ALGEBRAS GRADED BY DISCRETE DOI-HOPF DATA AND THE DRINFELD DOUBLE OF A HOPF GROUP-COALGEBRA

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ABSTRACT. We study Doi-Hopf data and Doi-Hopf modules for Hopf group-coalgebras. We introduce modules graded by a discrete Doi-Hopf datum; to a Doi-Hopf datum over a Hopf group coalgebra, we associate an algebra graded by the underlying discrete Doi-Hopf datum, using a smash product type construction. The category of Doi-Hopf modules is then isomorphic to the category of graded modules over this algebra. This is applied to the category of Yetter-Drinfeld modules over a Hopf group coalgebra, leading to the construction of the Drinfeld double. It is shown that this Drinfeld double is a quasitriangular $\mathbb{G}$-graded Hopf algebra.

INTRODUCTION

Hopf group coalgebras have been introduced by Turaev [11], and are important for the study of certain 3-manifolds. A purely algebraic study of Hopf group coalgebras and related structures was started in [13], and continued by several authors, see for example [14, 15, 16]. For a recent survey, we refer to [12]. A categorical explanation was presented in [2], where it was shown that group coalgebras, resp. Hopf group coalgebras, are coalgebras, resp. Hopf algebras in a suitable symmetric monoidal category $\mathcal{T}_k$. We will recall this construction in 1.4; it provides a natural method to generalize results of classical Hopf algebra theory to the setting of Hopf group coalgebras. For example, it is explained in [2, Sec. 4] how Yetter-Drinfeld modules over Hopf group coalgebras can be introduced: first we introduce the category of modules over a Hopf group coalgebra, and then we compute its center, which is a braided monoidal category, by construction.

A crucial result in the classical theory is now the following: to a finite dimensional Hopf algebra H, we can associate a new Hopf algebra $D(H)$, called the Drinfeld double of H. $D(H)$ is quasitriangular, and the category of modules over $D(H)$ is isomorphic to the category of Yetter-Drinfeld modules. The following natural question now arises: can this result be generalized to the setting of Hopf group coalgebras? At first glance, this looks like another straightforward application of the methods developed in [2]. This is not the case, and the underlying reason for this is the fact that the category $\mathcal{T}_k$ is

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not rigid. However, we have a duality functor on $\mathcal{T}_k$, but this takes values in a different category $\mathcal{Z}_k$.
To explain this, let us look at a less complicated situation. The dual of a finite dimensional coalgebra is an algebra, and the category of comodules over the coalgebra is isomorphic to the category of representations of the dual algebra. If we look at a group coalgebra, this is a coalgebra in $\mathcal{T}_k$, then the dual is an algebra in $\mathcal{Z}_k$, which turns out to be a G-graded algebra. The category of representations of such an algebra is then the category of modules graded by a (variable) G-set. We have two versions of this representation category, one with a forgetful functor to $\mathcal{T}_k$, and one with a forgetful functor to $\mathcal{Z}_k$. There are two corresponding duality results, which are explained in 1.5.
Before looking at Yetter-Drinfeld modules, we consider Doi-Hopf modules; these are more general, but the formalism is easier, see [3]. Doi-Hopf data and Doi-Hopf modules in $\mathcal{T}_k$ are discribed in 1.7 and 1.8. Our aim is then to describe the category of Doi-Hopf modules as a category of representations. Before we are able to do this, we have to introduce a new kind of graded algebra. Recall that a Doi-Hopf datum consists of a Hopf algebra H, an H-comodule algebra A and an H-module coalgebra C. This construction can be performed in any braided monoidal category, for example in the category of sets, leading to the notion of discrete Doi-Hopf datum, see 1.2. In Section 2, we introduce algebras graded by a discrete Doi-Hopf datum (G, Λ, X), and in Section 3, we discuss modules over a such a graded algebra, graded by a (G, Λ, X)-set. Now if we have a Doi-Hopf datum in $\mathcal{T}_k$, then the underlying algebra and the dual of the underlying coalgebra are both algebras, but in different monoidal categories. However, we can still form their smash product, and this turns out to be an algebra graded by the underlying Doi-Hopf datum (G, Λ, X), see Section 2. The main result of Section 3 states that the category of Doi-Hopf modules is isomorphic to the category of modules graded by (G, Λ, X)-sets, a result that comes in a $\mathcal{Z}$-version and in a $\mathcal{T}$-version, similar to the duality result in 1.5.
In Section 4, this result is applied to the category of Yetter-Drinfeld modules: we introduce the Drinfeld double; it can be constructed as a smash product, and is an algebra graded by a certain discrete Doi-Hopf datum, denoted $\mathbb{G}$ throughout the paper. In Sections 5 and 7, we introduce quasitriangular $\mathbb{G}$-graded Hopf algebras, and we show that the Drinfeld double is such a quasitriangular $\mathbb{G}$-graded Hopf algebra.

1. Preliminaries

1.1. Monoidal categories. Let $\mathcal{C}$ be a monoidal category. One can define algebras, coalgebras, (bi)modules and (bi)comodules in $\mathcal{C}$. If $\mathcal{C}$ is braided, then we can consider bialgebras and Hopf algebras in $\mathcal{C}$. Doi-Hopf data and Doi-Hopf modules can then be introduced; in the case where $\mathcal{C}$ is symmetric, this was done in [6], and it is easy to see that this can be extended to arbitrary braided monoidal categories. Let us briefly recall the definitions. Let H be a bialgebra in $\mathcal{C}$. The category $\mathcal{C}^H$ of right H-comodules is monoidal, and a right H-comodule algebra is an algebra in $\mathcal{C}^H$. The category of right H-modules $\mathcal{C}_H$ is also monoidal, and a coalgebra in this category is called a right H-module coalgebra. A (right-right) Doi-Hopf datum is a triple (H, A, C), where H is a bialgebra, A is a right H-comodule algebra and C is a right H-module coalgebra. An (H, A, C)-Doi-Hopf module is an object $M \in \mathcal{C}$ together with a right A-action ν_M and a right C-coaction ρ_M such that thecompatibility condition $\rho_M \nu_M = (\nu_M \otimes C)(M \otimes \psi)(\rho_M \otimes A)$, where $\psi = (A \otimes \nu_C)(c_{C,A} \otimes A)(C \otimes \rho_A) : C \otimes A \to A \otimes C$. ψ is called the entwining morphism. c is the braiding. We will denote the category of right (H, A, C)-Doi-Hopf modules and right A-linear right C-colinear morphisms by $\mathcal{C}(H)_A^C$.

1.2. Discrete Doi-Hopf data. Let us describe Doi-Hopf data in $\underline{\text{Sets}}$. An algebra in $\underline{\text{Sets}}$ is a monoid. Every set X is in a unique way a coalgebra in $\underline{\text{Sets}}$: the comultiplication is the diagonal map $X \to X \times X$, and the augmentation map is the unique map $X \to \{*\}$, where $\{*\}$ is a singleton. With this coalgebra structure, every monoid is a bialgebra in $\underline{\text{Sets}}$. A Hopf algebra in $\underline{\text{Sets}}$ is then a group. Let G be a monoid. A G-comodule algebra is a monoid Λ together with a morphism of monoids $\gamma : \Lambda \to G$. The corresponding G-coaction $\Lambda \to \Lambda \times G$ sends λ to $(\lambda, \gamma(\lambda))$. Finally, a G-module coalgebra is a right G-set. All these assertions are well-known; they can be proved as easy exercises, and details can be found in [2]. We conclude that a Doi-Hopf datum (G, Λ, X) in $\underline{\text{Sets}}$ consists of two monoids G and Λ, a monoid map $\gamma : \Lambda \to G$ and a right G-set X. We will call (G, Λ, X) a discrete Doi-Hopf datum.

Now it is easy to show that an object in $\underline{\text{Sets}}(G)_\Lambda^X$ is a right Λ-set Y together with a map $\beta : Y \to X$ such that $\beta(y\lambda) = \beta(y)\gamma(\lambda)$, for all $y \in Y$ and $\lambda \in \Lambda$. We call Y a (G, Λ, X)-set.

A morphism $Y \to Y'$ in $\underline{\text{Sets}}(G)_\Lambda^X$ is a map of right Λ-sets $\eta : Y \to Y'$ satisfying $\beta'(\eta(y)) = \beta(y)$, for all $y \in Y$.

An example of a (G, Λ, X)-set is $Y = \Lambda \times X$, with $\beta(\lambda, x) = x$ and $(\lambda, x)\lambda' = (\lambda\lambda', x\gamma(\lambda'))$.

1.3. The Fam-category. Let $\mathcal{C}$ be a braided monoidal category. To simplify the computations, we assume that $\mathcal{C}$ is strict; this assumptions is justified by the fact that every monoidal category is equivalent to a strict one, see for example [7]. A new braided monoidal category $\underline{\text{Fam}}(\mathcal{C})$ is introduced as follows: objects are families of objects in $\mathcal{C}$ indexed by a set X, which we denote as $\underline{M} = (X, (M_x)_{x \in X})$, where X is a set, and $M_x \in \mathcal{C}$, for all $x \in X$. A morphism $\underline{M} \to \underline{M'}$ is a couple $\underline{\varphi} = (f, (\varphi_x)_{x \in X})$, where $f : X \to X'$ is a map and $\varphi_x : M_x \to M'_{f(x)}$ is a morphism in $\mathcal{C}$. The composition of morphisms is defined in the obvious way. The tensor product on $\underline{\text{Fam}}(\mathcal{C})$ is given by

$$\underline{M} \otimes \underline{N} = (X \times X', (M_x \otimes M'_{x'})_{(x,x') \in X \times X'}).$$

The unit object is $(\{*\}, k)$, where $\{*\}$ is a singleton, and k is the unit object of $\mathcal{C}$. The braiding $\underline{c}$ is given by

$$\underline{c}_{\underline{M},\underline{M'}} = (t, c_{M_x, M'_{x'}}) : \underline{M} \otimes \underline{M'} \to \underline{M'} \otimes \underline{M},$$

where $t : X \times X' \to X' \times X$ is the switch map, and c is the braiding on $\mathcal{C}$. Obviously, we have a strictly monoidal functor $U : \underline{\text{Fam}}(\mathcal{C}) \to \underline{\text{Sets}}$, sending $(X, (M_x)_{x \in X})$ to X and $(f, (\varphi_x)_{x \in X})$ to f.

1.4. Group-coalgebras. Group-coalgebras and Hopf group-coalgebras have been introduced by Turaev in [11]. In [13], Virelizier studied Hopf group-coalgebras from an algebraic point of view. Group-coalgebras and related structures have been investigated by several authors, see for example [14, 15, 16]. For a recent survey, see [12].

Let k be a field (or, more generally, a commutative ring), and $\mathcal{M}_k$ the category of k-vector spaces (or, in the case where k is a commutative ring, k-modules). Now consider the categories

$$\mathcal{Z}_k = \underline{\text{Fam}}(\mathcal{M}_k) \text{ and } \mathcal{T}_k = \underline{\text{Fam}}(\mathcal{M}_k^{\text{op}})^{\text{op}}.$$

$\mathcal{Z}_k$ and $\mathcal{T}_k$ have the same objects $\underline{M} = (X, (M_x)_{x\in X})$, where X is a set, and M_x is a k-module, for all $x \in X$. For the description of the morphisms in $\mathcal{Z}_k$, see 1.3, with $\mathcal{C}$ replaced by $\mathcal{M}_k$. A morphism $\underline{M} = (X, (M_x)_{x\in X}) \to \underline{N} = (Y, (N_y)_{y\in Y})$ in $\mathcal{T}_k$ is a couple $(f, (\varphi_y)_{y\in Y})$, with $f : Y \to X$ a map, and $\varphi_y : M_{f(x)} \to N_y$ a k-linear map, for every $y \in Y$.

It was observed in [2] that a group-coalgebra (resp. a Hopf group-coalgebra) is a coalgebra (resp. a Hopf algebra) in $\mathcal{T}_k = \underline{\text{Fam}}(\mathcal{M}_k^{\text{op}})^{\text{op}}$. An algebra in $\mathcal{T}_k$ is a collection of k-algebras indexed by a set X.

In a similar way, a coalgebra in $\mathcal{Z}_k$ is a collection of k-coalgebras indexed by a set X. Algebras in $\mathcal{Z}_k$ are in one-to-one correspondence to algebras graded by a monoid.

1.5. Isomorphism of categories. Let $\mathcal{C}$ be as in 1.3, and consider the subcategory $\underline{\text{Fam}}^{bij}(\mathcal{C})$ of $\underline{\text{Fam}}(\mathcal{C})$, with the same objects as $\underline{\text{Fam}}(\mathcal{C})$, but with morphisms of the form $(f, (\varphi_x)_{x\in X})$, with f a bijection. Then we have an isomorphism F between the categories $\mathcal{Z}_k^{bij}$ and $\mathcal{T}_k^{bij}$, acting as the identity on objects. At the level of morphisms, F is defined by

$$F(f, (\varphi_x)_{x\in X}) = (f^{-1}, (\varphi_{f^{-1}(y)})_{y\in Y}).$$

1.6. A duality result. It is well-known that the dual $B = C^*$ of a k-coalgebra C is a k-algebra; we have a functor $\mathcal{M}^C \to \mathcal{M}_{B^{\text{op}}}$, which is an isomorphism of categories if C is finitely generated and projective as a k-module.

We will now discuss a similar result for group coalgebras. Actually, it is a special case of a more general duality result that will be discussed in the subsequent sections. But this special case might be illuminating, as it incorporates some of the subtleties that will reappear later in a more general situation, and this is why we decided to give an outline here.

Let A be a G-graded k-algebra, and $\mathcal{Z}_A$ the category of right modules over A, viewed as an algebra in $\mathcal{Z}_k$. The objects of $\mathcal{Z}_A$ are couples (X, M), where X is a right G-set, and $M = \oplus_{x\in X} M_x$ is a right A-module graded by X; we refer to [9] for detail on modules graded by G-sets. A morphism $(X, M) \to (Y, N)$ in $\mathcal{Z}_A$ is a couple (f, φ), where $f : X \to Y$ is a morphism of G-sets, and $\varphi : M \to N$ is a right A-module map such that $\varphi(M_x) \subset N_{f(x)}$, for all $x \in X$. It is obvious that we have a forgetful functor $\mathcal{Z}_A \to \mathcal{Z}_k$.

We have a second category $\mathcal{T}_A$, with the same objects as $\mathcal{Z}_A$, but morphisms defined in a different way: $(f, (\varphi_y)_{y\in Y}) : (X, M) \to (Y, N)$ consists of a morphisms of right G-sets $f : Y \to X$, and a bunch of k-linear maps $\varphi_y : M_{f(y)} \to N_y$ such that $\varphi_{yg}(ma) = \varphi_y(m)a$, for all $m \in M_{f(y)}$ and

$a \in A_g$. Observe that we have a forgetful functor $\mathcal{T}_A \to \mathcal{T}_k$.
In a similar way, we have two categories associated to a group coalgebra $\underline{C} = (G, (C_g)_{g\in G})$, one with a forgetful functor to $\mathcal{T}_k$, and the other one with a forgetful functor to $\mathcal{Z}_k$. The two categories $\mathcal{T}^{\underline{C}}$ and $\mathcal{Z}^{\underline{C}}$ have the same objects $(X, (M_x)_{x\in X})$, where X is a right G-set, M_x is a k-module, and

$$\rho_{x,g} :\ M_{xg} \to M_x \otimes C_g$$

are k-linear maps such that the following coassociativity and counit conditions hold:

$$(M_x \otimes \Delta_{g,h}) \circ \rho_{x,gh} = (\rho_{x,g} \otimes C_h) \circ \rho_{xg,h}\ ;\ (M_x \otimes \varepsilon) \circ \rho_{x,e} = M_x.$$

A morphism $\underline{M} \to \underline{N}$ in $\mathcal{T}^{\underline{C}}$ is a morphism $(f, (\varphi_y)_{y\in Y})$ in $\mathcal{T}_k$ such that f is a morphism of G-sets and

$$(\varphi_y \otimes C_g) \circ \rho_{f(y),g} = \rho_{y,g} \circ \varphi_{yg}.$$

A morphism $\underline{M} \to \underline{N}$ in $\mathcal{Z}^{\underline{C}}$ is a morphism $(f, (\varphi_x)_{x\in X})$ in $\mathcal{Z}_k$ such that f is a morphism of G-sets and

$$(\varphi_x \otimes C_g) \circ \rho_{x,g} = \rho_{f(x),g} \circ \varphi_{xg}.$$

Now let $\underline{C}$ be a group coalgebra, and suppose that the underlying monoid G is a group. Write $B_g = C^*_{g^{-1}}$. Then $B = \oplus_{g\in G} B_g$ is a G-graded k-algebra, with multiplication maps $B_g \otimes B_h \to B_{gh}$ given by opposite convolution: for $\xi \in C^*_{g^{-1}}$, $\xi' \in C^*_{h^{-1}}$ and $c \in C_{(gh)^{-1}}$, we have

$$(\xi\xi')(c) = \xi(c_{(2,g^{-1})})\xi'(c_{(1,h^{-1})}).$$

We have functors $T :\ \mathcal{T}^{\underline{C}} \to \mathcal{T}_B$ and $Z :\ \mathcal{Z}^{\underline{C}} \to \mathcal{Z}_B$ defined as follows: at the level of objects, T and F are defined in the same way:

$$T(X, (M_x)_{x\in X}) = Z(X, (M_x)_{x\in X}) = \bigoplus_{x\in X} M_x,$$

with the following right B-action: for $m \in M_x$ and $\xi \in B_g = C^*_{g^{-1}}$:

$$m\xi = \langle \xi, m_{[1,g^{-1}]}\rangle m_{[0,xg]}.$$

At the level of morphisms, T and Z are the identities. If every C_g is finitely generated and projective as k-modules, then T and F are isomorphisms of categories. The inverse functors are defined as follows: if M is graded by the G-set X, then $T^{-1}(M) = Z^{-1}(M) = (X, (M_x)_{x\in X})$, with coaction maps $\rho_{x,g} :\ M_{x,g} \to M_x \otimes C_g$ given by the formula

$$\rho_{x,g}(m) = m\xi^{(g)} \otimes c^{(g)}.$$

We implicitly introduced the following notation, which will be used throughout the rest of this paper. It is well known that C_g is finitely generated and projective if and only if there exists a unique $\xi^{(g)} \otimes c^{(g)} \in C^*_g \otimes C_g$, called finite dual basis of C_g (summation is implicitly understood) such that $c = \xi^{(g)}(c)c^{(g)}$ and $\xi = \xi(c^{(g)})\xi^{(g)}$ for all $c \in C_g$ and $\xi \in C^*_g$. Also observe that

$$h \rightharpoonup \xi^{(g)} \otimes c^{(g)} = \xi^{(g)} \otimes c^{(g)} h, \tag{1}$$

for all $h \in H_{\gamma(g)}$. Indeed, let $\xi^{(g)} \otimes c^{(g)} = \tilde{\xi}^{(g)} \otimes \tilde{c}^{(g)}$. Then

$$h \rightharpoonup \xi^{(g)} \otimes c^{(g)} = (h \rightharpoonup \xi^{(g)})(\tilde{c}^g)\tilde{\xi}^g \otimes c^{(g)} = \tilde{\xi}^g \otimes (h \rightharpoonup \xi^{(g)})(\tilde{c}^g)c^{(g)} = \xi^{(g)} \otimes c^{(g)}h.$$

1.7. Doi-Hopf data in $\mathcal{T}_k$. First, let $\underline{H} = (G, (H_g)_{g\in G})$ be a semi-Hopf group coalgebra, that is a bialgebra in $\mathcal{T}_k$. This means that we have the following data and properties:

- G is a monoid;
- H_g is a k-algebra, for every $g \in H$;
- we have k-algebra maps $\varepsilon : H_e \to k$ and $\Delta_{g,g'} : H_{gg'} \to H_g \otimes H_{g'}$, for all $g, g' \in G$.

