Arithmetic-Arboreal Residue Structures induced by Prüfer Extensions : An axiomatic approach

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Abstract

We present an axiomatic framework for the residue structures induced by Prüfer extensions with a stress upon the intimate connection between their arithmetic and arboreal theoretic properties. The main result of the paper provides an adjunction relationship between two naturally defined functors relating Prüfer extensions and superrigid directed commutative regular quasi-semirings.

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Introduction

Let K be a valued field with Krull valuation v, valuation ring A, maximal ideal $\mathbf{m}$, residue field $k := A/\mathbf{m}$, and value group $\Lambda := vK \cong K^*/A^*$. For any $\lambda \in \Lambda$, we consider the A-submodules $\mathbf{m}_\lambda := \{x \in K \,|\, v(x) > \lambda\} \subseteq A_\lambda := \{x \in K \,|\, v(x) \geq \lambda\}$ of K, whence $\mathbf{m}_0 = \mathbf{m}, A_0 = A$. We denote by $\mathcal{R}_\lambda := K/\mathbf{m}_\lambda, R_\lambda := K/A_\lambda$ the associated quotient A-modules. The disjoint union $\mathcal{R} := \bigsqcup_{\lambda \in \Lambda} \mathcal{R}_\lambda$ is identified with the set of all *open balls* $\mathcal{B}(a, \lambda) := \{x \in K \,|\, v(x - a) > \lambda\}$ for $a \in K, \lambda \in \Lambda$, while the disjoint union $R := \bigsqcup_{\lambda \in \Lambda} R_\lambda$ is identified with the set of all *closed balls* $B(a, \lambda) := \{x \in K \,|\, v(x - a) \geq \lambda\}$ for $a \in K, \lambda \in \Lambda$. Notice that $\mathcal{R} = R$ if and only if the maximal ideal $\mathbf{m} = \pi A$ is principal, so $v(\pi)$ is the smallest positive element of the totally ordered Abelian group Λ.

The arithmetic and geometric structure of the valued field K induces a suitable structure on the residue set $\mathcal{R} \cup R$ which can be interpreted as a *deformation* of the original structure of K. Fragments of this residue structure involving suitable families

of balls and relations between them play a basic role in various interconnected algebraic-geometric and model theoretic contexts (elimination of quantifiers, cell decomposition, p-adic and motivic integration) [3, 8, 10-15, 17, 18, 20, 23].

Such residue structures can be also considered for more general objects than valued fields. Thus we investigate in [6] the residue structure R associated to a Prüfer domain A with field of fractions K, where the totally ordered value group Λ above is replaced by the Abelian l-group of the non-zero fractional ideals of finite type of K. In particular, we introduce on R an arboreal structure which generalizes Tits' construction of the tree associated to a valued field [25]. The relation of this arboreal structure with the arithmetic operations induced from K as well as the action of $GL_2(K)$ are described.

In the present work we extend our paper [6] in two directions : on the one hand, we consider the larger category of *Prüfer extensions* [16, 19], and on the other hand we develop an axiomatic framework for the *arithmetic-arboreal residue structures* induced by such ring extensions. Roughly speaking these residue structures are algebras $(R, +, \bullet, -, {}^{-1}, \varepsilon)$ of signature $(2, 2, 1, 1, 0)$ satisfying a finite set of equational axioms which are mainly suitable deformations of ring axioms. Thus these algebras, called *directed commutative regular quasi-semirings* form a variety. The main result of the paper (Theorem 7.2.) provides a relationship of adjunction between two naturally defined functors relating the categories (with suitably defined morphisms) of Prüfer extensions and of directed commutative regular quasi-semirings satisfying an additional $\forall\exists$-axiom called *superrigidity* [1].

Let us briefly review the content of this long and technical paper. In Section 1 some basic notions and constructions to be used later in the paper are considered. Among them we mention the *commutative l-monoid extension* $\widehat{\Lambda}$ of an Abelian l-group Λ - a nontrivial generalization of the totally ordered monoid $\Lambda \cup \{\infty\}$ associated to a totally ordered Abelian group Λ, necessary to define the notion of *l-valuation* on a commutative ring $w : B \to \widehat{\Lambda}$ - a generalization of the wellknown notion of *valuation*. In particular, the basic notion of *Manis valuation* is generalized to the so called *Manis l-valuation*. In the last subsection 1.4. of Section 1, some basic facts on Prüfer extensions are recalled, and the *Prüfer-Manis l-valuation* associated to a Prüfer extension is defined.

The next four sections are devoted to the axiomatic framework for the arithmetic-arboreal residue structures and the investigation of the main properties of the structures $(R, +, \bullet, -, {}^{-1}, \varepsilon)$ of signature $(2, 2, 1, 1, 0)$ introduced step by step by suitable axioms. Thus, in Section 2, we introduce the class of *commutative regular semirings* which contains as proper subclasses the Abelian l-groups and the *commutative regular rings* (in the sense of von Neumann), in particular, the fields and the Boolean rings. By relaxing suitably the distributive law, we define the variety of *commutative regular quasi-semirings* which contains the class of commutative regular semirings as a proper subvariety. Notice that in any commutative regular quasi-semiring R, the subset E^+ of the idempotents of the commutative regular semigroup $(R, +)$ has a natural structure of Abelian l-group Λ with the group operation induced by $\bullet$ and the meet-semilattice operation defined by $+$. Another interesting subvariety, whose members are the so called *directed commutative regular quasi-semirings* is introduced and studied in Section

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[1] Not to be confounded with Margulis superrigidity.

3. Notice that the intersection of this subvariety with the subvariety of commutative regular semirings is the class of all Abelian l-groups. Section 4 is devoted to the Λ-metric structure of the directed commutative regular quasi-semirings as well as to other related questions (congruences, rigidity and superrigidity, subdirectly irreducible structures). Using the Λ-metric introduced in Section 4, we consider in Section 5 two classes of structures with nice arboreal theoretic properties : the class of *median directed commutative regular quasi-semirings* and its proper subclass of *locally faithfully full directed commutative regular quasi-semirings*. The underlying arboreal structure on the latter objects is a nontrivial generalization to Abelian l-groups Λ of the wellknown Λ-trees [1, 22], where Λ is a totally ordered Abelian group. The classes of structures above are also varieties in suitably extended signatures.

The last two sections of the paper are devoted to the study of the relationship between some ring theoretic structures (l-valued commutative rings, Prüfer extensions) and their deformations (the induced arithmetic-arboreal residue structures, axiomatized and investigated in the previous sections). In Section 6 we associate a l-valued commutative ring (B, w) to a superrigid directed commutative regular quasi-semiring R, and we characterize those R for which the associated ring B is a Prüfer extension of its l-valuation subring (Corollary 6.7.). Thus we obtain a covariant functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$ from an adequate category of directed commutative regular quasi-semirings $\mathfrak{R}$ to the category $\mathfrak{P}$ of Prüfer extensions, with suitably defined morphisms. In Section 7 we define the *deformation* functor $\mathcal{R} : \mathfrak{P} \longrightarrow \mathfrak{R}$ which is a left adjoint of the functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$ such that the endofunctor $\mathcal{B} \circ \mathcal{R} : \mathfrak{P} \longrightarrow \mathfrak{P}$ sends a Prüfer extension $A \subseteq B$ to its completion $\widehat{A} \subseteq \widehat{B}$ (Theorem 7.2.). Consequently, the full subcategory $\mathfrak{CP}$ of $\mathfrak{P}$ consisting of all *complete Prüfer extensions* is equivalent with a suitable full subcategory $\mathfrak{CR}$ of $\mathfrak{R}$, whose objects are explicitely described.

1 Notation and Preliminaries

1.1 The commutative l-monoid extension of an Abelian l-group

Let Λ be an Abelian l-group with a multiplicative group operation, the neutral element ε, the partial order $\leq$, and the (distributive) lattice operations $\wedge$ and $\vee$. For any $\alpha \in \Lambda$, put $\alpha_+ := \alpha \vee \varepsilon, \alpha_- := (\alpha^{-1})_+, |\alpha| := \alpha \vee \alpha^{-1} = \alpha_+ \alpha_-$. Let $\Lambda_+ := \{\alpha \,|\, \alpha \geq \varepsilon\}$ denote the commutative l-monoid of all nonnegative elements of Λ. The Abelian l-group Λ has a canonical subdirect representation into the product $\prod_{P \in \mathcal{P}(\Lambda)} \Lambda/P$ of its maximal totally ordered factors, where P ranges over the set $\mathcal{P}(\Lambda)$ of the minimal prime convex l-subgroups of Λ, in bijection with the set of the minimal prime convex submonoids of Λ_+ as well as with the set of the ultrafilters of the distributive lattice with a least element Λ_+.

In the present paper we shall use frequently the following construction providing a natural embedding of an Abelian l-group Λ into a commutative l-monoid $\widehat{\Lambda}$.

Setting $\alpha \downarrow := \{\gamma \in \Lambda \,|\, \gamma \leq \alpha\}$ for any $\alpha \in \Lambda$, we consider the inverse system consisting of the distributive lattices (with a last element) $\alpha \downarrow$ for $\alpha \in \Lambda_+$, with the natural connecting epimorphisms $\beta \downarrow \longrightarrow \alpha \downarrow, \gamma \mapsto \gamma \wedge \alpha$ for $\alpha \leq \beta$. The inverse limit

$\widehat{\Lambda} := \varprojlim \alpha \downarrow$ is a distributive lattice identified with the set of all maps $\varphi : \Lambda_+ \longrightarrow \Lambda$ satisfying $\varphi(\alpha) = \varphi(\beta) \wedge \alpha$ for $\alpha \leq \beta$, in particular, $\varphi(\alpha) \leq \alpha$, with the induced pointwise partial order and lattice operations, and with $\omega : \Lambda_+ \longrightarrow \Lambda, \alpha \mapsto \alpha$ as the last element. Notice that for any $\varphi \in \widehat{\Lambda}$, $\varphi(\alpha)_- = \varphi(\varepsilon)^{-1}$ for all $\alpha \in \Lambda_+$, and hence $\varphi(\alpha) = \varphi(\varepsilon) \leq \varepsilon$ whenever $\varepsilon \leq \alpha \leq \varphi(\varepsilon)^{-1}$. Set $\varphi_- := \varphi(\varepsilon)^{-1} \in \Lambda_+$ for any $\varphi \in \widehat{\Lambda}$.

The underlying unbounded distributive lattice of the Abelian l-group Λ is identified with an ideal, in particular, with a sublattice of $\widehat{\Lambda}$, trough the embedding $\Lambda \longrightarrow \widehat{\Lambda}, \gamma \mapsto \iota_\gamma$, where $\iota_\gamma(\alpha) = \gamma \wedge \alpha$ for all $\alpha \in \Lambda_+$, so $\iota_\gamma(\alpha) = \gamma$ for all $\alpha \geq \gamma_+$. Notice that $\varphi \wedge \iota_\gamma = \iota_{\gamma \wedge \varphi(\gamma_+)}$ for all $\varphi \in \widehat{\Lambda}, \gamma \in \Lambda$, in particular, $\varphi \wedge \iota_\varepsilon = \iota_{\varphi(\varepsilon)}$ for all $\varphi \in \widehat{\Lambda}$. In the following we shall write γ instead of ι_γ for any $\gamma \in \Lambda$. Λ_+ is identified with an ideal of the bounded distributive lattice $\widehat{\Lambda}_+ := \{\varphi \in \widehat{\Lambda} \,|\, \varphi \geq \varepsilon\} = \{\varphi \in \widehat{\Lambda} \,|\, \varphi(\varepsilon) = \varepsilon\}$ with the least element ε and the last element ω, the inverse limit of the inverse system consisting of the bounded distributive lattices $\alpha \downarrow_+ := [\varepsilon, \alpha] = \{\gamma \in \Lambda_+ \,|\, \gamma \leq \alpha\}$ for $\alpha \in \Lambda_+$, with the induced connecting epimorphisms.

Let $\partial \widehat{\Lambda}_+$ denote the boolean subalgebra of the bounded distributive lattice $\widehat{\Lambda}_+$ consisting of those $\theta \in \widehat{\Lambda}_+$ which admit a (unique) complement $\neg \theta$, i.e. $\theta \wedge \neg \theta = \varepsilon$ and $\theta \vee \neg \theta = \omega$. Thus $\partial \widehat{\Lambda}_+$ is the inverse limit of the inverse system consisting of the boolean algebras $\partial[\varepsilon, \alpha] := \{\gamma \in [\varepsilon, \alpha] \,|\, \gamma \wedge \frac{\alpha}{\gamma} = \varepsilon\}$ for $\alpha \in \Lambda_+$, with the natural connecting morphisms which, in general, are not necessarily surjective. Notice that $\partial \widehat{\Lambda}_+ \cap \Lambda_+ = \{\varepsilon\}$, each $\theta \in \partial \widehat{\Lambda}_+$ extends uniquely to an endomorphism $\widetilde{\theta}$ of the l-group Λ, and $\widetilde{\theta} \circ \widetilde{\zeta} = \widetilde{\zeta} \circ \widetilde{\theta} = \widetilde{\theta \wedge \zeta}$ for all $\theta, \zeta \in \partial \widehat{\Lambda}_+$, in particular, $\widetilde{\theta} \circ \widetilde{\theta} = \widetilde{\theta}$ for all $\theta \in \partial \widehat{\Lambda}_+$. It follows that $\widetilde{\theta}(\Lambda) \subseteq \widetilde{\zeta}(\Lambda) \Longleftrightarrow \theta \leq \zeta$, $\widetilde{\theta}(\Lambda) \cap \widetilde{\zeta}(\Lambda) = \widetilde{\theta \wedge \zeta}(\Lambda)$, and $\widetilde{\theta \vee \zeta}(\Lambda)$ is the convex l-subgroup of Λ generated by $\widetilde{\theta}(\Lambda) \cup \widetilde{\zeta}(\Lambda)$ for all $\theta, \zeta \in \partial \widehat{\Lambda}_+$. In particular, the Abelian l-group $\Lambda = \widetilde{\omega}(\Lambda)$ is the direct sum of its convex l-subgroups $\widetilde{\theta}(\Lambda)$ and $\mathrm{Ker}(\widetilde{\theta}) = \widetilde{\neg \theta}(\Lambda)$ for any $\theta \in \partial \widehat{\Lambda}_+$, where $\widetilde{\neg \theta}(\gamma) = \dfrac{\gamma}{\widetilde{\theta}(\gamma)}$ for all $\gamma \in \Lambda$.

The distributive lattice $\widehat{\Lambda}$ becomes a commutative l-monoid extending the Abelian l-group Λ, with the multiplication defined by

$$(\varphi \psi)(\alpha) := (\varphi(\alpha \psi_-) \psi(\alpha \varphi_-)) \wedge \alpha = \lim_{\gamma \to \infty} (\varphi(\gamma) \psi(\gamma) \wedge \alpha)$$

for $\varphi, \psi \in \widehat{\Lambda}, \alpha \in \Lambda_+$. In particular, the identity $\varphi \psi = (\varphi \wedge \psi)(\varphi \vee \psi)$ holds for all $\varphi, \psi \in \widehat{\Lambda}$. The last element ω of the distributive lattice $\widehat{\Lambda}$ is a zero element of the monoid $\widehat{\Lambda}$, i.e. $\omega \varphi = \omega$ for all $\varphi \in \widehat{\Lambda}$. Λ is identified with the subgroup $\widehat{\Lambda}^*$ of all invertible elements of the monoid $\widehat{\Lambda}$, and $\widehat{\Lambda} = \Lambda_+^{-1} \widehat{\Lambda}_+$ is the monoid of fractions of the monoid $\widehat{\Lambda}_+$ relative to its submonoid Λ_+. More precisely, any element $\varphi \in \widehat{\Lambda}$ is uniquely represented in the form $\varphi = \frac{\varphi_+}{\varphi_-}$ with $\varphi_+ \in \widehat{\Lambda}_+, \varphi_- \in \Lambda_+$ subject to $\varphi_+ \wedge \varphi_- = \varepsilon$: $\varphi_+(\alpha) = \varphi(\alpha)_+$ for $\alpha \in \Lambda_+$, while $\varphi_- = \varphi(\varepsilon)^{-1}$ as defined above. The idempotent elements of the monoid $\widehat{\Lambda}$ are exactly the elements of the boolean algebra $\partial \widehat{\Lambda}_+$, and $\varphi \psi = \varphi \vee \psi$ for all $\varphi, \psi \in \partial \widehat{\Lambda}_+$. Notice also that $\varphi \cdot \psi = \omega \Longleftrightarrow \varphi \vee \psi = \omega$ for $\varphi, \psi \in \widehat{\Lambda}$; in particular, $\varphi^n = \omega \Longleftrightarrow \varphi = \omega$ for $\varphi \in \widehat{\Lambda}, n \geq 1$.

If the Abelian l-group Λ is totally ordered then $\widehat{\Lambda} = \Lambda \cup \{\omega\}$ and $\partial \widehat{\Lambda}_+ = \{\varepsilon, \omega\}$, with $\varepsilon = \omega \Longleftrightarrow \Lambda = \{\varepsilon\}$.

Example 1.1. As a suggestive example, let $\Lambda := \mathbb{Q}_{>0}$ be the multiplicative Abelian

group of the positive rationals, freely generated by the subset $\mathbb{P}$ of all prime natural numbers. Λ is an *l*-group with the lattice operations

$$x \wedge y := \text{g.c.d.}(x,y) = \prod_{p \in \mathbb{P}} p^{\min(v_p(x),v_p(y))},$$

and

$$x \vee y := \text{l.c.m.}(x,y) = \prod_{p \in \mathbb{P}} p^{\max(v_p(x),v_p(y))},$$

where v_p denotes the *p*-adic valuation for $p \in \mathbb{P}$. One checks easily that $\widehat{\Lambda}$ consists of all formal products $\prod_{p \in \mathbb{P}} p^{n_p}$ with $n_p \in \mathbb{Z} \cup \{\infty\}$ such that the set $\{p \in \mathbb{P} \,|\, n_p < 0\}$ is finite (with the corresponding map $\Lambda_+ = \mathbb{Z}_{>0} \longrightarrow \Lambda, m \mapsto \prod_{p \in \mathbb{P}} p^{\min(v_p(m),n_p)}$), while $\widehat{\Lambda}_+$ consists of those formal products with $n_p \in \mathbb{N} \cup \{\infty\}$ for all $p \in \mathbb{P}$, the so called *supernatural numbers* [24]. The boolean algebra $\partial\widehat{\Lambda}_+$, identified with the power set $\mathbf{2}^{\mathbb{P}}$, consists of those formal products with $n_p \in \{0, \infty\}$ for all $p \in \mathbb{P}$.

The next statements provide some basic properties of the correspondence $\Lambda \mapsto \widehat{\Lambda}$.

