfrak{p}}$, there exists no cardinal number β such that $\aleph_0 < \beta < c$.*

Proof. This follows from 24, since every cardinal number is a CH-cardinal number. □

3.2 Disproving the Continuum Hypothesis

We now give two extended definitions of cardinality which will induce the falsity of the continuum hypothesis in the context of EZF* which will be defined below. The first extended definition is much weaker than the second in the sense that in the second there are many more ¬CH-cardinal numbers between $\mathbb{N}$ and $\mathfrak{c}$. The definitions are closely related to the lexicographic order. We will call the weaker extended definition of cardinality *¬CHW-cardinality* and the other *¬CH-cardinality*. We will see that those definitions give a richer theory of cardinality, which is similar to enriching the natural numbers with the rational numbers.

For the following, we extend the axiom of pairing of EZF to apply to sets of EZF instead of sets of ZF. The statement of the axiom is the same as in Zermelo-Fraenkel set theory. We will say that X is a *Zermelo form* if it is of the form of a set which arises from ZF, but which can have elements that are real sets, for examples $\{1, 2, X, Y, P^{-1}(Z), P^{-6}(1, 3, A, B)\}$ and $\{1, 2, 4, P^{-1}(Y), P^{-1}(Z)\}$. Note that most of the results presented above can be proved by using Zermelo forms instead of Zermelo sets.

For our purpose, we need to extend the axiom of union. We take C to be a Zermelo form, that is, a collection of sets of EZF which is constructed by the use of the extended axiom of pairing.

Axiom 35 (Extended Union)**.**

$$\exists U \exists U' \forall S[S \subseteq U \wedge S \in U' \Leftrightarrow \exists B(S \subseteq B \wedge B \in C)]$$

This definition is similar to the extended extensionality axiom where membership is replaced by the subset relation. We had to add the component $S \in U'$ in the extended definition, because the union of two subsets of U is a subset and this would imply that the union of those two subsets would have to be in C. Furthermore, without the component $S \in U'$, since $U \subseteq U$, we would have that $U \in C$. Thus, the component $S \in U'$ is used to remember which subset comes from C. We will denote U as $A_1 \cup A_2 \cup ... \cup A_n$ if $C = \{A_1, A_2, ..., A_n\}$. Note that the union is commutative and associative.

Proposition 36. *If we assume U, U', S, B and C are Zermelo sets which satisfy the axiom of extended union, then*

$$\exists U \forall x[x \in U \Leftrightarrow \exists B(x \in B \wedge B \in C)].$$

Proof. If $x \in B$, then $x \in B \subseteq B$ and this implies by the axiom of extended union that $x \in B \subseteq U$. Hence we have $\exists U \forall x[x \in U \Leftarrow \exists B(x \in B \wedge B \in C)]$.

If $x \in U$, then by construction of U, x must be in some S where $S \subseteq U$ and $S \in U'$. This implies by the axiom of extended union that $\exists B(x \in S \subseteq B \wedge B \in C)$. Hence, we have $\exists U \forall x[x \in U \Rightarrow \exists B(x \in B \wedge B \in C)]$. □

The axiom of extended union gives rise to a wider range of non-Zermelo sets, for example $P^{-1}(\{1,2,3,4,5\} \cup P^{-1}(P^{-1}(X))$ and $P^{-1}(X) \cup P^{-1}(Y) \cup Z \cup P^{-1}(P^{-1}(W))$.