The Sweedler notation for the comultiplication maps is the following:

$$\Delta_{g,g'}(h) = h_{(1,g)} \otimes h_{(2,g')}.$$

The following coassociativity and counit property have to be satisfied:

$$(H_g \otimes \Delta_{g',g''}) \circ \Delta_{g,g'g''} = (\Delta_{g,g'} \otimes H_{g''}) \circ \Delta_{gg',g''};$$

$$(H_g \otimes \varepsilon) \circ \Delta_{g,e} = (\varepsilon \otimes H_g) \circ \Delta_{e,g} = H_g.$$

Now let $\underline{A} = (X, (A_x)_{x\in X})$ be a right $\underline{H}$-comodule algebra. This means that we have the following data and properties:

- X is a right G-set;
- A_x is a k-algebra, for all $x \in X$;
- we have k-algebra maps $\rho_{x,g} : A_{xg} \to A_x \otimes H_g$;

The following coassociativity and counit properties have to hold:

$$(A_x \otimes \Delta_{g,h}) \circ \rho_{x,gh} = (\rho_{x,g} \otimes H_h) \circ \rho_{xg,h};$$

$$(A_x \otimes \varepsilon) \circ \rho_{x,e} = A_x.$$

We use the following Sweedler-type notation for the coaction maps:

$$\rho_{x,g}(a) = a_{[0,x]} \otimes a_{[1,g]}.$$

Finally, let $\underline{C} = (\Lambda, (C_\lambda)_{\lambda\in\Lambda})$ be a right $\underline{H}$-module coalgebra. This means that we have the following:

- Λ is a monoid, and we have a monoid morphism $\gamma : \Lambda \to G$;
- C_λ is a right $H_{\gamma(\lambda)}$-module, for every $\lambda \in \Lambda$;
- $\underline{C}$ is a group-coalgebra, that is, we have k-linear maps $\Delta_{\lambda,\lambda'} : C_{\lambda\lambda'} \to C_\lambda \otimes C_{\lambda'}$ and $\varepsilon : C_e \to k$ satisfying the appropriate coassociativity and counit properties;
- the following compatibility conditions have to be fulfilled: for all $c \in C_{\lambda\lambda'}$ and $h \in H_{\gamma(\lambda\lambda')}$, we have
$$\Delta_{\lambda,\lambda'}(ch) = c_{(1,\lambda)}h_{(1,\gamma(\lambda))} \otimes c_{(2,\lambda')}h_{(2,\gamma(\lambda'))},$$
and $\varepsilon(ch) = \varepsilon(c)\varepsilon(h)$, for all $c \in C_e$, $h \in H_e$.

Observe that (G, Λ, X) is a discrete Doi-Hopf datum; we call it the discrete Doi-Hopf datum underlying $(\underline{H}, \underline{A}, \underline{C})$.

1.8. Doi-Hopf modules in $\mathcal{T}_k$. Now we describe the objects of $\underline{M} = (Y, (M_y)_{y\in Y}) \in \mathcal{T}_k(\underline{H})^{\underline{C}}_{\underline{A}}$. These consist of the following data

- a (G, Λ, X)-set Y (see 1.2);
- for every $y \in Y$, a right $A_{\beta(y)}$-module M_y;

- k-linear maps $\rho_{y,\lambda} : M_{y\lambda} \to M_y \otimes C_\lambda$, satisfying the appropriate coassociativity and counit conditions;
- the following compatibility conditions have to be satisfied, for all $m \in M_{y\lambda}$ and $a \in A_{\beta(y\lambda)}$:

$$\rho_{y,\lambda}(ma) = m_{[0,y]}a_{[0,\beta(y)]} \otimes m_{[1,\lambda]}a_{[1,\gamma(\lambda)]}. \tag{2}$$

Here we use the following Sweedler-type notation for the coaction on $\underline{M}$: $\rho_{y,\lambda}(m) = m_{[0,y]} \otimes m_{[1,\lambda]}$, for $m \in M_{y\lambda}$.

A morphism $\underline{M} = (Y, (M_y)_{y\in Y}) \to \underline{M}' = (Y', (M'_{y'})_{y'\in Y'})$ in $\mathcal{T}_k(\underline{H})^{\underline{C}}_{\underline{A}}$ is a couple $(\eta, (\varphi_{y'})_{y'\in Y'})$, where

- $\eta : Y' \to Y$ is a morphism of (G, Λ, X)-sets;
- for every $y' \in Y'$, $\varphi_{y'} : M_{\eta(y')} \to M'_{y'}$ is a right $A_{\beta'(y')}$-linear map;
- for all $y' \in Y'$ and $\lambda \in \Lambda$, diagram (3) commutes.

$$\begin{array}{ccc} M_{\eta(y')\lambda} & \xrightarrow{\varphi_{y'\lambda}} & M'_{y'\lambda} \\ \Big\downarrow{\scriptstyle \rho_{\eta(y'),\lambda}} & & \Big\downarrow{\scriptstyle \rho'_{y',\lambda}} \\ M_{\eta(y')} \otimes C_\lambda & \xrightarrow{\varphi_{y'} \otimes C_\lambda} & M'_{y'} \otimes C_\lambda \end{array} \tag{3}$$

Example 1.9. Let G be a monoid; then (G, G, G) is a discrete Doi-Hopf datum. G is a right G-module by right multiplication, and the identity on G is a morphism of monoids. A (G, G, G)-set is a right G-set Y together with a map $\beta : Y \to G$ satisfying $\beta(yg) = \beta(y)g$, for all $y \in Y$ and $g \in G$. Let $\underline{H} = (G, (H_g)_{g\in G})$ be a semi-Hopf group coalgebra. $(\underline{H}, \underline{H}, \underline{H})$ is a Doi-Hopf datum in $\mathcal{T}_k$. A Doi-Hopf module $(Y, (M_y)_{y\in Y}) \in \mathcal{T}_k(\underline{H})^{\underline{H}}_{\underline{H}}$ consists of the following data: Y is (G, G, G)-set as above; every M_y is a right $H_{\beta(y)}$-module, and $\rho_{yg} : M_{yg} \to M_y \otimes H_g$ is a coassociative coaction. For every $m \in M_{yg}$ and $h \in H_{\beta(yg)}$, we have the compatibility relation

$$\rho_{y,g}(mh) = m_{[0,y]}h_{(1,\beta(y))} \otimes m_{[1,g]}h_{(2,g)}.$$

These Doi-Hopf modules are simply called Hopf modules, and have been considered in [2, Sec. 3.1], where the Structure Theorem for Doi-Hopf modules was discussed.

1.10. Let $\mathcal{Z}_k(\underline{H})^{\underline{C}}_{\underline{A}}$ be the category with the same objects as $\mathcal{T}_k(\underline{H})^{\underline{C}}_{\underline{A}}$, but with morphisms defined in a different way. A morphism $\underline{M} = (Y, (M_y)_{y\in Y}) \to \underline{M}' = (Y', (M'_{y'})_{y'\in Y'})$ in $\mathcal{Z}_k(\underline{H})^{\underline{C}}_{\underline{A}}$ is a couple $(\eta, (\varphi_y)_{y\in Y})$, where

- $\eta : Y \to Y'$ is a morphism of (G, Λ, X)-sets;
- for every $y \in Y$, $\varphi_y : M_y \to M'_{\eta(y)}$ is a morphism of $A_{\beta(y)} = A_{\beta'(\eta(y))}$-modules;
- for all $y \in Y$ and $\lambda \in \Lambda$, diagram (4) commutes.

$$\begin{array}{ccc} M_{y\lambda} & \xrightarrow{\varphi_{y\lambda}} & M'_{\eta(y)\lambda} \\ \Big\downarrow{\scriptstyle \rho_{y,\lambda}} & & \Big\downarrow{\scriptstyle \rho'_{\eta(y),\lambda}} \\ M_y \otimes C_\lambda & \xrightarrow{\varphi_y \otimes C_\lambda} & M'_{\eta(y)} \otimes C_\lambda \end{array} \tag{4}$$

2. Algebras graded by a discrete Doi-Hopf datum

Definition 2.1. Let (G,Λ,X) be a discrete Doi-Hopf datum. A (G,Λ,X)-graded algebra is an associative algebra A (not necessarily with unit) together with a direct sum decomposition

$$A = \oplus_{\lambda\in\Lambda}\oplus_{x\in X} A_{\lambda,x},$$

such that

$$A_{\lambda,x}A_{\lambda',x'} \subset \delta_{x',x\gamma(\lambda')}A_{\lambda\lambda',x'}, \tag{5}$$

where δ is the Kronecker symbol. Moreover, for every $x\in X$, there exists a $1_x\in A_{e,x}$ such that

$$a1_x = a, \text{ for all } \lambda\in\Lambda \text{ and } a\in A_{\lambda,x}; \tag{6}$$

$$1_x b = b, \text{ for all } \lambda\in\Lambda \text{ and } b\in A_{\lambda,x\gamma(\lambda)}. \tag{7}$$

Proposition 2.2. *Let A be a (G,Λ,X)-graded algebra, with either G or Λ a group, and put*

$$A_\lambda = \oplus_{x\in X}A_{\lambda,x},$$

for all $\lambda\in\Lambda$. Then $A=\oplus_{\lambda\in\Lambda}A_\lambda$ is a Λ-graded algebra with idempotent local units. If X is finite, then A is a Λ-graded algebra with unit $1=\sum_{x\in X}1_x$.

Proof. It follows from (5) that $A_\lambda A_{\lambda'}\subset A_{\lambda\lambda'}$. If G or Λ is a group, then $\gamma(\lambda)$ is invertible in G, for all $\lambda\in\Lambda$.
Take $a\in A_{\lambda,x}$. From (5-7), it follows that

$$a1_y = \delta_{x,y}a \text{ and } 1_y a = \delta_{y,x\gamma(\lambda)^{-1}}a.$$

In particular, $1_x1_y=\delta_{x,y}1_x$, so $\{1_x \mid x\in X\}$ is a set of orthgonal idempotents. This implies the following: for any finite subset $I\subset X$, we have the following implications:

$$x\in I \implies a(\sum_{y\in I}1_y)=a;$$

$$x\gamma(\lambda)^{-1}\in I \implies (\sum_{y\in I}1_y)a=a.$$

Now take a finite subset $B\subset A$. There exist $\lambda_1,\cdots,\lambda_n\in\Lambda$ and $x_1,\cdots,x_m\in X$ such that

$$B\subset \bigoplus_{i=1}^n\bigoplus_{j=1}^m A_{\lambda_i,x_j}.$$

We can always add e to $\{\lambda_1,\cdots,\lambda_n\}$, so it is no restriction to assume that $\lambda_1=e$. Let a be a homogeneous component of one of the elements of B. Then we find $i\in\{1,\cdots n\}$ and $j\in\{1,\cdots,m\}$ such that $a\in A_{\lambda_i,x_j}$. Consider $I=\{x_j\gamma(\lambda_i)^{-1} \mid i\in\{1,\cdots n\},\ j\in\{1,\cdots,m\}\}$. Since $x_j = x_j\gamma(\lambda_1)^{-1}$, $x_j\gamma(\lambda_i)^{-1}\in I$, we have

$$a(\sum_{y\in I}1_y)=(\sum_{y\in I}1_y)a=a,$$

and then it follows that

$$b(\sum_{y\in I}1_y)=(\sum_{y\in I}1_y)b=b,$$

for all $b \in B$. If X is finite, then the above arguments show that $1 = \sum_{x\in X} 1_x$ is a unit for A, and then A is a unital Λ-graded algebra. □

From (5) and (6), we deduce that $A_{e,x}A_{e,x'} = \delta_{x,x'}A_{e,x'}$. Therefore every $A_{e,x}$ is a k-algebra, with unit 1_x, and $(X, (A_{e,x})_{x\in X})$ is an algebra in $\mathcal{T}_k$.
Assume that X is finite, so that A is a unital Λ-graded algebra. Then $1 \in A_e$, see for example [8, Prop. I.1.1], and we can write $1 = \sum_{x\in X} 1_x \in \oplus_{x\in X} A_{e,x}$.

Proposition 2.3. *Take a discrete Doi-Hopf datum (G, Λ, X), with G or Λ a group, and assume that X is finite. The following assertions are equivalent:*

(a) $A = \oplus_{\lambda\in\Lambda} \oplus_{x\in X} A_{\lambda,x}$ *is a (G, Λ, X)-graded algebra;*
(b) $A = \oplus_{\lambda\in\Lambda}\big(\oplus_{x\in X} A_{\lambda,x}\big)$ *is a unital Λ-graded algebra, and, with $1_x \in A_{e,x}$ defined as above,*

$$A_{\lambda,x}1_{x'} = \delta_{x,x'}A_{\lambda,x}; \tag{8}$$

$$1_x A_{\lambda',x'} = \delta_{x',x\gamma(\lambda')}A_{\lambda',x'}. \tag{9}$$

Proof. $\underline{(a) \Rightarrow (b)}$. We have already seen above that A is a unital Λ-graded algebra; (8-9) follow immediately from (5-7).
$\underline{(b) \Rightarrow (a)}$. Take $a \in A_{\lambda,x}$. It follows from (8) that $a1_{x'} = 0$ if $x \neq x'$. Therefore $a = a1 = \sum_{x'\in X} a1_{x'} = a1_x$, proving (6). (7) is proved in a similar way: let $b \in A_{\lambda,x\gamma(\lambda)}$. It follows from (9) that $1_{x'}b = 0$ if $x' \neq x$, so $b = 1b = \sum_{x'\in X} 1_{x'}b = 1_x b$.
Let $a = 1_x \in A_{e,x}$. It follows that $1_x 1_{x'} = \delta_{x,x'}1_x$.
In order to show that (5) holds, take $a \in A_{\lambda,x}$ and $b \in A_{\lambda',x'}$. Then $ab \in A_{\lambda\lambda'}$, since A is a Λ-graded algebra. Now we have that

$$ab = (a1_x)(1_{x'\gamma(\lambda')^{-1}}b) = a(1_x 1_{x'\gamma(\lambda')^{-1}})b = 0,$$

if $x \neq x'\gamma(\lambda')^{-1}$, or, equivalently, $x' \neq x\gamma(\lambda')$. Now let $x' = x\gamma(\lambda')$. Since A is a Λ-graded algebra, $a \in A_\lambda$ and $b \in A_{\lambda'}$, we have that

$$ab \in A_{\lambda\lambda'} = \bigoplus_{y\in X} A_{\lambda\lambda',y} = \bigoplus_{y\in X} A_{\lambda\lambda',y}1_y,$$

hence we have that $ab = \sum_{y\in X} c_y 1_y$, with $c_y \in A_{\lambda\lambda',y}$. Now $b1_{x'} = b$, hence

$$ab = ab1_{x'} = \sum_{y\in X} c_y 1_y 1_{x'} = c_{x'}1_{x'} \in A_{\lambda\lambda',x'}1_{x'} = A_{\lambda\lambda',x'},$$

and this shows that (5) holds. □

2.4. 2-categorical interpretation. The definition of algebra graded by a discrete Doi-Hopf datum can be rephrased in terms of 2-categories. For more detail on 2-categories, we refer the reader to [1, Ch. 7].
To a discrete Doi-Hopf datum (G, Λ, X), we associate a 2-category $\mathcal{G}$, under the assumption that G or Λ is a group. The objects of $\mathcal{G}$ are the elements of X, and the morphisms are the elements of $\Lambda \times X$. $(\lambda, x) \in \Lambda \times X$ is a morphism with target x and source $x\gamma(\lambda)^{-1}$: $s(\lambda, x) = x\gamma(\lambda)^{-1}$ and $t(\lambda, x) = x$. Then the composition $(\lambda', z) \circ (\lambda, y)$ is defined if and only $y = t(\lambda, y) = s(\lambda', z) = z\gamma(\lambda')^{-1}$, and, in this case $(\lambda', z) \circ (\lambda, y) = (\lambda\lambda', z)$. It is easy to verify that the identity morphism on x is (e, x).
Like every category, $\mathcal{G}$ can be viewed as a 2-category: the 0-cells are the objects of $\mathcal{G}$, and, for all $x, y \in X$, $\mathrm{Hom}(x, y)$ is the discrete category with

objects the morphisms $x \to y$ in $\mathcal{G}$. The only 2-cells are then the identity 2-cells. For every $x \in X$, we have the unit functor $u_x : \mathbf{1} \to \mathrm{Hom}(x, x)$, sending the object 0 of $\mathbf{1}$ to (e, x), and the morphism 1 of $\mathbf{1}$ to the identity of (e, x). $\mathbf{1}$ is the category with one object 0 and one morphism 1.
The category of k-modules $\mathcal{M}_k$ is monoidal, so it can be viewed as a bicategory with one object $*$. To simplify notation, we will treat $\mathcal{M}_k$ as if it were a strict monoidal category, or a 2-category with one object. The unit functor $u_* : \mathbf{1} \to \mathcal{M}_k$ sends 0 to k and 1 to the identity of k.