Lemma 1.2. *Let $f : \Lambda \longrightarrow \Gamma$ be a morphism of Abelian l-groups. Then f extends canonically to a morphism $\widehat{f} : \widehat{\Lambda} \longrightarrow \widehat{\Gamma}$ of commutative l-monoids provided one of the following conditions is satisfied.*

(1) $\Gamma = \text{ch}_\Gamma(f(\Lambda))$, *the* convex hull *of the image $f(\Lambda)$ in Γ; in particular, if f is onto.*

(2) Γ *is totally ordered; in particular, $\widehat{f}$ is onto whenever f is onto.*

Proof. (1) Assuming that $\Gamma = \text{ch}(f(\Lambda))$, let $\varphi \in \widehat{\Lambda}, \gamma \in \Gamma_+$. By assumption there is $\alpha \in \Lambda_+$ such that $\gamma \le f(\alpha)$. For any two such elements α, β, we obtain

$$f(\varphi(\alpha)) \wedge \gamma = f(\varphi(\alpha \vee \beta) \wedge \alpha) = f(\varphi(\alpha \vee \beta)) \wedge f(\alpha) \wedge \gamma = f(\varphi(\alpha \vee \beta)) \wedge \gamma = f(\varphi(\beta)) \wedge \gamma,$$

therefore the element $\widehat{f}(\varphi)(\gamma) := f(\varphi(\alpha)) \wedge \gamma \in \Gamma$ for some (for all) α as above is well defined. One checks easily that the map $\widehat{f}(\varphi) : \Gamma_+ \longrightarrow \Gamma$ so defined belongs to $\widehat{\Gamma}$, and moreover the map $\widehat{f} : \widehat{\Lambda} \longrightarrow \widehat{\Gamma}$ is a morphism of commutative *l*-monoids extending f.

(2) Assume that Γ is totally ordered. If f is the null morphism then $\widehat{f}(\varphi) = \varepsilon$ for all $\varphi \in \widehat{\Lambda}$. If f is nontrivial, in particular, $\Gamma \ne \{\varepsilon\}$ and $\widehat{\Gamma} = \Gamma \sqcup \{\omega\}$, then for any $\varphi \in \widehat{\Lambda}$ we distinguish the following two cases :

(i) There is $\alpha \in \Lambda_+$ such that $f(\varphi(\alpha)) < f(\alpha)$. In this case, the element $\widehat{f}(\varphi) := f(\varphi(\alpha)) \in f(\Lambda) \subseteq \Gamma$ does not depend on the choice of α with the property above.

(ii) For all $\alpha \in \Lambda_+$, $f(\varphi(\alpha)) = f(\alpha)$, in particular, $\varphi \notin \Lambda$. In this case we define $\widehat{f}(\varphi) = \omega$.

The map $\widehat{f} : \widehat{\Lambda} \longrightarrow \widehat{\Gamma}$ is a morphism of *l*-monoids extending f, with image $\widehat{f(\Lambda)} = f(\Lambda) \sqcup \{\omega\}$. $\qquad\square$

Lemma 1.3. *Given a family $(\Lambda_i)_{i \in I}$ of nontrivial Abelian l-groups, let $\Lambda := \prod_{i \in I} \Lambda_i$. Then $\widehat{\Lambda} \cong \prod_{i \in I} \widehat{\Lambda_i}$.*

Proof. Let us denote by $\pi_i : \Lambda \longrightarrow \Lambda_i, i \in I$, the natural projections. By Lemma 1.2, π_i extends to a morphism $\widehat{\pi_i} : \widehat{\Lambda} \longrightarrow \widehat{\Lambda_i}$ of l-monoids for all $i \in I$, and hence we obtain a canonical morphism $\xi : \widehat{\Lambda} \longrightarrow \prod_{i \in I} \widehat{\Lambda_i}, \varphi \mapsto (\widehat{\pi_i}(\varphi))_{i \in I}$ that is the identity on Λ. It follows that ξ is an isomorphism of l-monoids and $\xi^{-1}((\psi_i)_{i \in I})(\alpha) = (\psi_i(\pi_i(\alpha)))_{i \in I}$ for $\psi_i \in \widehat{\Lambda_i}, i \in I$, and $\alpha \in \Lambda_+$. □

As a consequence of Lemmas 1.2, 1.3, we obtain

Corollary 1.4. *Let $(\Lambda_i)_{i \in I}$ be a family of nontrivial Abelian l-groups, and let $f : \Gamma \longrightarrow \Lambda := \prod_{i \in I} \Lambda_i$ be a morphism of Abelian l-groups such that $\mathrm{ch}_{\Lambda_i}(\pi_i(f(\Gamma))) = \Lambda_i$ for all $i \in I$. Then f extends canonically to a morphism $\widehat{f} : \widehat{\Gamma} \longrightarrow \widehat{\Lambda} \cong \prod_{i \in I} \widehat{\Lambda_i}$, and $\widehat{f}(\widehat{\Gamma}) \subseteq \{\varphi \in \widehat{\Lambda} \mid \forall \gamma \in \Gamma, \varphi \wedge f(\gamma) \in f(\Gamma)\} \cong \widehat{f(\Gamma)}$.*

Remark 1.5. Let Λ be an Abelian l-group. For each l-subgroup $\Gamma \subseteq \Lambda$, the set $\Omega(\Lambda \mid \Gamma) := \{\varphi \in \widehat{\Lambda} \mid \forall \gamma \in \Gamma, \varphi \wedge \gamma \in \Gamma\} = \{\varphi \in \widehat{\Lambda} \mid \forall \gamma \in \Gamma_+, \varphi(\gamma) \in \Gamma\}$ is a sub-l-monoid of $\widehat{\Lambda}$ with $\Omega(\Lambda \mid \Gamma)^* = \Gamma$, and the restriction map $r : \Omega(\Lambda \mid \Gamma) \longrightarrow \widehat{\Gamma}, \varphi \mapsto \varphi|_{\Gamma_+}$ is a well defined morphism of commutative l-monoids which is the identity on Γ. As in Corollary 1.4, if $\Lambda = \prod_{i \in I} \Lambda_i$ is a product of nontrivial Abelian l-groups, and $\mathrm{ch}(\pi_i(\Gamma)) = \Lambda_i$ for all $i \in I$ then r is an isomorphism. The simplest example with r onto but not injective is obtained by taking Λ a nontrivial Abelian l-group and $\Gamma = \{\varepsilon\}$, in which case, $\Omega(\Lambda \mid \Gamma) = \widehat{\Lambda}_+$ and r is the constant map ε.

Corollary 1.6. *For each nontrivial Abelian l-group Λ, its canonical subdirect representation into the product $\Gamma := \prod_{P \in \mathcal{P}(\Lambda)} \Lambda/P$ of its maximal totally ordered factors extends to a subdirect representation of the associated commutative l-monoid $\widehat{\Lambda}$ into the product $\prod_{P \in \mathcal{P}(\Lambda)} \widehat{\Lambda/P}$ of its maximal totally ordered factors, isomorphic to the commutative l-monoid $\widehat{\Gamma}$ associated to the Abelian l-group Γ.*

In particular, the boolean algebra $\partial \widehat{\Lambda}_+$ is canonically embedded into the power set $\mathbf{2}^{\mathcal{P}(\Lambda)}$, and the injective map $\mathcal{P}(\Lambda) \longrightarrow \mathrm{Max}(\partial \widehat{\Lambda}_+)$,

$$P \mapsto \{\theta \in \partial \widehat{\Lambda}_+ \mid \forall \alpha \in \Lambda_+, \theta(\alpha) \in P\} = \{\theta \in \partial \widehat{\Lambda}_+ \mid \exists \alpha \in \Lambda_+, \frac{\alpha}{\theta(\alpha)} \notin P\}$$

identifies $\mathcal{P}(\Lambda)$ with a dense subset of the profinite space $\mathrm{Max}(\partial \widehat{\Lambda}_+)$, the Stone dual of the boolean algebra $\partial \widehat{\Lambda}_+$, consisting of all maximal ideals of $\partial \widehat{\Lambda}_+$.

Remark 1.7. Let M be a commutative *semilattice-ordered monoid* (for short, *sl-monoid*), i.e., a commutative monoid, with neutral element ε, together with a meet-semilattice operation $\wedge$ defining the partial order $x \leq y \Longleftrightarrow x = x \wedge y$, which satisfies the compatibility condition $x(y \wedge z) = xy \wedge xz$ for all $x, y, z \in M$. Denote by M^* the subgroup of all invertible elements of the monoid M, and consider the submonoid $\widetilde{M} := \{x \in M \mid \exists y \in M, xy \leq \varepsilon\}$. Note that $M^* \cup M_- \subseteq \widetilde{M}$, where $M_- := \{x \in M \mid x \leq \varepsilon\}$, and $x \wedge y \in \widetilde{M}$ for all $x \in \widetilde{M}, y \in M$, i.e. $\widetilde{M}$ is a lower subset of M. It follows that

$$\widetilde{M} = M^* \Longleftrightarrow \forall x \in M^*, \forall y \in M, x \wedge y \in M^*,$$

so, in this case, M^* is an Abelian l-group and the embedding $M^* \longrightarrow \widehat{M^*}$ extends uniquely to the morphism of sl-monoids $\widehat{w} : M \longrightarrow \widehat{M^*}$, defined by $\widehat{w}(x)(\alpha) = x \wedge \alpha$ for $x \in M, \alpha \in M_+^*$. For any such *sl*-monoid M, the meet-semilattice operation $\wedge$ induces on the subset $E^\bullet(M) := \{x \in M \,|\, x^2 = x\}$ of idempotent elements a natural structure of distributive lattice with the least element ε and the join $x \vee y := x \cdot y$. The distributive lattice $E^\bullet(M)$ is bounded, with the last element ω, provided the element ω is a zero as well as a last element of M. The morphism $\widehat{w} : M \longrightarrow \widehat{M^*}$ induces by restriction a lattice morphism from the distributive lattice $E^\bullet(M)$ to the boolean algebra $E^\bullet(\widehat{M^*}) = \partial \widehat{M^*}_+$.

The simplest way to extend the correspondence $\Lambda \mapsto \widehat{\Lambda}$ to a covariant functor is to define it on the subcategory $\mathcal{C}$ of the category of Abelian l-groups whose morphisms $f : \Gamma \longrightarrow \Lambda$ satisfy the condition $\mathrm{ch}_\Lambda(f(\Gamma)) = \Lambda$. On the other hand, let $\mathcal{D}$ be the category having as objects the *sl*-monoids M which satisfy the condition from Remark 1.7 : $x \in M, y \in M^* \Longrightarrow x \wedge y \in M^*$, with morphisms $F : M \longrightarrow N$ of *sl*-monoids for which $\mathrm{ch}_{N^*}(F(M^*)) = N^*$. By Lemma 1.2.(1), the correspondence $\Lambda \mapsto \widehat{\Lambda}$ extends to a covariant functor $\widehat{} : \mathcal{C} \longrightarrow \mathcal{D}$. The next statement is immediate.

Proposition 1.8. (1) *The functor* $\widehat{} : \mathcal{C} \longrightarrow \mathcal{D}$ *is a right adjoint of the covariant functor* $^* : \mathcal{D} \longrightarrow \mathcal{C}, M \mapsto M^*$.

   (2) *The* counit $^* \circ \widehat{} \longrightarrow 1_\mathcal{C}$ *of the adjunction is a natural isomorphism, while the* unit $1_\mathcal{D} \longrightarrow \widehat{} \circ {}^*$ *is the natural transformation sending each M in $\mathcal{D}$ to the canonical morphism* $\widehat{w} : M \longrightarrow \widehat{M^*}$ *as defined in* Remark 1.7. *In particular, the category $\mathcal{C}$ is equivalent with the full subcategory of $\mathcal{D}$ consisting of those M for which the canonical morphism* $\widehat{w} : M \longrightarrow \widehat{M^*}$ *is an isomorphism.*

Remark 1.9. As a right adjoint, the functor $\widehat{} : \mathcal{C} \longrightarrow \mathcal{D}$ is *continuous*, in particular, $\widehat{\prod_{i \in I} \Lambda_i} \cong \prod_{i \in I} \widehat{\Lambda_i}$ for each finite family $(\Lambda_i)_{i \in I}$ of Abelian l-groups. Though, by Lemma 1.3, the isomorphism above holds for arbitrary index sets I, we cannot deduce this fact from the continuity of the functor $\widehat{}$ since the category $\mathcal{C}$ does not have arbitrary products. Indeed, given a family $\Lambda_i, i \in I$, of Abelian l-groups, and a family $f_i : \Gamma \longrightarrow \Lambda_i$ of morphisms in $\mathcal{C}$, the canonical morphism $f : \Gamma \longrightarrow \prod_{i \in I} \Lambda_i$ of Abelian l-groups is not necessarily a morphism in $\mathcal{C}$. However f is a morphism in $\mathcal{C}$ provided the index set I is finite.

Lemma 1.10. *For any $\theta \in \partial\widehat{\Lambda}_+$, put $\widehat{\Lambda}\theta := \{\varphi \cdot \theta \,|\, \varphi \in \widehat{\Lambda}\} = \{\varphi \in \widehat{\Lambda} \,|\, \varphi \cdot \theta = \varphi\}$, and let $\widetilde{\theta}$ be the endomorphism of the Abelian l-group Λ induced by the endomorphism θ of the commutative l-monoid Λ_+, with $\mathrm{Ker}\,(\widetilde{\theta}) = \widetilde{\neg\theta}(\Lambda) = \{\frac{\alpha}{\theta(\alpha)} \,|\, \alpha \in \Lambda\}$. Then :*

   (1) *For all $\varphi \in \widehat{\Lambda}$, the following assertions are equivalent.*

       (i) $\varphi \in \widehat{\Lambda}\theta$.

       (ii) $\varphi \cdot \neg\theta = \omega$.

       (iii) $\widetilde{\theta} \circ \varphi = \theta$.

       (iv) $\theta \leq \varphi_+$.

(v) $\varphi(\alpha) = \varphi(\frac{\alpha}{\theta(\alpha)}) \vee (\theta(\alpha) \cdot \varphi(\varepsilon))$ *for all* $\alpha \in \Lambda_+$.

(vi) $\varphi(\alpha) = \theta(\alpha) \cdot \varphi(\frac{\alpha}{\theta(\alpha)})$ *for all* $\alpha \in \Lambda_+$.

Thus $\widehat{\Lambda}\theta = \{\varphi \in \widehat{\Lambda} \,|\, \varphi_+ \geq \theta\}$ *is a filter of* $\widehat{\Lambda}$ *and also a commutative l-monoid with neutral element* θ, *and* $(\widehat{\Lambda}\theta)_+ = \widehat{\Lambda}_+\theta = \{\varphi \in \widehat{\Lambda} \,|\, \varphi \geq \theta\}$.

(2) *The restriction map* $\varphi \mapsto \varphi|_{\mathrm{Ker}\,(\widetilde{\theta})_+}$ *is an isomorphism of commutative l-monoids* $\widehat{\Lambda}\theta \longrightarrow \widehat{\mathrm{Ker}\,(\widetilde{\theta})}$, *whose inverse sends* $\psi \in \widehat{\mathrm{Ker}\,(\widetilde{\theta})}$ *to* $\varphi \in \widehat{\Lambda}\theta$ *defined by* $\varphi(\alpha) = \theta(\alpha) \cdot \psi(\frac{\alpha}{\theta(\alpha)})$ *for* $\alpha \in \Lambda_+$. *In particular,* $(\widehat{\Lambda}\theta)^* = \Lambda\theta \cong \mathrm{Ker}\,(\widetilde{\theta})$, *and* $\partial(\widehat{\Lambda}\theta)_+ = (\partial\widehat{\Lambda}_+)\theta = \{\zeta \in \partial\widehat{\Lambda}_+ \,|\, \zeta \geq \theta\} \cong \partial\widehat{\mathrm{Ker}\,(\widetilde{\theta})}_+$.

(3) *For* $\theta, \zeta \in \partial\widehat{\Lambda}_+$, $\widehat{\Lambda}\theta \subseteq \widehat{\Lambda}\zeta \iff \theta \geq \zeta \iff \theta = \theta \cdot \zeta$, $\widehat{\Lambda}\theta \cap \widehat{\Lambda}\zeta = \widehat{\Lambda}(\theta \cdot \zeta) = \widehat{\Lambda}(\theta \vee \zeta)$, *and* $\widehat{\Lambda}\theta \vee \widehat{\Lambda}\zeta := \{\varphi \wedge \psi \,|\, \varphi \in \widehat{\Lambda}\theta, \psi \in \widehat{\Lambda}\zeta\} = \widehat{\Lambda}(\theta \wedge \zeta)$. *In particular,* $\widehat{\Lambda}\theta \cap \widehat{\Lambda}\neg\theta = \widehat{\Lambda}\omega = \{\omega\}$, $\widehat{\Lambda}\theta \vee \widehat{\Lambda}\neg\theta = \widehat{\Lambda}\varepsilon = \widehat{\Lambda}$, *and the map* $\widehat{\Lambda} \longrightarrow \widehat{\Lambda}\theta \times \widehat{\Lambda}\neg\theta, \varphi \mapsto (\varphi \cdot \theta, \varphi \cdot \neg\theta)$ *is an isomorphism of commutative l-monoids, with the inverse* $(\varphi, \psi) \mapsto \varphi \wedge \psi$.

Proof. (1). (i) $\Longrightarrow$ (ii) : $\varphi \cdot \neg\theta = \varphi \cdot \theta \cdot \neg\theta = \varphi \cdot \omega = \omega$.

(ii) $\Longrightarrow$ (iii) : Let $\alpha \in \Lambda_+$. By assumption,

$$\alpha = \omega(\alpha) = (\varphi \cdot \neg\theta)(\alpha) \leq \varphi(\alpha) \, \neg\theta(\alpha\varphi(\varepsilon)^{-1}) = \frac{\alpha\varphi(\alpha)\widetilde{\theta}(\varphi(\varepsilon))}{\theta(\alpha)\varphi(\varepsilon)},$$

therefore $\theta(\alpha)\varphi(\varepsilon) \leq \varphi(\alpha)\widetilde{\theta}(\varphi(\varepsilon))$. In particular, setting $\alpha = \varepsilon$, we obtain $\widetilde{\theta}(\varphi(\varepsilon)) = \varepsilon$, and hence $\theta(\alpha)\varphi(\varepsilon) \leq \varphi(\alpha) \leq \alpha$. Applying the endomorphism $\widetilde{\theta}$ to the inequalities above, we obtain the required identity $\widetilde{\theta}(\varphi(\alpha)) = \theta(\alpha)$ for all $\alpha \in \Lambda_+$.