In EZF plus the axiom of extended union (along with the axiom of pairing extended to sets of EZF), we have sets which can have an infinite number of operators P and P^{-1}, hence, we will restrict the domain. We will take EZF* to be EZF plus the axiom of extended union restricted to all the Zermelo sets, powered non-Zermelo sets and the sets which arise from the finite union of a combination of Zermelo and powered non-Zermelo sets, for example $X \cup Y \cup P^4(Z) \cup P^{-3}(U) \cup P^{-3}(V) \cup P^{-4}(W)$. Since the union is commutative and associative and since the union of two Zermelo sets is a Zermelo set, we can combine all the Zermelo sets together. Also, since $P^{-1}(P(X)) = X$, we have that every set of EZF* can be written as

$$X \cup P^{-1}(Y_1) \cup P^{-1}(Y_2) \cup ... \cup P^{-1}(Y_{n_1}) \cup ... \cup P^{-k}(Z_1) \cup P^{-k}(Z_2) \cup ... \cup P^{-k}(Z_{n_k}),$$

with $X, Y_1, Y_2, ..., Y_{n_1}, ..., Z_1, Z_2, ..., Z_{n_k}$ each a Zermelo set which is not a powered set (Zermelo or non-Zermelo). Formally, we define EZF* as follows.

Definition 37. *We define the collection EZF* of real sets from EZF with the extended axiom of union as:*

1. *If U is a Zermelo set, then $U \in$ EZF*,*
2. *If U is a powered non-Zermelo set, then $U \in$ EZF*,*
3. *If U is a Zermelo set and V_i is a powered non-Zermelo set for all i, then $U \cup V_1 \cup ... \cup V_k \in$ EZF*.*

We define two functions which will be used in the extended definitions of cardinality.

Definition 38. *Let Y_1 be Zermelo and let Y_2, Y_3 be non-empty non-powered Zermelo sets and let m be a positive integer. We define ρ, a function which takes as input a set X of EZF* and returns an integer or $-\infty$, as follows:*

$$\rho(X) = \begin{cases} 0 & \text{if } X = Y_1 \\ -m & \text{if } X = P^{-m}(Y_3) \\ -\infty & \text{if } X = \emptyset \end{cases}$$

Definition 39. *Let Y_1 be Zermelo and let Y_2, Y_3 be non-powered Zermelo sets and let m be a positive integer. We define τ, a function which takes as input a set X of EZF* and returns a cardinal number, as follows:*

$$\tau(X) = \begin{cases} |Y_1| & \text{if } X = Y_1 \\ |Y_3| & \text{if } X = P^{-m}(Y_3) \end{cases}$$

We now give a useful formal definition, but in a few words it is equivalent to ordering the components of the union in the following way:

$$X \cup P^{-1}(Y_1) \cup ... \cup P^{-1}(Y_{n_1}) \cup ... \cup P^{-k}(Z_1) \cup ... \cup P^{-k}(Z_{n_k})$$

with $|Y_1| \leq |Y_2| \leq ... \leq |Y_{n_1}|$ and ... and $|Z_1| \leq ... \leq |Z_{n_k}|$.

Definition 40. *Let X be a set of EZF* such that $X = X_1 \cup X_2 \cup ... \cup X_n$, then X is well-represented if and only if*

$$\rho(X_1) \geq ... \geq \rho(X_n)$$
$$\textit{and}$$
$$\textit{if } \tau(X_i) = ... = \tau(X_{i+k}) \textit{ then } \tau(X_i) \geq ... \geq \tau(X_{i+k}) \ .$$

3.3 Weak $\neg$ Continuum Hypothesis

In analogy with numbers, the definition of the weaker extended definition of cardinality $\neg$*CHW-cardinality* can be seen as ordering numbers in the following manner.

$$0.223 < 0.224 < 0.225 < ... < 0.255 < ...$$
$$< 0.324 < 0.3241 < 0.3242 < 1 < ...$$
$$< 1.111 < ... < 1.99999 < ...$$

Definition 41 ($\neg CHW$-cadinality)**.** *Let X, Y be sets of EZF*. Let $X = X_1 \cup X_2 \cup ... \cup X_n$ and let $Y = Y_1 \cup Y_2 \cup ... \cup Y_m$ where only X_1, Y_1 are Zermelo (possibly empty) and where $X_2, ..., X_n, Y_2, ..., Y_m$ are all non-empty and non-Zermelo with $X_2 \cup X_3 \cup ... \cup X_n$ and $Y_2 \cup Y_3 \cup ... \cup Y_n$ both well-represented. Then,*

