A SIMPLE PROOF OF TYURIN'S BABYLONIAN TOWER THEOREM

IUSTIN COANDĂ

Abstract. Using the method of Coandă and Trautmann (2006), we give a simple proof of the following theorem due to Tyurin (1976) in the smooth case: if a vector bundle E on a c-codimensional locally Cohen-Macaulay closed subscheme X of the projective space $\mathbb{P}^n$ extends to a vector bundle F on a similar closed subscheme Y of $\mathbb{P}^N$, for every $N > n$, then E is the restriction to X of a direct sum of line bundles on $\mathbb{P}^n$. Using the same method, we also provide a proof of the Babylonian tower theorem for locally complete intersection subschemes of projective spaces.

Let $\mathbb{P}^n$ be the projective n-space over an algebraically closed field k of arbitrary characteristic and $S = k[X_1, \ldots, X_n]$ its projective coordinate ring. For $m > 0$, embed $\mathbb{P}^n$ into the projective $(n+m)$-space $\mathbb{P}^{n+m}$ with coordinate ring $R = k[X_1, \ldots, X_{n+m}]$ as the linear subspace L of equations $X_{n+1} = \cdots = X_{n+m} = 0$. One says that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ *extends* to a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^{n+m}$ if $\mathcal{G} \,|\, \mathbb{P}^n \simeq \mathcal{F}$ and $\mathcal{T}or_i^{\mathcal{O}_{\mathbb{P}^{n+m}}}(\mathcal{G}, \mathcal{O}_L) = 0$, $\forall i > 0$. Since L is defined locally in $\mathbb{P}^{n+m}$ by a regular sequence, the later condition is equivalent to $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^{n+m}}}(\mathcal{G}, \mathcal{O}_L) = 0$ (see Matsumura (1986, Thm. 16.5)). A closed subscheme X of $\mathbb{P}^n$ *extends* to a closed subscheme Y of $\mathbb{P}^{n+m}$ if the structure sheaf $\mathcal{O}_X$ extends to $\mathcal{O}_Y$. It is easy to see that in this case the ideal sheaf $\mathcal{I}_{X,\mathbb{P}^n}$ extends to $\mathcal{I}_{Y,\mathbb{P}^{n+m}}$.

In this note, using the method of Coandă and Trautmann (2006), we shall provide simple, elementary proofs of the following two results:

Theorem 1. *Let E be a vector bundle (= locally free sheaf) on a locally Cohen-Macaulay closed subscheme X of $\mathbb{P}^n$, of pure codimension c. If, for every $m > 0$, E extends to a vector bundle F on a locally Cohen-Macaulay closed subscheme Y of $\mathbb{P}^{n+m}$, of pure codimension c, then E is isomorphic to a direct sum of line bundles of the form $\mathcal{O}_X(a)$, $a \in \mathbb{Z}$.*

Theorem 2. *Let X be a locally complete intersection closed subscheme of $\mathbb{P}^n$, of pure codimension c. If, for every $m > 0$, X extends to a locally complete intersection closed subscheme Y of $\mathbb{P}^{n+m}$, of pure codimension c, then X is a complete intersection.*

Notice that, under the hypothesis of Theorem 1, E extends to F if and only if $F \,|\, \mathbb{P}^n \simeq E$, and, under the hypothesis of Theorem 2, X extends to Y if and only if $Y \cap \mathbb{P}^n = X$ (as schemes).

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In the case where X and Y are assumed to be smooth, Theorem 1 is due to Tyurin (1976), and Theorem 2 is due to Barth and Van de Ven (1974), Barth (1975) and Tyurin (1976). The more general version of Theorem 2 stated above is due to Flenner (1985).

The proofs of these theorems are based on the next three lemmas. The first two of them are very elementary and the proof of the third one uses the idea of Coandă and Trautmann (2006). Before stating and proving them, we recall the following notation: for $i \geq 0$, $\mathrm{H}^i_*(\mathcal{F})$ denotes the graded S-module $\bigoplus_{d \in \mathbb{Z}} \mathrm{H}^i(\mathcal{F}(d))$, and, for Z closed subscheme of $\mathbb{P}^{n+m}$, $\mathcal{G}_Z$ denotes the sheaf $\mathcal{G} \otimes_{\mathcal{O}_{\mathbb{P}^{n+m}}} \mathcal{O}_Z$.