(iii) $\Longrightarrow$ (iv) : Let $\alpha \in \Lambda_+$. We obtain

$$\theta(\alpha) = \widetilde{\theta}(\varphi(\alpha)) \leq \theta((\varphi(\alpha))_+) \leq \varphi_+(\alpha),$$

therefore $\theta \leq \varphi_+$ as desired.

(iv) $\Longrightarrow$ (v) and (vi) : Since $\theta \leq \varphi_+$ by assumption, it follows that

$$\varphi_+ = \theta \vee (\varphi_+ \wedge \neg\theta) = \theta \cdot (\varphi_+ \wedge \neg\theta),$$

and hence

$$\varphi_+(\alpha) = \theta(\alpha) \vee \varphi_+\left(\frac{\alpha}{\theta(\alpha)}\right) = \theta(\alpha) \cdot \varphi_+\left(\frac{\alpha}{\theta(\alpha)}\right)$$

for all $\alpha \in \Lambda_+$. The assertions (v) and (vi) are now immediate by multiplication with $\varphi(\varepsilon) = \varphi(\alpha)_-^{-1} = \varphi(\frac{\alpha}{\theta(\alpha)})_-^{-1}$.

The implication (v) $\Longrightarrow$ (iv) is obvious.

(vi) $\Longrightarrow$ (i) : As $\varphi(\alpha) = \varphi(\varepsilon)$ provided $\varepsilon \leq \alpha \leq \varphi(\varepsilon)^{-1}$, it follows by assumption that $\theta(\varphi(\varepsilon)^{-1}) = \varepsilon$, therefore

$$(\varphi \cdot \theta)(\alpha) = \varphi(\alpha)\theta(\alpha\varphi(\varepsilon)^{-1}) \wedge \alpha = \theta(\alpha)(\varphi(\alpha) \wedge \frac{\alpha}{\theta(\alpha)}) = \theta(\alpha)\varphi(\frac{\alpha}{\theta(\alpha)}) = \varphi(\alpha)$$

for all $\alpha \in \Lambda_+$, i.e. $\varphi \cdot \theta = \varphi$ as desired.

(2) By (1), (i) $\Longrightarrow$ (iii), (i) $\Longrightarrow$ (vi), it follows that the restriction map $\widehat{\Lambda\theta} \longrightarrow \mathrm{Ker}\,(\widetilde{\theta}), \varphi \mapsto \varphi|_{\mathrm{Ker}\,(\widetilde{\theta})_+}$ is an injective morphism of commutative *l*-monoids. To prove that it is an isomorphism, it suffices by (1), (vi) $\Longrightarrow$ (i), to show that for any $\psi \in \widehat{\mathrm{Ker}\,(\widetilde{\theta})}$, the map

$$\varphi : \Lambda_+ \longrightarrow \Lambda, \alpha \mapsto \theta(\alpha)\psi(\frac{\alpha}{\theta(\alpha)})$$

belongs to $\widehat{\Lambda}$, i.e. $\varphi(\alpha) = \varphi(\beta) \wedge \alpha$ for all $\alpha, \beta \in \Lambda_+$ such that $\alpha \le \beta$. Since $\psi \in \widehat{\mathrm{Ker}\,(\widetilde{\theta})}$, it follows that $\widetilde{\theta}(\varphi(\alpha)) = \theta(\alpha), \neg\widetilde{\theta}(\varphi(\alpha)) = \psi(\frac{\alpha}{\theta(\alpha)})$ for all $\alpha \in \Lambda_+$. Consequently, assuming that $\varepsilon \le \alpha \le \beta$, we obtain

$$\widetilde{\theta}(\frac{\varphi(\beta) \wedge \alpha}{\varphi(\alpha)}) = \widetilde{\neg\theta}(\frac{\varphi(\beta) \wedge \alpha}{\varphi(\alpha)}) = \varepsilon,$$

and hence the required identity.

The proof of the statement (3) is straightforward. $\qquad\qquad\square$

Corollary 1.11. $\overline{\Lambda} := \{\gamma \cdot \theta \,|\, \gamma \in \Lambda, \theta \in \partial\widehat{\Lambda}_+\}$ *is the smallest l-submonoid of* $\widehat{\Lambda}$ *containing the union* $\Lambda \cup \partial\widehat{\Lambda}_+$, *and* $\overline{\Lambda}_+ = \{\gamma \cdot \theta \,|\, \gamma \in \Lambda_+, \theta \in \partial\widehat{\Lambda}_+\}$. *In addition, the commutative monoid* $\overline{\Lambda}$ *is regular* [2], *with the quasi-inverse naturally defined by* $(\gamma \cdot \theta)^{-1} := \gamma^{-1} \cdot \theta$ *for* $\gamma \in \Lambda, \theta \in \partial\widehat{\Lambda}_+$.

Proof. By Lemma 1.10.(2), the surjective map $\Lambda \times \partial\widehat{\Lambda}_+ \longrightarrow \overline{\Lambda}, (\gamma, \theta) \mapsto \gamma \cdot \theta$ identifies $\overline{\Lambda}$ with the disjoint union $\bigsqcup_{\theta \in \partial\widehat{\Lambda}} \mathrm{Ker}\,(\widetilde{\theta})$ of convex *l*-subgroups of Λ. For $\theta, \zeta \in \partial\widehat{\Lambda}_+, \gamma \in \mathrm{Ker}\,(\widetilde{\theta}), \rho \in \mathrm{Ker}\,(\widetilde{\zeta})$, we obtain

$$(\gamma\theta) \cdot (\rho\zeta) = (\gamma\rho) \cdot (\theta\zeta) = (\frac{\gamma\rho}{\widetilde{\zeta}(\gamma)\widetilde{\widetilde{\theta}}(\rho)}) \cdot (\theta\zeta),$$

$$\gamma\theta \le \rho\zeta \Longleftrightarrow \theta \le \zeta, \gamma \le \rho\zeta(\gamma_+) \Longleftrightarrow \theta \le \zeta, (\neg\zeta)(\gamma_+) \le \rho_+, \rho_- \le \gamma_-,$$

$$(\gamma\theta) \wedge (\rho\zeta) = (\gamma\theta(\rho_+) \wedge \rho\zeta(\gamma_+)) \cdot (\theta \wedge \zeta),$$

and

$$(\gamma\theta) \vee (\rho\zeta) = (\frac{\gamma}{\zeta(\gamma_+)} \vee \frac{\rho}{\theta(\rho_+)}) \cdot (\theta\zeta).$$

$\qquad\qquad\square$

For instance, in the case $\Lambda = \mathbb{Q}_{>0}$ as in Example 1.1, $\overline{\Lambda}$ consists of those formal products $x := \prod_{p \in \mathbb{P}} p^{n_p}$, with $n_p \in \mathbb{Z} \cup \{\infty\}$, for which the set $\{p \in \mathbb{P} \,|\, n_p \in \mathbb{Z} \setminus \{0\}\}$ is finite, and $x^{-1} = \prod_{p \in \mathbb{P}} p^{m_p}$, where $m_p = -n_p$ if $n_p \in \mathbb{Z}$, and $m_p = \infty$ if $n_p = \infty$.

---

[2] A commutative (multiplicative) semigroup S is *regular* if for all $x \in S$ there is $y \in S$ such that $x^2 y = x$. A regular commutative semigroup S is an *inverse* semigroup, i.e. for all $x \in S$ there is a unique element $x^{-1} \in S$, called the *quasi-inverse* of x, satisfying the identities $x^2 x^{-1} = x, (x^{-1})^2 x = x^{-1}$.

1.2   The sl-monoid associated to a commutative ring extension

Let $A \subseteq B$ be a commutative unital ring extension. We denote by $M := M(A, B)$ the set of all finitely generated (in short f.g.) A-submodules of B. M becomes a commutative multiplicative sl-monoid with the usual multiplication and the meet-semilattice operation $I \wedge J := I + J$ induced by the partial order $I \leq J \iff J \subseteq I$. The neutral element is A, while $\{0\}$ is a zero element as well as the last element of the sl-monoid M. The sl-submonoid M_+ of the nonnegative elements of M consists of the f.g. ideals of A.

The subgroup M^* of the invertible elements of the monoid M is exactly the Abelian multiplicative group of those A-submodules $I \subseteq B$ which are B-*invertible*, i.e. $IJ = A$ for some A-submodule J of B. Indeed, every B-invertible A-submodule I is f.g., and the A-submodule J satisfying $IJ = A$ is unique, i.e. $J = I^{-1} := [A : I] := \{x \in B \mid xI \subseteq A\}$, and f.g. In particular, $Ax \in M^*$ and $(Ax)^{-1} = Ax^{-1}$ for all $x \in B^*$, so the factor group B^*/A^* is identified with a subgroup of M^*. With the notation from Remark 1.7, the lower submonoid $\widetilde{M} := \{I \in M \mid \exists J \in M, A \subseteq IJ\}$ of M, containing the ordered subgroup M^* and the lower submonoid $M_- := \{I \in M \mid A \subseteq I\}$, consists of those f.g. A-submodules I of B which are B-*regular*, i.e. $IB = B$.

1.3 l-valuations on commutative rings

Let B be a commutative unital ring and Λ a (multiplicative) Abelian l-group, extended as in 1.1. to the commutative l-monoid $\widehat{\Lambda}$, with its boolean algebra $\partial\widehat{\Lambda}_+$ of idempotent elements.

Definition 1.12. *A map $w : B \longrightarrow \widehat{\Lambda}$ is called a l-valuation whenever the following conditions are satisfied.*

(1) $w(xy) = w(x)w(y)$ *for all* $x, y \in B$.

(2) $w(x + y) \geq w(x) \wedge w(y)$ *for all* $x, y \in B$.

(3) $w(1) = \varepsilon$ *and* $w(0) = \omega$.

From the axioms above we deduce that $w(-x) = w(x)$ for all $x \in B$, $w(B^*)$ is a subgroup of Λ, and the map $E^\bullet(B) := \{x \in B \mid x^2 = x\} \longrightarrow \partial\widehat{\Lambda}_+, x \mapsto w(1 - x)$ is a morphism of boolean algebras.

In particular, if $\Lambda = \{\varepsilon\}$ is trivial, so $\widehat{\Lambda} = \Lambda$, i.e. $\omega = \varepsilon$, then the l-valuation w is the constant map $x \mapsto \varepsilon$. If Λ is a nontrivial totally ordered Abelian group, so $\widehat{\Lambda} = \Lambda \sqcup \{\omega\}$, then we obtain the usual notion of *valuation* [9, VI.3.1.], [19, I.1.]

Some notions and basic facts about valuations extend to l-valuations as follows. Let $w : B \longrightarrow \widehat{\Lambda}$ be a l-valuation, and assume that $\Lambda \neq \{\varepsilon\}$.

The set $w^{-1}(\omega)$ is a radical ideal of B. It is called the *support* of w and is denoted by $\mathrm{supp}(w)$. The subring $A_w := w^{-1}(\widehat{\Lambda}_+)$ of B is called the *l-valuation ring* of w. Thus $\mathrm{supp}(w)$ is a radical ideal of the ring A_w too, contained in the *conductor* of A_w in B, the biggest ideal $\{x \in B \mid Bx \subseteq A_w\}$ of B contained in A_w.

Setting $\overline{B} := B/\mathrm{supp}(w)$, with the canonical projection $\pi : B \longrightarrow \overline{B}$, there exists a unique l-valuation $\overline{w} : \overline{B} \longrightarrow \widehat{\Lambda}$ such that $\overline{w} \circ \pi = w$, with $\mathrm{supp}(\overline{w}) = \{0\}$, so the factor ring $\overline{B}$ is reduced, and $A_{\overline{w}} = \overline{A_w} := A_w/\mathrm{supp}(w)$. In particular, if w is a valuation then $\mathrm{supp}(w)$ is a prime ideal, $\mathbf{p}_w := \{x \in B \mid w(x) > \varepsilon\}$ is a prime ideal of A_w, the *center* of w, and $\overline{w}$ extends uniquely to a Krull valuation $\widetilde{w}$ on the quotient field of the domain $\overline{B}$, so $A_{\overline{w}} \subseteq A_{\widetilde{w}}, \mathbf{p}_{\widetilde{w}} \cap A_{\overline{w}} = \mathbf{p}_{\overline{w}} = \mathbf{p}_w/\mathrm{supp}(w)$.

Consider the subdirect representation $\widehat{\Lambda} \longrightarrow \prod_{P \in \mathcal{P}(\Lambda)} \widehat{\Lambda/P}$, where $\mathcal{P}(\Lambda)$ consists of all minimal prime convex l-subgroups of Λ, cf. Corollary 1.6.. Composing the l-valuation $w : B \longrightarrow \widehat{\Lambda}$ with the projection $\widehat{\Lambda} \longrightarrow \widehat{\Lambda/P} = (\Lambda/P) \sqcup \{\omega_P\}$ for $P \in \mathcal{P}(\Lambda)$, we obtain a family of valuations $(w_P : B \longrightarrow \widehat{\Lambda/P})_{P \in \mathcal{P}(\Lambda)}$. It follows that $\mathrm{supp}(w) = \bigcap_{P \in \mathcal{P}(\Lambda)} \mathrm{supp}(w_P)$, $A_w = \bigcap_{P \in \mathcal{P}(\Lambda)} A_{w_P}$, and $\overline{B}$ is identified to a subdirect product of domains through the injective ring morphism $\overline{B} \longrightarrow \prod_{P \in \mathcal{P}(\Lambda)} \overline{B_P}$, where $\overline{B_P} := B/\mathrm{supp}(w_P)$ for $P \in \mathcal{P}(\Lambda)$.

For any l-valuation $w : B \longrightarrow \widehat{\Lambda}$ with $A := A_w$, let $M := M(A, B)$ be the commutative sl-monoid associated as in 1.2 to the commutative ring extension $A \subseteq B$. The map $\widehat{w} : M \longrightarrow \widehat{\Lambda}$ sending a f.g. A-submodule $I = \sum_{1 \leq i \leq n} Ax_i \subseteq B$ to the element $\widehat{w}(I) := \bigwedge_{1 \leq i \leq n} w(x_i) \in \widehat{\Lambda}$ is a well-defined morphism of sl-monoids with a last element. Its image

$$\widehat{w}(M) = \{ \bigwedge_{1 \leq i \leq n} \varphi_i \mid n \in \mathbb{N}, \varphi_i \in w(B)\}$$

is a sl-submonoid of $\widehat{\Lambda}$, generated as meet-semilattice by $w(B)$, with

$$\widehat{w}(M)_+ = \widehat{w}(M_+) = \{ \bigwedge_{1 \leq i \leq n} \varphi_i \mid n \in \mathbb{N}, \varphi_i \in w(A)\} \subseteq \widehat{\Lambda}_+,$$

$$\widehat{w}(M)_- = \widehat{w}(M_-) = \{ \bigwedge_{1 \leq i \leq n} (\varphi_i)_-^{-1} \mid n \in \mathbb{N}, \varphi_i \in w(B \setminus A)\} \subseteq \Lambda_-.$$

The subset $U := \widehat{w}^{-1}(\Lambda) \subseteq M$ is closed under multiplication, and $M^* \subseteq \widetilde{M} \subseteq U$. Notice that $I \leq J$ (i.e. $J \subseteq I$) $\Longleftrightarrow \widehat{w}(I) \leq \widehat{w}(J)$ for all $I \in M^*, J \in M$, i.e. $I = \{x \in B \mid w(x) \geq \widehat{w}(I)\}$ for all $I \in M^*$. Indeed, a implication holds for all $I, J \in M$, while, assuming $I \in M^*$ and $\widehat{w}(I) \leq \widehat{w}(J)$, it follows that $\varepsilon \leq \widehat{w}(I^{-1}J)$, and hence $I^{-1}J \subseteq A$, so $J \subseteq I$ as desired. Consequently, M^* is torsion free, and the morphism $\widehat{w} : M \longrightarrow \widehat{\Lambda}$ induces by restriction a monomorphism of ordered groups $\widehat{w}|_{M^*} : M^* \longrightarrow \Lambda$.

Notice also that $I \in U, J \in M, J \leq I \Longrightarrow J \in U$ since Λ is a lower subset of $\widehat{\Lambda}$, therefore the monoid of fractions $\mathbf{M}_w := U^{-1}M$ is a sl-monoid with $[\frac{I}{J}] \wedge [\frac{I'}{J'}] = [\frac{IJ'+I'J}{JJ'}]$, and $\mathbf{M}_w^*$ is a lower subset of $\mathbf{M}_w$ and hence a l-group. Denote also by $\widehat{w}$ the extended morphism of sl-monoids $\mathbf{M}_w \longrightarrow \widehat{\Lambda}, [\frac{I}{J}] \mapsto \widehat{w}(I)\widehat{w}(J)^{-1}$.

Definition 1.13. *The image* $\widehat{w}(\mathbf{M}_w) = \{\varphi \in \widehat{\Lambda} \mid \exists \psi \in \widehat{w}(M) \cap \Lambda, \varphi\psi \in \widehat{w}(M)\}$ *is called the* value sl-monoid *of the l-valuation w and is denoted by $\mathcal{M}_w$. The l-subgroup $\mathcal{M}_w^* = \mathcal{M}_w \cap \Lambda = \widehat{w}(\mathbf{M}_w^*)$ of Λ is called the* value l-group *of w.*

Notice that $\mathcal{M}_w^*$ contains the l-subgroup generated by $w(B) \cap \Lambda$ as well as the l-subgroup generated by the subset $\{\varphi_- = \varphi(\varepsilon)^{-1} \mid \varphi \in w(B \setminus A_w)\}$. In particular, if w

is a valuation then $\mathcal{M}_w^* = \{\varphi \cdot \psi^{-1} \mid \varphi, \psi \in w(B) \cap \Lambda\}$ is the totally ordered value group of w. If in addition w is a Krull valuation, i.e. B is a field, then $\mathbf{M}_w^* \cong \mathcal{M}_w^*$.

The coarsening and equivalence relations for valuations extends to l-valuations as follows.