$|X|_{\neg chw} \leq |Y|_{\neg chw}$ if and only if $|X_1| < |Y_1|$ or each of the following are simultaneously valid:

$$|X_1| = |Y_1|,$$
$$n \leq m,$$
$$\rho(X_i) \leq \rho(Y_i) \textit{ for all } i \textit{ such that } n \geq i \geq 2,$$
$$\tau(X_i) \leq \tau(Y_i) \textit{ for all } i \textit{ such that } n \geq i \geq 2,$$

We have equality on the left hand side if and only if $|X_1| = |Y_1|$, $n = m$, $\rho(X_i) = \rho(Y_i)$ and $\tau(X_i) = \tau(Y_i)$.

Since the union is commutative and since the union of two Zermelo sets is a Zermelo set, each set of EZF* can be written as required in the definition. We have introduced the function τ because EZF* includes all Zermelo sets and we cannot assume that Zermelo sets can be written as a powered set or a set of the same cardinality of $\mathbb{N}$. Note that the transitivity of $\leq$ on the $\neg CHW$-cardinal numbers follows from the transitivity of $\leq$ on the integers and $\leq$ on the classical cardinal numbers.

Proposition 42. *Let X and Y be Zermelo, then $|X|_{\neg chw} \leq |Y|_{\neg chw}$ if and only if $|X| \leq |Y|$.*

Proof. Since X and Y are Zermelo, we have that $X = X_1$, $Y = Y_1$ and $m = n = 1$. By definition of $\neg CHW$-cardinality we have that $|X|_{\neg chw} \leq |Y|_{\neg chw}$ if and only if $[|X_1| < |Y_1|$ or $(|X_1| = |Y_1|$ and $m = n)]$. But $|X_1| < |Y_1|$ or $|X_1| = |Y_1|$ is equivalent to $|X| \leq |Y|$ and hence we find $|X|_{\neg chw} \leq |Y|_{\neg chw} \Leftrightarrow |X| \leq |Y|$. ☐

We are now in a context where we can prove the falsity of the continuum hypothesis.

Theorem 43 (Weak $\neg$ Continuum Hypothesis)**.** *Let Z be a Zermelo set such that $|Z|_{\neg chw} < |P(Z)|_{\neg chw}$, then $|Z|_{\neg chw} < |Z \cup P^{-n}(Z)|_{\neg chw} < |P(Z)|_{\neg chw}$ where n is any integer greater than or equal to* 1.

Proof. First, we must show that $|Z|_{\neg chw} < |Z \cup P^{-n}(Z)|_{\neg chw}$. Taking $X = Z$ and $Y = Z \cup P^{-n}(Z)$, then since $|X_1| = |Z| = |Y_1|$, $n = 1 \leq 2 = m$, $\rho(X_1) = \rho(Y_1)$ and $\tau(X_1) = \tau(Y_1)$, we have by definition 41 that $|Z|_{\neg chw} < |Z \cup P^{-n}(Z)|_{\neg chw}$.

Secondly, we must show that $|Z \cup P^{-n}(Z)|_{\neg chw} < |P(Z)|_{\neg chw}$. Taking $X = Z \cup P^{-n}(Z)$ and $Y = P(Z)$, then since $|Z| = |X_1| < |Y_1| = |P(Z)|$ by Cantor's theorem, we have by definition of $\neg CHW$-cardinality, $|Z \cup P^{-n}(Z)|_{\neg chw} < |P(Z)|_{\neg chw}$. ☐