Proposition 2.5. *Assume that X is finite, and that Λ or G is a group. Then we have a bijective correspondence between (G, Λ, X)-graded algebras and lax functors $F : \mathcal{G} \to \mathcal{M}_k$.*

Proof. According to [1, Def. 7.5.1], a lax functor $F : \mathcal{G} \to \mathcal{M}_k$ consists of the following data:
(a) for every $x \in X$, a 0-cell $F(x)$ of $\mathcal{M}_k$. Since $\mathcal{M}_k$ has only one 0-cell, so there is only one way to define F at the level of 0-cells;
(b) for every $x, y \in X$, a functor $F_{x,y} : \mathrm{Hom}(x, y) \to \underline{\mathrm{Hom}}(F(x), F(y)) = \mathcal{M}_k$. Since $\mathrm{Hom}(x, y)$ is discrete, it suffices to give $F_{x\gamma(\lambda)^{-1},x}(\lambda, x)$, for every morphism (λ, x) in $\mathcal{G}$. Write $F_{x\gamma(\lambda)^{-1},x}(\lambda, x) = A_{(\lambda,x)}$.
(c) for $x, y, z \in X$, we have to give a natural transformation $\mu : F_{x,y} \Rightarrow F_{y,z} \to F_{x,z}$. This means that for every $(\lambda, y) \in \mathrm{Hom}(x, y)$ and $(\lambda', z) \in \mathrm{Hom}(y, z)$, we have to give a k-linear map

$$\mu_{(\lambda,y),(\lambda',z)} : A_{(\lambda,y)} \otimes A_{(\lambda',z)} \to A_{(\lambda\lambda',z)}.$$

Since $y = z\gamma(\lambda')^{-1}$, we find k-linear maps $A_{(\lambda,y)} \otimes A_{(\lambda',y\gamma(\lambda'))} \to A_{(\lambda\lambda',z)}$, which is precisely what is needed to define on $A = \oplus_{(\lambda,x)\in\Lambda\times X} A_{(\lambda,x)}$ a multiplication that satisfies (5). The naturality of μ is automatically fulfilled since $\mathrm{Hom}(x, y)$ is discrete. The associativity of the multiplication on A follows from the functorial properties of F.
(d) For all $x \in X$, we need a natural transformation

$$\delta_x : u_* \Rightarrow F_{x,x} \circ u_x.$$

This natural transformation is determined by a linear map

$$\delta_x(0) : u_*(0) = k \to F_{x,x}(u_x(0)) = A_{e,x}.$$

The diagrams (7.12) in [1] have to commute. In our particular situation, this means that the diagrams

$$\begin{array}{ccc} k \otimes A_{\lambda,x\gamma(\lambda)} & \xrightarrow{\delta_x(0)\otimes A_{\lambda,x\gamma(\lambda)}} & A_{e,x} \otimes A_{\lambda,x\gamma(\lambda)} \\ & {\scriptstyle =} \searrow & \downarrow {\scriptstyle \mu_{(e,x),(\lambda,x\gamma(\lambda))}} \\ & & A_{\lambda,x\gamma(\lambda)} \end{array}$$

$$\begin{array}{ccc} A_{\lambda,x} \otimes k & \xrightarrow{A_{\lambda,x}\otimes\delta_x(0)} & A_{\lambda,x} \otimes A_{e,x} \\ & {\scriptstyle =} \searrow & \downarrow {\scriptstyle \mu_{(\lambda,x),(e,x)}} \\ & & A_{\lambda,x} \end{array}$$

commute. Now write $1_x = \delta_x(0)(1_k)$. The commutativity of the above diagrams is equivalent to (6-7). □

2.6. The smash product. We propose a first method to construct algebras graded by a discrete Doi-Hopf datum (G, Λ, X), in the situation where G or Λ is a group. Let $\underline{A}$ be a right $\underline{H}$-comodule algebra, as in 1.7. Let B be a Λ-graded algebra, and assume that every B_λ is a left $H_{\gamma(\lambda)^{-1}}$-module; the action of $h \in H_{\gamma(\lambda)^{-1}}$ on $b \in B_\lambda$ is denoted by $h \rightharpoonup b$. Moreover, assume that

$$h \rightharpoonup (bb') = (h_{(2,\gamma(\lambda)^{-1})} \rightharpoonup b)(h_{(1,\gamma(\lambda')^{-1})} \rightharpoonup b'), \tag{10}$$

for all $b \in B_\lambda$, $b' \in B_{\lambda'}$, $h \in H_{\gamma(\lambda\lambda')^{-1}}$, and $h \rightharpoonup 1 = \varepsilon(h)1$, for all $h \in H_e$. Now define

$$B\#\underline{A} = \bigoplus_{\lambda\in\Lambda} \bigoplus_{x \in X} B_\lambda \# A_x.$$

Here $B_\lambda \# A_x = B_\lambda \otimes A_x$ as a k-module. We define a multiplication map on $B\#\underline{A}$, making it a (G, Λ, X)-graded algebra. We need to define multiplication maps

$$(B_\lambda \# A_x) \otimes (B_{\lambda'} \# A_{x'}) \to B\#\underline{A}.$$

If $x' \neq x\gamma(\lambda')$, then we let this multiplication map be zero. For $x' = x\gamma(\lambda')$,

$$\mu : \ (B_\lambda \# A_x) \otimes (B_{\lambda'} \# A_{x'}) \to B_{\lambda\lambda'} \# A_{x'}$$

is given by the formula

$$(b\#a)(b'\#a') = b(a_{[1,\gamma(\lambda')^{-1}]} \rightharpoonup b')\#a_{[0,x\gamma(\lambda')]}a'.$$

Proposition 2.7. *With notation as above, $B\#\underline{A}$ is an algebra graded by (G, Λ, X).*

Proof. We have to show that the mulitplication is associative. Take $a \in A_x$, $a' \in A_{x'}$, $a'' \in A_{x''}$, $b \in B_\lambda$, $b' \in B_{\lambda'}$ and $b'' \in B_{\lambda''}$. Also assume that $x' = x\gamma(\lambda')$ and $x'' = x'\gamma(\lambda'')$.

$$\begin{aligned}
\big((b\#a)(b'\#a')\big)(b''\#a'') &= \big(b(a_{[1,\gamma(\lambda')^{-1}]} \rightharpoonup b')\#a_{[0,x']}a'\big)(b''\#a'') \\
&= b(a_{[2,\gamma(\lambda')^{-1}]} \rightharpoonup b')\big((a_{[1,\gamma(\lambda'')^{-1}]}a'_{[1,\gamma(\lambda'')^{-1}]}) \rightharpoonup b''\big)\#a_{[0,x'']}a'_{[0,x'']}a''; \\
(b\#a)\big((b'\#a')(b''\#a'')\big) &= (b\#a)\big(b'(a'_{[1,\gamma(\lambda'')^{-1}]} \rightharpoonup b'')\#a'_{[0,x'']}a''\big) \\
&= b\Big(a_{[1,\gamma(\lambda'\lambda'')^{-1}]} \rightharpoonup \big(b'(a'_{[1,\gamma(\lambda'')^{-1}]} \rightharpoonup b'')\big)\Big)\#a_{[0,x'']}a'_{[0,x'']}a'' \\
&= b(a_{[2,\gamma(\lambda')^{-1}]} \rightharpoonup b')\big((a_{[1,\gamma(\lambda'')^{-1}]}a'_{[1,\gamma(\lambda'')^{-1}]}) \rightharpoonup b''\big)\#a_{[0,x'']}a'_{[0,x'']}a''.
\end{aligned}$$

With the same notation, we easily compute that

$$\begin{aligned}
(1\#1_x)(b'\#a') &= 1(1_{\gamma(\lambda')^{-1}}) \rightharpoonup b')\#1_{x'}a' = b'\#a'; \\
(b\#a)(1\#1_x) &= b(a_{[1,\gamma(\lambda')^{-1}]} \rightharpoonup 1)\#a_{[0,x']}1_{x'} \\
&= b\#\varepsilon(a_{[1,\gamma(\lambda')^{-1}]})a_{[0,x']} = b\#a.
\end{aligned}$$

□

2.8. The Koppinen smash product. Now we introduce a second method to construct algebras graded by a discrete Doi-Hopf datum. Let $(\underline{H}, \underline{A}, \underline{C})$ be a Doi-Hopf datum in $\mathcal{T}_k$, with Λ a group, and let

$$\mathcal{A} = \bigoplus_{\lambda\in G} \bigoplus_{x\in X} \mathrm{Hom}(C_{\lambda^{-1}}, A_x) = \bigoplus_{\lambda\in G} \bigoplus_{x\in X} \mathcal{A}_{\lambda,x}.$$

We define multiplication maps

$$\mathcal{A}_{\lambda,x}\otimes\mathcal{A}_{\lambda',x'}\to\mathcal{A};$$

for $x'\neq x\gamma(\lambda')$, this multiplication map is 0. For $x'=x\gamma(\lambda')$, then we describe

$$\mu:\ \mathcal{A}_{\lambda,x}\otimes\mathcal{A}_{\lambda',x'}\to\mathcal{A}_{\lambda\lambda',x'}.$$

For $f\in\mathcal{A}_{\lambda,x}$ and $g\in\mathcal{A}_{\lambda',x'}$, $\mu(f\otimes g)=f\#g\in\mathcal{A}_{\lambda\lambda',x'}$, is given by the following formula, for $c\in C_{(\lambda\lambda')^{-1}}$:

$$(f\#g)(c)=f(c_{(2,\lambda^{-1})})_{[0,x']}g\big(c_{(1,(\lambda')^{-1})}f(c_{(2,\lambda^{-1})})_{[1,\gamma(\lambda')^{-1}]}\big)\in A_{x'}.$$

Proposition 2.9. *$\mathcal{A}$, as defined in 2.8 is an algebra graded by (G,Λ,X).*

Proof. Take $f\in\mathcal{A}_{\lambda,x}$, $g\in\mathcal{A}_{\lambda',x'}$ and $h\in\mathcal{A}_{\lambda'',x''}$. Assume also that $x'=x\gamma(\lambda')$ and $x''=x'\gamma(\lambda'')$. We have to show that $(f\#g)\#h=f\#(g\#h)$. For $c\in C_{(\lambda\lambda'\lambda'')^{-1}}$, we have

$$\begin{aligned}
&((f\#g)\#h)(c)\\
&=\ \big((f\#g)(c_{(2,(\lambda\lambda')^{-1})})\big)_{[0,x'']}h\Big(c_{(1,(\lambda'')^{-1})}\big((f\#g)(c_{(2,(\lambda\lambda')^{-1})})\big)_{[1,\gamma(\lambda'')^{-1}]}\Big)\\
&=\ f(c_{(3,\lambda^{-1})})_{[0,x'']}g\Big(c_{(2,(\lambda')^{-1})}f(c_{(3,\lambda^{-1})})_{[2,\gamma(\lambda')^{-1}]}\Big)_{[0,x'']}h\Big(c_{(1,(\lambda'')^{-1})}\\
&\qquad f(c_{(3,\lambda^{-1})})_{[1,\gamma(\lambda'')^{-1}]}g\big(c_{(2,(\lambda')^{-1})}f(c_{(3,\lambda^{-1})})_{[2,\gamma(\lambda')^{-1}]}\big)_{[1,\gamma(\lambda'')^{-1}]}\Big)\\
&=\ f(c_{(2,\lambda^{-1})})_{[0,x'']}(g\#h)\big(c_{(1,(\lambda'\lambda'')^{-1})}f(c_{(2,\lambda^{-1})})_{[1,\gamma(\lambda'\lambda'')^{-1}]}\big)\\
&=\ (f\#(g\#h))(c).
\end{aligned}$$

Define $e_x:C_e\to A_x$ by $e_x(c)=\varepsilon(c)1_x$. Then it is easy to compute that $e_x\#g=g$ and $f\#e_{x'}=f$. □

2.10. Let $(\underline{H},\underline{A},\underline{C})$ be a Doi-Hopf datum in $\mathcal{T}_k$, with Λ a group, and put

$$B=\bigoplus_{\lambda\in\Lambda}B_\lambda,$$

with $B_\lambda=C^*_{\lambda^{-1}}$. In 1.6, we showed that B is a Λ-graded algebra. B_λ is a left $H_{\gamma(\lambda)^{-1}}$-module: for $\xi\in B_\lambda$, $h\in H_{\gamma(\lambda)^{-1}}$and $c\in C_{\lambda^{-1}}$, let

$$(h\rightharpoonup\xi)(c)=\xi(ch).$$

It is easy to verify that (10) is satisfied: for $\xi'\in B_{\lambda'}$, $h\in H_{\gamma(\lambda\lambda')^{-1}}$ and $c\in C_{(\lambda\lambda')^{-1}}$, we have

$$\begin{aligned}
(h\rightharpoonup(\xi\xi'))(c)=(\xi\xi')(ch)&=\xi\big((ch)_{(2,\lambda^{-1})}\big)\xi'\big((ch)_{(1,(\lambda')^{-1})}\big)\\
&=\ \xi\big(c_{(2,\lambda^{-1})}h_{(2,\gamma(\lambda)^{-1})}\big)\xi'\big(c_{(1,(\lambda')^{-1})}h_{(1,\gamma(\lambda')^{-1})}\big)\\
&=\ \big((h_{(2,\gamma(\lambda)^{-1})}\rightharpoonup\xi)(h_{(1,\gamma(\lambda')^{-1})}\rightharpoonup\xi')\big)(c).
\end{aligned}$$

Now we can consider the smash product $B\#\underline{A}$, as in 2.6. Consider the maps

$$\alpha_{\lambda,x}:C^*_{\lambda^{-1}}\#A_x\to\mathcal{A}_{\lambda,x}=\mathrm{Hom}(C_{\lambda^{-1}},A_x),\ \ \alpha_{\lambda,x}(\xi\#a)(c)=\xi(c)a,$$

for $\xi\in C^*_{\lambda^{-1}}$, $a\in A_x$, $c\in C_{\lambda^{-1}}$. It is well-known that $\alpha_{\lambda,x}$ is an isomorphism of k-modules if C_λ is finitely generated and projective as a k-module. For later use, we describe $\alpha^{-1}_{\lambda,x}$, using the notation introduced in 1.6 for the dual basis of C_λ:

$$\alpha^{-1}_{\lambda,x}(f) = \xi^{(\lambda^{-1})}\#f(c^{(\lambda^{-1})}). \tag{11}$$

Proposition 2.11. *With notation as in 2.10,*

$$\alpha = \bigoplus_{\lambda\in\Lambda}\bigoplus_{x\in X}\alpha_{\Lambda,x} :\ B\#\underline{A} \to \mathcal{A}$$

is a morphism of algebras graded by (G,Λ,X). If every C_λ is finitely generated and projective as a k-module, then it is an isomorphism.

Proof. Take $\xi\in B_\lambda$, $\xi'\in B_{\lambda'}$, $a\in A_x$, $a'\in A_{x'}$, and assume that $x'=x\gamma(\lambda')$. For all $c\in C_{(\lambda\lambda')^{-1}}$, we have

$$\begin{aligned}\alpha_{\lambda\lambda',x'}\big((\xi\#a)(\xi'\#a')\big)(c) &= \alpha_{\lambda\lambda',x'}\big(\xi(a_{[1,\gamma(\lambda')^{-1}]}\rightharpoonup\xi')\#(a_{[0,x']}a')\big)(c)\\ &= \xi(c_{(2,\lambda^{-1})})\xi'(c_{(1,(\lambda')^{-1})}a_{[1,\gamma(\lambda')^{-1}]})a_{[0,x']}a'\\ &= \big(\alpha_{\lambda,x}(\xi\#a)\alpha_{\lambda',x'}(\xi'\#a')\big)(c).\end{aligned}$$

It is also obvious that $\alpha_{e,x}(\varepsilon\#1_x)=e_x$. □

3. Modules graded by (G,Λ,X)-sets

Definition 3.1. Let (G,Λ,X) be a discrete Doi-Hopf datum, and A a (G,Λ,X)-graded algebra. Let Y be a (G,Λ,X)-set, see 1.2. A right A-module M is graded by the (G,Λ,X)-set Y if

$$M = \bigoplus_{y\in Y} M_y$$

with

$$M_yA_{\lambda,x} \subset \delta_{x,\beta(y\lambda)}M_{y\lambda} \tag{12}$$

and

$$m1_{\beta(y)} = m, \tag{13}$$

for all $m\in M_y$.

Example 3.2. Let Y be a (G,Λ,X)-set. $Z\subset Y$ is a (G,Λ,X)-subset of Y if $z\lambda\in Z$, for all $\lambda\in\Lambda$ and $z\in Z$.
Now suppose that $M=\oplus_{y\in Y}M_y$ is a right A-module graded by the (G,Λ,X)-set Y. Then $N=\oplus_{z\in Z}M_z$ is a right A-module graded by the (G,Λ,X)-set Z. Indeed, for all $z\in Z$, $\lambda\in\Lambda$ and $x\in X$, we have

$$N_zA_{\lambda,x} = M_zA_{\lambda,x}\subset\delta_{x,\beta(z\lambda)}M_{z\lambda} = \delta_{x,\beta(z\lambda)}N_{z\lambda}.$$

Example 3.3. Recall from 1.2 that $Y=\Lambda\times X$ is a (G,Λ,X)-set. Let A be a (G,Λ,X)-graded algebra; then A viewed as a right A-module is graded by the (G,Λ,X)-set $\Lambda\times X$. We need to verify that

$$A_{\lambda,x}A_{\lambda',x'}\subset\delta_{x',\beta((\lambda,x)\lambda')}A_{(\lambda,x)\lambda'}. \tag{14}$$

We have seen in 1.2 that $(\lambda,x)\lambda' = (\lambda\lambda', x\gamma(\lambda'))$ and $\beta((\lambda,x)\lambda') = x\gamma(\lambda')$, and then (14) reduces to (5). It is also easy to check the unit condition: for

$a \in A_{\lambda,x}$, we have that $a1_{\beta(\lambda,x)} = a1_x = a$.
Now fix $x \in X$. Then

$$Z_x = \{(\lambda, x\gamma(\lambda)) \mid \lambda \in \Lambda\}$$

is a (G, Λ, X)-subset of $\Lambda \times X$. Indeed, for all $\lambda' \in \Lambda$, $(\lambda, x\gamma(\lambda))\lambda' = (\lambda\lambda', x\gamma(\lambda\lambda')) \in Z_x$. It follows from Example 3.2 that $A^{(x)} = \oplus_{\lambda\in\Lambda} A_{\lambda,x\gamma(\lambda)}$ is a right A-module graded by the G-set Z_x.

Assume now that X is finite; then we know that A is an algebra with unit $1 = \sum_{x\in X} 1_x$. If M is a right A-module graded by a (G, Λ, X)-set Y, then we have for all $m \in M_y$ that $m1_x = 0$ if $x \neq \beta(y)$, hence $m1 = \sum_{x\in X} m1_x = m1_{\beta(y)} = m$, so M is a unital A-module. It also follows from (12) that $M_y A_\lambda \subset M_{y\lambda}$, hence M is a right A-module graded by the Λ-set Y. We refer to [9] for a discussion of modules graded by G-sets.
Conversely, let M be a right A-module graded by a Λ-set Y (which is not necessarily a (G, Λ, X)-set). Since $1 = \sum_{x\in X} 1_x$, we have, for all $y \in Y$, $M_y = M_y 1 = \sum_{x\in X} M_y 1_x$. Let $x \neq x' \in X$, and assume that $m \in M_y 1_x \cap M_y 1_{x'}$. Then $m = n1_{x'}$ for some $n \in M$, and $m = m1_x = n1_{x'}1_x = 0$. Hence $M_y 1_x \cap M_y 1_{x'} = \{0\}$ and

$$M_y = \oplus_{x\in X} M_y 1_x. \tag{15}$$

If M is graded by a (G, Λ, X)-set Y, then it follows from (12) that $M_y 1_x = \{0\}$ if $x \neq \beta(y)$, and then we find that $M_y 1_{\beta(y)} = M_y$. Hence at most one direct summand in (15) is nontrivial.