Definition 1.14. *A l-valuation $w' : B \longrightarrow \widehat{\Lambda}'$ is called* coarser *than the l-valuation $w : B \longrightarrow \widehat{\Lambda}$ (or a* coarsening *of w) if the following equivalent conditions are satisfied.*

(i) *There exists a morphism [3] of sl-monoids $F : \mathcal{M}_w \longrightarrow \mathcal{M}_{w'}$ such that, for all $x \in B$, $w'(x) = F(w(x))$, and $F(\mathcal{M}_w^*) = \mathcal{M}_{w'}^*$.*

(ii) *There exists a morphism of sl-monoids $f : \widehat{w}(M) \longrightarrow \widehat{w'}(M)$ such that, for all $x \in B$, $w'(x) = f(w(x))$, and $f(\widehat{w}(M) \cap \Lambda) = \widehat{w'}(M) \cap \Lambda'$.*

The relation above is a preordering inducing the equivalence relation :

Definition 1.15. *Two l-valuations w, w' on the commutative ring B are said to be* equivalent *(for short, $w \sim w'$) if the following equivalent conditions are satisfied :*

(i) *There is an isomorphism $F : \mathcal{M}_w \longrightarrow \mathcal{M}_{w'}$ of sl-monoids such that, for all $x \in B$, $w'(x) = F(w(x))$.*

(ii) *For all $x_1, \ldots, x_n, y \in B$, $\bigwedge_{1 \le i \le n} w(x_i) \le w(y) \iff \bigwedge_{1 \le i \le n} w'(x_i) \le w'(y)$, in particular, $A := A_w = A_{w'}$, and the sl-monoids $\mathbf{M}_w, \mathbf{M}_{w'}$ are isomorphic over $M = M(A, B)$.*

Remarks 1.16.   (1) If w' is coarser than w then $\text{supp}(w) \subseteq \text{supp}(w')$ and $A_w \subseteq A_{w'}$. If, in addition, w is a valuation then w' is a valuation too, and $\text{supp}(w) = \text{supp}(w')$.

(2) Let $w : B \longrightarrow \widehat{\Lambda}$ be a l-valuation. For any convex l-subgroup H of $\mathcal{M}_w^*$, the natural projection $\mathcal{M}_w^* \longrightarrow \Gamma_H := \mathcal{M}_w^*/H$ extends to a morphism of sl-monoids $h : \mathcal{M}_w \longrightarrow \widehat{\Gamma_H}$ (cf. Remark 1.7 and Proposition 1.8). The map $w/H = h \circ w : B \longrightarrow \widehat{\Gamma_H}$ is a l-valuation with $\mathcal{M}_{w/H} = h(\mathcal{M}_w)$, which is, up to equivalence, the minimal coarsening w' of w with $\mathcal{M}_{w'}^* \cong \Gamma_H$. In particular, taking $H = \{\varepsilon\}$, we obtain the minimal, up to equivalence, coarsening $\widetilde{w} := w/\{\varepsilon\}$ of w with $\mathcal{M}_{\widetilde{w}}^* \cong \mathcal{M}_w^*$, and hence $A_{\widetilde{w}} = A_w$.

If w is a valuation then the coarsenings w' of w correspond uniquely, up to equivalence, to the convex subgroups H of the totally ordered Abelian group $\mathcal{M}_w^*$ via $w' = w/H$; in particular, $\widetilde{w} \sim w$.

The following definition provides a generalisation of the basic notion of *Manis valuation* [21].

Definition 1.17. *Let $w : B \longrightarrow \widehat{\Lambda}$ be a l-valuation, $M := M(A_w, B)$ the sl-monoid associated to the ring extension $A_w \subseteq B$, and $\widehat{w} : M \longrightarrow \widehat{\Lambda}$ the canonical morphism of sl-monoids. w is said to be a* Manis l-valuation *if $\widehat{w}(M) \cap \Lambda$ is a l-group generated by the subset $\{\varphi_- \mid \varphi \in w(B \setminus A_w)\}$.*

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[3] The morphism F is necessarily surjective.

Assuming that w is a Manis l-valuation, it follows that $\mathcal{M}_w = \widehat{w}(M)$, $\mathcal{M}_w^* = \widehat{w}(M) \cap \Lambda$, and $(\mathcal{M}_w^*)_+ = \{\bigvee_{1 \leq i \leq n} (\varphi_i)_- \mid n \in \mathbb{N}, \varphi_i \in w(B \setminus A_w)\}$.

1.4 Prüfer extensions

A commutative unital ring extension $A \subseteq B$ is said to be a *Prüfer extension* if A is a B-Prüfer ring in the sense of Griffin [16]. These extensions relate to Manis valuations in much the same way as Prüfer domains with Krull valuations. They admit various characterizations as shown for instance in [19, I, Theorem 5.2.] We mention only the following two useful criteria for a commutative ring extension $A \subseteq B$ to be a Prüfer extension :

(P1) Every subring of B containing A is integrally closed in B.

(P2) For every element $x \in B$ there exists $y \in A$ such that $xy \in A$ and $x(1 - xy) \in A$.

In particular, a commutative ring A is a *Prüfer ring* if and only if the ring extension $A \subseteq B$ is Prüfer, where $B = Q(A)$ is the total ring of quotients of A.

Thus the Prüfer extensions are the models of an inductive ($\forall \exists$) theory in the first order language $(+, -, \cdot, 0, 1)$ of rings augmented with a unary predicate standing for a subring A of B.

The Prüfer extensions have a good "multiplicative ideal theory" [19, II].

Given a commutative ring extension $A \subseteq B$, let $M := M(A, B), M^*$, and $\widetilde{M}$ be as defined in 1.2. By [19, II, Theorem 1.13.], the ring extension $A \subseteq B$ is Prüfer if and only if I is B-invertible for all $I \in \widetilde{M}$, i.e. $M^* = \widetilde{M}$. Assuming that the ring extension $A \subseteq B$ is Prüfer, it follows by Remark 1.7. (see also [19, II, Corollary 1.14.]) that M^* is an Abelian l-group with $I \wedge J = I + J, I \vee J = I \cap J = (I^{-1} + J^{-1})^{-1}$ for all $I, J \in M^*$, so $M_+^* = \{I \in M^* \mid I \subseteq A\}$, and $I_+ = I \cap A, I_- = (A + I)^{-1} = I^{-1} \cap A, |I| = I \cap I^{-1} = I_+ I_- = I_+ \cap I_-$ for $I \in M^*$. By Remark 1.7. again, the canonical embedding of the Abelian l-group M^* into its commutative l-monoid extension $\widehat{M^*}$, as defined in 1.1, extends uniquely to the morphism of sl-monoids $\widehat{\mathfrak{w}} : M \longrightarrow \widehat{M^*}$, defined by $\widehat{\mathfrak{w}}(I)(\alpha) = I + \alpha$ for $I \in M, \alpha \in M_+^*$. Composing the morphism $\widehat{\mathfrak{w}} : M \longrightarrow \widehat{M^*}$ with the map $B \longrightarrow M, x \mapsto Ax$, we obtain the l-valuation $\mathfrak{w} : B \longrightarrow \widehat{M^*}$, called the *$l$-valuation associated to the Prüfer extension* $A \subseteq B$. Notice that $\mathfrak{w}$ is trivial (the constant map $x \mapsto A$) $\Longleftrightarrow A = B$.

We obtain $\{x \in B \mid \mathfrak{w}(x) \geq \gamma\} = \gamma$ for all $\gamma \in M^*$, in particular, $A_{\mathfrak{w}} = A$, and $\mathfrak{w}^{-1}(\gamma) = \gamma \setminus \bigcup_{\alpha \in M^*, \alpha \subsetneq \gamma} \alpha$ (may be empty) for $\gamma \in M^*$. It follows also that $\text{supp}(\mathfrak{w}) = \bigcap_{\alpha \in M_+^*} \alpha$ is the conductor of A in B. Indeed, assuming that $x \in B \setminus \text{supp}(\mathfrak{w})$, let $\alpha \in M_+^*$ be such that $x \notin \alpha$, so $\beta := \alpha + Ax \in M^*$ and $\beta^{-1} \subsetneq \alpha^{-1}$. Thus $xy \notin A$ for all $y \in \alpha^{-1} \setminus \beta^{-1}$, therefore $Bx \nsubseteq A$ as desired.

The morphism $\widehat{\mathfrak{w}} : M \longrightarrow \widehat{M^*}$ is induced by the l-valuation $\mathfrak{w} : B \longrightarrow \widehat{M^*}$, i.e. $\widehat{\mathfrak{w}}(I) = \wedge_{1 \leq i \leq n} \mathfrak{w}(x_i)$ for $I = \sum_{1 \leq i \leq n} Ax_i \in M$. Consequently, the sl-submonoid $\widehat{\mathfrak{w}}(M)$ of the l-monoid $\widehat{M^*}$ contains the Abelian l-group M^* and is generated as meet-semilattice by its submonoid $\mathfrak{w}(B)$. Thus $\mathcal{M}_{\mathfrak{w}} = \widehat{\mathfrak{w}}(M)$ is the value sl-monoid, while $\mathcal{M}_{\mathfrak{w}}^* = M^*$ is the value l-group of the l-valuation $\mathfrak{w}$ associated to the Prüfer extension $A \subseteq B$. As the l-group M^* is generated by the invertible ideals $\mathfrak{w}(x)_- = (A + Ax)^{-1}$

for $x \in B \setminus A$, $\mathfrak{w}$ is a Manis l-valuation (cf. Definition 1.17), and hence a Prüfer-Manis l-valuation according to the next definition which extends to l-valuations the basic notion of *Prüfer-Manis valuation* [19, I, 6, Definition 1]).

Definition 1.18. *Let $w : B \longrightarrow \widehat{\Lambda}$ be a l-valuation.*

  (1) *w is said to be a* Prüfer *l-valuation if the ring extension $A_w \subseteq B$ is Prüfer.*

  (2) *w is said to be a* Prüfer-Manis *l-valuation if w is a Prüfer as well as a Manis l-valuation.*

Remark 1.19. Let $w : B \longrightarrow \widehat{\Lambda}$ be a l-valuation with $A_w = B$. Let M denote the sl-monoid of the f.g. ideals of B. Then w is obviously Prüfer. It is Prüfer-Manis $\Longleftrightarrow \mathcal{M}_w^* = \widehat{w}(M) \cap \Lambda = \{\varepsilon\}$. As an example of such a l-valuation, let Λ be a nontrivial Abelian l-group, and let B be the underlying boolean ring of the boolean algebra with support $\partial\widehat{\Lambda}_+$ and opposite order. Then the inclusion $w : B \longrightarrow \widehat{\Lambda}$ is a l-valuation with $A_w = B, \mathcal{M}_w = w(B) = \partial\widehat{\Lambda}_+, \mathcal{M}_w^* = \{\varepsilon\}$. In particular, if Λ is totally ordered then $B = \mathbb{F}_2$, the field with 2 elements.

Lemma 1.20. *Let $A \subseteq B$ be a Prüfer extension, M-the sl-monoid of f.g. A-submodules of B, M^*-the l-group of invertible A-submodules of B, and $\mathfrak{w} : B \longrightarrow \widehat{M^*}$-the Prüfer-Manis l-valuation associated as above to the Prüfer extension $A \subseteq B$. Let $w : B \longrightarrow \widehat{\Lambda}$ be a Prüfer l-valuation with $A_w = A$. Then :*

  (1) *The morphism of sl-monoids $\widehat{w} : M \longrightarrow \widehat{\Lambda}$ induces by restriction a monomorphism of Abelian l-groups $M^* \longrightarrow \Lambda$, identifying M^* with $\widehat{w}(M^*) = \widehat{w}(M)^*$, the l-group of invertible elements of the sl-monoid $\widehat{w}(M)$.*

  (2) *$\widehat{\mathfrak{w}} = f \circ \widehat{w}$, $\mathfrak{w} = f \circ w$, and $\mathcal{M}_{\mathfrak{w}} = f(\widehat{w}(M))$, where $f : \widehat{w}(M) \longrightarrow \widehat{M^*}$ is the canonical morphism of sl-monoids extending the isomorphism $\widehat{w}(M)^* \cong M^*$ of Abelian l-groups.*

  (3) *$\widehat{w}(M) \cap \Gamma = \widehat{w}(M)^*$, where Γ is the convex hull of $\widehat{w}(M)^* \cong M^*$ in Λ. In particular, $w \sim \mathfrak{w}$ whenever $\Gamma = \Lambda$.*

Proof. The morphism $M^* \longrightarrow \Lambda, \alpha \mapsto \widehat{w}(\alpha)$ of Abelian l-groups is injective cf. 1.3. To show that $\widehat{w}(M^*) = \widehat{w}(M)^*$, let $\gamma \in \widehat{w}(M)^*$. By assumption, $\gamma = \widehat{w}(I), \gamma^{-1} = \widehat{w}(J)$ for some $I, J \in M$, and hence $\gamma = (\gamma \wedge \varepsilon)(\gamma^{-1} \wedge \varepsilon)^{-1} = \widehat{w}(I + A)\widehat{w}(J + A)^{-1} = \widehat{w}(K)$, where $K = (I + A)(J + A)^{-1} \in M^*$, as desired.

The assertion (2) of the lemma follows in a straightforward way.

To prove the assertion (3), let $\gamma := \widehat{w}(I) \in \widehat{w}(M) \cap \Gamma$. We have to show that $\gamma \in \widehat{w}(M)^* = \widehat{w}(M^*)$. By assumption there exists $\mathbf{a} \in M^*$ such that $\gamma \leq \widehat{w}(\mathbf{a})$, therefore $I + \mathbf{a} \in M^*$ and $\gamma = \widehat{w}(I + \mathbf{a}) \in \widehat{w}(M^*)$ as desired. Assuming that $\Gamma = \Lambda$, it follows that the morphism $f : \widehat{w}(M) \longrightarrow \widehat{M^*}$ of sl-monoids is injective, and hence $\mathcal{M}_w = \widehat{w}(M) \cong f(\widehat{w}(M)) = \mathcal{M}_{\mathfrak{w}}$, so $w \sim \mathfrak{w}$ as required. $\square$

Corollary 1.21. *Let $A \subseteq B$ be a Prüfer ring extension. Then $\mathfrak{w} : B \longrightarrow \widehat{M^*}$ is, up to equivalence, the minimal Prüfer-Manis l-valuation on B with l-valuation ring A, i.e. $\mathfrak{w}$ is coarser than any Prüfer-Manis l-valuation $w : B \longrightarrow \widehat{\Lambda}$ with $A_w = A$. Moreover for any such w, $\mathcal{M}_w^* \cong \mathcal{M}_{\mathfrak{w}}^* = M^*$.*

Corollary 1.22. ([19, I, Proposition 5.1]) *Let $A \subseteq B$ be a Prüfer extension such that $A \neq B$. The necessary and sufficient condition for the Prüfer-Manis l-valuation $\mathfrak{w} : B \longrightarrow \widehat{M^*}$ to be a valuation, i.e. M^* is totally ordered, is that the set $B \setminus A$ is multiplicatively closed.*

Proof. Assuming that the Abelian group M^* is totally ordered, let $x, y \in B \setminus A$ be such that $xy \in A$. As $A + Ax, A + Ay \in M^*$, we may assume without loss that $x \in A + Ay$, so $x = \lambda + \mu y$ for some $\lambda, \mu \in A$, and hence $P(x) = 0$, where $P(X) = X^2 - \lambda X - \mu xy \in A[X]$. Since A is integrally closed in B, it follows that $x \in A$, i.e. a contradiction. Conversely, assume that $B \setminus A$ is multiplicatively closed and there exists $\alpha \in M^*$ such that $\alpha \not\subseteq A$ and $\alpha^{-1} \not\subseteq A$. Choose $x \in \alpha \setminus A, y \in \alpha^{-1} \setminus A$. As $B \setminus A$ is multiplicatively closed by assumption, it follows that $xy \notin A$, contrary to the fact that $xy \in \alpha\alpha^{-1} = A$. Consequently, M^* is totally ordered, and hence $\mathfrak{w}$ is a valuation, as required. $\qquad\square$

In the following four sections (2 - 5) we provide an abstract axiomatic framework for the residue structures induced by Prüfer extensions.

2   Commutative regular semirings and quasi-semirings

Definition 2.1. *By a **commutative regular semiring**, abbreviated a **cr-sring**, we understand an algebra $(R, +, \bullet, -, ^{-1}, \varepsilon)$ of signature $(2, 2, 1, 1, 0)$, satisfying the following conditions*

(1) *$(R, +)$ and $(R, \bullet)$ are commutative regular semigroups, with $-x$ and x^{-1} the corresponding quasi-inverses of any element $x \in R$;*

(2) *$E^+ \cap E^\bullet = \{\varepsilon\}$, where $E^+ := \{x \in R \mid 2x := x + x = x\} = \{e^+(x) := x + (-x) \mid x \in R\}, E^\bullet := \{x \in R \mid x^2 := x \bullet x = x\} = \{e^\bullet(x) := x \bullet x^{-1} \mid x \in R\}$;*

(3) ***Distributive law :*** *$x \bullet y + x \bullet z = x \bullet (y + z)$ for all $x, y, z \in R$.*

Remarks 2.2. (1) The *commutative*, not necessarily unital, *regular rings* (in the sense of von Neumann) are exactly those cr-srings R for which $(R, +)$ is an Abelian group. In this case, the neutral element $\varepsilon = 0$ is the unique element of E^+, while $E^\bullet$ is a quasiboolean lattice with the least element 0 under the operations $x \wedge y = x \bullet y, x \vee y = x + y - x \bullet y, x \setminus y = x - x \bullet y$. $E^\bullet$ is a Boolean algebra $\Longleftrightarrow R$ has a unit 1, in which case $\neg x = 1 - x$. Thus the fields and the Boolean rings are amongst the simplest examples of cr-srings.

(2) On the opposite side, the *Abelian l-groups* are identified with those cr-srings R for which $(R, \bullet)$ is an Abelian group. In this case, the neutral element $\varepsilon = 1$ is the unique element of $E^\bullet$, while $E^+ = R$, so $(R, +)$ is a semilattice. Indeed, we obtain by 2.1.(3) and 2.1.(2) the identities

$$x + x = x \bullet \varepsilon + x \bullet \varepsilon = x \bullet (\varepsilon + \varepsilon) = x \bullet \varepsilon = x$$

for all $x \in R$. As the group and the semilattice operations $\bullet$ and $+$ are compatible by 2.1.(3), it follows that $(R, \bullet, \wedge, \vee)$ is an Abelian l-group, where

$$x \wedge y = x + y, x \vee y = (x^{-1} + y^{-1})^{-1}$$

Notice that the trivial cr-sring (the singleton) is the only common member of the two remarkable subclasses of cr-srings considered above.

(3) In the presence of the axioms 2.1.(1) and 2.1.(3), the axiom 2.1.(2) is equivalent with the conjunction of the following two equational axioms

2.1.(2') $\varepsilon \in E^+$, i.e. $2\varepsilon = \varepsilon$, and

2.1.(2") $E^+ \subseteq R_\varepsilon^\bullet := \{x \in R \mid e^\bullet(x) = \varepsilon\}$, i.e. $e^\bullet(e^+(x)) = \varepsilon$ for all $x \in R$.

The implications 2.1.(2) $\Longrightarrow$ 2.1.(2'), and (2.1.(2') $\bigwedge$ 2.1(2")) $\Longrightarrow$ 2.1.(2) are obvious, so it remains to prove the implication 2.1.(2) $\Longrightarrow$ 2.1.(2"). Assuming that $x \in E^+$, we obtain by 2.1.(3)

$$e^\bullet(x) = x^{-1} \bullet x = x^{-1} \bullet (x + x) = x^{-1} \bullet x + x^{-1} \bullet x = 2e^\bullet(x),$$

so $e^\bullet(x) \in E^+ \cap E^\bullet$, and hence $e^\bullet(x) = \varepsilon$ by 2.1.(2), as desired.