Since n is an arbitrary integer, the statement of the theorem tells us that there are infinitely many $\neg CHW$-cardinal numbers between $|\mathbb{N}|_{\neg chw}$ and $|P(\mathbb{N})|_{\neg chw}$. Moreover, strictly between $|Z \cup P^{-n}(Z)|_{\neg chw}$ and $|P(Z)|_{\neg chw}$ there are also infinitely many $\neg CHW$-cardinal numbers. Taking $n \geq 2$, examples are:

$$|Z \cup P^{-n+1}(Z)|_{\neg chw},\ |Z \cup P^{-n}(Z) \cup P^{-n-1}(Z)|_{\neg chw},$$
$$|Z \cup P^{-n}(Z) \cup P^{-n-2}(Z)|_{\neg chw},\ |Z \cup P^{-n}(Z) \cup P^{-n-1}(Z) \cup P^{-n-1}(Z)|_{\neg chw},$$
$$|Z \cup P^{-n}(Z) \cup P^{-n-1}(Z) \cup P^{-n-2}(Z)|_{\neg chw} \text{ and so forth.}$$

We also have infinitely many $\neg CHW$-cardinal numbers between $|Z|_{\neg chw}$ and $|Z \cup P^{-n}(Z)|_{\neg chw}$, for example $|Z \cup P^{-n-1}(Z)|_{\neg chw}$, $|Z \cup P^{-n-2}(Z)|_{\neg chw}$ and so forth. An interesting question is to ask if there is a $\neg CHW$-cardinal number between $|Z \cup P^{-n-1}(Z)|_{\neg chw}$ and $|Z \cup P^{-n}(Z)|_{\neg chw}$. This question, which relies on the definition of $\neg CHW$-cardinality, might also be undecidable.

We will see in the following that it is possible to give an extension to the definition of cardinality which can then answer this question. Furthermore, to ask if there is always an extended cardinal number between two extended cardinal numbers could be seen as the real question behind the falsity of the continuum hypothesis. Seen as an analogy with numbers, it is similar to ask if between each pair of rational numbers there is a rational number.

We can also prove an extended version of Shroeder-Bernstein theorem for the $\neg CHW$-cardinality.

Proposition 44. *If* $|X|_{\neg chw} \leq |Y|_{\neg chw}$ *and* $|Y|_{\neg chw} \leq |X|_{\neg chw}$ *then* $|X|_{\neg chw} = |Y|_{\neg chw}$.

Proof. When $|X|_{\neg chw} \leq |Y|_{\neg chw}$ and $|Y|_{\neg chw} \leq |X|_{\neg chw}$, the only possibility is when $|X_1| = |Y_1|$. Since, if $|X_1| < |Y_1|$ this implies that $|Y|_{\neg chw} \nleq |X|_{\neg chw}$. By assumption, we have $m \leq n$, $n \leq m$, $\rho(X_i) \leq \rho(Y_i)$, $\tau(Y_i) \leq \tau(X_i)$, $\rho(X_i) \geq \rho(Y_i)$ and $\tau(Y_i) \geq \tau(X_i)$ for all i and thus we have $|X_1| = |Y_1|$, $n = m$, $\rho(X_i) = \rho(Y_i)$ and $\tau(X_i) = \tau(Y_i)$. Therefore we can conclude that $|X|_{\neg chw} = |Y|_{\neg chw}$. $\square$

3.4 ¬ Continuum Hypothesis

Before giving the $\neg CH$-cardinality definition, note that if $X = X_1 \cup X_2 \cup ... \cup X_n$, $Y = Y_1 \cup Y_2 \cup ... \cup Y_m$ and $n > m$, we can write Y as $Y_1 \cup Y_2 \cup ... \cup Y_n$, by adding a certain amount of union components which are empty sets.

Definition 45 ($\neg CH$-cardinality)**.** *Let X, Y be sets of EZF*. Let* $X = X_1 \cup X_2 \cup ... \cup X_n$ *and let* $Y = Y_1 \cup Y_2 \cup ... \cup Y_n$ *where* X_1, Y_1 *are Zermelo (possibly empty) and where* $X_2, ..., X_n, Y_2, ..., Y_n$ *are non-Zermelo or empty with* $X_2 \cup X_3 \cup ... \cup X_n$ *and* $Y_2 \cup Y_3 \cup ... \cup Y_n$ *both well-represented. Then,* $|X|_{\neg ch} < |Y|_{\neg ch}$ *if and only if*

for some k such that $k \leq n$, *we have* $\rho(X_i) = \rho(Y_i)$, $\tau(X_i) = \tau(Y_i)$, $\rho(X_k) < \rho(Y_k)$ *and* $\tau(X_k) \leq \tau(Y_k)$ *for all* $i < k$.