Proposition 3.4. *Let A be a (G, Λ, X)-graded algebra, with X finite, and Y a (G, Λ, X)-set. For a (unital) right A-module $M = \oplus_{y\in Y} M_y$, the following assertions are equivalent*

- *M is graded by the (G, Λ, X)-set Y;*
- *M is graded by the Λ-set Y and $M_y 1_x = \delta_{x,\beta(y)} M_y$.*

Proof. 1) $\Rightarrow$ 2): see the arguments preceding Proposition 3.4.
2) $\Rightarrow$ 1). Take $m \in M_y$. If $x \neq \beta(y)$, then $m1_x = 0$, hence $m = m1 = \sum_{x\in X} m1_x = m1_{\beta(y)}$.
Take $m \in M_y$ and $a \in A_{\lambda,x}$. Then

$$ma = (m1_{\beta(y)})a = m(1_{\beta(y)}a) = m(\delta_{x,\beta(y)\gamma(\lambda)}a).$$

If $x \neq \beta(y)\gamma(\lambda)$, then $ma = 0$. In any case $ma \in M_{y\lambda}$, so we conclude that (12) holds, since $\beta(y)\gamma(\lambda) = \beta(y\lambda)$. □

Now we introduce the category $\mathcal{Z}_A^{(G,\Lambda,X)}$ of right A-modules graded by (G, Λ, X)-sets. The objects are couples (Y, M), where Y is a (G, Λ, X)-set, and M is a right A-module graded by the (G, Λ, X)-set Y. A morphism $(Y, M) \to (Y', M')$ in $\mathcal{Z}_A^{(G,\Lambda,X)}$ is a couple (η, φ), where $\eta : Y \to Y'$ is a morphism of (G, Λ, X)-sets, and $\varphi : M \to M'$ is a right A-linear map such that $\varphi(M_y) \subset M'_{\eta(y)}$. If the condition $\varphi(M_y) \subset M'_{\eta(y)}$ is satisfied, then the condition that φ is right A-linear is equivalent to the commutativity of the diagrams

$$
\begin{array}{ccc}
M_y \otimes A_{\lambda,\beta(y\lambda)} & \longrightarrow & M_{y\lambda} \\
\downarrow{\scriptstyle \varphi_y\otimes \mathrm{id}} & & \downarrow{\scriptstyle \varphi_{y\lambda}} \\
M'_{\eta(y)} \otimes A_{\lambda,\beta'(\eta(y)\lambda)} & \longrightarrow & M'_{\eta(y\lambda)} = M'_{\eta(y)\lambda}
\end{array} \tag{16}
$$

$\mathcal{T}_A^{(G,\Lambda,X)}$ has the same objects as $\mathcal{Z}_A^{(G,\Lambda,X)}$. A morphism $(Y,M)\to(Y',M')$ in $\mathcal{T}_A^{(G,\Lambda,X)}$ is a couple $(\eta,(\varphi_{y'})_{y'\in Y'})$, where $\eta:\ Y'\to Y$ is a morphism of (G,Λ,X)-sets, and $\varphi_{y'}:\ M_{\eta(y')}\to M'_{y'}$ are k-linear maps such that the diagram

$$
\begin{array}{ccc}
M_{\eta(y')} \otimes A_{\lambda,\beta(\eta(y')\lambda)} & \longrightarrow & M_{\eta(y')\lambda} = M_{\eta(y'\lambda)} \\
\downarrow{\scriptstyle \varphi_{y'}\otimes \mathrm{id}} & & \downarrow{\scriptstyle \varphi_{y'\lambda}} \\
M_{y'} \otimes A_{\lambda,\beta'(y'\lambda)} & \longrightarrow & M'_{y'\lambda}
\end{array} \tag{17}
$$

commutes, for all $y'\in Y'$ and $\lambda\in\Lambda$.
Observe that these definitions are designed in such a way that we have forgetful functors

$$\mathcal{Z}_A^{(G,\Lambda,X)}\to \mathcal{Z}_k \text{ and } \mathcal{T}_A^{(G,\Lambda,X)}\to\mathcal{T}_k.$$

Proposition 3.5. *Let $(\underline{H},\underline{A},\underline{C})$ be a Doi-Hopf datum in $\mathcal{T}_k$. Then we have fully faithful functors*

$$T:\ \mathcal{T}_k(\underline{H})_{\underline{A}}^{\underline{C}}\to \mathcal{T}_{\mathcal{A}}^{(G,\Lambda,X)} \text{ and } Z:\ \mathcal{Z}_k(\underline{H})_{\underline{A}}^{\underline{C}}\to \mathcal{Z}_{\mathcal{A}}^{(G,\Lambda,X)}.$$

At the level of objects, the functors are defined in the same way: $T(\underline{M})=Z(\underline{M})=(Y,\oplus_{y\in Y}M_y)$, with multiplication maps $M_y\otimes\mathcal{A}_{\lambda,\beta(y\lambda)}\to M_{y\lambda}$ given by the formula

$$mf=m_{[0,y\lambda]}f(m_{[1,\lambda^{-1}]}). \tag{18}$$

At the level of morphisms, T and Z are defined by

$$T(\eta,(\varphi_{y'})_{y'\in Y'})=(\eta,(\varphi_{y'})_{y'\in Y'}) \text{ and } Z(\eta,(\varphi_y)_{y\in Y})=(\eta,\bigoplus_{y\in Y}\varphi_y).$$

Proof. We will show that the action (18) is associative and satisfies the unit property. Take $f\in\mathcal{A}_{\lambda,x}$, $f'\in\mathcal{A}_{\lambda',x'}$, with $x'=x\gamma(\lambda')$, so that $f\#f'\in\mathcal{A}_{\lambda\lambda',x'}$. Let $m\in M_y$, and take $x=\beta(y)$. Now

$$
\begin{aligned}
(mf)f' &= (m_{[0,y\lambda]}f(m_{[1,\lambda^{-1}]}))f'\\
&= m_{[0,x']}f(m_{[2,\lambda^{-1}]})_{[0,x']}f'\big(m_{[1,(\lambda')^{-1}]}f(m_{[2,\lambda^{-1}]})_{[1,\gamma(\lambda')^{-1}]}\big)\\
&= m(f\#f').
\end{aligned}
$$

The unit property is handled as follows. For $m\in M_y$, we have

$$me_{\beta(y)}=m_{[0,y]}e_{\beta(y)}(m_{[1,e]})=m_{[0,y]}\varepsilon(m_{[1,e]})1_{\beta(y)}=m.$$

Now we look at the morphisms. Let $(\eta,(\varphi_{y'})_{y'\in Y'})$ be a morphism $\underline{M}\to\underline{M'}$ in $\mathcal{T}_k(\underline{H})_{\underline{A}}^{\underline{C}}$. We then have to show that it is also a morphism $(Y,\oplus_{y\in Y}M_y)\to$

$(Y', \oplus_{y'\in Y'} M'_{y'})$ in $\mathcal{Z}_{\mathcal{A}}^{(G,\Lambda,X)}$. To this end, it suffices to show that the diagrams (17) commute. Take $m \in M_{\eta(y')}$ and $f \in \mathcal{A}_{\lambda,\beta(\eta(y')\lambda)}$. Then

$$\begin{aligned}\varphi_{y'}(m)f = \varphi_{y'}(m)_{[0,y'\lambda]}f(\varphi_{y'}(m)_{[1,\lambda^{-1}]}) &\overset{(3)}{=} \varphi_{y'\lambda}(m_{[0,\eta(y')\lambda]})f(m_{[1,\lambda^{-1}]})\\ &= \varphi_{y'\lambda}\big(m_{[0,\eta(y')\lambda]}f(m_{[1,\lambda^{-1}]})\big) = \varphi_{y'\lambda}(mf).\end{aligned}$$

Finally, take a morphism $(\eta, (\varphi_y)_{y\in Y}) : \ \underline{M} \to \underline{M}'$ in $\mathcal{Z}_k(\underline{H})_{\underline{A}}^{\underline{C}}$. We have to show that $(\eta, \oplus_{y\in Y}$ is a morphism in $\mathcal{Z}_{\mathcal{A}}^{(G,\Lambda,X)}$. To this end, we have to show that (16) commutes. For $m \in M_y$ and $f \in \mathcal{A}_{\lambda,\beta(y\lambda)}$, we have

$$\begin{aligned}\varphi_y(m)f = \varphi_y(m)_{[0,\eta(y\lambda)]}f(\varphi_y(m)_{[1,\lambda^{-1}]}) &\overset{(4)}{=} \varphi_{y\lambda}(m_{[0,y\lambda]})f(m_{[1,\lambda^{-1}]})\\ &= \varphi_{y\lambda}\big(m_{[0,y\lambda]}f(m_{[1,\lambda^{-1}]})\big) = \varphi_{y\lambda}(mf).\end{aligned}$$

□

Theorem 3.6. *Let $(\underline{H}, \underline{A}, \underline{C})$ be a Doi-Hopf datum in $\mathcal{T}_k$, and assume that every C_λ is finitely generated and projective as a k-module. Then the functors T and Z from Proposition 3.5 are isomorphisms of categories.*

Proof. We will construct a functor $G : \ \mathcal{T}_{\mathcal{A}}^{(G,\Lambda,X)} \to \mathcal{T}_k(\underline{H})_{\underline{A}}^{\underline{C}}$ and show that it is the inverse of T. Take $(Y, M) \in \mathcal{M}_{\mathcal{A}}^{(G,\Lambda,X)}$. Let $G(M) = (Y, (M_y)_{y\in Y})$, with structure described as below.

a) M_y is a right $A_{\beta(y)}$-module: $ma = m\alpha_{e,\beta(y)}(\varepsilon\#a)$, for $m \in M_y$, $a \in A_{\beta(y)}$. Let $x = \beta(y)$. It is straightforward to see that this action is associative.

b) Coaction maps $\rho_{y,\lambda} : \ M_{y\lambda} \to M_y \otimes C_\lambda$ are defined as follows:

$$\rho_{y,\lambda}(m) = m\alpha_{\lambda^{-1},\beta(y)}(\xi^{(\lambda)}\#1_{\beta(y)}) \otimes c^{(\lambda)},$$

where we use the notation introduced in 1.6. We have to show that this coaction is coassociative. For $m \in M_{y\lambda\lambda'}$, we have that

$$\begin{aligned}&(\rho_{y,\lambda} \otimes C_{\lambda'})(\rho_{y\lambda,\lambda'}(m))\\ &\quad= (\rho_{y,\lambda} \otimes C_{\lambda'})\big(m\alpha_{(\lambda')^{-1},\beta(y\lambda)}(\xi^{(\lambda')}\#1_{\beta(y\lambda)})\big) \otimes c^{(\lambda')}\\ &\quad= m\alpha_{(\lambda')^{-1},\beta(y\lambda)}(\xi^{(\lambda')}\#1_{\beta(y\lambda)})\alpha_{\lambda^{-1},\beta(y)}(\xi^{(\lambda)}\#1_{\beta(y)}) \otimes c^{(\lambda)} \otimes c^{(\lambda')}\\ &\quad= m\alpha_{(\lambda\lambda')^{-1},\beta(y)}(\xi^{(\lambda')}\xi^{(\lambda)}\#1_{\beta(y)}) \otimes c^{(\lambda)} \otimes c^{(\lambda')};\\ &(M_y \otimes C_{\lambda,\lambda'})(\rho_{y,\lambda\lambda'}(m))\\ &\quad= m\alpha_{(\lambda\lambda')^{-1},\beta(y)}(\xi^{(\lambda\lambda')}\#1_{\beta(y)}) \otimes \Delta_{\lambda,\lambda'}(c^{(\lambda\lambda')}).\end{aligned}$$

These expressions are equal since

$$\begin{aligned}\xi^{(\lambda')}\xi^{(\lambda)} \otimes c^{(\lambda)} \otimes c^{(\lambda')} &= (\xi^{(\lambda')}\xi^{(\lambda)})(c^{(\lambda\lambda')})\xi^{(\lambda\lambda')} \otimes c^{(\lambda)} \otimes c^{(\lambda')}\\ &= \xi^{(\lambda')}(c^{(\lambda\lambda')}_{(2,\lambda')})\xi^{(\lambda)}(c^{(\lambda\lambda')}_{(1,\lambda)})\xi^{(\lambda\lambda')} \otimes c^{(\lambda)} \otimes c^{(\lambda')}\\ &= \xi^{(\lambda\lambda')} \otimes \xi^{(\lambda)}(c^{(\lambda\lambda')}_{(1,\lambda)})c^{(\lambda)} \otimes \xi^{(\lambda')}(c^{(\lambda\lambda')}_{(2,\lambda')})c^{(\lambda')}\\ &= \xi^{(\lambda\lambda')} \otimes c^{(\lambda\lambda')}_{(1,\lambda)} \otimes c^{(\lambda\lambda')}_{(2,\lambda')} = \xi^{(\lambda\lambda')} \otimes \Delta_{\lambda,\lambda'}(c^{(\lambda\lambda')}).\end{aligned}$$

Let us prove that the counit property holds. For $m \in M_y$, we have

$$(M_y \otimes \varepsilon)\rho_{y,e}(m) = m\alpha_{e,\beta(y)}(\xi^{(e)}\#1_{\beta(y)})\varepsilon(c^{(e)}) = m\alpha_{e,\beta(y)}(\varepsilon\#1_{\beta(y)}) = m.$$

Finally, we need to prove that the action and coaction on $\underline{M}$ are compatible, that is,

$$\rho_{y,\lambda}(ma) = m_{[0,y]}a_{[0,\beta(y)]} \otimes m_{[1,\lambda]}a_{[1,\gamma(\lambda)]},$$

for $m \in M_{y\lambda}$ and $a \in A_{\beta(y\lambda)}$.

$$\begin{aligned}
\rho_{y,\lambda}(ma) &= ma\alpha_{\lambda^{-1},\beta(y)}(\xi^{(\lambda)}\#1_{\beta(y)}) \otimes c^{(\lambda)} \\
&= m\alpha_{e,\beta(y\lambda)}(\varepsilon\#a)\alpha_{\lambda^{-1},\beta(y)}(\xi^{(\lambda)}\#1_{\beta(y)}) \otimes c^{(\lambda)} \\
&= m\alpha_{\lambda^{-1},\beta(y)}\big((\varepsilon\#a)(\xi^{(\lambda)}\#1_{\beta(y)})\big) \otimes c^{(\lambda)} \\
&= m\alpha_{\lambda^{-1},\beta(y)}\big((a_{[1,\gamma(\lambda)]}\rightharpoonup\xi^{(\lambda)})\#a_{[0,\beta(y)]}\big) \otimes c^{(\lambda)} \\
&\overset{(1)}{=} m\alpha_{\lambda^{-1},\beta(y)}\big(\xi^{(\lambda)}\#a_{[0,\beta(y)]}\big) \otimes c^{(\lambda)}a_{[1,\gamma(\lambda)]} \\
&= m\alpha_{\lambda^{-1},\beta(y)}(\xi^{(\lambda)}\#1_{\beta(y)})\alpha_{e,\beta(y)}(\varepsilon\#a_{[0,\beta(y)]}) \otimes c^{(\lambda)}a_{[1,\gamma(\lambda)]} \\
&= m_{[0,y]}a_{[0,\beta(y)]} \otimes m_{[1,\lambda]}a_{[1,\gamma(\lambda)]}.
\end{aligned}$$

Let us now show that T and G are inverses. Take $(Y,(M_y)_{y\in Y}) \in \mathcal{T}_k(\underline{H})^{\underline{C}}_{\underline{A}}$. Then $GT(Y,(M_y)_{y\in Y}) = (Y,(M_y)_{y\in Y})$, where every M_y is a right $A_{\beta(y)}$-module; this new action is denoted $\cdot$, and we prove that it coincides with the original one: for $m \in M_y$ and $a \in A_{\beta(y)}$, we have

$$m{\cdot}a = m\alpha_{e,\beta(y)}(\varepsilon\#a) = m_{[0,y]}(\alpha_{e,\beta(y)}(\varepsilon\#a))(m_{[1,e]}) = m_{[0,y]}\varepsilon(m_{[1,e]})a = ma.$$

We also have to show that the coaction maps $\tilde\rho_{y,\lambda}$ on $GT(Y,(M_y)_{y\in Y})$ coincide with the original $\rho_{y,\lambda}$ on $(Y,(M_y)_{y\in Y})$. For all $m \in M_{y\lambda}$, we have

$$\begin{aligned}
\tilde\rho_{y,\lambda}(m) &= m\alpha_{\lambda^{-1},\beta(y)}(\xi^{(\lambda)}\#1_{\beta(y)}) \otimes c^{(\lambda)} \\
&= m_{[0,y]}\alpha_{\lambda^{-1},\beta(y)}(\xi^{(\lambda)}\#1_{\beta(y)})(m_{[1,\lambda]}) \otimes c^{(\lambda)} \\
&= m_{[0,y]}\xi^{(\lambda)}(m_{[1,\lambda]})1_{\beta(y)} \otimes c^{(\lambda)} = m_{[0,y]} \otimes m_{[1,\lambda]} = \rho_{y,\lambda}(m).
\end{aligned}$$

Now let $(Y,M) \in \mathcal{T}_{\mathcal{A}}^{(G,\Lambda,X)}$. Then $TG(Y,M) = (Y,M)$, with new right $\mathcal{A}$-action denoted $\cdot$. In order to show that this new action coincides with the original one, it suffices to show that $m \cdot f = mf$, for all $m \in M_y$ and $f \in \mathcal{A}_{\lambda,\beta(y\lambda)}$ of the form

$$f = \alpha_{\lambda,\beta(y\lambda)}(\xi\#a),$$

where $\xi \in C^*_{\lambda^{-1}}$ and $a \in A_{\beta(y\lambda)}$.

$$\begin{aligned}
m \cdot f &= m_{[0,y\lambda]}f(m_{[1,\lambda^{-1}]}) = m\alpha_{\lambda,\beta(y\lambda)}(\xi^{(\lambda^{-1})}\#1_{\beta(y)})f(c^{(\lambda^{-1})}) \\
&= m\alpha_{\lambda,\beta(y\lambda)}(\xi^{(\lambda^{-1})}\#1_{\beta(y)})\xi(c^{(\lambda^{-1})})a = m\alpha_{\lambda,\beta(y\lambda)}(\xi\#1_{\beta(y)})\alpha_{e,\beta(y\lambda)}(\varepsilon\#a) \\
&= m\alpha_{\lambda,\beta(y\lambda)}(\xi\#a) = mf.
\end{aligned}$$

It is left to the reader to show the result at the level of morphisms. The inverse H of Z is constructed in a similar way. □

4. Yetter-Drinfeld modules and the Drinfeld double

4.1. Crossed G-sets. Let G be a group. Recall that a right crossed G-set is a G-set V together with a map $\nu: V \to G$ such that

$$\nu(vg) = g^{-1}\nu(v)g = \nu(v)^g,$$

for all $v \in V$ and $g \in G$. This notion goes back to Whitehead, and it can be reformulated as follows. Observe first that G is a right $G \times G$-set, with action

$$l \cdot (g, g') = g^{-1} l g'.$$

The diagonal map $\gamma :\ G \to G \times G$ is clearly a morphism of monoids. Hence $\mathbb{G} = (G \times G, G, G)$ is a discrete Doi-Hopf datum. Then it is easy to see that a right crossed G-set is the same thing as a $\mathbb{G}$-set. The category $\mathcal{X}_G^G$ of right crossed G-modules is a braided monoidal category: for two crossed G-modules (V, ν) and (V, ν'), $(V \times V', \omega)$, with $\omega(v, v') = \nu(v)\nu'(v')$, and $(v, v')g = (vg, v'g)$ is again a crossed G-set. The unit object is the singleton $\{*\}$, as a trivial right G-set, together with the map sending $*$ to the unit element $e \in G$.

The braiding $c_{V,V'} :\ V \times V' \to V' \times V$ and its inverse are given by the following formulas:

$$c_{V,V'}(v, v') = (v', v\nu'(v'))\ ;\ c^{-1}_{V,V'}(v', v) = (v\nu'(v')^{-1}, v').$$

This can be verified directly, see [5] or [7, XIII.1.4]. It is also a consequence of the (folklore) fact that the category of crossed G-sets can be obtained from the category of G-sets using the centre construction, see [2, Sec. 4] for a detailed explanation.

4.2. Hopf group coalgebras. Recall that a Hopf group coalgebra is a semi-Hopf group coalgebra $\underline{H}$ (as in 1.7), such that the underlying monoid G is a group, together with maps S_g, $\overline{S}_g :\ H_{g^{-1}} \to H_g$ ($g \in G$) such that

$$\begin{aligned} S_g(h_{(1,g^{-1})})h_{(2,g)} &= h_{(1,g)}S_g(h_{(2,g^{-1})}) = \varepsilon(h)1_g, \\ h_{(2,g)}\overline{S}_g(h_{(1,g^{-1})}) &= \overline{S}_g(h_{(2,g^{-1})})h_{(1,g)} = \varepsilon(h)1_g, \end{aligned}$$

for all $g \in G$ and $h \in H_e$. The S_g are called the antipode maps, while the $\overline{S}_g$ are called the twisted antipode maps. The $\overline{S}_g$ are then the antipode maps of the opposite Hopf group coalgebra $\underline{H}^{\rm op}$, which is defined as follows: $H_g^{\rm op} = H_g$, with opposite multiplication, and $\Delta^{\rm op}_{g,g'} = \Delta_{g,g'}$. For all $g \in G$, $\overline{S}_g$ is the inverse of $S_{g^{-1}}$ and, according to [13], they always exist in the case when each H_g is finite dimensional (G is arbitrary).