Thus the class of all cr-srings is a variety of algebras of signature (2, 2, 1, 1, 0), i.e. it is closed under homomorphic images, subalgebras and direct products.

We introduce a larger class of algebras of the same signature as above by relaxing suitably the distributive law 2.1.(3) and adding three new natural axioms as follows.

Definition 2.3. *By a **commutative regular quasi-semiring**, abbreviated a **cr-qsring**, we understand an algebra $(R, +, \bullet, -, ^{-1}, \varepsilon)$ of signature $(2, 2, 1, 1, 0)$, satisfying the axioms $2.1.(1), (2)$, and the following new axioms*

(3') ***Quasidistributive law :** $x \bullet y + x \bullet z \leq x \bullet (y + z)$ for all $x, y, z \in R$, where $\leq$ denotes the canonical partial order on $(R, +) : x \leq y \Longleftrightarrow x = y + e^+(x)$;*

(4) *$y \leq z \Longrightarrow x \bullet y \leq x \bullet z$ for all $x, y, z \in R$;*

(5) *$-(x \bullet y) = x \bullet (-y)$ for all $x, y \in R$;*

(6) *$e^+(x + y) \leq e^+(x \bullet y)$ for all $x, y \in E^\bullet$.*

The next lemmas collect some basic properties of the cr-qsrings.

Lemma 2.4. *Let R be a cr-qsring. With the notation above, the following statements hold.*

(1) *The unary operations $x \mapsto -x$ and $x \mapsto x^{-1}$ commute, i.e. $(-x)^{-1} = -(x^{-1})$ for all $x \in R$.*

(2) *The multiplication induces an action of the multiplicative semigroup $(R, \bullet)$ on the additive idempotent semigroup (semilattice) $(E^+, +)$, i.e. $x \bullet f \in E^+$, and $x \bullet (f + g) = x \bullet f + x \bullet g$ for all $x \in R, f, g \in E^+$.*

(3) $E^+ = R_\varepsilon^\bullet := \{x \in R \mid e^\bullet(x) = \varepsilon\}$.

(4) $(E^+, \bullet, \wedge, \vee)$ *is a multiplicative Abelian l-group, with the lattice operations* $e \wedge f = e + f, e \vee f = (e^{-1} + f^{-1})^{-1}$.

(5) *The map* $v : R \longrightarrow E^+, x \mapsto v(x) := \varepsilon \bullet x$ *is a* quasivaluation *(abreviatted a* qvaluation*) on the cr-qsring R with values in the multiplicative Abelian l-group* E^+, *i.e.* $v : (R, \bullet) \longrightarrow (E^+, \bullet)$ *is a surjective morphism of semigroups, and* $v(x + y) \geq v(x) \wedge v(y) = v(x) + v(y)$ *for all* $x, y \in R$. *In addition,* $v(-x) = v(x)$ *for all* $x \in R$, *and* $v(x) \leq v(y)$ *provided* $x \leq y$.

(6) *The epimorphism* $v : (R, \bullet) \longrightarrow (E^+, \bullet)$ *splits, and the embedding* $E^+ \hookrightarrow R$ *is the unique section s of v satisfying in addition* $s(f + g) = s(f) + s(g)$ *for all* $f, g \in E^+$. *Moreover the qvaluation v satisfies the following universal property: for every Abelian l-group* $(\Lambda, +, \wedge, \vee)$ *and for every morphism of semigroups* $w : (R, \bullet) \longrightarrow (\Lambda, +)$ *satisfying* $w(f + g) = w(f) \wedge w(g)$ *for all* $f, g \in E^+$, *there exists a unique morphism of l-groups* $\varphi : E^+ \longrightarrow \Lambda$ *such that* $\varphi \circ v = w$. *In particular, any such map w is a qvaluation provided it is surjective.*

Proof. (1) follows easily from 2.3.(5).

(2) For $x \in R, f \in E^+$, we obtain by 2.3.(3')

$$x \bullet f = x \bullet (f + f) \geq x \bullet f + x \bullet f,$$

so

$$x \bullet f + x \bullet f = x \bullet f + e^+(x \bullet f + x \bullet f) = x \bullet f + e^+(x \bullet f) = x \bullet f,$$

and hence $x \bullet f \in E^+$, as desired.

For $x \in R, f, g \in E^+$, it follows by 2.3.(3') again that $x \bullet (f + g) \geq x \bullet f + x \bullet g$. On the other hand, $f, g \in E^+ \implies f + g = f \wedge g \leq f$, and hence $x \bullet (f + g) \leq x \bullet f$ by 2.3.(4). Similarly, we obtain $x \bullet (f + g) \leq x \bullet g$, therefore $x \bullet (f + g) \leq x \bullet f \wedge x \bullet g = x \bullet f + x \bullet g$, since $x \bullet f, x \bullet g \in E^+$ as shown above. Consequently, we obtain the required equality.

Thus we have obtained a morphism of semigroups $\psi : (R, \bullet) \longrightarrow \text{End}(E^+, +)$ defined by $\psi(x)(f) = x \bullet f$ for $x \in R, f \in E^+$. We shall see later that the image of ψ is contained in $\text{Aut}(E^+, +)$.

(3, 4) We already know from (2) above that E^+ is closed under multiplication. It remains to show that $E^+ = R_\varepsilon^\bullet$ to conclude that E^+ is an Abelian group under multiplication, with ε as neutral element. Assuming that $f \in E^+$, it follows by (2) that $e^\bullet(f) = f^{-1} \bullet f \in E^+ \cap E^\bullet$, and hence $e^\bullet(f) = \varepsilon$, i.e. $f \in R_\varepsilon^\bullet$, since $E^+ \cap E^\bullet = \{\varepsilon\}$ by 2.1.(2). Conversely, assuming that $x \in R_\varepsilon^\bullet$, we obtain $x = x \bullet e^\bullet(x) = x \bullet \varepsilon \in E^+$ by (2) since $\varepsilon \in E^+$. Thus $E^+ = R_\varepsilon^\bullet$, as desired.

Since the group operation $\bullet$ and the semilattice operation $+$ on E^+ are compatible by (2), we obtain the required structure of Abelian l-group on E^+. In particular, the multiplicative Abelian group $(E^+, \bullet)$ is torsion free.

(5, 6) Setting $v(x) := \varepsilon \bullet x \in E^+$ for $x \in R$, we obtain for arbitrary $x, y \in R$:

$$v(x) \bullet v(y) = (\varepsilon \bullet x) \bullet (\varepsilon \bullet y) = \varepsilon^2 \bullet (x \bullet y) = \varepsilon \bullet (x \bullet y) = v(x \bullet y),$$

and
$$v(x + y) = \varepsilon \bullet (x + y) \geq \varepsilon \bullet x + \varepsilon \bullet y = v(x) + v(y)$$
by 2.3.(3').

Notice that $v(-x) = v(x)$ for all $x \in R$ since the Abelian group $(E^+, \bullet)$ is torsion free, and $(-x)^2 = x^2$ (by 2.3.(4)) $\implies v(-x)^2 = v(x)^2$, while $x \leq y \implies v(x) = \varepsilon \bullet x \leq \varepsilon \bullet y = v(y)$ by 2.3.(4).

For $x \in E^+$, we obtain $v(x) = \varepsilon \bullet x = x$ by (3). Thus v is a qvaluation, and the embedding $i : (E^+, \bullet) \hookrightarrow (R, \bullet)$ is a section of the epimorphism $v : (R, \bullet) \longrightarrow (E^+, \bullet)$ satisfying the supplementary property $i(f + g) = i(f) + i(g)$ for all $f, g \in E^+$. To prove its uniqueness, let $s : E^+ \longrightarrow R$ be a map satisfying $v \circ s = 1_{E^+}$ and $s(f+g) = s(f)+s(g)$ for $f, g \in E^+$. We have to show that $s(f) = f$ for all $f \in E^+$. It follows that $s(f) \in E^+$ for all $f \in E^+$ since $s(f) + s(f) = s(f + f) = s(f)$, therefore $s(f) = v(s(f)) = f$, as required.

To prove the universal property of the qvaluation v, let Λ and $w : R \longrightarrow \Lambda$ be as in the statement (6) above. First notice that if a map $\varphi : E^+ \longrightarrow \Lambda$ satisfies $\varphi \circ v = w$, then $\varphi(f) = \varphi(v(f)) = w(f)$ for all $f \in E^+$, so $\varphi = w \mid_{E^+}$ is unique. By assumption, the map φ as defined above is a morphism of l-groups, and $\varphi(v(x)) = w(v(x)) = w(\varepsilon \bullet x) = w(\varepsilon) + w(x) = 0 + w(x) = w(x)$ for all $x \in R$, so $\varphi \circ v = w$ as desired. As a composition of a qvaluation with a morphism of l-groups, the map w is a qvaluation too provided it is surjective.

Finally notice that the morphism $\psi : (R, \bullet) \longrightarrow \mathrm{End}(E^+, +)$ as defined in (2) is the composition of the epimorphism $v : (R, \bullet) \longrightarrow (E^+, \bullet)$ with the canonical monomorphism $\iota : (E^+, \bullet) \longrightarrow \mathrm{Aut}(E^+, +) \subseteq \mathrm{End}(E^+, +)$ defined by $\iota(f)(g) = f \bullet g$ for $f, g \in E^+$. Indeed, we obtain
$$\psi(x)(f) = x \bullet f = x \bullet (\varepsilon \bullet f) = (x \bullet \varepsilon) \bullet f = v(x) \bullet f = \iota(v(x))(f)$$
for $x \in R, f \in E^+$, i.e. $\psi = \iota \circ v$ as desired. In particular, the image of ψ is a subgroup of $\mathrm{Aut}(E^+, +)$, identified with the multiplicative Abelian group $(E^+, \bullet)$. □

Remark 2.5. Though very similar, the notions of valuation and qvaluation are quite different. Thus, the constant map $x \mapsto 0$, the unique qvaluation on any field F, is distinct from the trivial valuation on F. However the qvaluations are naturally related to the valued fields and the Prüfer domains [6].

Lemma 2.6. *The surjective maps* $e^+ : R \longrightarrow E^+, e^\bullet : R \longrightarrow E^\bullet$, *and* $v : R \longrightarrow E^+$ *have the following properties.*

(1) $v \leq e^+$, *i.e.* $v(x) \leq e^+(x)$ *for all* $x \in R$. *In particular, the restriction* $e^+ \mid_{E^\bullet}$ *takes values in* $E_+^+ := \{f \in E^+ \mid f \geq \varepsilon\}$, *the positive cone of the Abelian l-group* $(E^+, \bullet, + = \wedge, \vee)$.

(2) $e^+ = v \bullet (e^+ \circ e^\bullet)$, *i.e.* $e^+(x) = v(x) \bullet e^+(e^\bullet(x)) = x \bullet e^+(e^\bullet(x))$ *for all* $x \in R$. *In particular,* $e^+(x^{-1}) = v(x)^{-2} \bullet e^+(x) = x^{-2} \bullet e^+(x)$ *for all* $x \in R$.

(3) *The restriction* $e^+ \mid_{E^\bullet} : (E^\bullet, \bullet) \longrightarrow (E_+^+, +)$ *is a morphism of semilattices, i.e.* $e^+(x \bullet y) = e^+(x) + e^+(y)$ *for all* $x, y \in E^\bullet$, *which preserves the common least element* ε.

(4) **Leibniz's rule :** *The additive map* $e^+ : R \longrightarrow E^+$ *is a "derivation", i.e.*
$e^+(x \bullet y) = x \bullet e^+(y) + y \bullet e^+(x)$ *for all* $x, y \in R$.

(5) $\{x \in E^\bullet \mid e^+(x) = \varepsilon\} = \{x + \varepsilon \mid x \in E^\bullet\}$.

(6) $e^+(e^\bullet(x)) \leq e^+(e^\bullet(y))$ *provided* $x \leq y$.

Proof. (1) For any $x \in R$, it follows by Lemma 2.4.(5, 6) that

$$e^+(x) = v(e^+(x)) = v(x + (-x)) \geq v(x) + v(-x) = v(x) + v(x) = v(x),$$

while for $x \in E^\bullet$, we obtain $e^+(x) \geq v(x) = v(x \bullet x^{-1}) = \varepsilon$, as required.

(2) For any $x \in R$, it follows by 2.3.(3') and 2.3.(5) that

$$v(x)^{-1} \bullet e^+(x) = x^{-1} \bullet (x + (-x)) \geq x^{-1} \bullet x + (-(x^{-1} \bullet x)) = e^+(e^\bullet(x))$$

and

$$v(x) \bullet e^+(e^\bullet(x)) = x \bullet (e^\bullet(x) + (-e^\bullet(x))) \geq x + (-x) = e^+(x),$$

and hence the desired identity.

(3) Let $x, y \in E^\bullet$, in particular, $v(x) = v(y) = \varepsilon$. It follows by 2.3.(3') and 2.3.(5) again that

$$e^+(x \bullet y) = x \bullet y + x \bullet (-y) \leq x \bullet e^+(y) = v(x) \bullet e^+(y) = e^+(y),$$

and similarly, $e^+(x \bullet y) \leq e^+(x)$. As $(E^+, +)$ is a meet-semilattice, we deduce the inequality $e^+(x \bullet y) \leq e^+(x) + e^+(y)$. To get an equality, we have to appeal for the first time to the axiom 2.3.(6).

(4) Let $x, y \in R$. By (2) and (3) above, we obtain $e^+(x \bullet y) = x \bullet y \bullet e^+(e^\bullet(x \bullet y)) = x \bullet y \bullet e^+(e^\bullet(x) \bullet e^\bullet(y)) = x \bullet y \bullet (e^+(e^\bullet(x)) + e^+(e^\bullet(y))) = y \bullet e^+(x) + x \bullet e^+(y)$, as required.

(5) Assuming that $x \in E^\bullet$ and $e^+(x) = \varepsilon$, it follows that $x = x + \varepsilon$ since $x + \varepsilon \leq x$ and $e^+(x + \varepsilon) = e^+(x) + \varepsilon = \varepsilon = e^+(x)$. Conversely, for any $x \in E^\bullet$, we obtain $e^+(x + \varepsilon) = e^+(x) + \varepsilon = \varepsilon$ since $x \in E^\bullet \Longrightarrow e^+(x) \geq \varepsilon$, so it remains to show that $x + \varepsilon \in E^\bullet$. As $v(x) = \varepsilon \bullet x = \varepsilon$, it follows by 2.3.(4) that $(x + \varepsilon)^2 \geq x + \varepsilon$, and hence, again by 2.3.(4), $e^\bullet(x + \varepsilon) = (x + \varepsilon) \bullet (x + \varepsilon)^{-1} \leq (x + \varepsilon)^2 \bullet (x + \varepsilon)^{-1} = x + \varepsilon$, therefore $e^\bullet(x + \varepsilon) = x + (\varepsilon + e^+(e^\bullet(x + \varepsilon))) = x + \varepsilon$, i.e. $x + \varepsilon \in E^\bullet$, as desired.

(6) Let $x, y \in R$ be such that $x \leq y$, in particular, $e^+(x) \vee v(y) \leq e^+(y)$ by (1). By (2) it follows that $e^+(e^\bullet(x)) = e^+(x) \bullet v(x)^{-1} = e^+(x) \bullet v(y + e^+(x))^{-1} \leq e^+(x) \bullet (v(y) + e^+(x))^{-1} = (e^+(x) \vee v(y)) \bullet v(y)^{-1} \leq e^+(y) \bullet v(y)^{-1} = e^+(e^\bullet(y))$, as required. $\square$

A characterization of those cr-qsrings which are cr-srings is given by the next lemma.

Lemma 2.7. *The following statements are equivalent for an algebra* $(R, +, \bullet, -, ^{-1}, \varepsilon)$ *of signature* $(2, 2, 1, 1, 0)$.

(1) R *is cr-sring.*

(2) R *is a cr-qsring satisfying the following equivalent conditions.*

(i) $e^+ = v$;

(ii) $e^+(x) = \varepsilon$ for all $x \in E^\bullet$;

(iii) $e^+(x \bullet y) = e^+(x) \bullet e^+(y)$ for all $x, y \in R$;

(iv) $v(x + y) = v(x) + v(y)$ for all $x, y \in R$.

Proof. First notice that in a cr-qsring R, the conditions (i) - (iv) above are equivalent since (i) $\Longleftrightarrow$ (ii) by Lemma 2.6.(2), (i) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iv) are obvious, (iii) $\Longrightarrow$ (ii) as $x \in E^\bullet \Longrightarrow e^+(x) = e^+(x^2) = e^+(x)^2$ (by assumption) $\Longrightarrow e^+(x) = \varepsilon$ for all $x \in E^\bullet$, while (iv) $\Longrightarrow$ (i) as $e^+(x) = v(e^+(x)) = v(x - x) = v(x) + v(-x)$ (by assumption), so $e^+(x) = 2v(x) = v(x)$ for all $x \in R$, as desired.

(2) $\Longrightarrow$ (1) : We have to show that R satisfies 2.1.(3). As $x \bullet y + x \bullet z \leq x \bullet (y + z)$ by 2.3.(3'), the desired equality is a consequence of the identity

$$e^+(x \bullet y + x \bullet z) = e^+(x \bullet (y + z)),$$

which is immediate by (iii) and the identity $f \bullet g + f \bullet h = f \bullet (g + h)$ for $f, g, h \in E^+$.

(1) $\Longrightarrow$ (2) : Assuming that R is a cr-sring, we have to show that R satisfies the axioms 2.3.(4), (5), (6), and the equivalent conditions (i) - (iv) above. First notice that for all $x \in R$, $f \in E^+$, it follows by 2.1.(3) that $x \bullet f = x \bullet (f + f) = x \bullet f + x \bullet f$, i.e. $x \bullet f \in E^+$. Consequently, assuming $y \leq z$, i.e. $y = z + f$ for some $f \in E^+$, it follows again by 2.1.(3) that $x \bullet y = x \bullet z + x \bullet f$, so $x \bullet y \leq x \bullet z$ as $x \bullet f \in E^+$. Thus 2.3.(4) is true on R. On the other hand, 2.3.(5) is an immediate consequence of 2.1.(3), while 2.3.(6) is obvious since R satisfies the condition (ii) : assuming that $x \in E^\bullet$, it follows by 2.1.(3) that $e^+(x)^2 = (x - x) \bullet (x - x) = 2(x^2 - x^2) = x - x = e^+(x)$, therefore $e^+(x) \in E^+ \cap E^\bullet = \{\varepsilon\}$ by 2.1.(2).   □

Remarks 2.8. (1) Since all the axioms involving the binary relation $\leq$ are equivalent with identities, while, as in the case of cr-srings (see Remarks 2.2.(3)), the axiom 2.1.(2) can be replaced by a conjunction of identities, the class of all cr-qsrings is a variety of algebras of signature $(2, 2, 1, 1, 0)$, and the cr-srings form a subvariety of it.