or

for some k such that $k \leq n$, *we have* $\rho(X_i) = \rho(Y_i)$, $\tau(X_i) = \tau(Y_i)$, $\rho(X_k) \leq \rho(Y_k)$ *and* $\tau(X_k) < \tau(Y_k)$ *for all* $i < k$.

Definition 46. *Using the same assumptions as in definition 45.*
$|X|_{\neg ch} = |Y|_{\neg ch}$ *if and only if*

$$\text{for all } j \leq n, \text{ we have } \rho(X_j) = \rho(Y_j) \text{ and } \tau(X_j) = \tau(Y_j).$$

The definition of $\neg CH$-cardinality is basically the lexicographic order or in analogy with numbers, it is similar to ordering of rational numbers in decimal notation. Note that the transitivity of $<$ on the $\neg CH$-cardinal numbers follows from the transitivity of $<$ on the integers and $<$ on the classical cardinal numbers.

Proposition 47. *If* $|X|_{\neg ch} \leq |Y|_{\neg ch}$ *and* $|Y|_{\neg ch} \leq |Z|_{\neg ch}$*, then* $|X|_{\neg ch} \leq |Z|_{\neg ch}$.

Proof. Let X, Y, Z be sets of EZF*. Let $X = X_1 \cup X_2 \cup ... \cup X_n$, $Y = Y_1 \cup Y_2 \cup ... \cup Y_n$ and let $Z = Z_1 \cup Z_2 \cup ... \cup Z_n$ where X_1, Y_1, Z_1 are Zermelo sets (possibly empty) and where $X_2, ..., X_n, Y_2, ..., Y_n, Z_2, ..., Z_n$ are non-Zermelo or empty with $X_2 \cup X_3 \cup ... \cup X_n$, $Y_2 \cup Y_3 \cup ... \cup Y_n$ and $Z_2 \cup Z_3 \cup ... \cup Z_n$ are each well-represented. This can be done by adding empty union components to make sure that each X, Y, Z have exactly n components.

If $|X|_{\neg ch} = |Y|_{\neg ch}$ and $|Y|_{\neg ch} = |Z|_{\neg ch}$ then by definition 46, we must have that $|X|_{\neg ch} = |Z|_{\neg ch}$.

If $|X|_{\neg ch} < |Y|_{\neg ch}$ and $|Y|_{\neg ch} = |Z|_{\neg ch}$ then there is a k_1 such that for all $i < k_1$, $\rho(X_i) = \rho(Y_i)$, $\tau(X_i) = \tau(Y_i)$ and such that

$$[\rho(X_{k_1}) < \rho(Y_{k_1}) \wedge \tau(X_{k_1}) \leq \tau(Y_{k_1})] \vee \rho(X_{k_1}) \leq \rho(Y_{k_1}) \wedge \tau(X_{k_1}) < \tau(Y_{k_1}).$$

Since $\tau(Y_j) = \tau(Z_j)$ and $\rho(Y_j) = \rho(Z_j)$ for all $j < n$, we can take $k = k_1$ and conclude, using the $\neg CH$-cardinality definition, that $|X|_{\neg ch} < |Z|_{\neg ch}$. Similarly, we prove that if $|X|_{\neg ch} = |Y|_{\neg ch}$ and $|Y|_{\neg ch} < |Z|_{\neg ch}$, then $|X|_{\neg ch} < |Z|_{\neg ch}$.

If $|X|_{\neg ch} < |Y|_{\neg ch}$ and $|Y|_{\neg ch} < |Z|_{\neg ch}$, then we take k_1 for the k which appears in the $\neg CH$-cardinality definition for $|X|_{\neg ch} < |Y|_{\neg ch}$ and take k_2 for the k which comes from $|Y|_{\neg ch} < |Z|_{\neg ch}$. Let $m = \min\{k_1, k_2\}$.