Lemma 3. *Assume that a coherent sheaf $\mathcal{A}$ on $\mathbb{P}^N$ extends to a coherent sheaf $\mathcal{B}$ on $\mathbb{P}^{N+1}$ with the property that $\mathrm{H}^0(\mathcal{B}(-t)) = 0$ for $t >> 0$. Let $\cdots \to G^{-1} \to G^0 \to \mathrm{H}^0_*(\mathcal{B}) \to 0$ be a graded minimal free resolution of $\mathrm{H}^0_*(\mathcal{B})$ over $k[X_0, \ldots, X_{N+1}]$. If $\mathrm{H}^0_*(\mathcal{B}) \to \mathrm{H}^0_*(\mathcal{A})$ is surjective then $G^\bullet/X_{N+1}G^\bullet$ is a minimal free resolution of $\mathrm{H}^0_*(\mathcal{A})$ over $k[X_0, \ldots, X_N]$.*

Proof. Using the exact sequence:

$$0 \longrightarrow \mathcal{B}(-1) \xrightarrow{X_{N+1}} \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow 0$$

one deduces, firstly, that X_{N+1} is $\mathrm{H}^0_*(\mathcal{B})$-regular, hence $G^\bullet/X_{N+1}G^\bullet$ is a minimal free resolution of $\mathrm{H}^0_*(\mathcal{B})/X_{N+1}\mathrm{H}^0_*(\mathcal{B})$ over $k[X_0, \ldots, X_N]$ and, then, that $\mathrm{H}^0_*(\mathcal{B})/X_{N+1}\mathrm{H}^0_*(\mathcal{B}) \simeq \mathrm{H}^0_*(\mathcal{A})$. □

Lemma 4. *Assume that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ extends to a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^{n+m}$ with the property that $\mathrm{H}^i(\mathcal{G}(-t)) = 0$ for $t >> 0$, $i = 0, \ldots, m-1$. Let $\cdots \to G^{-1} \to G^0 \to \mathrm{H}^0_*(\mathcal{G}) \to 0$ be a minimal free resolution of the graded R-module $\mathrm{H}^0_*(\mathcal{G})$. If there exists an $(n+1)$-dimensional linear subspace P of $\mathbb{P}^{n+m}$ containing $L = \mathbb{P}^n$ such that $\mathrm{H}^0_*(\mathcal{G}_P) \to \mathrm{H}^0_*(\mathcal{F})$ is surjective, then $G^\bullet \otimes_R S$ is a minimal free resolution of the graded S-module $\mathrm{H}^0_*(\mathcal{F})$.*

Proof. Consider a saturated flag of linear subspaces $P^{(0)} = L \subset P = P^{(1)} \subset \ldots \subset P^{(m)} = \mathbb{P}^{n+m}$. By decreasing induction on $j = m, \ldots, 1$ one shows easily that $\mathrm{H}^i(\mathcal{G}_{P^{(j)}}(-t)) = 0$ for $t >> 0$, $i = 0, \ldots, j-1$. Since $\mathrm{H}^0_*(\mathcal{G}_P) \to \mathrm{H}^0_*(\mathcal{F})$ is surjective and $\mathrm{H}^1(\mathcal{G}_P(t)) = 0$ for $t >> 0$, it follows that $\mathrm{H}^1_*(\mathcal{G}_P) = 0$. One shows now, by increasing induction on $j = 1, \ldots, m$, that $\mathrm{H}^i_*(\mathcal{G}_{P^{(j)}}) = 0$, $i = 1, \ldots, j$. But $\mathrm{H}^1_*(\mathcal{G}_{P^{(j)}}) = 0$ implies that $\mathrm{H}^0_*(\mathcal{G}_{P^{(j)}}) \to \mathrm{H}^0_*(\mathcal{G}_{P^{(j-1)}})$ is surjective, $j = 2, \ldots, m$, and one finally applies Lemma 3. □