4.3. Yetter-Drinfeld modules. Let $\underline{H}$ be a semi-Hopf group coalgebra. Right-right $\underline{H}$-Yetter-Drinfeld modules were introduced in [2, Def. 4.4]. We recall this definition in the special case where $\underline{H}$ is a Hopf group coalgebra. We need an object $\underline{M} = (V, (M_v)_{v\in V}) \in \mathcal{T}_k$, with V a crossed right G-set (G a group), together with the following structure:

- every M_v is a right $H_{\nu(v)}$-module;
- $\underline{M}$ is a right $\underline{H}$-comodule, with coaction maps $\rho_{v,g} :\ M_{vg} \to M_v \otimes H_g$.

The following compatibility condition has to be satisfied

$$\rho_{v,g}(mh) = m_{[0,v]}h_{(2,\nu(v))} \otimes S_g(h_{(1,g^{-1})})m_{[1,g]}h_{(3,g)}, \tag{19}$$

for all $m \in M_{vg}$ and $h \in H_{\nu(vg)} = H_{g^{-1}\nu(v)g}$. $\mathcal{YDT}^{\underline{H}}_{\underline{H}}$ is the category of right-right $\underline{H}$-Yetter-Drinfeld modules and morphisms that are morphisms in $\mathcal{T}^{\underline{H}}$ and $\mathcal{T}_{\underline{H}}$.

The category $\mathcal{YDZ}_{\underline{H}}^{\underline{H}}$ is introduced in a similar way; the objects coincide with the objects of $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$, and the morphisms have to be morphisms $\mathcal{Z}^{\underline{H}}$ and $\mathcal{Z}_{\underline{H}}$.

4.4. A Doi-Hopf datum. Let $\underline{H}$ be a Hopf group coalgebra. Then $\underline{H}^{\rm op}\otimes\underline{H}$ is also a Hopf group coalgebra. Then $\underline{H}$ is a right $\underline{H}^{\rm op}\otimes\underline{H}$-comodule algebra, with structure maps

$$\rho_{l,(g,g')}:\ H_{g^{-1}lg'}\to H_l\otimes(H_g\otimes H_{g'})$$

given by

$$\rho_{l,(g,g')}(h)=h_{(2,l)}\otimes S_g(h_{(1,g^{-1})})\otimes h_{(3,g')}.$$

A technical but straightforward computation shows that the coassociativity and counit properties hold.
$\underline{H}$ is a right $\underline{H}^{\rm op}\otimes\underline{H}$-module coalgebra. Indeed, for $g\in G$, $\gamma(g)=g\otimes g$ and H_g is a right $H_g^{\rm op}\otimes H_g$-module, with action $k(h\otimes h')=hkh'$.
We conclude that $(\underline{H}^{\rm op}\otimes\underline{H},\underline{H},\underline{H})$ is a Doi-Hopf datum in $\mathcal{T}_k$.

Proposition 4.5. *For a Hopf group coalgebra $\underline{H}$, the categories $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$ (resp. $\mathcal{YDZ}_{\underline{H}}^{\underline{H}}$) and $\mathcal{T}_k(\underline{H}^{\rm op}\otimes\underline{H})_{\underline{H}}^{\underline{H}}$ (resp. $\mathcal{Z}_k(\underline{H}^{\rm op}\otimes\underline{H})_{\underline{H}}^{\underline{H}}$) are isomorphic.*

Proof. Objects in $\mathcal{T}_k(\underline{H}^{\rm op}\otimes\underline{H})_{\underline{H}}^{\underline{H}}$ and $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$ are objects $\underline{M}\in\mathcal{T}_k$ with a right $\underline{H}$-action and a right $\underline{H}$-coaction. We have to show that the compatibility relations in both categories are the same.
Let $\underline{M}=(V,(M_v)_{v\in V})\in\mathcal{T}_k$, and assume that V is a right crossed G-set, M_v is a right $H_{\nu(v)}$-module, for all $v\in V$, and $\underline{\rho}:\ \underline{M}\to\underline{M}\otimes\underline{H}$ is a right $\underline{H}$-coaction. Then we have maps $\rho_{v,g}:\ M_{vg}\to M_v\otimes H_g$. The compatibility relation (2) now takes the following form: for all $m\in M_{vg}$ and $h\in H_{\nu(vg)}$, we have

$$\rho_{v,g}(mh)=m_{[0,v]}h_{[0,\nu(v)]}\otimes m_{[1,g]}h_{[1,\gamma(g)]}. \tag{20}$$

Now

$$\rho_{\nu(v),\gamma(g)}(h)=h_{(2,\nu(v))}\otimes(S_g(h_{(1,g^{-1})})\otimes h_{(3,g)}),$$

so (20) is equivalent to (19), as needed. The statement at the level of morphisms is left to the reader. □

4.6. Now assume that H_λ is finitely generated and projective as a k-module, for every $\lambda\in G$. Combining Proposition 4.5 and Theorem 3.6, we find a $\mathbb{G}$-graded algebra $D(\underline{H})$ such that the categories $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$ and $\mathcal{T}_{D(\underline{H})}^{\mathbb{G}}$, resp. $\mathcal{YDZ}_{\underline{H}}^{\underline{H}}$ and $\mathcal{Z}_{D(\underline{H})}^{\mathbb{G}}$, are isomorphic. $D(\underline{H})$ is called the Drinfeld double of $\underline{H}$, and can be described in two isomorphic ways: as a smash product or as a Koppinen smash product. A straightforward computation based on our previous results leads to these constructions.

<u>Smash product.</u>

$$D(\underline{H})=\bigoplus_{\lambda\in G}\bigoplus_{g\in G}H^*_{\lambda^{-1}}\#H_g.$$

We describe the multiplication on $D(\underline{H})$. Let $\xi\in H^*_{\lambda^{-1}}$, $\xi'\in H^*_{\lambda'^{-1}}$, $h\in H_g$ and $h'\in H_{g'}$. Also assume that $g'=g\gamma(\lambda')=\lambda'^{-1}g\lambda'=g^{\lambda'}$.

First recall the following notation. For $k, k', l \in H_{\lambda^{-1}}$, $k \rightharpoonup \xi \leftharpoonup k' \in H^*_{\lambda^{-1}}$ is defined by

$$(k \rightharpoonup \xi \leftharpoonup k')(l) = \xi(k' l k).$$

Then

$$(\xi \# h)(\xi' \# h') = \xi(h_{(3,\lambda'^{-1})} \rightharpoonup \xi' \leftharpoonup S_{\lambda'^{-1}}(h_{(1,\lambda')}))\# h_{(2,g')} h'.$$

<u>Koppinen smash product.</u>

$$D(\underline{H}) = \bigoplus_{\lambda \in G} \bigoplus_{g \in G} \mathrm{Hom}(H_{\lambda^{-1}}, H_g).$$

Take $f : H_{\lambda^{-1}} \to H_g$, $f' : H_{\lambda'^{-1}} \to H_{g'}$, with $g' = g^{\lambda'}$. Then $f \# f' : H_{(\lambda\lambda')^{-1}} \to H_{g'}$ is defined by

$$\begin{aligned}(f\#f')(h) &= f(h_{(2,\lambda^{-1})})_{(2,g')} \\ &\quad f'\Big(S_{\lambda'^{-1}}\big(f(h_{(2,\lambda^{-1})})_{(1,\lambda')}\big) h_{(1,\lambda'^{-1})} f(h_{(2,\lambda^{-1})})_{(3,(\lambda')^{-1})}\Big).\end{aligned}$$

5. $\mathbb{G}$-graded bialgebras

Definition 5.1. Let $A = \oplus_{\lambda, g \in G} A_{\lambda,g}$ be a $\mathbb{G}$-graded algebra. We call A a $\mathbb{G}$-graded bialgebra if we have the following additional structure on A: for every $\lambda \in G$, $(G, (A_{\lambda,g})_{g\in G})$ is a semi-Hopf group coalgebra, with structure maps

$$\Delta_{\lambda,g,g_1} : A_{\lambda,gg_1} \to A_{\lambda,g} \otimes A_{\lambda,g_1} \ ; \ \varepsilon_\lambda : A_{\lambda,e} \to k,$$

such that the following compatibility conditions hold:

$$\Delta_{\lambda\lambda', g^{\lambda'}, g_1^{\lambda'}}(aa') = a_{(1,g)} a'_{(1,g^{\lambda'})} \otimes a_{(2,g_1)} a'_{(2,g_1^{\lambda'})}, \tag{21}$$

for all $a \in A_{\lambda,gg_1}$, $a' \in A_{\lambda', g^{\lambda'} g_1^{\lambda'}}$;

$$\varepsilon_{\lambda\lambda'}(aa') = \varepsilon_\lambda(a)\varepsilon_{\lambda'}(a'), \tag{22}$$

for all $a \in A_{\lambda,e}$, $a' \in A_{\lambda',e}$;

$$\Delta_{e,gg_1}(1_{gg_1}) = 1_g \otimes 1_{g_1}; \tag{23}$$

$$\varepsilon_e(1_e) = 1. \tag{24}$$

Definition 5.1 has a monoidal justification, similar to the monoidal justification of the definition of a bialgebra. Let $\mathcal{C}$ be the category with objects of the form $(V, M = \oplus_{v\in V} M_v)$, with V a crossed G-set, and every M_v a k-module. A morphism $(V, M) \to (V', M')$ in $\mathcal{C}$ is a couple (η, f), where $\eta : V \to V'$ is a morphism of crossed G-sets and $f : M \to M'$ is a k-linear map such that $f(M_v) \subset M'_{\eta(v)}$. Now $\mathcal{C}$ is a monoidal category. The tensor product is defined as follows

$$(V, M) \otimes (V', M') = (V \times V', M \otimes M'),$$

with $M \otimes M' = \oplus_{(v,v')\in V\times V'} M_v \otimes M_{v'}$. The unit object is $(\{*\}, k)$.
Now let A be a $\mathbb{G}$-graded algebra, and consider the forgetful functor

$$U : \mathcal{Z}_A^{\mathbb{G}} \to \mathcal{C}.$$

Theorem 5.2. *Let G be a group, and A a $\mathbb{G}$-graded algebra. We have a bijective correspondence between*

- *monoidal structures on $\mathcal{Z}_A^{\mathbb{G}}$ such that the forgetful functor U is strictly monoidal;*
- *$\mathbb{G}$-graded bialgebra structures on A.*

Proof. Assume that we have a monoidal structure on $\mathcal{Z}_A^{\mathbb{G}}$ such that U is strictly monoidal. We first describe the structure maps ε_λ and Δ_{λ,g,g_1}. Then the unit object is k, with a certain right A-module structure. From Definition 3.1, we know that this action is determined by maps

$$\varepsilon_\lambda : \ k \otimes A_{\lambda,e} = A_{\lambda,e} \to k,$$

otherwise stated

$$\varepsilon_\lambda(a) = 1_k \cdot a, \tag{25}$$

for $a \in A_{\lambda,e}$. From Example 3.3, we know that

$$(G \times G, \bigoplus_{\lambda,g\in G} A_{\lambda,g}) \in \mathcal{Z}_A^{\mathbb{G}}.$$

As an object in $\mathcal{C}$,

$$(G \times G, A) \otimes (G \times G, A) = (G \times G \times G \times G, \bigoplus_{\lambda,\lambda',g,g'\in G} A_{\lambda,g} \otimes A_{\lambda',g'}).$$

The crossed G-set structure on $G \times G \times G \times G$ is the following:

$$\omega(\lambda, g, \lambda', g') = gg' \ ; \ (\lambda, g, \lambda', g')\lambda'' = (\lambda\lambda'', g^{\lambda''}, \lambda'\lambda'', g'^{\lambda''}).$$

Now we have a right A-module structure on $A \otimes A$. According to Definition 3.1, this is given by multiplication maps

$$(A_{\lambda,g} \otimes A_{\lambda',g'}) \otimes A_{\lambda'',g^{\lambda''}g'^{\lambda''}} \to A_{\lambda\lambda'',g^{\lambda''}} \otimes A_{\lambda'\lambda'',g'^{\lambda''}}.$$

Take $\lambda = \lambda' = e$, and replace λ'' by λ; this gives multiplication maps

$$\psi_{\lambda,g,g'} : \ (A_{e,g} \otimes A_{e,g'}) \otimes A_{\lambda,g^\lambda g'^\lambda} \to A_{\lambda,g^\lambda} \otimes A_{\lambda,g'^\lambda}.$$

Now we define $\Delta_{\lambda,g,g'} : \ A_{\lambda,gg'} \to A_{\lambda,g} \otimes A_{\lambda,g'}$ as follows:

$$\Delta_{\lambda,g,g'}(a) = \psi_{\lambda,g^{\lambda^{-1}},g'^{\lambda^{-1}}}\big((1_{g^{\lambda^{-1}}} \otimes 1_{g'^{\lambda^{-1}}}) \otimes a\big) = (1_{g^{\lambda^{-1}}} \otimes 1_{g'^{\lambda^{-1}}})a. \tag{26}$$

For later use, observe that, for $a \in A_{\lambda,g^\lambda,g'^\lambda}$,

$$\Delta_{\lambda,g^\lambda,g'^\lambda}(a) = (1_g \otimes 1_{g'})a. \tag{27}$$

We now have to show that the maps ε_λ and $\Delta_{\lambda,g,g'}$ satisfy the conditions of Definition 5.1. Before we do this, we show that the right A-action on $M \otimes N$ is completely determined by the maps $\Delta_{\lambda,g,g'}$, for all $M, N \in \mathcal{Z}_A^{\mathbb{G}}$. We proceed as follows.

Let $(V, M) \in \mathcal{Z}_A^{\mathbb{G}}$, and fix elements $v \in V$ and $m \in M_v$. Recall from 1.2 that $G \times G$ is a crossed G-set, with structure maps

$$(\lambda, g)\lambda' = (\lambda\lambda', g^{\lambda'}) \ ; \ \beta(\lambda, g) = g.$$

In Example 3.3, we have seen that

$$Z_{\nu(v)} = \{(\lambda, \nu(v)^\lambda) \mid \lambda \in G\}$$

is a crossed G-subset of $G \times G$ and that

$$\big(Z_{\nu(v)}, A^{(\nu(v))} = \bigoplus_{\lambda\in G} A_{\lambda,\nu(v)^\lambda}\big) \in \mathcal{Z}_A^{\mathbb{G}}.$$

Now $\eta:\ Z_{\nu(v)}\to V,\ \eta(\lambda,\nu(v)^\lambda)=v\lambda$ is a morphism of crossed G-sets. Indeed,

$$\eta((\lambda,\nu(v)^\lambda)\lambda')=\eta(\lambda\lambda',\nu(v)^{\lambda\lambda'})=v\lambda\lambda'=\eta(\lambda,\nu(v)^\lambda)\lambda',$$

and

$$(\nu\circ\eta)(\lambda,\nu(v)^\lambda)=\nu(v\lambda)=\nu(v)^\lambda=\beta(\lambda,\nu(v)^\lambda).$$

Now we define

$$f_m:\ A^{(\nu(v))}\to M,\ f_m(a)=ma.$$

It follows from (12) that $M_vA_{\lambda,\nu(v)^\lambda}\subset M_{v\lambda}$, so

$$f_m(A_{\lambda,\nu(v)^\lambda})\subset M_{v\lambda},$$

and $(\eta,f_m):\ (Z_{\nu(v)},A^{(\nu(v))})\to(V,M)$ is a morphism in $\mathcal{M}_A^{\mathbb{G}}$.
Now take $(V',N)\in\mathcal{M}_A^{\mathbb{G}}$, fix $v'\in V'$ and $n\in N_{v'}$, and repeat the above construction. We obtain a morphism $(\eta',g_n):\ (Z_{\nu'(v')},A^{(\nu'(v'))})\to(V',N)$ in $\mathcal{M}_A^{\mathbb{G}}$.
From the functoriality of the tensor product, it follows that $(\eta\eta',f_m\otimes g_n)$ is a morphism in $\mathcal{Z}_A^{\mathbb{G}}$, in particular, $f_m\otimes g_n$ is right A-linear. Now take $a\in A_{\lambda,\nu(v)^\lambda\nu'(v')^\lambda}$. Since $f_m(1_{\nu(v)})=m$ and $g_n(1_{\nu'(v')})=n$, we find

$$\begin{aligned}(m\otimes n)a &= \big((f_m\otimes g_n)(1_{\nu(v)}\otimes 1_{\nu'(v')})\big)a\\ &= (f_m\otimes g_n)\big((1_{\nu(v)}\otimes 1_{\nu'(v')})a\big)\\ &\overset{(27)}{=} (f_m\otimes g_n)(\Delta_{\lambda,\nu(v)^\lambda,\nu'(v')^\lambda}(a))\\ &= ma_{(1,\nu(v)^\lambda)}\otimes na_{(2,\nu'(v')^\lambda)}.\end{aligned}\tag{28}$$

We are now ready to show that each A_λ is a semi-Hopf group coalgebra. The (trivial) associativity constraint $a_{A,A,A}$

$$\big((G\times G,A)\otimes(G\times G,A)\big)\otimes(G\times G,A)\to(G\times G,A)\otimes\big((G\times G,A)\otimes(G\times G,A)\big)$$

is a morphism in $\mathcal{Z}_A^{\mathbb{G}}$; in particular $a_{A,A,A}$ is right A-linear. For all $a\in A_{\lambda,gg'g''}$, we have that

$$\begin{aligned}\big(1_{g^{\lambda^{-1}}}\otimes(1_{g'^{\lambda^{-1}}}\otimes 1_{g''^{\lambda^{-1}}})\big)a &\overset{(28)}{=} a_{(1,g)}\otimes(1_{g'^{\lambda^{-1}}}\otimes 1_{g''^{\lambda^{-1}}})a_{(2,g'g'')}\\ &= a_{(1,g)}\otimes\Delta_{\lambda,g'g''}(a_{(2,g'g'')})\end{aligned}$$

equals

$$\begin{aligned}&a_{A,A,A}\big((1_{g^{\lambda^{-1}}}\otimes 1_{g'^{\lambda^{-1}}})\otimes 1_{g''^{\lambda^{-1}}}\big)a\\ &\quad= a_{A,A,A}\Big(\big((1_{g^{\lambda^{-1}}}\otimes 1_{g'^{\lambda^{-1}}})\otimes 1_{g''^{\lambda^{-1}}}\big)a\Big)\\ &\quad\overset{(28)}{=} a_{A,A,A}\big((1_{g^{\lambda^{-1}}}\otimes 1_{g'^{\lambda^{-1}}})a_{(1,gg')}\otimes a_{(2,g'')}\big)\\ &\quad= a_{A,A,A}(\Delta_{\lambda,g,g'}(a_{(1,gg')})\otimes a_{(2,g'')}),\end{aligned}$$

which is precisely the required coassociativity condition. Now we prove the counit conditions. The (trivial) left counit constraint $l_A:\ (\{*\},k)\otimes(G\times G,A)\to(G\times G,A)$ is a morphism in $\mathcal{Z}_A^{\mathbb{G}}$, hence l_A is right A-linear. For $a\in A_{\lambda,g}$, we have

$$\begin{aligned}a &= l_A(1_k\otimes 1_{g^\lambda})a=l_A((1_k\otimes 1_{g^\lambda})a)\overset{(28)}{=}l_A(1_k.a_{(1,e)}\otimes 1_{g^\lambda}a_{(2,g)})\\ &= l_A(\varepsilon_\lambda(a_{(1,e)})\otimes a_{(2,g)})=\varepsilon_\lambda(a_{(1,e)})a_{(2,g)},\end{aligned}$$

The right counit property is handled in a similar way. Now let $a \in A_{\lambda,gg_1}$ and $a' \in A_{\lambda',g^{\lambda'}g_1^{\lambda'}}$. Then $aa' \in A_{\lambda\lambda',g^{\lambda'}g_1^{\lambda'}}$ and

$$\begin{aligned}\Delta_{\lambda\lambda',g^{\lambda'},g_1^{\lambda'}}(aa') &= (1_{g^{\lambda^{-1}}} \otimes 1_{g_1^{\lambda^{-1}}})aa' \\ &= (a_{(1,g)} \otimes a_{(2,g_1)})a' \overset{(28)}{=} a_{(1,g)}a'_{(1,g^{\lambda'})} \otimes a_{(2,g_1)}a'_{(2,g_1^{\lambda'})}.\end{aligned}$$

This proves that (21) holds. Now take $a \in A_{\lambda,e}$ and $a' \in A_{\lambda',e}$. Then

$$\varepsilon_{\lambda\lambda'}(aa') = 1_k \cdot (aa') = (1_k \cdot a) \cdot a' = \varepsilon_\lambda(a) \cdot a' = \varepsilon_\lambda(a)\varepsilon_{\lambda'}(a'),$$

proving (22). Finally

$$\Delta_{e,gg'}(1_{gg'}) = (1_g \otimes 1_{g'})1_{gg'} \overset{(13)}{=} 1_g \otimes 1_{g'},$$

and

$$\varepsilon_e(1_e) = 1_k \cdot 1_e \overset{(13)}{=} 1_k.$$

Conversely, assume that A is a $\mathbb{G}$-graded bialgebra. Let $(V, M),\ (V', M') \in \mathcal{Z}_A^{\mathbb{G}}$. We have already seen that $V \times V'$ is again a crossed G-set. Now we define a right A-module structure on $M \otimes M' = \oplus_{(v,v')\in V\times V'} M_v \otimes M'_{v'}$ using (28), which is designed in such a way that $(V \times V', M \otimes M') \in \mathcal{Z}_A^{\mathbb{G}}$. Also $(\{*\}, k) \in \mathcal{Z}_A^{\mathbb{G}}$, using (25). Then straightforward computations show that this makes $\mathcal{Z}_A^{\mathbb{G}}$ into a monoidal category such that the forgetful functor to $\mathcal{C}$ is strictly monoidal.