If $m = k_1$, this means that $k_2 > m$ and that $\tau(Y_i) = \tau(Z_i)$ and $\rho(Y_i) = \rho(Z_i)$ for all $i < k_2$. Therefore, we have $\tau(X_m) < \tau(Y_m) = \tau(Z_m)$ or $\rho(X_m) < \rho(Y_m) = \rho(Z_m)$ which implies by the $\neg CH$-cardinality definition that $|X|_{\neg ch} < |Z|_{\neg ch}$. In a similar manner, if $m = k_2$, then we find that $\tau(X_m) = \tau(Y_m) < \tau(Z_m)$ or $\rho(X_m) = \rho(Y_m) < \rho(Z_m)$ which implies that $|X|_{\neg ch} < |Z|_{\neg ch}$. $\square$

We now prove the stronger version of the negative of the continuum hypothesis.

Theorem 48 ($\neg$ Continuum Hypothesis)**.** *Let X, Y be sets of EZF* such that $|X|_{\neg ch} < |Y|_{\neg ch}$, then there is a set U of EZF* such that $|X|_{\neg ch} < |U|_{\neg ch} < |Y|_{\neg ch}$.*

Proof. Let X, Y be sets of EZF*. Let $X = X_1 \cup X_2 \cup ... \cup X_n$ and let $Y = Y_1 \cup Y_2 \cup ... \cup Y_n$ where X_1, Y_1 are Zermelo (possibly empty) and where $X_2, ..., X_n, Y_2, ..., Y_n$ are non-Zermelo or empty with $X_2 \cup X_3 \cup ... \cup X_n$ and $Y_2 \cup Y_3 \cup ... \cup Y_n$ both well-represented.

Since $|X|_{\neg ch} < |Y|_{\neg ch}$, there is a k such that $[\rho(X_k) \leq \rho Y_k$ and $\tau(X_k) < \tau Y_k]$ or $[\rho(X_k) < \rho Y_k$ and $\tau(X_k) \leq \tau Y_k]$. Let r be the minimum integer (different from $-\infty$) of $\{\rho(X_1), ..., \rho(X_n), \rho(Y_1), ..., \rho(Y_n)\}$. Suppose that the smallest cardinal number (different from $|\emptyset|$) of $\tau(X_1), ..., \tau(X_n), \tau(Y_1), ..., \tau(Y_n)$ is $\tau(Z)$ such that $\tau(Z) = |K|$ with K a non-powered Zermelo set.

Take $U = X \cup P^{r-1}(K)$. We will show that $|X|_{\neg ch} < |X \cup P^{r-1}(K)|_{\neg ch}$ and $|X \cup P^{r-1}(K)|_{\neg ch} < |Y|_{\neg ch}$.

In $X = X_1 \cup X_2 \cup ... \cup X_n$ there are finitely many empty X_h components. Let $X = X_1 \cup X_2 \cup ... \cup X_m$ where $m \leq n$ and where all X_i are not empty. Since $\rho(P^{r-1}(K)) < \rho(X_j)$ for all $j \leq m$, we have that $X \cup P^{r-1}(K) = X_1 \cup X_2 \cup ... \cup X_m \cup P^{r-1}(K)$ is well-presented. Thus, we have that $X_i = X_i$ for all $i \leq m$, $0 < \rho(P^{r-1}(K))$ and $0 < \tau(P^{r-1}(K))$ for $k = m + 1$ which means, by definition 45 of $\neg CH$-cardinality, that $|X|_{\neg ch} < |X \cup P^{r-1}(K)|_{\neg ch}$.