(2) For any cr-qsring R, the subset

$$\widetilde{R} := \{x \in R \mid e^+(x) = v(x)\} = \{x \in R \mid e^+(e^\bullet(x)) = \varepsilon\}$$

is the support of the largest cr-sring contained in R. By Lemma 2.7., it suffices to show that $\widetilde{R}$ is a substructure of R. The closure under the unary operations $-$ and $^{-1}$ is obvious, the closure under the multiplication follows by Lemma 2.6.(3), while for $x, y \in \widetilde{R}$, it follows by Lemma 2.6.(1) and Lemma 2.5.(5) that

$$v(x + y) \leq e^+(x + y) = e^+(x) + e^+(y) = v(x) + v(y) \leq v(x + y),$$

therefore $x + y \in \widetilde{R}$. Notice that $E^+ \subseteq \widetilde{R}$, and $\widetilde{R}^+_\varepsilon := \{x \in \widetilde{R} \mid e^+(x) = \varepsilon\}$ is the largest regular ring contained in R.

3 Directed commutative regular quasi-semirings

We introduce an interesting class of cr-qsrings whose intersection with the class of all cr-srings is the class of all Abelian l-groups.

Proposition 3.1. *The following properties are equivalent for a cr-qsring R.*

(1) *$\varepsilon \leq x$ for all $x \in E^\bullet$.*

(2) *$v \leq 1_R$, i.e. $v(x) \leq x$ for all $x \in R$.*

(3) *$E^+ \cap x \downarrow = v(x) \downarrow$ for all $x \in R$, where $x \downarrow := \{y \in R \mid y \leq x\}$.*

(4) *$x + v(x - y) = y + v(x - y)$ for all $x, y \in R$.*

(5) *There exists the meet $x \wedge y$ with respect to the partial order $\leq$ for any pair (x, y) of elements of R, and the semilattice operation $\wedge$ is compatible with the multiplication, i.e. $x \bullet (y \wedge z) = (x \bullet y) \wedge (x \bullet z)$ for all $x, y, z \in R$.*

(6) *The partial orders $\leq$ and $\preceq$ coincide on $E^\bullet$, where $x \preceq y \iff x = y \bullet e^\bullet(x)$ is the partial order induced by multiplication.*

Proof. (1) $\implies$ (2) : Let $x \in R$. By assumption, $\varepsilon \leq e^\bullet(x)$, and hence $v(x) = x \bullet \varepsilon \leq x \bullet e^\bullet(x) = x$ by 2.3.(4).

(2) $\implies$ (3) : Let $x \in R, f \in E^+ \cap x \downarrow$. By Lemma 2.4.(5, 6), it follows that $f = v(f) \leq v(x)$. Conversely, let $y \in v(x) \downarrow$, i.e. $y \leq v(x)$, in particular, $y = v(x) + e^+(y) \in E^+$ since $v(x) \in E^+$. As $v(x) \leq x$ by assumption, we deduce also that $y \in x \downarrow$ as desired.

(3) $\implies$ (1) is obvious since $v(x) = \varepsilon$ for all $x \in E^\bullet$.

(2) $\implies$ (4) : Let $x, y \in R$. By assumption $v(x - y) \leq x - y$, so $v(x - y) = v(y - x) = x - y + v(x - y)$. Consequently, $y + v(x - y) = y + x - y + v(x - y) = x + (e^+(y) + v(x - y)) = x + v(x - y)$ since $v(x - y) \leq e^+(x - y) = e^+(x) + e^+(y) \leq e^+(y)$.

(4) $\implies$ (5) : Let $x, y \in R$. By assumption, $z := x + v(x - y) = y + v(x - y)$, so $z \leq x$ and $z \leq y$. It remains to show that $u \leq x, u \leq y \implies u \leq z$ to conclude that $z = x \wedge y$ is the meet of the pair (x, y) with respect to the partial order $\leq$. For such an element u, we obtain $-u \leq -y$, so $e^+(u) = u - u \leq x - y$, and hence $e^+(u) = v(e^+(u)) \leq v(x - y)$ by Lemma 2.4.(5, 6). Consequently, $u = u + e^+(u) \leq x + v(x - y) = z$, as required.

It remains to verify the compatibility with the multiplication of the semilattice operation $\wedge$. Let $x, y, z \in R$. The inequality $x \bullet (y \wedge z) \leq x \bullet y \wedge x \bullet z$ follows by 2.3.(4). On the other hand, $x \bullet y - x \bullet z \leq x \bullet (y - z)$ (by 2.3.(4, 5)) implies by Lemma 2.4.(5) and 2.3.(4) the opposite inequality

$$x \bullet y \wedge x \bullet z = x \bullet y + v(x \bullet y - x \bullet z) \leq x \bullet y + v(x \bullet (y - z)) =$$

$$x \bullet y + x \bullet v(y - z) \leq x \bullet (y + v(y - z)) = x \bullet (y \wedge z).$$

(5) $\implies$ (6) : Since for all $x, y \in E^\bullet$, $x \bullet y \in E^\bullet$ is the meet of the pair (x, y) with respect to the partial order $\preceq$, we have to show that $x \bullet y = x \wedge y$, the meet of the same pair with respect to the partial order $\leq$, which exists by assumption.

Let us show that $x \wedge y \in E^\bullet$ provided $x, y \in E^\bullet$. Since $x, y \in E^\bullet$ and the operations $\wedge$ and $\bullet$ are compatible by assumption, it follows that $(x \wedge y)^2 = x \wedge y \wedge (x \bullet y)$, in particular, $(x \wedge y)^2 \leq x \wedge y$. Moreover the last inequality becomes an equality since, using Lemma 2.6.(4) and again the compatibility above, we obtain

$$e^+((x \wedge y)^2) = (x \wedge y) \bullet e^+(x \wedge y) = (x \bullet e^+(x \wedge y)) \wedge (y \bullet e^+(x \wedge y)) = e^+(x \wedge y),$$

as $x \bullet f = f$ for $x \in E^\bullet, f \in E^+$.

Consequently, $x \wedge y \in E^\bullet$, and $x \wedge y = x \wedge y \wedge (x \bullet y)$, i.e. $x \wedge y \leq x \bullet y$, so it remains to show that $e^+(x \bullet y) \leq e^+(x \wedge y)$ to obtain the required equality $x \wedge y = x \bullet y$. As $e^+ \mid_{E^\bullet} : (E^\bullet, \bullet) \longrightarrow (E^+, +)$ is a morphism of semilattices by Lemma 2.6.(3), it suffices to check that $x \bullet y \preceq x \wedge y$. Using again the compatibility of the operations $\wedge$ and $\bullet$, we obtain $(x \bullet y) \bullet (x \wedge y) = x^2 \bullet y \wedge x \bullet y^2 = x \bullet y$, i.e. $x \bullet y \preceq x \wedge y$, as desired.

$(6) \Longrightarrow (1)$: Let $x \in E^\bullet$. Since $v(x) = \varepsilon \bullet x = \varepsilon$, i.e. $\varepsilon \preceq x$, it follows by assumption that $\varepsilon \leq x$, as required.   □

Definition 3.2. *A cr-qsring R is said to be* directed *if R satisfies the equivalent conditions $(1) - (6)$ from* Proposition 3.1.

Remarks 3.3. (1) A directed cr-qsring R is *trivial*, i.e. $R = \{\varepsilon\}$, if and only if $E^+ = \{\varepsilon\}$.

(2) By Lemma 2.7., Proposition 3.1. and Remarks 2.2.(2), it follows that the Abelian l-groups are identified with those cr-srings which are also directed cr-qsrings.

(3) By Remarks 2.8.(1) and Proposition 3.1.(4), the class of all directed cr-qsrings is a subvariety of the variety of cr-qsrings.

The next lemmas collect some basic properties of the directed cr-qsrings.

Lemma 3.4. *Let R be a directed cr-qsring. Then the following assertions hold.*

(1) $e^+(x \wedge y) = v(x - y) = \varepsilon \bullet (x - y)$, *and* $v(x \wedge y) = v(x) \wedge v(y) = v(x) + v(y)$ *for all $x, y \in R$. In particular, $v(x) = x \wedge e^+(x)$ for all $x \in R$.*

(2) $v(x - y) = (v(x) + v(y)) \bullet e^+(e^\bullet(x \wedge y))$ *for all $x, y \in R$.*

In particular, $v(x \pm y) \leq v(x) \bullet e^+(e^\bullet(y)) = v(x) \bullet v(y)^{-1} \bullet e^+(y)$ *for all $x, y \in R$.*

(3) *The semilattice operation $\wedge$ is compatible with the operations $+$ and $-$, i.e.*
$x + (y \wedge z) = (x + y) \wedge (x + z)$ *and* $(-y) \wedge (-z) = -(y \wedge z)$ *for all $x, y, z \in R$.*

(4) *The following assertions are equivalent for $x, y \in R$.*

   (i) $e^+(x \wedge y) = e^+(x) + e^+(y)$;

   (ii) $e^+(x \wedge y) = x - y$;

   (iii) $x - y \in E^+$.

In particular, $e^+(x \wedge y) = e^+(x) + e^+(y) = x - y = y - x$ provided the pair (x, y) is bounded above with respect to the partial order $\leq$, i.e. $x \leq z$ and $y \leq z$ for some $z \in R$. Moreover $x \wedge y = x + y - z$, $x + y = x \wedge y + z$, and $(x + y) \wedge z \leq x \wedge y$ provided $x \leq z$ and $y \leq z$; in particular, $x \wedge y = x + y \Longleftrightarrow x + y \leq z$.

(5) $e^+(x \wedge y \wedge z) = e^+(x \wedge z) + e^+(y \wedge z)$ *for all* $x, y, z \in R$.

Proof. (1) By Proposition 3.1.(4, 5), $x \wedge y = x + v(x - y)$, and hence $e^+(x \wedge y) = e^+(x) + v(x - y) = v(x - y)$ since $v(x - y) \leq e^+(x - y) = e^+(x) + e^+(y) \leq e^+(x)$. To show that $v(x \wedge y) = v(x) + v(y)$, notice that

$$(x \wedge y \leq x) \bigwedge (x \wedge y \leq y) \implies v(x \wedge y) \leq v(x) \wedge v(y) = v(x) + v(y)$$

by Lemma 2.4.(5), while $v(x) \leq x$ and $v(y) \leq y$ (by Proposition 3.1.(2)) imply $v(x) + v(y) = v(x) \wedge v(y) \leq x \wedge y$, therefore $v(x) + v(y) = v(v(x) + v(y)) \leq v(x \wedge y)$, again by Lemma 2.4.(5).

The identity $v(x) = x \wedge e^+(x)$ is a consequence of the inequality $v(x) \leq x \wedge e^+(x)$ and of the equality $e^+(x \wedge e^+(x)) = v(x - e^+(x)) = v(x) = e^+(v(x))$.

(2) The identity is immediate by (1) and Lemma 2.6.(2), while the inequality is a consequence of the identity since $v(x) + v(y) \leq v(x)$ and

$$x \wedge y \leq y \implies e^+(e^\bullet(x \wedge y)) \leq e^+(e^\bullet(y))$$

by Lemma 2.6.(6).

(3) The inequality $x + (y \wedge z) \leq (x + y) \wedge (x + z)$ is obvious. To get an identity, we note that

$$e^+((x+y) \wedge (x+z)) = v((x+y)-(x+z)) = v(y-z+e^+(x)) \leq e^+(x)+v(y-z) = e^+(x+(y \wedge z))$$

since $v(y - z + e^+(x)) \leq e^+(y - z + e^+(x)) \leq e^+(x)$ and

$$y - z + e^+(x) \leq y - z \implies v(y - z + e^+(x)) \leq v(y - z).$$

The compatibility of $\wedge$ with $-$ is obvious.

(4) (ii) $\implies$ (i) is obvious.

(iii) $\implies$ (ii) : We get $e^+(x \wedge y) = v(x - y) = x - y$ since $x - y \in E^+$ by assumption, and v is the identity on E^+.

(i) $\implies$ (iii) : As $x \wedge y \leq x, y$, it follows by assumption that

$$x \wedge y = x + e^+(x \wedge y) = x + e^+(x) + e^+(y) = x + e^+(y),$$

and similarly, $x \wedge y = y + e^+(x)$. Consequently,

$$x - y = (x + e^+(y)) - (y + e^+(x)) = e^+(x \wedge y),$$

so $x - y \in E^+$ as desired.

Assuming that $x \leq z$ and $y \leq z$ for some $z \in R$, we get

$$x - y = (z + e^+(x)) - (z + e^+(y)) = e^+(x) + e^+(y) + e^+(z) \in E^+,$$

i.e. the elements x and y satisfy condition (iii) above, so (i) and (ii) are satisfied too. Consequently,

$$x + y - z = (z + e^+(x)) + (z + e^+(y)) - z = z + e^+(x \wedge y) = x \wedge y,$$

therefore $x \wedge y + z = x + y + e^+(z) = x + y$, and

$$(x + y) \wedge z = z + v(x + y - z) = z + v(x \wedge y) \le z + e^+(x \wedge y) = x \wedge y.$$

In particular, $x + y \le z$ implies $x + y \le x \wedge y$, and hence $x + y = x \wedge y$ since $e^+(x + y) = e^+(x \wedge y)$.

Finally notice that (5) follows from (4) since $x \wedge y \wedge z = (x \wedge z) \wedge (y \wedge z)$, and the pair $(x \wedge z, y \wedge z)$ is bounded above by z.   □

Lemma 3.5. *Let R be a directed cr-qsring. Then the following assertions hold.*

(1) *The semilattice morphism $e^+ \mid_{E^\bullet} : (E^\bullet, \bullet = \wedge) \longrightarrow (E_+^+, +)$ is a monomorphism, and $y \in E^\bullet$ whenever $\varepsilon \le y \le x$ and $x \in E^\bullet$.*

(2) $e^+(x \bullet y) = x - y = y - x$ *for all $x, y \in E^\bullet$.*

(3) $x \bullet y \le x$ *for all $x \in R, y \in E^\bullet$, and for all $x \in R, \{y \in R \mid x \bullet y \ge x\} = e^\bullet(x) \uparrow :=$ $\{y \in R \mid e^\bullet(x) \le y\}$, and $\{y \in R \mid x \bullet y = x\} = \{y \in e^\bullet(x) \uparrow \mid v(y) = \varepsilon\}$. In particular, $x \bullet y = x \iff e^\bullet(x) \le y$, for $x \in R, y \in E^\bullet$, and $x \le y \implies e^\bullet(x) \le e^\bullet(y)$, for $x, y \in R$.*

(4) $x \bullet y \bullet (x^{-1} \wedge y^{-1}) = x \wedge y$ *for all $x, y \in R$. In particular, $x \wedge y \le (x^{-1} \wedge y^{-1})^{-1}$ (with equality if and only if $v(x) = v(y)$), $(x \wedge y) \bullet (x^{-1} \wedge y^{-1})^{-1} \le x \bullet y$, $v(x \bullet y \bullet (x^{-1} - y^{-1})) = v(x - y)$, and $e^\bullet(x \wedge y) = e^\bullet(x^{-1} \wedge y^{-1})$ for all $x, y \in R$. In addition, $e^\bullet(x \wedge y) = e^\bullet(x \bullet y) = e^\bullet(x) \wedge e^\bullet(y)$ provided $x - y \in E^+$.*

Proof. (1) Let $x, y \in E^\bullet$ be such that $e^+(x) \le e^+(y)$, so $e^+(x \bullet y) = e^+(x) + e^+(y) = e^+(x)$. As $x \ge x \wedge y = x \bullet y$ by Proposition 3.1.(6), it follows that $x = x \bullet y$, as required.

Now assume that $x \in E^\bullet$ and $\varepsilon \le y \le x$. Applying v to the previous chain of inequalities, we get $v(y) = \varepsilon$, therefore $e^+(e^\bullet(y)) = e^+(y) \le e^+(x)$ by Lemma 2.6.(2), so $e^\bullet(y) \le x$ as shown above. Consequently, $e^\bullet(y) = x + e^+(e^\bullet(y)) = x + e^+(y) = y$, and hence $y \in E^\bullet$ as desired. In other words, $E^\bullet$ is identified through the embedding $e^+ \mid_{E^\bullet}$ with an ideal of the poset $(E_+^+, \le)$.

(2) Let $x, y \in E^\bullet$. As $x \bullet y = x \wedge y$, and $e^+(x \bullet y) = e^+(x) + e^+(y)$, the identity $e^+(x \bullet y) = x - y$ follows by Lemma 3.4.(4), (i) $\implies$ (ii).

(3) Let $x \in R, y \in E^\bullet$. We get $x \bullet y = x \bullet (e^\bullet(x) \bullet y) = x \bullet (e^\bullet(x) \wedge y) \le x \bullet e^\bullet(x) = x$. If $e^\bullet(x) \le y$ then $x \le x \bullet y$ by multiplication with x, and hence $x \bullet y = x$. Conversely, assuming that $x \bullet y = x$, it follows by multiplication with x^{-1} that $e^\bullet(x) \wedge y = e^\bullet(x) \bullet y = e^\bullet(x)$, so $e^\bullet(x) \le y$.

Assuming now that $x \le y$, it follows by Lemma 2.6.(2, 4) that

$$e^+(x \bullet e^\bullet(y)) = e^+(x) + x \bullet e^+(e^\bullet(y)) = e^+(x) + (y + e^+(x)) \bullet e^+(e^\bullet(y)) \ge$$

$$e^+(x) + y \bullet e^+(e^\bullet(y)) + e^+(x) \bullet e^+(e^\bullet(y)) = e^+(x) + e^+(y) = e^+(x),$$

and hence $x \bullet e^\bullet(y) = x$, i.e. $e^\bullet(x) \le e^\bullet(y)$, since $x \bullet e^\bullet(y) \le x$, as shown above.