Lemma 5. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, $n \geq 2$, with the property that $\mathrm{H}^i(\mathcal{F}(-t)) = 0$ for $t >> 0$, $i = 0, 1$. For $i \in \mathbb{Z}$, let μ_i denote the number of minimal generators of degree i of the graded S-module $\mathrm{H}^0_*(\mathcal{F})$. If, for some $m > \sum_{i>j} \mu_i \, \mathrm{h}^1(\mathcal{F}(j))$, $\mathcal{F}$ extends to a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^{n+m}$ with $\mathrm{H}^i(\mathcal{G}(-t)) = 0$ for $t >> 0$, $i = 1, \ldots, m$, then there exists an $(n+1)$-dimensional linear subspace P of $\mathbb{P}^{n+m}$ containing $L = \mathbb{P}^n$ such that $\mathrm{H}^0_*(\mathcal{G}_P) \to \mathrm{H}^0_*(\mathcal{F})$ is surjective.*

Proof. We recall that $\mathrm{h}^1(\mathcal{F}(j))$ denotes $\dim_k \mathrm{H}^1(\mathcal{F}(j))$. Now, for $i \geq 0$, let L_i be the i th infinitesimal neighbourhood of L in $\mathbb{P}^{n+m}$, defined by the ideal sheaf $\mathcal{I}^{i+1}_{L,\mathbb{P}^{n+m}}$. Let L'

be the linear subspace of $\mathbb{P}^{n+m}$ of equations $X_0 = \ldots = X_n = 0$, with coordinate ring $S' = k[X_{n+1}, \ldots, X_{n+m}]$. If $\pi : \mathbb{P}^{n+m} \setminus L' \to L$ is the linear projection then $\pi \,|\, L_i \to L$ is a retract of the inclusion $L \hookrightarrow L_i$ and endows $\mathcal{O}_{L_i}$ with a structure of $\mathcal{O}_L$-algebra. As an $\mathcal{O}_L$-module:

$$\mathcal{O}_{L_i} \simeq \mathcal{O}_L \oplus \mathcal{O}_L(-1) \otimes_k S'_1 \oplus \cdots \oplus \mathcal{O}_L(-i) \otimes_k S'_i\,.$$

Moreover, if $J \subset S'$ is a homogeneous ideal then:

$$\mathcal{O}_L(-1) \otimes_k J_1 \oplus \cdots \oplus \mathcal{O}_L(-i) \otimes_k J_i$$

is an ideal sheaf of $\mathcal{O}_{L_i}$, hence defines a closed subscheme Y_i of $\mathbb{P}^{n+m}$ with $L \subseteq Y_i \subseteq L_i$. Using the exact sequence:

$$0 \longrightarrow \mathcal{O}_L(-i-1) \otimes_k (S'_{i+1}/J_{i+1}) \longrightarrow \mathcal{O}_{Y_{i+1}} \longrightarrow \mathcal{O}_{Y_i} \longrightarrow 0$$

one deduces easily, by induction on i, that $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^{n+m}}}(\mathcal{G}, \mathcal{O}_{Y_i}) = 0$, $\forall i \geq 0$. It follows that, tensorizing the above exact sequence by $\mathcal{G}$, one gets an exact sequence:

$$0 \longrightarrow \mathcal{F}(-i-1) \otimes_k (S'_{i+1}/J_{i+1}) \longrightarrow \mathcal{G}_{Y_{i+1}} \longrightarrow \mathcal{G}_{Y_i} \longrightarrow 0\,.$$

Consider, now, a section $s \in \mathrm{H}^0(\mathcal{F})$. Using an argument similar to that used in the proof of Coandă (2010, Lemma 5), one can show that there exists a homogeneous ideal $J(s) \subset S'$, with at most $\mathrm{h}^1(\mathcal{F}(-j))$ minimal generators in degree j, $\forall j \geq 1$, with the property that if $Y_i(s)$ is the closed subscheme of L_i defined by the ideal sheaf:

$$\mathcal{O}_L(-1) \otimes_k J(s)_1 \oplus \cdots \oplus \mathcal{O}_L(-i) \otimes_k J(s)_i$$

then s can be lifted to a global section of $\mathcal{G}_{Y_i(s)}$, $\forall i \geq 1$. Choosing, next, a minimal system of generators of the graded S-module $\mathrm{H}^0_*(\mathcal{F})$ one deduces the existence of an ideal $J \subset S'$ generated by at most $\sum_{i>j} \mu_i \,\mathrm{h}^1(\mathcal{F}(j))$ homogeneous elements such that $\mathrm{H}^0_*(\mathcal{G}_{Y_i}) \to \mathrm{H}^0_*(\mathcal{F})$ is surjective, $\forall i \geq 1$.