Similarly, we have $X \cup P^{r-1}(K) = X_1 \cup X_2 \cup ... \cup X_m \cup P^{r-1}(K)$ where all X_i are non-empty. In $Y = Y_1 \cup Y_2 \cup ... \cup Y_n$ there are a finitely many empty components. Let $Y = Y_1 \cup Y_2 \cup ... \cup Y_{m'}$ where $m' \leq n$ and where all Y_i are not empty. There are two cases to consider, either $m < m'$ or $m \geq m'$.

Suppose that $m \geq m'$. Since $|X|_{\neg ch} < |Y|_{\neg ch}$ there is some k which satisfies a statement of definition 45. Thus, since $\rho(P^{r-1}(K)) < \rho(X_j)$ for all j, adding the union component $P^{r-1}(K)$ to X cannot make $|X \cup P^{r-1}(K)|_{\neg ch} \geq |Y|_{\neg ch}$, therefore we have that $|X \cup P^{r-1}(K)|_{\neg ch} < |Y|_{\neg ch}$.

Suppose $m < m'$, since $\rho(P^{r-1}(K)) < \rho(Y_i)$ for all $i \leq n$ we can take $k = m + 1$ and have that $\rho(P^{r-1}(K)) < \rho(Y_k)$ and $\tau(P^{r-1}(K)) \leq \tau(Y_k)$. Thus, by definition 45 of $\neg CH$-cardinality we have that $|X \cup P^{r-1}(K)|_{\neg ch} < |Y|_{\neg ch}$. $\square$

4 Further Investigations

The full extension of ZF with the concept of the inverse power set has not been fully investigated here since we restricted our context to EZF*. We have called the sets which arise from EZF the real sets. The reason for this is that the sets of EZF* come from a finite number of applications of the operators P and P^{-1}. Those can be called *rational sets* by analogy with the rational numbers and the sets which arise from an infinite number of applications of the operators P and P^{-1} can be called *irrational sets*. It would be interesting to find extended definitions of cardinality in the context EZF instead of only EZF*, in particular what will be the cardinality of an irrational set $P^{-1}(P^{-1}(...P^{-1}(z)...))$ where P^{-1} occurs an infinite number of times. We could further extend the power set axiom to apply to sets such as $P^{-1}(\{1, 2, 3, 4, 5\} \cup P^{-1}(P^{-1}(X))$ and $P^{-1}(X) \cup P^{-1}(Y) \cup Z \cup P^{-1}(P^{-1}(W))$.

An idea related to EZF* and well-represented sets would be to generalize the concept of well-represented for rational sets and to find a 'fundamental theorem of set theory' where each rational set A can be represented uniquely as

$$X \cup P^{-1}(Y_1) \cup P^{-1}(Y_2) \cup ... \cup P^{-1}(Y_{n_1}) \cup ... \cup P^{-k}(Z_1) \cup P^{-k}(Z_2) \cup ... \cup P^{-k}(Z_{n_k}),$$

where A is 'well-represented', where $X, Y_1, Y_2, ..., Y_{n_1}, ..., Z_1, Z_2, ..., Z_{n_k}$ are 'well-represented' rational sets, where the sequence of nested rational sets eventually ends with Zermelo sets and where uniqueness of representation is up to the choice of representative for the cardinality of the Zermelo sets. This theorem would imply, in the realm of $\neg$CH-cardinality, something similar to the uniqueness of representation of a rational number in decimal notation.

The continuum hypothesis is undecidable in ZF and in ZF with the axiom of choice, but extending to EZF* permitted us to decide the continuum hypothesis. A key feature about EZF is that it can be seen as 'closed' under the powerset. To which extent can we diminish the quantity of undecidable statements by setting those statements in the context of the 'algebraic closure' of a certain theory? Could the number of undecidable statements be reduced to a finite amount?

Since most of mathematics relies on set theory, further investigations could be to consider introducing the non-Zermelo sets in the context of different mathematical theories. It might also be interesting to generalize the power set operation to an operation which takes as input a set X of ZF (or

EZF) and outputs a certain collection of subsets of X. In this generalization, the power set is the operation which returns the collection of every subset of X. Thus, we could introduce its inverse operation and consider its implication on cardinal arithmetic.

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