(4) By Proposition 3.1.(5) and statement (3) above, it follows that $x \bullet y \bullet (x^{-1} \wedge y^{-1}) = x \bullet e^\bullet(y) \wedge y \bullet e^\bullet(x) \le x \wedge y$, in particular, $e^\bullet(x \wedge y) = e^\bullet(x^{-1} \wedge y^{-1}) \le e^\bullet(x \bullet y)$ by (3)

again. We have to show that $e^+(x \bullet y \bullet (x^{-1} \wedge y^{-1})) = e^+(x \wedge y)$. By Lemma 2.6.(4) and Lemma 3.4.(1, 2), we get

$$e^+(x \bullet y \bullet ((x^{-1} \wedge y^{-1}) = e^+(x \bullet y) \bullet (x^{-1} \wedge y^{-1}) + x \bullet y \bullet e^+(x^{-1} \wedge y^{-1}) =$$

$$e^+(x \bullet y) \bullet (v(x) \vee v(y))^{-1} + v(x \bullet y \bullet (x^{-1} - y^{-1})) =$$

$$(v(x) + v(y)) \bullet (e^+(e^\bullet(x \bullet y)) + e^+(e^\bullet(x^{-1} \wedge y^{-1}))) =$$

$$(v(x) + v(y)) \bullet e^+(e^\bullet(x \wedge y)) = v(x - y) = e^+(x \wedge y),$$

as required. In particular, we obtain the identity $v(x \bullet y \bullet (x^{-1} - y^{-1})) = v(x - y)$. By multiplication with $(x^{-1} \wedge y^{-1})^{-1}$, it follows that

$$(x \wedge y) \bullet (x^{-1} \wedge y^{-1})^{-1} = x \bullet y \bullet e^\bullet(x \wedge y) \le x \bullet y,$$

as desired. On the other hand, $x \wedge y \le (x^{-1} \wedge y^{-1})^{-1}$ since

$$x \wedge y \wedge (x^{-1} \wedge y^{-1})^{-1} = x \wedge y \wedge (x \bullet y \bullet (x \wedge y)^{-1}) = ((x \wedge y)^2 \bullet (x \wedge y)^{-1}) \wedge (x \bullet y \bullet (x \wedge y)^{-1}) =$$

$$(x \wedge y)^{-1} \bullet ((x \wedge y)^2 \wedge (x \bullet y)) = (x \wedge y)^{-1} \bullet (x \wedge y)^2 = x \wedge y.$$

Finally, assuming that $x - y \in E^+$, we obtain $e^\bullet(x \wedge y) = e^\bullet(x \bullet y)$ since $e^\bullet(x \wedge y) \le e^\bullet(x \bullet y)$ and

$$e^+(e^\bullet(x \wedge y)) = \frac{e^+(x \wedge y)}{v(x \wedge y)} = \frac{e^+(x) + e^+(y)}{v(x) + v(y)} = \frac{(e^+(x) + e^+(y)) \bullet (v(x) \vee v(y))}{v(x \bullet y)} \ge$$

$$\frac{e^+(x) \bullet v(y) + e^+(y) \bullet v(x)}{v(x \bullet y)} = \frac{e^+(x \bullet y)}{v(x \bullet y)} = e^+(e^\bullet(x \bullet y))$$

by Lemmas 2.6.(4) and 3.4.(4). $\qquad\qquad\square$

Lemma 3.6. *Let R be a directed cr-qsring. Then the following assertions hold.*

(1) *There exists the join $x \vee y$ provided the elements $x, y \in R$ are* incident, *i.e. the pair (x, y) is bounded above with respect to the partial order $\le$. Set by convention $x \vee y = \infty$ whenever the elements x and y are not incident.*

(2) *$e^+(x \vee y) = e^+(x) \vee e^+(y)$ and $v(x \vee y) = v(x) \vee v(y)$ provided $x \vee y \ne \infty$.*

(3) *The partial binary operation $\vee$ is compatible with the operations $+, -,$ and $\bullet$, i.e. for all $x, y, z \in R$ such that $y \vee z \ne \infty$, the following identities hold.*

   (i) *$(x + y) \vee (x + z) = x + (y \vee z)$;*
   (ii) *$(-y) \vee (-z) = -(y \vee z)$;*
   (iii) *$(x \bullet y) \vee (x \bullet z) = x \bullet (y \vee z)$.*

(4) *$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$ for all $x, y, z \in R$, with $y \vee z \ne \infty$. In particular, for all $x \in R, x \downarrow$ is a distributive lattice with a last element.*

(5) *For all $x, y \in R$, satisfying $x \vee y \neq \infty$,*

$$x \bullet y = (x \wedge y) \bullet (x \vee y), \quad e^{\bullet}(x \wedge y) = e^{\bullet}(x \bullet y) = e^{\bullet}(x) \wedge e^{\bullet}(y),$$

$$e^{\bullet}(x \vee y) = e^{\bullet}(x) \vee e^{\bullet}(y), \quad \text{and} \quad (x^{-1} \wedge y^{-1})^{-1} = (x \vee y) \bullet e^{\bullet}(x \bullet y) \leq x \vee y.$$

In particular, $x \vee y \in E^{\bullet}$ provided $x, y \in E^{\bullet}$ and $x \vee y \neq \infty$.

(6) *For all $x, y \in R$, the following assertions are equivalent.*

  (i) $(x^{-1} \wedge y^{-1})^{-1} = x \vee y$.

  (ii) $x - y \in E^{+}$ and $e^{\bullet}(x) = e^{\bullet}(y)$.

  (iii) $e^{\bullet}(x \wedge y) = e^{\bullet}(x) = e^{\bullet}(y)$.

(7) *For all $x, y \in R$ satisfying $x \wedge y \in E^{+}$, i.e. $x \wedge y = v(x) + v(y)$, the following assertions are equivalent.*

  (i) $x \vee y \neq \infty$.

  (ii) $x - y \in E^{+}$.

  (iii) $x - y = x \wedge y$.

  In particular, for all $x \in R, y \in E^{+}, x \vee y \neq \infty \iff e^{+}(x) + y \leq v(x)$, and $x \vee y \neq \infty$ for all $x, y \in E^{\bullet}$ which are orthogonal, i.e. $x \bullet y = \varepsilon$.

Proof. (1) Let $x, y, z \in R$ be such that $x \leq z$ and $y \leq z$. We show that the element $u := z + (e^{+}(x) \vee e^{+}(y))$ is the join of the pair (x, y). Obviously, $e^{+}(u) = e^{+}(z) + (e^{+}(x) \vee e^{+}(y)) = e^{+}(x) \vee e^{+}(y)$. As $x \leq z$, we get $x = z + e^{+}(x) = z + e^{+}(x) + (e^{+}(x) \vee e^{+}(y)) = u + e^{+}(x)$, so $x \leq u$, and similarly, $y \leq u$. Assuming that $x \leq t$ and $y \leq t$ for some $t \in R$, in particular, $e^{+}(u) \leq e^{+}(t)$, it remains to check that $u \leq t$. Since $x, y \leq z$, it follows that $x, y \leq z \wedge t$, so we may assume without loss that $t \leq z$. Then we get $t + e^{+}(u) = z + e^{+}(t) + e^{+}(u) = z + e^{+}(u) = u$, and hence $u \leq t$ as desired.

(2) Let $x, y \in R$ be such that $x \vee y \neq \infty$. We already know from (1) that $e^{+}(x \vee y) = e^{+}(x) \vee e^{+}(y)$, so it remains only to evaluate $v(x \vee y)$. As $x, y \leq x \vee y$, it follows that $v(x) \vee v(y) \leq v(x \vee y)$. On the other hand, setting $u = x \vee y$, we get $v(x) = v(u + e^{+}(x)) \geq v(u) + e^{+}(x)$, and similarly, $v(y) \geq v(u) + e^{+}(y)$, therefore $v(x) \vee v(y) \geq v(u) + (e^{+}(x) \vee e^{+}(y)) = v(u) + e^{+}(u) = v(u)$, so $v(u) = v(x) \vee v(y)$ as desired.

(3) Let $x, y, z \in R$ be such that $u := y \vee z \neq \infty$. As $y, z \leq u$, it follows that $x + y, x + z \leq x + u$ and $x \bullet y, x \bullet z \leq x \bullet u$, and hence $(x + y) \vee (x + z) \leq x + u$ and $(x \bullet y) \vee (x \bullet z) \leq x \bullet u$. Since $e^{+}((x + y) \vee (x + z)) = e^{+}(x + y) \vee e^{+}(x + z) = e^{+}(x) + (e^{+}(y) \vee e^{+}(z)) = e^{+}(x + u)$, we conclude that $(x + y) \vee (x + z) = x + (y \vee z)$. On the other hand, we obtain $x \bullet y = x \bullet (u + e^{+}(y)) \geq x \bullet u + x \bullet e^{+}(y)$, and similarly, $x \bullet z \geq x \bullet u + x \bullet e^{+}(z)$, therefore $x \bullet y \vee x \bullet z \geq (x \bullet u + x \bullet e^{+}(y)) \vee (x \bullet u + x \bullet e^{+}(z)) = x \bullet u + (x \bullet e^{+}(y) \vee x \bullet e^{+}(z)) = x \bullet u + x \bullet e^{+}(u) = x \bullet u$ since $e^{+}(x \bullet u) \leq x \bullet e^{+}(u)$.

The proof of the identity (ii) is straightforward.

(4) Let $x, y, z \in R$ be such that $u := y \vee z \neq \infty$. As $y, z \leq u$, it follows that $(x \wedge y) \vee (x \wedge z) \leq x \wedge u$. We get equality since $e^{+}(x \wedge u) = v(u - x) = v((y - x) \vee (z - x)) = v(y - x) \vee v(z - x) = e^{+}(x \wedge y) \vee e^{+}(x \wedge z) = e^{+}((x \wedge y) \vee (x \wedge z))$.

(5) Let $x, y \in R$ be such that $u := x \vee y \neq \infty$. The identity $x \bullet y = (x \wedge y) \bullet u$ is an immediate consequence of the compatibility of the multiplication with $\wedge$ and $\vee$, while the identity $e^\bullet(x \wedge y) = e^\bullet(x \bullet y)$ follows by Lemma 3.5.(4), since $x \vee y \neq \infty \Longrightarrow x - y \in E^+$.

On the other hand, $e^\bullet(x), e^\bullet(y) \leq e^\bullet(u)$ by Lemma 3.5.(3), therefore $t := e^\bullet(x) \vee e^\bullet(y) \neq \infty$ and $t \leq e^\bullet(u)$. To obtain the identity $t = e^\bullet(u)$, we have to show that $e^+(e^\bullet(u)) \leq e^+(t)$. By Lemma 2.6.(2, 4) and statement (2) of the lemma, we get

$$e^+(e^\bullet(u)) = e^+(u) \bullet v(u)^{-1} = (e^+(x) \vee e^+(y)) \bullet (v(x) \vee v(y))^{-1} =$$

$$e^+(x) \bullet e^+(y) \bullet ((e^+(x) + e^+(y)) \bullet (v(x) \vee v(y)))^{-1} \leq e^+(x) \bullet e^+(y) \bullet e^+(x \bullet y)^{-1} =$$

$$e^+(e^\bullet(x)) \bullet e^+(e^\bullet(y)) \bullet e^+(e^\bullet(x \bullet y))^{-1} = e^+(e^\bullet(x)) \vee e^+(e^\bullet(y)) \leq e^+(t),$$

as required.

Using Lemma 3.5.(3, 4), we obtain

$$(x^{-1} \wedge y^{-1})^{-1} = x \bullet y \bullet (x \wedge y)^{-1} = (x \vee y) \bullet (x \wedge y) \bullet (x \wedge y)^{-1} = (x \vee y) \bullet e^\bullet(x \bullet y) \leq x \vee y.$$

Finally, for $x, y \in E^\bullet$ with $x \vee y \neq \infty$, we get $e^\bullet(x \vee y) = e^\bullet(x) \vee e^\bullet(y) = x \vee y$, and hence $x \vee y \in E^\bullet$.

(6) (i) $\Longrightarrow$ (ii) : Assume that $(x^{-1} \wedge y^{-1})^{-1} = x \vee y$. As $x \vee y \neq \infty$, we get $x - y \in E^+$. On the other hand, it follows from (5) that $(x \vee y) \bullet e^\bullet(x \bullet y) = x \vee y$, therefore $e^\bullet(x \bullet y) \geq e^\bullet(x \vee y) = e^\bullet(x) \vee e^\bullet(y)$ by Lemma 3.5.(3), so $e^\bullet(x) = e^\bullet(y)$ as desired.

(ii) $\Longrightarrow$ (iii) : Assuming that $x - y \in E^+$ and $e^\bullet(x) = e^\bullet(y)$, it follows by Lemmas 2.6.(2) and 3.4.(1, 4) that

$$e^+(e^\bullet(x)) \bullet (v(x) + v(y)) = e^+(x) + e^+(y) = e^+(x \wedge y) = e^+(e^\bullet(x \wedge y)) \bullet (v(x) + v(y)),$$

therefore $e^\bullet(x) = e^\bullet(x \wedge y)$ by Lemma 3.5.(1).

(iii) $\Longrightarrow$ (i) : Assuming that $e^\bullet(x) = e^\bullet(y) = e^\bullet(x \wedge y)$, it follows by Lemma 3.5.(4) that

$$x \wedge (x^{-1} \wedge y^{-1})^{-1} = x \bullet (x^{-1} \wedge y^{-1})^{-1} \bullet (x^{-1} \wedge y^{-1}) = x \bullet e^\bullet(x \wedge y) = x \bullet e^\bullet(x) = x,$$

i.e. $x \leq (x^{-1} \wedge y^{-1})^{-1}$, and, similarly, $y \leq (x^{-1} \wedge y^{-1})^{-1}$. Consequently, $x \vee y \neq \infty$, so $x \vee y = (x^{-1} \wedge y^{-1})^{-1}$ by (5).

(7) (i) $\Longrightarrow$ (ii) and (ii) $\Longleftrightarrow$ (iii) are obvious, so it remains to prove the implication (ii) $\Longrightarrow$ (i). It suffices to show that $z := x \vee e^+(y) \neq \infty$ and $u := y \vee e^+(x) \neq \infty$, since then we obtain $x = x + e^+(x) \leq z + u, y = e^+(y) + y \leq z + u$, and $e^+(z + u) = e^+(z) = e^+(u) = e^+(x) \vee e^+(y)$, and hence $x \vee y = z + u \neq \infty$ as desired. Thus we are reduced to the case when $y \in E^+$ and $e^+(x) + y = v(x) + y \leq v(x)$. Let us show that $x \vee y = t^{-1}$, where $t := x^{-1} + \delta$, with $\delta := \frac{e^+(e^\bullet(x))}{v(x) \vee y}$. We get $e^+(t) = \delta, v(t) = v(x)^{-1} + \delta = (v(x) \vee y)^{-1}, e^+(e^\bullet(t)) = e^+(e^\bullet(x))$, and hence $e^\bullet(t) = e^\bullet(x), y \leq v(t^{-1}) \leq t^{-1}$ and $e^+(t^{-1}) = e^+(x) \vee y$. To conclude that $t^{-1} = x \vee y$, it remains to note that

$$x \wedge t^{-1} = x \bullet t^{-1} \bullet (x^{-1} \wedge t) = x \bullet t^{-1} \bullet t = x \bullet e^\bullet(t) = x \bullet e^\bullet(x) = x.$$

$\square$

Remarks 3.7. (1) Since the directed cr-qsrings form a variety of algebras of signature $(2, 2, 1, 1, 0)$, any substructure R' of a directed cr-qsring R is a directed cr-qsring, in particular, $x \wedge y \in R'$ provided $x, y \in R'$. However, for $x \in R, y \in R', x \leq y$ does not imply $x \in R'$, and also, for $x, y \in R', \infty \neq x \vee y \in R$ does not imply $x \vee y \in R'$ (see Remark 4.9.)

(2) As in the case of commutative regular rings, the map $E^\bullet \times E^\bullet \longrightarrow E^\bullet, (x, y) \mapsto x + y - x \bullet y$ is well defined in any directed cr-qsring R, but by contrast with the former case, where the map above defines the join $x \vee y$ of any pair of idempotents (x, y), in the latter case we obtain $x + y - x \bullet y = (x - x \bullet y) + (y - x \bullet y) + x \bullet y = x \bullet y = x \wedge y$ for all $x, y \in E^\bullet$ since $x - x \bullet y = y - x \bullet y = e^+(x \bullet y)$ by Lemma 3.5.(2). Notice also that in an arbitrary cr-qsring R, we have only the inequality $x + y - x \bullet y \leq (x + y - x \bullet y)^2$ for all $x, y \in E^\bullet$, so, in general, $E^\bullet$ is not necessarily closed under the map above.

4   The metric structure of a directed commutative regular quasi-semiring

Let R be a directed cr-qsring. As we have shown in Section 2, the subset E^+ of the idempotents of the commutative regular semigroup $(R, +)$ has a natural structure of Abelian l-group with multiplication $\bullet$ as the group operation, neutral element ε, partial order $\leq$, and addition $+$ as the corresponding meet-semilattice operation $\wedge$. Let us denote by Λ this structure of Abelian l-group, and by $\Lambda_+ = \{\alpha \in \Lambda \mid \alpha \geq \varepsilon\}$, the monoid of the nonnegative elements of Λ. For any $\alpha \in \Lambda$, set $\alpha_+ := \alpha \vee \varepsilon = (\alpha^{-1} + \varepsilon)^{-1}, \alpha_- := (\alpha^{-1})_+$, and $\mid \alpha \mid := \alpha \vee \alpha^{-1} = (\alpha + \alpha^{-1})^{-1} = \alpha_+ \bullet \alpha_-$. In this section we shall define a distance map on the directed cr-qsring R with values in Λ_+, and we shall investigate the main properties of this metric structure.

For any pair (x, y) of elements of R, set

$$[x, y] := \{z \in R \mid x \wedge y \leq z = (x \wedge z) \vee (y \wedge z)\}$$

Notice that $[x, x] = \{x\}, [x, y] = [y, x], x, y, x \wedge y \in [x, y]$, and $x \vee y \in [x, y]$ provided $x \vee y \neq \infty$. Notice also that $[x, y] = \{z \mid x \leq z \leq y\}$ (the interval) whenever $x \leq y$. Thus $[x, y]$ coincides with the interval $[x \wedge y, x \vee y]$ provided $x \vee y \neq \infty$.

Call *cell* or *simplex* any subset of R of the form $[x, y]$. Given a cell $C \subseteq R$, any $x \in R$ for which there exists $y \in R$ such that $C = [x, y]$ is called an *end* of the cell C. The (non-empty) subset of all ends of a cell C is denoted by ∂C and called the *boundary* of C.

To provide equivalent descriptions for the cells as introduced above, we define two maps $\lambda : R \times R \longrightarrow \Lambda_+$ and $d : R \times R \longrightarrow \Lambda_+$ as follows :

$$\lambda(x, y) := \frac{e^+(x)}{e^+(x \wedge y)} = \frac{e^+(x)}{v(x - y)} = e^+(x) \bullet (x - y)^{-1}$$

and

$$d(x, y) = \lambda(x, y) \bullet \lambda(y, x) = \frac{e^+(x) \bullet e^+(y)}{(e^+(x \wedge y))^2}$$

Notice that $\lambda(x,y) = (x \bullet y^{-1})_+$ and $d(x,y) = |x \bullet y^{-1}|$ provided $x, y \in E^+$. The next lemmas collect some basic properties of the maps λ and d.