Since $m > \sum_{i>j} \mu_i \,\mathrm{h}^1(\mathcal{F}(j))$, there exists a point $p \in L' \simeq \mathbb{P}^{m-1}$ such that all the elements of J vanish at p. Let $P \subset \mathbb{P}^{n+m}$ be the linear span of L and p. One has $P \cap L_i \subseteq Y_i$ hence $\mathrm{H}^0_*(\mathcal{G}_{P \cap L_i}) \to \mathrm{H}^0_*(\mathcal{F})$ is surjective, $\forall i \geq 1$. But $P \cap L_i$ is the i th infinitesimal neighbourhood in P of the hyperplane L of P. Tensorizing by $\mathcal{G}(d)$, for a *fixed* $d \in \mathbb{Z}$, the exact sequence:

$$0 \longrightarrow \mathcal{O}_P(-i-1) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_{P \cap L_i} \longrightarrow 0$$

one gets an exact sequence:

$$\mathrm{H}^0(\mathcal{G}_P(d)) \longrightarrow \mathrm{H}^0(\mathcal{G}_{P \cap L_i}(d)) \longrightarrow \mathrm{H}^1(\mathcal{G}_P(d-i-1))\,.$$

Since $\mathrm{H}^i(\mathcal{G}(-t)) = 0$ for $t >> 0$, $i = 1, \ldots, m$, one deduces, as in the proof of Lemma 4, that $\mathrm{H}^1(\mathcal{G}_P(d-i-1)) = 0$, for $i >> 0$, hence $\mathrm{H}^0(\mathcal{G}_P(d)) \to \mathrm{H}^0(\mathcal{G}_{P \cap L_i}(d))$ is surjective for $i >> 0$. This implies that $\mathrm{H}^0_*(\mathcal{G}_P) \to \mathrm{H}^0_*(\mathcal{F})$ is surjective. $\square$

Proof of Theorem 1. Let μ_i be the number of minimal generators of degree i of the graded S-module $\mathrm{H}^0_*(E)$ and consider a minimal free S-resolution $\cdots \to L^{-1} \to L^0 \to \mathrm{H}^0_*(E) \to 0$ with, of course, $L^0 \simeq \bigoplus_{i\in\mathbb{Z}} S(-i)^{\mu_i}$. Let $r_i := \mathrm{rk}\, L^{-i}$, $i \geq 0$. We want to show that $r_0 = \mathrm{rk}\, E =: r$.

Now, from the hypothesis, $\mathrm{H}^i(E(-t)) = 0$ for $t >> 0$, $i = 0, \ldots, n-c-1$ and $\mathrm{H}^i(F(-t)) = 0$ for $t >> 0$, $i = 0, \ldots, n+m-c-1$. We may assume that $n - c \geq 2$. Let $\cdots \to K^{-1} \to K^0 \to \mathrm{H}^0_*(F) \to 0$ be a minimal free resolution of the graded R-module $\mathrm{H}^0_*(F)$. It follows from Lemma 4 and Lemma 5 that if $m > \sum_{i>j} \mu_i\, \mathrm{h}^1(E(j))$ then $K^\bullet \otimes_R S \simeq L^\bullet$.