Let us show that the tensor on $\mathcal{Z}_A^{\mathbb{G}}$ is functorial. Consider morphisms

$$(\eta, \varphi) :\ (V, M) \to (W, N)\ ;\ (\eta', \varphi') :\ (V', M') \to (W', N')$$

in $\mathcal{Z}_A^{\mathbb{G}}$. The diagram (16) takes the form

$$\begin{array}{ccc} M_v \otimes M'_{v'} \otimes A_{\lambda,\nu(v)^\lambda\nu'(v')^\lambda} & \longrightarrow & M_{v\lambda} \otimes M'_{v'\lambda} \\ \downarrow{\scriptstyle \varphi_v\otimes\varphi'_{v'}\otimes id} & & \downarrow{\scriptstyle \varphi_{v\lambda}\otimes\varphi'_{v'\lambda}} \\ N_{\eta(v)} \otimes N'_{\eta'(v')} \otimes A_{\lambda,\nu(v)^\lambda\nu'(v')^\lambda} & \longrightarrow & N_{\eta(v\lambda)} \otimes N'_{\eta'(v'\lambda)} \end{array}$$

and we have to show that it commutes. Since (η, φ) and (η', φ') are morphisms in $\mathcal{Z}_A^{\mathbb{G}}$, we have, for $m \in M_v$, $a_1 \in A_{\lambda,\nu(v\lambda)}$, $m' \in M'_{v'}$ and $a_2 \in A_{\lambda,\nu'(v'\lambda)}$ that

$$\varphi_{v\lambda}(ma_1) = \varphi_v(m)a_1 \text{ and } \varphi'_{v'\lambda}(m'a_2) = \varphi'_{v'}(m)a_2.$$

Now take $a \in A_{\lambda,\nu(v)^\lambda\nu'(v')^\lambda}$. Then we have

$$\begin{aligned}&\varphi_{v\lambda}\big(ma_{(1,\nu(v)^\lambda)}\big) \otimes \varphi'_{v'\lambda}\big(m'a_{(2,\nu'(v')^\lambda)}\big) \\ &\quad= \varphi_v(m)a_{(1,\nu(v)^\lambda)} \otimes \varphi'_{v'}(m')a_{(2,\nu'(v')^\lambda)} = \big((\varphi_v \otimes \varphi'_{v'})(m \otimes m')\big)a,\end{aligned}$$

as needed. □

It would be nice to have a result similar to Theorem 5.2, with the category $\mathcal{Z}_A^{\mathbb{G}}$ replaced by $\mathcal{T}_A^{\mathbb{G}}$. Unfortunately, we were only able to prove it in one direction. First, we need to introduce the category $\mathcal{D}$, which can be viewed as the $\mathcal{T}$-version of $\mathcal{C}$: it has the same objects as $\mathcal{C}$, and a morphism $(V, M) \to (W, N)$ is a couple $(\eta, (\varphi_w)_{w\in W})$, with $\eta :\ W \to V$ a morphism of crossed G-sets, and $\varphi_w :\ M_{\eta(w)} \to N_w$ k-linear, for all $w \in W$.

Proposition 5.3. *Let G be a group, and A a $\mathbb{G}$-graded bialgebra. Then we have a monoidal structure on $\mathcal{T}_A^{\mathbb{G}}$ such that the forgetful functor $\mathcal{T}_A^{\mathbb{G}} \to \mathcal{D}$ is monoidal.*

Proof. As in the proof of Theorem 5.2, we define a right A-module structure on $(V \times V', M \otimes M')$ using (28), and on $(\{*\}, k)$ using (25). Let us show that the tensor product on $\mathcal{T}_A^{\mathbb{G}}$ is functorial. Take morphisms $(\eta, (\varphi_w)_{w\in W}) : (V, M) \to (W, N)$ and $(\eta', (\varphi'_{w'})_{w'\in W'}) : (V', M') \to (W', N')$ in $\mathcal{T}_A^{\mathbb{G}}$. The diagram (17) takes the form

$$\begin{array}{ccc}
M_{\eta(w)} \otimes M'_{\eta'(w')} \otimes A_{\lambda,\nu(\eta(w))^\lambda \nu'(\eta'(w'))^\lambda} & \longrightarrow & M_{\eta(w\lambda)} \otimes M'_{\eta'(w'\lambda)} \\
\downarrow{\scriptstyle \varphi_w \otimes \varphi'_{w'} \otimes id} & & \downarrow{\scriptstyle \varphi_{w\lambda} \otimes \varphi'_{w'\lambda'}} \\
N_w \otimes N'_{w'} \otimes A_{\lambda,\omega(w)^\lambda \omega'(w')^\lambda} & \longrightarrow & N_{w\lambda} \otimes N'_{w'\lambda}
\end{array}$$

and we have to show that it commutes. $(\eta, (\varphi_w)_{w\in W})$ and $(\eta', (\varphi'_{w'})_{w'\in W'})$ are morphisms in $\mathcal{T}_A^{\mathbb{G}}$, so, for all $m \in M_{\eta(w)}$, $a_1 \in A_{\lambda,\nu(\eta(w))^\lambda}$, $m' \in M'_{\eta'(w')}$ and $a_2 \in A_{\lambda,\nu'(\eta'(w'))^\lambda}$, we have that

$$\varphi_{w\lambda}(ma_1) = \varphi_w(m)a_1 \text{ and } \varphi'_{w'\lambda}(m'a_2) = \varphi_{w'}(m')a_2.$$

For $a \in A_{\lambda,\nu(\eta(w))^\lambda \nu'(\eta'(w'))^\lambda}$, we now compute

$$\begin{aligned}
&(\varphi_{w\lambda} \otimes \varphi'_{\omega'\lambda})((m \otimes m')a) \\
&\quad= \varphi_{w\lambda}(ma_{(1,\nu(\eta(w))^\lambda)}) \otimes \varphi'_{w'\lambda}(m'a_{(2,\nu'(\eta'(w'))^\lambda)}) \\
&\quad= \varphi_w(m)a_{(1,\omega(w)^\lambda)} \otimes \varphi'_{w'}(m')a_{(2,\omega'(w')^\lambda)} = (\varphi_w(m) \otimes \varphi'_{w'}(m'))a,
\end{aligned}$$

as needed. □

Let $\underline{H}$ be a Hopf group coalgebra. The category of Yetter-Drinfeld modules $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$ is obtained from the category $\mathcal{T}_{\underline{H}}$ using the center construction, see [2, Sec. 4], and therefore $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$ is a braided monoidal category. For detail on the centre construction, we refer to [7, XIII.4]. We first describe the monoidal structure. Take (V, M), $(V', N) \in \mathcal{YDT}_{\underline{H}}^{\underline{H}}$.

$$(V, M) \otimes (V', N) = (V \times V', (M_v \otimes N_{v'})_{(v,v')\in V\times V'}),$$

with the following structure. We have already seen that $V \times V'$ is a right crossed G-set, with $\omega : V \times V' \to G$, $\omega(v, v') = \nu(v)\nu'(v')$ and $(v, v')g = (vg, v'g)$.

$M_v \otimes N_{v'}$ is a right $H_{\nu(v)\nu'(v')}$-module, with

$$(m \otimes n)h = mh_{(1,\nu(v))} \otimes nh_{(2,\nu'(v'))}; \tag{29}$$

The coaction maps $\rho_{(v,v'),g} : M_{vg} \otimes N_{v'g} \to M_v \otimes N_{v'} \otimes H_g$ are given by

$$\rho_{(v,v'),g}(m \otimes n) = m_{[0,v]} \otimes n_{[0,v']} \otimes m_{[1,g]}n_{[1,g]}. \tag{30}$$

$(\{*\}, k) \in \mathcal{YDT}_{\underline{H}}^{\underline{H}}$; we already know that the singleton $\{*\}$ is a right crossed G-set; furthermore k is an H_e-module via ε, and the coaction maps $\rho_{*,g} : k \to k \otimes H_g$ are given by $\rho_{*,g}(1_k) = 1_k \otimes 1_g$.

Now we describe the braiding. The braiding isomorphism

$$(V \times V', (M_v \otimes N_{v'})_{(v,v')\in V\times V'}) \to (V' \times V, (N_{v'} \otimes M_v)_{(v',v)\in V'\times V})$$

is given by the following data:

$$c_{V',V}:\ V'\times V\to V\times V',\ c_{V',V}(v',v)=(v,v'\nu(v)).$$

$t_{M,N,v',v}:\ M_v\otimes N_{v'\nu(v)}\to N_{v'}\otimes M_v,\ t_{M,N,v',v}(m\otimes n)=n_{[0,v']}\otimes mn_{[1,\nu(v)]}$.
As we have mentioned, this monoidal structure can be deduced from the center construction, but it can also be verified directly that this defines a monoidal structure on $\mathcal{YDT}^{\underline{H}}_{\underline{H}}$.
$\mathcal{YDZ}^{\underline{H}}_{\underline{H}}$ is also a braided monoidal category. The tensor product is defined using (29-30). The braiding isomorphism

$$(V\times V',(M_v\otimes N_{v'})_{(v,v')\in V\times V'})\to(V'\times V,(N_{v'}\otimes M_v)_{(v',v)\in V'\times V})$$

is given by the following data:

$$c^{-1}_{V',V}:\ V\times V'\to V'\times V,\ c_{V',V}(v,v')=(v'\nu(v)^{-1},v);$$

$\tilde{t}_{M,N,v,v'}:\ M_v\otimes N_{v'}\to N_{v'\nu(v)^{-1}}\otimes M_v$ is given by

$$\tilde{t}_{M,N,v,v'}(m\otimes n)=n_{[0,v'\nu(v)^{-1}]}\otimes mn_{[1,\nu(v)]}. \tag{31}$$

We will also need the inverse of the braiding of $(c^{-1},\tilde{t})$

$$(V'\times V,(N_{v'}\otimes M_v)_{(v',v)\in V'\times V})\to(V\times V',(M_v\otimes N_{v'})_{(v,v')\in V\times V'}).$$

This is described by the data

$$c_{V',V}:\ V'\times V\to V\times V,\ c_{V',V}(v',v)=(v,v'\nu(v));$$

$\tilde{q}_{N,M,v',v}:\ N_{v'}\otimes M_v\to M_v\otimes N_{v'\nu(v)}$ is given by the formula

$$\tilde{q}_{N,M,v',v}(n\otimes m)=m\overline{S}_{\nu(v)}(n_{[1,\nu(v)^{-1}]})\otimes n_{[0,v'\nu(v)]}. \tag{32}$$

If $(V,(M_v)_{v\in V})\in\mathcal{YDZ}^{\underline{H}}_{\underline{H}}$, then it is easy to see that $(V,M=\oplus_{v\in V}M_v)\in\mathcal{C}$: every M_v is a k-module. Thus we have a forgetful functor $U':\ \mathcal{YDZ}^{\underline{H}}_{\underline{H}}\to\mathcal{C}$, and it is clear that U' is strictly monoidal.
Now assume that every H_g is finitely generated and projective as a k-module. Then we have an isomorphism of categories Z (Theorem 3.6) and a forgetful functor U as in Theorem 5.2 such that the diagram of functors

$$\begin{array}{ccc}\mathcal{YDZ}^{\underline{H}}_{\underline{H}} & \xrightarrow{\ Z\ } & \mathcal{Z}^{\mathbb{G}}_{D(\underline{H})}\\ \big\downarrow{\scriptstyle U'} & \swarrow{\scriptstyle U} & \\ \mathcal{C} & & \end{array}$$

commutes. It follows from all these observations that $\mathcal{Z}^{\mathbb{G}}_{D(\underline{H})}$ is a monoidal category and that U is strictly monoidal. Then it follows from Theorem 5.2 that $D(\underline{H})$ is a $\mathbb{G}$-graded bialgebra.
Our aim is now to construct the comultiplication and counit maps on $D(\underline{H})$. We know that $(G\times G,D(\underline{H}))\in\mathcal{Z}^{\mathbb{G}}_{D(\underline{H})}$, see Example 3.3. From Theorem 3.6 and Proposition 4.5, we know that $H(G\times G,D(\underline{H}))=(G\times G,D(\underline{H})_{(\lambda,g)\in G\times G})\in\mathcal{YDZ}^{\underline{H}}_{\underline{H}}$. We compute the structure maps, using the proof of Theorem 5.2.
First, every $D(\underline{H})_{(\lambda,g)}=H^*_{\lambda^{-1}}\#H_g$ is a right H_g-module in the obvious way:

$(\xi\# h)h' = \xi\# hh'$.
The coaction maps

$$\rho_{(\lambda,g),\lambda'}:\ H^*_{(\lambda\lambda')^{-1}}\# H_{g^{\lambda'}} \to (H^*_{\lambda^{-1}}\# H_g)\otimes H_{\lambda'}$$

are given by

$$\rho_{(\lambda,g),\lambda'}(\xi\# h) = (\xi\# h)(\xi^{(\lambda')}\# 1_g)\otimes h^{(\lambda')}, \tag{33}$$

where we use the notation introduced in 2.10: $\xi^{(\lambda)}\otimes h^{(\lambda)}$ is a finite dual basis of H_λ. H_λ is a finitely projective algebra, hence H^*_λ is a coalgebra, with comultiplication

$$\Delta(\xi) = \xi_{(1)}\otimes\xi_{(2)} = \langle \xi, h^{(\lambda)}\overline{h}^{(\lambda)}\rangle \xi^{(\lambda)}\otimes\overline{\xi}^{(\lambda)}, \tag{34}$$

where $\overline{\xi}^{(\lambda)}\otimes\overline{h}^{(\lambda)} = \xi^{(\lambda)}\otimes h^{(\lambda)}$ is a second copy of the dual basis of H_λ. Since $\mathcal{YDZ}^{\underline{H}}_{\underline{H}}$ is monoidal, we have that

$$(G\times G\times G\times G, (D(\underline{H})_{\lambda,g}\otimes D(\underline{H})_{\lambda',g'})\in \mathcal{YDZ}^{\underline{H}}_{\underline{H}},$$

and we compute

$$\begin{aligned}&\rho_{(\lambda,g,\lambda,g'),\lambda^{-1}}\big((\varepsilon\# 1_{g^{\lambda^{-1}}})\otimes(\varepsilon\# 1_{g'^{\lambda^{-1}}})\big)\\ &\overset{(30)}{=} (\xi^{(\lambda^{-1})}\# 1_g)\otimes(\overline{\xi}^{(\lambda^{-1})}\# 1_{g'})\otimes h^{(\lambda^{-1})}\overline{h}^{(\lambda^{-1})}.\end{aligned}$$

Now we apply (26) to compute $\Delta_{\lambda,g,g'}:\ D(\underline{H})_{\lambda,gg'}\to D(\underline{H})_{\lambda,g}\otimes D(\underline{H})_{\lambda,g'}$: for $\xi\in H^*_{\lambda^{-1}}$ and $h\in H_{gg'}$, we have

$$\begin{aligned}\Delta_{\lambda,g,g'}(\xi\# h) &\overset{(26)}{=} \big((\varepsilon\# 1_{g^{\lambda^{-1}}})\otimes(\varepsilon\# 1_{g'^{\lambda^{-1}}})\big)(\xi\# h)\\ &= (\xi^{(\lambda^{-1})}\# 1_g)\otimes(\overline{\xi}^{(\lambda^{-1})}\# 1_{g'})\langle\xi, h^{(\lambda^{-1})}\overline{h}^{(\lambda^{-1})}\rangle h\\ &\overset{(29,34)}{=} (\xi_{(1)}\# h_{(1,g)})\otimes(\xi_{(2)}\# h_{(2,g')}).\end{aligned}$$

Now we compute the counit maps $\varepsilon_\lambda:\ H^*_{\lambda^{-1}}\# H_e\to k$. k is a right $H^*_{\lambda^{-1}}\# H_e$-module, and

$$\varepsilon_\lambda(\xi\# h)\overset{(25)}{=} 1_k\cdot(\xi\# h) = \xi(1_{\lambda^{-1}})\varepsilon(h),$$

since $\rho_{*,\lambda^{-1}}(1_k) = 1_k\otimes 1_{\lambda^{-1}}$. We conclude our computations as follows.

Proposition 5.4. *Let $\underline{H}$ be a Hopf group coalgebra, and assume that every H_g is finitely generated and projective as a k-module. Then $D(\underline{H})$ is a $\mathbb{G}$-graded bialgebra, with structure maps*

$$\begin{aligned}&\Delta_{\lambda,g,g'}:\ D(\underline{H})_{\lambda,gg'}\to D(\underline{H})_{\lambda,g}\otimes D(\underline{H})_{\lambda,g'},\\ &\qquad\Delta_{\lambda,g,g'}(\xi\# h) = (\xi_{(1)}\# h_{(1,g)})\otimes(\xi_{(2)}\# h_{(2,g')});\\ &\varepsilon_\lambda:\ H^*_{\lambda^{-1}}\# H_e\to k,\ \varepsilon_\lambda(\xi\# h) = \xi(1_{\lambda^{-1}})\varepsilon(h).\end{aligned}$$

Recall from 4.6 that $D(\underline{H})$ can also be written as a Koppinen smash product. Then the comultiplication maps

$$\Delta_{\lambda,g,g'}:\ \mathrm{Hom}(H_{\lambda^{-1}}, H_{gg'})\to \mathrm{Hom}(H_{\lambda^{-1}}, H_g)\otimes\mathrm{Hom}(H_{\lambda^{-1}}, H_{g'})$$

can be characterized as follows: $\Delta_{\lambda,g,g'}(f) = f_{(1)}\otimes f_{(2)}$ if and only if

$$\Delta_{g,g'}(f(h_1h_2)) = f(h_1)_{(1,g)}\otimes f(h_2)_{(2,g')},$$

for all $h_1,h_2\in H^*_{\lambda^{-1}}$. The counit maps are the following:

$$\varepsilon_\lambda:\ \mathrm{Hom}(H_{\lambda^{-1}}, H_e)\to k,\ \varepsilon_\lambda(f) = (\varepsilon\circ f)(1_{\lambda^{-1}}).$$

Let A be $\mathbb{G}$-graded bialgebra. From Theorem 5.2 and Proposition 5.3, it follows that a monoidal structure on $\mathcal{Z}_A^{\mathbb{G}}$ such that U is strictly monoidal induces a monoidal structure on $\mathcal{T}_A^{\mathbb{G}}$. The tensor product of objects coincides in both categories.

On $\mathcal{Z}_{D(\underline{H})}^{\mathbb{G}}$, we have the monoidal structure transported using the category isomorphism with $\mathcal{YDZ}_{\underline{H}}^{\underline{H}}$. We have also a monoidal structure arising from the $\mathbb{G}$-graded bialgebra structure on $D(\underline{H})$, using Theorem 5.2. These two monoidal structures coincide, actually this is the way the $\mathbb{G}$-graded bialgebra structure on $D(\underline{H})$ is constructed in the proof of Proposition 5.4.

The monoidal structure on $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$ can be transported to a monoidal structure on $\mathcal{T}_{D(\underline{H}}^{\mathbb{G}}$. It follows easily from our previous constructions that this monoidal structure is induced from the monoidal structure on $\mathcal{Z}_{D(\underline{H}}^{\mathbb{G}}$. Hence this monoidal structure coincide with the monoidal structures arising from the $\mathbb{G}$-graded bialgebra structure on $D(\underline{H})$, using Proposition 5.3. We summarize these observations as follows.

Theorem 5.5. *The $\mathbb{G}$-graded bialgebra structure on $D(\underline{H})$ from Proposition 5.4 defines a monoidal algebra structure on $\mathcal{Z}_{D(\underline{H})}^{\mathbb{G}}$ and $\mathcal{T}_{D(\underline{H})}^{\mathbb{G}}$ (Theorem 5.2 and Proposition 5.3) that are such that the category isomorphisms $\mathcal{Z}_{D(\underline{H})}^{\mathbb{G}} \cong \mathcal{YDZ}_{\underline{H}}^{\underline{H}}$ and $\mathcal{T}_{D(\underline{H})}^{\mathbb{G}} \cong \mathcal{YDT}_{\underline{H}}^{\underline{H}}$ are isomorphisms of monoidal categories.*

6. $\mathbb{G}$-graded Hopf algebras

Definition 6.1. Let $A = \oplus_{\lambda,g\in G} A_{\lambda,g}$ be a $\mathbb{G}$-graded bialgebra. We call A a $\mathbb{G}$-graded Hopf algebra if there exist maps

$$S_{\lambda,g}, \overline{S}_{\lambda,g} :\ A_{\lambda,g^{-1}} \to A_{\lambda^{-1},g^{\lambda^{-1}}}$$

such that

$$a_{(1,g)}S_{\lambda,g}(a_{(2,g^{-1})}) = a_{(2,g)}\overline{S}_{\lambda,g}(a_{(1,g^{-1})}) = \varepsilon_\lambda(a)1_{g^{\lambda^{-1}}}; \tag{35}$$

$$S_{\lambda,g}(a_{(1,g^{-1})})a_{(2,g)} = \overline{S}_{\lambda,g}(a_{(2,g^{-1})})a_{(1,g)} = \varepsilon_\lambda(a)1_g, \tag{36}$$

for all $a \in A_{\lambda,e}$. The $S_{\lambda,g}$ ($\overline{S}_{\lambda,g}$) are called the (twisted) antipode maps.

Proposition 6.2. *Let $\underline{H}$ be a Hopf group coalgebra, and assume that every H_g is finitely generated and projective as a k-module. Then $D(\underline{H})$ is a $\mathbb{G}$-graded Hopf algebra, with (twisted) antipode maps*

$$S_{\lambda,g}, \overline{S}_{\lambda,g} :\ H^*_{\lambda^{-1}}\#H_{g^{-1}} \to H^*_\lambda\#H_{g^{\lambda^{-1}}},$$

$$\begin{aligned} S_{\lambda,g}(\xi\#h) &= (\varepsilon\#S_g(h))(\xi\circ\overline{S}_{\lambda^{-1}}\#1_{g^{\lambda^{-1}}}); \\ \overline{S}_{\lambda,g}(\xi\#h) &= (\varepsilon\#\overline{S}_g(h))(\xi\circ S_{\lambda^{-1}}\#1_{g^{\lambda^{-1}}}). \end{aligned}$$

Proof. For $\xi \in H^*_{\lambda^{-1}}$ and $h \in H_e$, we have

$$\begin{aligned} &(\xi_{(1)}\#h_{(1,g)})S_{\lambda,g}(\xi_{(2)}\#h_{(2,g^{-1})}) \\ &= (\xi_{(1)}\#h_{(1,g)})(\varepsilon\#S_g(h_{(2,g^{-1})}))(\xi_{(2)}\circ\overline{S}_{\lambda^{-1}}\#1_{g^{\lambda^{-1}}}) \\ &= (\xi_{(1)}\#h_{(1,g)}S_g(h_{(2,g^{-1})}))(\xi_{(2)}\circ\overline{S}_{\lambda^{-1}}\#1_{g^{\lambda^{-1}}}) \end{aligned}$$

$$\begin{aligned} &= \varepsilon(h)(\xi_{(1)}\#1_g)(\xi_{(2)}\circ\overline{S}_{\lambda^{-1}}\#1_{g^{\lambda^{-1}}})\\ &= \varepsilon(h)\xi(1_{\lambda^{-1}})(\varepsilon\#1_{g^{\lambda^{-1}}}) = \varepsilon_\lambda(\xi\#h)(\varepsilon\#1_{g^{\lambda^{-1}}}),\end{aligned}$$

where we used the following property, for all $h \in H_e$:

$$\begin{aligned}(\xi_{(1)}(\xi_{(2)}\circ S_\lambda^{-1}))(h) &= \xi_{(1)}(h_{(2,\lambda^{-1})})\xi_{(2)}(S_\lambda^{-1}(h_{(1,\lambda)}))\\ &= \xi(h_{(2,\lambda^{-1})}S_\lambda^{-1}(h_{(1,\lambda)})) = \xi(1_{\lambda^{-1}})\varepsilon(h).\end{aligned}$$

This proves one equality of (35); the proof of three other equalities is similar and is left to the reader. □

7. Braidings and quasitriangular $\mathbb{G}$-graded Hopf algebras

Definition 7.1. Let A be a $\mathbb{G}$-graded bialgebra. A is called quasitriangular if it comes equipped with the following additional structure: for all $g, g' \in G$, we have

$$\begin{aligned} R_{g,g'} &= R^1_{g,g'}\otimes R^2_{g,g'} \in A_{g^{-1},gg'g^{-1}}\otimes A_{e,g};\\ Q_{g,g'} &= Q^1_{g,g'}\otimes Q^2_{g,g'} \in A_{g,g^{-1}g'g}\otimes A_{e,g},\end{aligned}$$

such that the following conditions are fulfilled:

$$R_{g,g'}Q_{gg'g^{-1},g} = Q_{g',g}R_{g,g^{-1}g'g} = 1_{g'}\otimes 1_g; \tag{37}$$

$$\Delta_{g,g'^g,g''^g}(R^1_{g,g'g''})\otimes R^2_{g,g'g''} = R^1_{g,g'}\otimes\tilde{R}^1_{g,g''}\otimes R^2_{g,g'}\tilde{R}^2_{g,g''} \tag{38}$$

in $A_{g,g'^g}\otimes A_{g,g''^g}\otimes A_{e,g}$;

$$R^1_{gg',g''}\otimes\Delta_{e,g,g'}(R^2_{gg',g''}) = R^1_{g',g''}\tilde{R}^1_{g,g'g''g'^{-1}}\otimes\tilde{R}^2_{g,g'g''g'^{-1}}\otimes R^2_{g',g''} \tag{39}$$

in $A_{(gg')^{-1},g''^{(gg')^{-1}}}\otimes A_{e,g}\otimes A_{e,g'}$.
In addition, we have for all $a \in A_{\lambda,g^\lambda g'^\lambda}$ that

$$\tau(\Delta_{\lambda,g^\lambda,g'^\lambda})R_{g^\lambda,g'^\lambda} = R_{g,g'}\Delta_{\lambda,g'^{g^{-1}\lambda},g^\lambda}(a). \tag{40}$$

Here τ is the switch map.

Definition 7.1 has a monoidal categorical justification. Let G be a group, and A a $\mathbb{G}$-graded bialgebra. We know that $\mathcal{Z}_A^{\mathbb{G}}$ is a monoidal category, and that the forgetful functor $U:\ \mathcal{Z}_A^{\mathbb{G}}\to\mathcal{X}_G^G$ is monoidal. Let $\mathcal{X}_G^{G\mathrm{inv}}$ be $\mathcal{X}_G^G$ with the inverse braiding c^{-1}. Then we can look at braidings on $\mathcal{Z}_A^{\mathbb{G}}$ such that U preserves the braiding. Such a braiding is of the form $(c^{-1},\tilde{t})$, where c is the braiding on $\mathcal{X}_G^G$ as described in 4.1. In Proposition 5.3, we have seen that we have a monoidal structure on $\mathcal{T}_A^{\mathbb{G}}$ such that $V:\ \mathcal{T}_A^{\mathbb{G}}\to\mathcal{X}_G^G$ is monoidal, and we can consider braidings on $\mathcal{T}_A^{\mathbb{G}}$ of the form (c,t), i.e. they are such that V preserves the braiding.

Theorem 7.2. *Let G be a group, and A a $\mathbb{G}$-graded bialgebra. There is a bijective correspondence between the following data:*

- *braidings on $\mathcal{Z}_A^{\mathbb{G}}$ of the form $(c^{-1},\tilde{t})$;*
- *braidings on $\mathcal{T}_A^{\mathbb{G}}$ of the form (c,t);*
- *quasitriangular structures on A as defined in Definition 7.1.*

Proof. Given a braiding (c,t) on $\mathcal{T}_A^{\mathbb{G}}$, a braiding $(c^{-1},\tilde{t})$ on $\mathcal{Z}_A^{\mathbb{G}}$ is given by the formula

$$\tilde{t}_{M,N,v,v'} = t_{M,N,v,v'\nu(v)^{-1}}, \tag{41}$$

and vice versa.

Next assume that we have a braiding $(c^{-1},\tilde{t})$ on $\mathcal{Z}_A^{\mathbb{G}}$. For $\underline{M}=(V,M)$, $\underline{N}=(V',N)\in\mathcal{Z}_A^{\mathbb{G}}$, we have the braiding morphism

$$(c^{-1}_{V',V},\tilde{t}_{M,N,v,v'}):\ (V\times V', M\otimes N)\to (V'\times V, N\otimes M).$$

Then we have

$$\tilde{t}_{M,N}(M_v\otimes N_{v'})\subset N_{v'\nu(v)^{-1}}\otimes M_v. \tag{42}$$

Let $V=V'=G^2$, $M=N=A$. Then

$$c^{-1}_{G^2,G^2}(\lambda,g,\lambda',g')=(\lambda' g^{-1}, gg'g^{-1},\lambda,g),$$

and $\tilde{t}_{A,A}:\ A\otimes A\to A\otimes A$ satisfies

$$\tilde{t}_{A,A}(A_{\lambda,g}\otimes A_{\lambda',g'})\subset A_{\lambda' g^{-1},gg'g^{-1}}\otimes A_{\lambda,g}.$$

Now let

$$R_{g,g'}=R^1_{g,g'}\otimes R^2_{g,g'}=\tilde{t}_{A,A}(1_g\otimes 1_{g'})\in A_{g^{-1},gg'g^{-1}}\otimes A_{e,g}. \tag{43}$$

We will show that the braiding $\tilde{t}$ is completely determined by the $R_{g,g'}$. Take $m\in M_v$, $n\in N_{v'}$. We have seen in the proof of Theorem 5.2 that we have morphisms

$$(\eta,f_m):\ (Z_{\nu(v)},A^{(\nu(v))})\to (V,M),\ (\eta',g_n):\ (Z_{\nu'(v')},A^{(\nu'(v'))})\to(V',N)$$

in $\mathcal{Z}_A^{\mathbb{G}}$. From the naturality of $(c^{-1},\tilde{t})$, we have the following commutative diagram

$$\begin{array}{ccc}
(Z_{\nu(v)}\times Z_{\nu'(v')}, A^{(\nu(v))}\otimes A^{(\nu'(v'))}) & \xrightarrow{(c^{-1},\tilde{t})} & (Z_{\nu'(v')}\times Z_{\nu(v)}, A^{(\nu'(v'))}\otimes A^{(\nu(v))})\\
\Big\downarrow{\scriptstyle (\eta,\eta',f_m\otimes g_n)} & & \Big\downarrow{\scriptstyle (\eta',\eta,g_n\otimes f_m)}\\
(V\times V', M\otimes M') & \xrightarrow{(c^{-1}_{V',V},\tilde{t}_{M,N})} & (V'\otimes V, N\otimes M)
\end{array}$$

Observe that $(e,\nu(v))\in Z_{\nu(v)}$, $(e,\nu'(v'))\in Z_{\nu'(v')}$, $1_{\nu(v)}\in A^{(\nu(v))}$, $1_{\nu'(v')}\in A^{(\nu'(v'))}$. From the commutativity of the diagram, it then follows that

$$\begin{aligned}
\tilde{t}_{M,N}(m\otimes n) &= (\tilde{t}_{M,N}\circ(f_m\otimes g_n))(1_{\nu(v)}\otimes 1_{\nu'(v')})\\
&= ((g_n\otimes f_m)\circ\tilde{t}_{A^{(\nu(v))},A^{(\nu'(v'))}})(1_{\nu(v)}\otimes 1_{\nu'(v')})\\
&= (g_n\otimes f_m)(R_{\nu(v),\nu'(v')})\\
&= nR^1_{\nu(v),\nu'(v')}\otimes mR^2_{\nu(v),\nu'(v')}.
\end{aligned}\tag{44}$$

The inverse braiding can be described in a similar way: by assumption, $\tilde{t}_{M,N}$ is invertible, and

$$\tilde{t}^{-1}_{M,N}(N_{\nu'}\otimes M_{\nu})\subset M_v\otimes N_{v'\nu(v)}. \tag{45}$$

In fact the inclusions in (42) and (45) are equalities, since $\tilde{t}_{M,N}$ is bijective. In particular, we find that

$$\tilde{t}^{-1}_{A,A}(A_{\lambda',g'}\otimes A_{\lambda,g})=A_{\lambda,g}\otimes A_{\lambda' g,g'g}.$$

Now let

$$\tilde{t}^{-1}_{A,A}(1_{g'}\otimes 1_g)=Q_{g,g'}=Q^2_{g',g}\otimes Q^1_{g',g}\in A_{e,g}\otimes A_{g,g'^g}. \tag{46}$$

The $Q_{g',g}$ describe the inverse braiding completely. Arguments similar to the ones above show that, for $m\in M_v$ and $n\in N_{v'}$:

$$\tilde{t}^{-1}_{M,N}(n\otimes m)=mQ^2_{\nu'(v'),\nu(v)}\otimes nQ^1_{\nu'(v'),\nu(v)}. \tag{47}$$

Then we compute that

$$\begin{aligned}
1_g\otimes 1_{g'}&=(\tilde{t}^{-1}_{A,A}\circ\tilde{t}_{A,A})(1_g\otimes 1_{g'})\overset{(43)}{=}\tilde{t}^{-1}_{A,A}(R^1_{g,g'}\otimes R^2_{g,g'})\\
&\overset{(47)}{=} R^2_{g,g'}Q^2_{gg'g^{-1},g}\otimes R^1_{g,g'}Q^2_{gg'g^{-1},g};\\
1_{g'}\otimes 1_g&=(\tilde{t}_{A,A}\circ\tilde{t}^{-1}_{A,A})(1_{g'}\otimes 1_g)\overset{(46)}{=}\tilde{t}_{A,A}(Q^2_{g',g}\otimes Q^1_{g',g})\\
&\overset{(44)}{=} Q^1_{g',g}R^1_{g,g^{-1}g'g}\otimes Q^2_{g',g}R^2_{g,g^{-1}g'g}.
\end{aligned}$$

This shows that (37) holds.
From the fact that $(c^{-1},\tilde{t})$ is a braiding, it follows that

$$\tilde{t}_{A,A\otimes A} = (A\otimes\tilde{t}_{A,A})\circ(\tilde{t}_{A,A}\otimes A); \tag{48}$$

$$\tilde{t}_{A\otimes A,A} = (\tilde{t}_{A,A}\otimes A)\circ(A\otimes\tilde{t}_{A,A}). \tag{49}$$

Now we compute that

$$\begin{aligned}
&\tilde{t}_{A,A\otimes A}(1_g\otimes 1_{g'}\otimes 1_{g''})\overset{(44)}{=}(1_{g'}\otimes 1_{g''})R^1_{g,g'g''}\otimes 1_gR^2_{g,g'g''}\\
&\overset{(27)}{=} \Delta_{g,g'^g,g''^g}(R^1_{g,g'g''})\otimes R^2_{g,g'g''};\\
&\big((A\otimes\tilde{t}_{A,A})\circ(\tilde{t}_{A,A}\otimes A)\big)(1_g\otimes 1_{g'}\otimes 1_{g''})\\
&\overset{(28)}{=} (A\otimes\tilde{t}_{A,A})(R^1_{g,g'}\otimes R^2_{g,g'}\otimes 1_{g''})\\
&\overset{(44)}{=} R^1_{g,g'}\otimes 1_{g''}\tilde{R}^1_{g,g''}\otimes R^2_{g,g'}\tilde{R}^2_{g,g''}\\
&= R^1_{g,g'}\otimes\tilde{R}^1_{g,g''}\otimes R^2_{g,g'}\tilde{R}^2_{g,g''}.
\end{aligned}$$

This shows that (38) holds. (39) can be proved in a similar way:

$$\begin{aligned}
&\tilde{t}_{A\otimes A,A}(1_g\otimes 1_{g'}\otimes 1_{g''})\overset{(44)}{=}1_{g''}R^1_{gg',g''}\otimes(1_g\otimes 1_{g'})R^2_{gg',g''}\\
&\overset{(27)}{=} R^1_{gg',g''}\otimes\Delta_{e,g,g'}(R^2_{gg',g''});\\
&\big((\tilde{t}_{A,A}\otimes A)\circ(A\otimes\tilde{t}_{A,A})\big)(1_g\otimes 1_{g'}\otimes 1_{g''})\\
&\overset{(27)}{=} (\tilde{t}_{A,A}\otimes A)(1_g\otimes R^1_{g',g''}\otimes R^2_{g',g''})\\
&\overset{(44)}{=} R^1_{g',g''}\tilde{R}^1_{g,g'g''g'^{-1}}\otimes 1_g\tilde{R}^2_{g,g'g''g'^{-1}}\otimes R^2_{g',g''}\\
&= R^1_{g',g''}\tilde{R}^1_{g,g'g''g'^{-1}}\otimes\tilde{R}^2_{g,g'g''g'^{-1}}\otimes R^2_{g',g''}.
\end{aligned}$$