The sheafified morphism $\widetilde{K}^{-1} \to \widetilde{K}^0$ has constant corank r along Y. If $r_0 > r$ then the $(r_0 - r) \times (r_0 - r)$ minors of the matrix defining this morphism vanish on a closed subscheme of $\mathbb{P}^{n+m}$ of codimension $\leq (r_1 - (r_0 - r) + 1)(r_0 - (r_0 - r) + 1) = (r_1 - r_0 + r + 1)(r + 1)$. If $\dim Y \geq (r_1 - r_0 + r + 1)(r + 1)$, i.e., if $m \geq (r_1 - r_0 + r + 1)(r + 1) - n + c$, then one gets a contradiction. □

Proof of Theorem 2. Let $\cdots \to F^{-1} \to F^0 \to I(X) \to 0$ be a minimal free resolution of the homogeneous ideal $I(X) := \mathrm{H}^0_*(\mathcal{I}_{X,\mathbb{P}^n})$ of S. One has $F^0 \simeq \bigoplus_{i\in\mathbb{Z}} S(-i)^{\mu_i}$, where μ_i is the number of homogeneous minimal generators of degree i of $I(X)$. Let $r_i := \mathrm{rk}\, F^{-i}$, $i \geq 0$. We want to show that $r_0 = c$.

Now, it follows from the hypothesis that $\mathrm{H}^i(\mathcal{I}_{X,\mathbb{P}^n}(-t)) = 0$ for $t >> 0$, $i = 0, \ldots n - c$, and that $\mathrm{H}^i(\mathcal{I}_{Y,\mathbb{P}^{n+m}}(-t)) = 0$ for $t >> 0$, $i = 0, \ldots, n + m - c$. We may assume that $n - c \geq 1$. Let $\cdots \to G^{-1} \to G^0 \to I(Y) \to 0$ be a minimal free resolution of the homogeneous ideal $I(Y)$ of R. From Lemma 4 and Lemma 5 one deduces that if $m > \sum_{i>j} \mu_i \mathrm{h}^1(\mathcal{I}_{X,\mathbb{P}^n}(j))$ then $G^\bullet \otimes_R S \simeq F^\bullet$.

The sheafified morphism $\widetilde{G}^{-1} \to \widetilde{G}^0$ has constant corank c along Y. If $r_0 > c$ then the $(r_0 - c) \times (r_0 - c)$ minors of the matrix defining this morphism vanish on a closed subscheme of $\mathbb{P}^{n+m}$ of codimension $\leq (r_1 - (r_0 - c) + 1)(r_0 - (r_0 - c) + 1) = (r_1 - r_0 + c + 1)(c + 1)$. If $\dim Y \geq (r_1 - r_0 + c + 1)(c + 1)$, i.e., if $m \geq (r_1 - r_0 + c + 1)(c + 1) - n + c$, then one gets a contradiction. □

Using Lemma 4 and Lemma 5 one can also prove the following result, that answers a question the hypothesis of Theorem 1 might raise.

Theorem 6. *Let X be a locally Cohen-Macaulay closed subscheme of $\mathbb{P}^n$, of pure codimension $c \geq 2$. If, for every $m > 0$, X extends to a locally Cohen-Macaulay closed subscheme of $\mathbb{P}^{n+m}$, of pure codimension c, then X is arithmetically Cohen-Macaulay.*

Proof. Consider, as in the above proof of Theorem 2, minimal free resolutions $\cdots \to F^{-1} \to F^0 \to I(X) \to 0$ and $\cdots \to G^{-1} \to G^0 \to I(Y) \to 0$. If m is sufficiently large then $G^\bullet \otimes_R S \simeq F^\bullet$. Put $F^{-1} := S$ and $G^{-1} := R$. One deduces that the vector bundle $E := \mathrm{Ker}(\widetilde{F}^{-c+2} \to \widetilde{F}^{-c+3})$ on $\mathbb{P}^n$ extends to the vector bundle $E^{(m)} := \mathrm{Ker}(\widetilde{G}^{-c+2} \to \widetilde{G}^{-c+3})$ on $\mathbb{P}^{n+m}$. By the Babylonian tower theorem for vector bundles on projective spaces of Barth and Van de Ven (1974), E. Sato (1977), (1978) and Tyurin (1976) (which is, of course, a particular case of Theorem 1) E is a direct sum of line bundles on $\mathbb{P}^n$. It follows that $\mathrm{H}^i_*(\mathcal{I}_{X,\mathbb{P}^n}) = 0$, $i = 1, \ldots, n-c = \dim X$, hence X is arithmetically Cohen-Macaulay. $\square$

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Institute of Mathematics of the Romanian Academy, P. O. Box 1-764, RO-014700, Bucharest, Romania

E-mail address: Iustin.Coanda@imar.ro