Now take $a\in A_{\lambda,g^\lambda g'^\lambda}$. Since $\tilde{t}_{A,A}$ is right A-linear, we have

$$\tilde{t}_{A,A}((1_g\otimes 1_{g'})a)=\tilde{t}_{A,A}(1_g\otimes 1_{g'})a.$$

Now

$$\begin{aligned}
&\tilde{t}_{A,A}((1_g\otimes 1_{g'})a)\overset{(27)}{=}\tilde{t}_{A,A}(\Delta_{\lambda,g^\lambda,g'^\lambda}(a))\\
&\quad= \tilde{t}_{A,A}(a_{(1,g^\lambda)}\otimes a_{(2,g'^\lambda)})\overset{(44)}{=}a_{(2,g'^\lambda)}R^1_{g^\lambda,g'^\lambda}\otimes a_{(1,g^\lambda)}R^2_{g^\lambda,g'^\lambda};\\
&\tilde{t}_{A,A}(1_g\otimes 1_{g'})a\overset{(43)}{=}(R^1_{g,g'}\otimes R^2_{g,g'})a\overset{(28)}{=}R^1_{g,g'}a_{(1,g'^{g^{-1}\lambda})}\otimes R^2_{g,g'}a_{(2,g^\lambda)}.
\end{aligned}$$

(40) follows, and we have shown that the $R_{g,g'}$ define a quasitriangular structure on A.
Conversely, if A is quasitriangular. Then we define $\tilde{t}_{M,N}$ using (44). A lengthy but straightforward computation shows that $(c^{-1},\tilde{t})$ is a braiding on $\mathcal{Z}_A^{\mathbb{G}}$. □

Now let $\underline{H}$ be a Hopf group coalgebra, and assume that every H_g is finitely generated and projective as a k-module. Then the category $\mathcal{YDZ}_{\underline{H}}^{\underline{H}}$ is braided monoidal, and is isomorphic to $\mathcal{Z}_{D(\underline{H})}^{\mathbb{G}}$. We know from Proposition 6.2 that $D(\underline{H})$ is a $\mathbb{G}$-graded Hopf algebra, and it follows from Theorem 7.2 that we have a quasitriangular structure on $D(\underline{H})$. The corresponding R-matrices can be computed easily. The coaction map

$$\rho_{(g^{-1},gg'g^{-1}),g}:\ H_e^*\#H_{g'}\to (H_g^*\#H_{gg'g^{-1}})\otimes H_g$$

can be computed using (33). In particular

$$\rho_{(g^{-1},gg'g^{-1}),g}(\varepsilon\#1_{g'})=(\xi^{(g)}\#1_{gg'g^{-1}})\otimes h^{(g)}. \tag{50}$$

Then

$$\begin{aligned}R_{g,g'} &\overset{(43)}{=} \tilde{t}_{D(\underline{H}),D(\underline{H})}\big((\varepsilon\#1_g)\otimes(\varepsilon\#1_{g'})\big)\\ &\overset{(31,50)}{=} (\xi^{(g)}\#1_{gg'g^{-1}})\otimes(\varepsilon\#1_g)h^{(g)}=(\xi^{(g)}\#1_{gg'g^{-1}})\otimes(\varepsilon\#h^{(g)}).\end{aligned}$$

In a similar way, we can compute the Q-matrices. Using (33), we compute $\rho_{(g,g^{-1}g'g),g^{-1}}:\ H_e^*\#H_{g'}\to (H_{g^{-1}}^*\#H_{g^{-1}g'g})\otimes H_{g^{-1}}$:

$$\rho_{(g,g^{-1}g'g),g^{-1}}(\varepsilon\#1_{g'})=(\xi^{(g^{-1})}\#1_{g^{-1}g'g})\otimes h^{(g^{-1})}.$$

Then we find

$$\begin{aligned}Q_{g,g'} &\overset{(45)}{=} \tilde{q}_{D(\underline{H},\underline{H}}\big((\varepsilon\#1_{g'})\otimes(\varepsilon\#1_g)\big)\overset{(32)}{=}(\varepsilon\#1_g)\overline{S}_g(h^{(g^{-1})})\otimes(\xi^{(g^{-1})}\#1_{g^{-1}g'g})\\ &= (\varepsilon\#\overline{S}_g(h^{(g^{-1})}))\otimes(\xi^{(g^{-1})}\#1_{g^{-1}g'g}).\end{aligned}$$

We summarize our results.

Theorem 7.3. *Let $\underline{H}$ be a Hopf group coalgebra, and assume that every H_g is finitely generated and projective as a k-module. Then $D(\underline{H})$ is a quasitriangular $\mathbb{G}$-graded Hopf algebra, with R- and S-matrices*

$$R_{g,g'}=(\xi^{(g)}\#1_{gg'g^{-1}})\otimes(\varepsilon\#h^{(g)})\ ;\ Q_{g,g'}=(\varepsilon\#\overline{S}_g(h^{(g^{-1})}))\otimes(\xi^{(g^{-1})}\#1_{g^{-1}g'g}).$$

The isomorphisms between the categories $\mathcal{YDZ}_{\underline{H}}^{\underline{H}}$ and $\mathcal{Z}_{D(\underline{H})}^{\mathbb{G}}$ and between $\mathcal{YDT}_{\underline{H}}^{\underline{H}}$ and $\mathcal{T}_{D(\underline{H})}^{\mathbb{G}}$ (see 4.6) are isomorphisms of braided monoidal categories.

8. Appendix: generalized Yetter-Drinfeld modules

A generalization of Yetter-Drinfeld modules was proposed in [3], see also [4]. First one has to introduce Yetter-Drinfeld data. There is a functor from Yetter-Drinfeld data to Doi-Hopf data, and the corresponding categories of Yetter-Drinfeld modules and Doi-Hopf modules are isomorphic. This construction was carried out in the category of vector spaces, but can be generalized to symmetric monoidal categories. Let us give the definition of Yetter-Drinfeld data in $\mathcal{T}_k$.

First we discuss discrete Yetter-Drinfeld data, these are Yetter-Drinfeld data in $\underline{\text{Sets}}$. This is a four-tuple (L, G, Λ, X), where L and G are groups, Λ is a monoid, $\psi : \Lambda \to L$ and $\gamma : \Lambda \to G$ are monoid maps, and X is a set with compatible left L-action and right G-action.
A crossed (L, G, Λ, X)-set is a right Λ-set V together with a map $\nu : V \to X$ such that $\nu(y\lambda) = \psi(\lambda)^{-1}\nu(y)\gamma(\lambda)$.
If (L, G, Λ, X) is a discrete Yetter-Drinfeld datum, then we have a discrete Doi-Hopf datum $(L\times G, \Lambda, X)$, with $(\psi, \gamma) : \Lambda \to L\times G$, and $x(l, g) = l^{-1}xg$. An $(L \times G, \Lambda, X)$-set is the same as a crossed (L, G, Λ, X)-set.
An example of a discrete Yetter-Drinfeld datum is $\mathbb{G} = (G, G, G)$, as discussed in the previous Sections.
A Yetter-Drinfeld datum in $\mathcal{T}_k$ is a fourtuple $(\underline{K}, \underline{H}, \underline{A}, \underline{C})$, where

- $\underline{K} = (L, (K_l)_{l\in L})$ and $\underline{H} = (G, (H_g)_{g\in G})$ are Hopf group-coalgebras;
- $\underline{A} = (X, (A_x)_{x\in X})$ is a $(\underline{K}, \underline{H})$-bicomodule algebra;
- $\underline{C} = (\Lambda, (C_\lambda)_{\lambda\in\Lambda})$ is a $(\underline{K}, \underline{H})$-bimodule coalgebra.

Then (L, G, Λ, X) is a discrete Yetter-Drinfeld datum; we have coaction maps

$$\rho_{l,x,g} : A_{lxg} \to K_l \otimes A_x \otimes H_g,\ \rho_{l,x,g}(a) = a_{[-1,l]} \otimes a_{[0,x]} \otimes a_{[1,g]},$$

(Sweedler notation). C_λ is a $(K_{\psi(\lambda)}, H_{\gamma(\lambda)})$-bimodule, for every $\lambda \in \Lambda$.
A Yetter-Drinfeld module is a couple $(V, (M_v)_{v\in V})$, where V is a crossed (L, G, Λ, X)-set, every M_v is a right $A_{\nu(v)}$-module, and $\underline{M}$ is a right $\underline{C}$-comodule, with structure maps $\rho_{v,\lambda} : M_{v\lambda} \to M_v \otimes C_\lambda$ such that the compatibility relation

$$\rho_{v,\lambda}(ma) = m_{[0,v]}a_{[0,\nu(v)]} \otimes S_{\psi(\lambda)}(a_{[-1,\psi(\lambda)^{-1}]})m_{[1,\lambda]}a_{[2,\gamma(\lambda)]}$$

holds for all $m \in M_{v\lambda}$ and $a \in A_{\nu(v\lambda)} = A_{\psi(\lambda)^{-1}\nu(v)\gamma(\lambda)}$.
If $(\underline{K}, \underline{H}, \underline{A}, \underline{C})$ is a Yetter-Drinfeld datum in $\mathcal{T}_k$, then $(\underline{K}^{\rm op} \otimes \underline{H}, \underline{A}, \underline{C})$ is a Doi-Hopf datum in $\mathcal{T}_k$: $\underline{A}$ is a right $\underline{K}^{\rm op} \otimes \underline{H}$-comodule algebra with coaction maps

$$\rho_{x,(l,g)} : A_{l^{-1}xg} \to A_x \otimes K_l \otimes H_g,\ \rho_{x,(l,g)}(a) = a_{[0,x]} \otimes S_l(a_{[-1,l^{-1}]}) \otimes a_{[1,g]}.$$

$\underline{C}$ is a right $\underline{K}^{\rm op} \otimes \underline{H}$-module coalgebra, since every C_λ is a right $K^{\rm op}_{\psi(\lambda)} \otimes H_\gamma(\lambda)$-module. Yetter-Drinfeld modules over $(\underline{K}, \underline{H}, \underline{A}, \underline{C})$ then coincide with Doi-Hopf modules over $(\underline{K}^{\rm op} \otimes \underline{H}, \underline{A}, \underline{C})$. We can then consider the categories $\mathcal{YDT}(\underline{K}, \underline{H})^{\underline{C}}_{\underline{A}}$ and $\mathcal{YDZ}(\underline{K}, \underline{H})^{\underline{C}}_{\underline{A}}$ which are respectively isomorphic to the categories $\mathcal{T}_k(\underline{K}^{\rm op} \otimes \underline{H})^{\underline{C}}_{\underline{A}}$ and $\mathcal{Z}_k(\underline{K}^{\rm op} \otimes \underline{H})^{\underline{C}}_{\underline{A}}$. Now the duality results from Section 3 can be applied.

Example 8.1. $(\underline{H}, \underline{H}, \underline{H}, \underline{H})$ is a Yetter-Drinfeld datum in $\mathcal{T}_k$, and the corresponding Yetter-Drinfeld modules are the Yetter-Drinfeld modules that we considered in Section 4.

Example 8.2. Let (L, G, Λ, X) be a discrete Yetter-Drinfeld datum. The crossed (L, G, Λ, X)-structures on a singleton $\{*\}$ are in bijective correspondence with $X_0 = \{x_0 \in X \mid x_0\gamma(\lambda) = \psi(\lambda)x_0,\ \text{for all } \lambda \in \Lambda\}$. The right Λ-action on $\{*\}$ is the trivial one, and $\nu(*) = x_0$. In the case where $L = G = \Lambda = X$, X_0 is just the center of G.

Let $(\underline{K},\underline{H},\underline{A},\underline{C})$ a Yetter-Drinfeld datum in $\mathcal{T}_k$, and fix $x_0 \in X_0$. An x_0-Yetter-Drinfeld module is an $(\underline{K},\underline{H},\underline{A},\underline{C})$-Yetter-Drinfeld module of the form $(\{*\}, M)$, with $\nu(*) = x_0$. The full subcategory of $\mathcal{YDT}(\underline{K},\underline{H})^{\underline{C}}_{\underline{A}}$ consisting of x_0-Yetter-Drinfeld modules will be denoted by $\mathcal{YDT}_{x_0}(\underline{K},\underline{H})^{\underline{C}}_{\underline{A}}$.

Example 8.3. We consider a particular instance of Example 8.2. At the discrete level, take $L = G = \Lambda = X$. The left and right G-action on $X = G$ are given by multiplication. We fix $x_0 \in X = G$, and define $\psi,\ \gamma:\ \Lambda \to G$ by $\psi(g) = x_0 g x_0^{-1}$ and $\gamma(g) = g$. It is then easy to see that $x_0 \in X$. We thus have a discrete Yetter-Drinfeld datum, which we will denote by $(G, G, G, {}_{x_0}G)$.
Let $\underline{H}$ be a Hopf group coalgebra, with underlying group G; we construct a Yetter-Drinfeld datum in $\mathcal{T}_k$ with underlying discrete Yetter-Drinfeld datum $(G, G, G, {}_{x_0}G)$. Let $\underline{K} = \underline{H}$ and $\underline{A} = \underline{H}$, with $\underline{H}$-bicomodule algebra structure induced by the comultiplication maps. Now we make $\underline{C} = \underline{H}$ into an H-module coalgebra. Every H_λ is a right H_λ-module, by multiplication. Consider a family of algebra maps $\varphi = (\varphi_\lambda :\ H_{x_0\lambda x_0^{-1}} \to H_\lambda)_{\lambda\in G}$. φ_λ defines a left $H_{x_0\lambda x_0^{-1}}$-module structure on H_λ by restriction of scalars, and this makes H_λ a $(H_{x_0\lambda x_0^{-1}}, H_\lambda)$-bimodule. This defines a left $\underline{H}$-bimodule coalgebra structure on $\underline{H}$ if and only if

$$\varepsilon_{\underline{H}}\varphi_e = \varepsilon_{\underline{H}} \text{ and } \Delta_{\lambda,\lambda'} \circ \varphi_{\lambda\lambda'} = (\varphi_\lambda \otimes \varphi_{\lambda'}) \circ \Delta_{x_0\lambda x_0^{-1}, x_0\lambda' x_0^{-1}}, \tag{51}$$

for all $\lambda, \lambda' \in G$. The resulting $\underline{H}$-bimodule coalgebra will be denoted ${}_{x_0,\varphi}\underline{H}$, and $\mathcal{YDT}_{x_0,\varphi}{}^{\underline{H}}_{\underline{H}}$ will be a shorter notation for the category $\mathcal{YD}_{x_0}(\underline{H},\underline{H})^{x_0,\varphi\underline{H}}_{\underline{H}}$. This definition of an x_0-$(\underline{H},\underline{H},\underline{H},{}_{x_0,\varphi}\underline{H})$-Yetter-Drinfeld module agrees with the right version of x_0-Yetter-Drinfeld module over a T-coalgebra $\underline{H}$ as introduced by Zunino in [16]. Recall that a T-coalgebra is a Hopf group-coalgebra $\underline{H} = (G, (H_g)_{g\in G})$ together with a family of k-algebra isomorphisms $\varphi = (\varphi^\sigma_\tau :\ H_\sigma \to H_{\tau\sigma\tau^{-1}})_{\sigma,\tau\in G}$ satisfying, among other, the conditions

$$\varepsilon_{\underline{H}}\varphi^e_\tau = \varepsilon_{\underline{H}} \text{ and } \Delta_{\theta\sigma\theta^{-1},\theta\tau\theta^{-1}} \circ \varphi^{\sigma\tau}_\theta = (\varphi^\sigma_\theta \otimes \varphi^\tau_\theta) \circ \Delta_{\sigma,\tau}.$$

for all $\sigma, \tau, \theta \in G$. Fix $x_0 \in G$, and define $\varphi_\lambda = \varphi^{x_0\lambda x_0^{-1}}_{x_0^{-1}}$, for any $\lambda \in G$. Then $\varepsilon_{\underline{H}}\varphi_e = \varepsilon_{\underline{H}}\varphi^e_{x_0^{-1}} = \varepsilon_{\underline{H}}$ and the family $\underline{\varphi} := (\varphi_\lambda = \varphi^{x_0\lambda x_0^{-1}}_{x_0} :\ H_{x_0\lambda x_0^{-1}} \to H_\lambda)_{\lambda\in G}$ satisfies (51). To see this take $\theta = x_0^{-1}$, $\sigma = x_0\lambda x_0^{-1}$ and $\tau = x_0\lambda' x_0^{-1}$, where $\lambda, \lambda' \in G$ in the above equality. In this situation $\mathcal{YDT}_{x_0,\varphi}{}^{\underline{H}}_{\underline{H}}$ is precisely the category of right x_0-Yetter-Drinfeld modules over a T-coalgebra, in the spirit of [16].

Example 8.4. We present a variation of Example 8.3. At the discrete level, let ψ be the identity on G, and let γ be conjugation by a fixed $x_0 \in G$: $\gamma(g) = x_0^{-1} g x_0$. An $\underline{H}$-bimodule structure on $\underline{H}$ can be obtained using a family of algebra maps $\varphi' = (\varphi'_\lambda :\ H_{x_0^{-1}\lambda x_0} \to H_\lambda)_{\lambda\in G}$. The $(H_\lambda, H_{x_0^{-1}\lambda x_0})$-bimodule structure on H_λ is obtained via restriction of scalars, using the identity on the left and φ_λ on the right hand side. This defines an $\underline{H}$-bimodule coalgebra structure on $\underline{H}$ if and only if

$$\varepsilon_{\underline{H}}\varphi'_e = \varepsilon_{\underline{H}} \text{ and } \Delta_{\lambda,\lambda'} \circ \varphi'_{\lambda\lambda'} = (\varphi'_\lambda \otimes \varphi'_{\lambda'}) \circ \Delta_{x_0^{-1}\lambda x_0, x_0^{-1}\lambda' x_0}, \tag{52}$$

for all $\lambda, \lambda' \in G$. The resulting $\underline{H}$-bimodule coalgebra is denoted by $\underline{H}_{x_0,\varphi'}$, and we use the shorter notation $\mathcal{YDT}^{\underline{H}}_{\underline{H}_{x_0,\varphi}}$ for the category $\mathcal{YD}_{x_0}(\underline{H},\underline{H})^{\underline{H}_{x_0,\varphi}}_{\underline{H}}$. Particular examples can be deduced from T-coalgebras. More precisely, let $\underline{H}$ be a T-coalgebra and $\varphi = (\varphi^\sigma_\tau : H_\sigma \to H_{\tau\sigma\tau^{-1}})_{\sigma,\tau\in G}$ the conjugation of $\underline{H}$. We have a family of algebra morphisms $\underline{\varphi'} = (\varphi'_\lambda = \varphi^{x_0^{-1}\lambda x_0}_{x_0} : H_{x_0^{-1}\lambda x_0} \to H_\lambda)_{\lambda\in G}$. A simple inspection shows that $\underline{\varphi'}$ satisfies (52). Thus it is possible to define the notion of x_0-Yetter-Drinfeld module in a way that is different from the one in [16].

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