Lemma 4.1. (1) $\lambda(x,y) = \varepsilon \Longleftrightarrow x \leq y$.

(2) $\lambda(x,y) = \lambda(x, x \wedge y) = d(x, x \wedge y)$.

(3) *The ternary map* $\widehat{\lambda} : R^3 \longrightarrow \Lambda_+$, *defined by*

$$\widehat{\lambda}(x,y,z) = \lambda(x,y) \bullet \lambda(y,z) \bullet \lambda(z,x) = \frac{e^+(x) \bullet e^+(y) \bullet e^+(z)}{e^+(x \wedge y) \bullet e^+(y \wedge z) \bullet e^+(z \wedge x)}$$

is symmetric in the variables x, y, z.

(4) $\lambda(x,y) \leq \lambda(x,z) \bullet \lambda(z,y)$ *for all* $x, y, z \in R$.

(5) $\lambda(x,y) = \lambda(v(x), v(y)) \bullet d(e^\bullet(x), e^\bullet(x \wedge y))$. *In particular,* $\lambda(x,y) = \lambda(v(x), v(y)) = v(x \bullet y^{-1})_+$ *if and only if* $e^\bullet(x) = e^\bullet(x \wedge y)$.

(6) $d(x,y) = d(v(x), v(y)) \bullet d(e^\bullet(x), e^\bullet(y)) \bullet d(e^\bullet(x \bullet y), e^\bullet(x \wedge y))^2$. *In particular,* $d(x,y) = d(v(x), v(y)) = |v(x \bullet y^{-1})|$ *if and only if* $e^\bullet(x) = e^\bullet(y) = e^\bullet(x \wedge y)$, *and* $d(x,y) = d(e^+(x), e^+(y)) = |e^+(x) \bullet e^+(y)^{-1}|$ *provided* $x, y \in E^\bullet$.

Proof. The statemements (1)-(3) are obvious. Notice that (4) is equivalent with the inequality

$$e^+(x \wedge z) \bullet e^+(y \wedge z) \leq e^+(z) \bullet e^+(x \wedge y)$$

Indeed, setting $u := (x \wedge z) \vee (y \wedge z) \leq z$, it follows by Lemma 3.4.(4) and Lemma 3.6.(2) that

$$\text{LHS} = (e^+(x \wedge z) \vee e^+(y \wedge z)) \bullet (e^+(x \wedge z) + e^+(y \wedge z)) = e^+(u) \bullet e^+(x \wedge y \wedge z) \leq \text{RHS}$$

as desired.

(5) By Lemmas 2.6.(2), 3.4.(1, 2), 3.5.(3), we get

$$\lambda(x,y) = \frac{v(x) \bullet e^+(e^\bullet(x))}{(v(x) + v(y)) \bullet e^+(e^\bullet(x \wedge y))} = \lambda(v(x), v(y)) \bullet d(e^\bullet(x), e^\bullet(x \wedge y)),$$

as required. The necessary and sufficient condition to have the equality $\lambda(x,y) = \lambda(v(x), v(y))$ is immediate by Lemma 3.5.(1). Finally, notice that (5) $\Longrightarrow$ (6). □

Lemma 4.2. (1) *Any directed cr-qsring* R *is a* Λ-metric space *with the* Λ-valued distance map $d : R \times R \longrightarrow \Lambda$ *satisfying*

(i) $d(x,y) = \varepsilon \Longleftrightarrow x = y$,

(ii) $d(x,y) = d(y,x)$ *for all* $x, y \in R$, *and*

(iii) ***Triangle inequality*** : $d(x,y) \leq d(x,z) \bullet d(z,y)$ *for all* $x, y, z \in R$.

(2) $d(x,y) = d(x, x \wedge y) \bullet d(y, x \wedge y)$ *for all* $x, y \in R$.

Proof. The triangle inequality follows by Lemma 4.1.(4), while the rest of assertions are obvious. □

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Lemma 4.3. *Let R be a directed cr-qsring. Then the following assertions hold.*

(1) *For all $x, y \in R$,*

$$[x, y] = \{z \in R \mid \lambda(x, z) \bullet \lambda(z, y) = \lambda(x, y)\} = \{z \in R \mid d(x, z) \bullet d(z, y) = d(x, y)\}.$$

(2) *The ternary relation $B(x, y, z) \Longleftrightarrow y \in [x, z]$ is a betweenness relation, i.e. the following hold for all $x, y, z, u \in R$.*

   (i) $B(x, x, y)$,

   (ii) $B(x, y, x) \Longrightarrow x = y$, *and*

   (iii) $B(x, y, z), B(x, u, y) \Longrightarrow B(z, u, x)$.

(3) *The betweenness relation B is compatible with the semilattice operation $\wedge$, i.e. $B(x, y, z) \Longrightarrow B(x \wedge u, y \wedge u, z)$ for all $x, y, z, u \in R$.*

(4) *For all $a \in R$, the map $a \downarrow \longrightarrow \Lambda_+, x \mapsto d(a, x) = \frac{e^+(a)}{e^+(x)}$ is an antiisomorphism of distributive lattices, with the inverse $\alpha \in \Lambda_+ \mapsto a + \frac{e^+(a)}{\alpha} \in a \downarrow$.*

Proof. (1) Let us denote by M the set involving the map λ, and by N the set involving d. Let M' be the set obtained from M by interchanging the elements x and y, and notice that $M \cap M' = N$ by Lemma 4.1.(4) and the definition of d. Consequently, it suffices to prove that $[x, y] = M$ since $[x, y] = [y, x]$. However the equality $[x, y] = M$ follows easily with the same argument as in the proof of Lemma 4.1.(4).

(2) is straightforward, while (3) and (4) follow by Lemma 3.6.(4). □

Lemma 4.4. (1) $d(\varphi(x), \varphi(y)) \leq d(x, y)$, *where $\varphi(x)$ is defined by one of the following formulas, for some $a \in R$:*

   (i) $\varphi(x) = a + x$;

   (ii) $\varphi(x) = a \bullet x$;

   (iii) $\varphi(x) = a \wedge x$;

   (iv) $\varphi(x) = a \vee x$ *provided $a \vee x \neq \infty$.*

(2) $\lambda(-x, -y) = \lambda(x, y)$ *and* $d(-x, -y) = d(x^{-1}, y^{-1}) = d(x, y)$ *for all $x, y \in R$.*

Proof. (1) Since $x \geq y \Longrightarrow \varphi(x) \geq \varphi(y)$, it suffices to prove the inequality $d(\varphi(x), \varphi(y)) \leq d(x, y)$ in the case $x \geq y$, and hence we have to check that $e^+(\varphi(x)) \bullet e^+(y) \leq e^+(\varphi(y)) \bullet e^+(x)$ under the assumption $x \geq y$. The verification in the cases (i) and (iv) is straightforward. In the case (ii), by Lemma 2.6.(2, 4, 6), we obtain

$$e^+(a \bullet x) \bullet e^+(y) = e^+(x) \bullet e^+(y) \bullet v(a) \bullet (\varepsilon + \frac{e^+(e^\bullet(a))}{e^+(e^\bullet(x))}) \leq$$

$$e^+(x) \bullet e^+(y) \bullet v(a) \bullet (\varepsilon + \frac{e^+(e^\bullet(a))}{e^+(e^\bullet(y))}) = e^+(a \bullet y) \bullet e^+(x),$$

as desired. In the case (iii), $e^+(a \wedge y) \vee e^+(y) \leq e^+(x)$ implies by Lemma 3.4.(5) that

$$e^+(a \wedge x) \bullet e^+(y) \leq (e^+(a \wedge x) + e^+(y)) \bullet e^+(x) = (e^+(a \wedge x) + e^+(x \wedge y)) \bullet e^+(x) =$$

$$e^+(a \wedge x \wedge y) \bullet e^+(x) = e^+(a \wedge y) \bullet e^+(x),$$

as required.

(2) The identities $\lambda(-x, -y) = \lambda(x, y)$ and $d(-x, -y) = d(x, y)$ are obvious since $(-x) \wedge (-y) = -(x \wedge y)$ by Lemma 3.4.(3). By Lemma 2.4.(5), Lemma 2.6.(2), Lemma 3.4.(1) and Lemma 3.5.(4), we get

$$\lambda(x^{-1}, y^{-1}) = \frac{e^+(x^{-1})}{e^+(x^{-1} \wedge y^{-1})} = \frac{v(x)^{-2} \bullet e^+(x)}{v(x \bullet y)^{-1} \bullet e^+(x \wedge y)} = v(x^{-1} \bullet y) \bullet \lambda(x, y),$$

therefore $d(x^{-1}, y^{-1}) = d(x, y)$ as desired. $\qquad\square$

Remark 4.5. By Lemma 4.4.(2), the commuting involutions $x \mapsto -x$ and $x \mapsto x^{-1}$ are both automorphisms of the Λ-metric space (R, d). Notice that the involution $x \mapsto -x$ is an automorphism of the structure $(R, \wedge, B)$, while the involution $x \mapsto x^{-1}$ is only an automorphism of the structure (R, B), and also an isomorphism from the structure $(R, \wedge, B)$ onto the structure $(R, \sqcap, B)$, where the semilattice operation $\sqcap$, defined by $x \sqcap y := (x^{-1} \wedge y^{-1})^{-1}$, is the meet of any pair (x, y) with respect to the partial order $x \sqsubseteq y \iff x^{-1} \leq y^{-1} \iff x \leq x^2 \bullet y^{-1}$. The semilattice operations $\wedge$ and $\sqcap$ are both compatible with the betweenness relation B, and, by Lemmas 3.5.(4) and 3.6.(6), $x \wedge y = x \sqcap y \iff v(x) = v(y)$, and $x \sqcap y = x \vee y \iff e^\bullet(x \wedge y) = e^\bullet(x) = e^\bullet(y)$. Notice also that the corresponding partial join operation $x \sqcup y$ is defined if and only if $x^{-1} \vee y^{-1} \neq \infty$, and $x \sqcup y = (x^{-1} \vee y^{-1})^{-1}$.

4.1 Congruences on directed commutative regular quasi-semirings

Lemma 4.6. *The map $\equiv \mapsto \{\alpha \in \Lambda_+ \mid \alpha \equiv \varepsilon\}$ is an isomorphism from the lattice of all congruences of the directed cr-qsring R onto the lattice of all convex submonoids of Λ_+. Its inverse sends a convex submonoid $S \subseteq \Lambda_+$ to the congruence relation $\{(x, y) \in R \times R \mid d(x, y) \in S\}$.*

Proof. Let $\equiv$ be a congruence of the directed cr-qsring R. The relation induced by $\equiv$ on the substructure Λ is obviously a congruence on the commutative l-group Λ, and hence $\{\alpha \in \Lambda_+ \mid \alpha \equiv \varepsilon\}$ is a convex submonoid of Λ_+. First we have to show that for all $x, y \in R$, $x \equiv y \iff d(x, y) \equiv \varepsilon$, i.e. $e^+(x) \bullet e^+(y) \equiv e^+(x \wedge y)^2$. The implication $\implies$ is a consequence of the obvious implications $x \equiv y \implies x \equiv x \wedge y$, $x \equiv y \implies e^+(x) \equiv e^+(y)$, and $(x \equiv z, y \equiv z) \implies x \bullet y \equiv z^2$. Conversely, assuming that $d(x, y) \equiv \varepsilon$, it follows that $\lambda(x, y) \equiv \lambda(y, x) \equiv \varepsilon$ since $\varepsilon \leq \lambda(x, y), \lambda(y, x) \leq d(x, y)$. Consequently, $e^+(x) \equiv e^+(x \wedge y) \equiv e^+(y)$, and hence $x = x + e^+(x) \equiv x + e^+(x \wedge y) = x \wedge y$, and, similarly, $y \equiv x \wedge y$, so $x \equiv y$ as desired.

Next, assuming that S is a convex submonoid of Λ_+, we deduce that the binary relation $\{(x, y) \in R \times R \mid d(x, y) \in S\}$ is a congruence of the directed cr-qsring R thanks to Lemmas 4.2. and 4.4. $\qquad\square$

Lemma 4.7. *Let $S = \mathcal{S}(E^\bullet)$ be the smallest convex submonoid of Λ_+ containing $e^+(E^\bullet)$, $G = \mathcal{G}(E^\bullet) = \{\gamma \in \Lambda \mid |\gamma| \in S\}$ the corresponding convex l-subgroup of Λ, and $\equiv\,=\,\equiv_{E^\bullet}$ the congruence on the directed cr-qsring R defined by S. Then the following assertions hold.*

(1) *$S = \bigcup_{x \in E^\bullet, n \geq 1} [\varepsilon, e^+(x)^n]$ provided $x \vee y \neq \infty$ for all $x, y \in E^\bullet$.*

(2) *For all $x, y \in R$, $x \equiv y \Longleftrightarrow v(x \bullet y^{-1}) \in G$.*

(3) *G is the smallest convex l-subgroup of Λ for which the surjective map $\widetilde{v} : R \longrightarrow \Lambda/G$ induced by the q-valuation $v : R \longrightarrow \Lambda$ is a morphism of cr-qsrings.*

(4) *G is the smallest convex l-subgroup of Λ for which $\mathrm{Ker}(\widetilde{v}) := \{x \in R \mid v(x) \in G\}$ is a directed cr-qsring, a substructure of R. Moreover $x \vee y \in \mathrm{Ker}\,(\widetilde{v})$ provided $x, y \in \mathrm{Ker}\,(\widetilde{v})$ and $x \vee y \neq \infty$.*

(5) *$\equiv$ is the smallest congruence on the directed cr-qsring R for which the quotient $R/\equiv$ is an Abelian l-group.*

Proof. (1) Let $\alpha, \beta \in \Lambda_+$ be such that $\alpha \leq e^+(x)^n, \beta \leq e^+(y)^m$ for $x, y \in E^\bullet, n, m \geq 1$. By assumption, $z := x \vee y \in E^\bullet$, therefore $\alpha \bullet \beta \leq e^+(z)^{n+m}$, as required.

(2) For all $x, y \in E^\bullet$, we get $d(x, y) \leq d(x, \varepsilon) \bullet d(y, \varepsilon) = e^+(x) \bullet e^+(y) \in S$, and hence $d(x, y) \in S$, and $x \equiv \varepsilon$ for all $x \in E^\bullet$. Then, by Lemma 4.1.(6), it follows that for all $x, y \in R$,

$$x \equiv y \Longleftrightarrow d(x, y) \in S \Longleftrightarrow d(v(x), v(y)) = \mid v(x \bullet y^{-1}) \mid \in S \Longleftrightarrow v(x \bullet y^{-1}) \in G,$$

as desired.

(3) By (2), the quotient directed cr-qsring $R/\equiv$ is isomorphic to the quotient Abelian l-group Λ/G, and the surjective map $\widetilde{v} : R \longrightarrow \Lambda/G$ induced by v is obviously a morphism of cr-qsrings. On the other hand, let H be a convex l-subgroup of Λ with the property that the surjective map $\widetilde{v}_H : R \longrightarrow \Lambda/H$ induced by v is a morphism of cr-qsrings. As $v(x) = \varepsilon \in H$ provided $x \in E^\bullet$, it follows that $\widetilde{v}_H(e^+(x)) = \widetilde{v}_H(x - x) = \widetilde{v}_H(x) - \widetilde{v}_H(x) = \varepsilon \bmod H$, i.e. $e^+(x) = v(e^+(x)) \in H$ for all $x \in E^\bullet$, and hence $G \subseteq H$, as required.

(4) Obviously, $\varepsilon \in \mathrm{Ker}\,(\widetilde{v})$, and $\mathrm{Ker}\,(\widetilde{v})$ is closed under multiplication and the unary operations $x \mapsto -x$ and $x \mapsto x^{-1}$, so it remains to note that $v(x - y) = e^+(x \wedge y) = e^+(e^\bullet(x \wedge y)) \bullet (v(x) + v(y)) \in G$, i.e. $x - y \in \mathrm{Ker}\,(\widetilde{v})$, provided $x, y \in \mathrm{Ker}\,(\widetilde{v})$. Note that $E^+(\mathrm{Ker}\,(\widetilde{v})) = G, E^\bullet(\mathrm{Ker}\,(\widetilde{v})) = E^\bullet$, and the convex monoid $S = G_+$ is obviously generated by $e^+(E^\bullet)$. On the other hand, let H be a convex l-subgroup of Λ such that $\mathrm{Ker}\,(\widetilde{v}_H) = \{x \in R \mid v(x) \in H\}$ is a substructure of R. With the same argument as in the proof of (3), it follows that $e^+(x) = v(e^+(x)) \in H$ for all $x \in E^\bullet$, and hence $G \subseteq H$ as desired. Notice also that for any such H, in particular for G, $x \vee y \in \mathrm{Ker}\,(\widetilde{v}_H)$ provided $x, y \in \mathrm{Ker}\,(\widetilde{v}_H)$ and $x \vee y \neq \infty$, since $v(x \vee y) = v(x) \vee v(y) \in H$ by Lemma 3.6.(2).

Finally notice that (5) is a reformulation of (3). $\square$

4.2 Superrigid directed commutative regular quasi-semirings

Call *rigid* a directed cr-qsring R for which $\mathcal{S}(E^\bullet) = \Lambda_+$, i.e., by Lemma 4.7., R does not have proper Abelian l-group quotients. In particular, R is rigid provided the injective map $e^+|_{E^\bullet} \colon E^\bullet \longrightarrow \Lambda_+$ is surjective. Call such an R *superrigid*, and for any $\alpha \in \Lambda_+$, denote by 1_α the unique element of $E^\bullet$ satisfying $e^+(1_\alpha) = \alpha$. Moreover, for any $\alpha \in \Lambda$, put $1_\alpha := 1_{\alpha_+} + \alpha = \lim_{\gamma \to \infty}(1_\gamma + \alpha)$, and notice that $e^+(1_\alpha) = \alpha$, $v(1_\alpha) = \alpha_-^{-1} \leq \varepsilon$, $e^\bullet(1_\alpha) = 1_{\alpha_+}$, $1_\alpha^{-1} = 1_{\alpha_+} \vee \alpha_- \geq \varepsilon$ (by Lemma 3.6.(7)), and the map $\alpha \mapsto 1_\alpha$ is an isomorphism $(\Lambda, +, \vee) \longrightarrow (\{1_\alpha \mid \alpha \in \Lambda\}, \wedge, \vee)$ of distributive lattices.

By Lemma 4.7., any directed cr-qsring can be seen as an *extension of an Abelian l-group by a rigid directed cr-qsring.* On the other hand, the category of Abelian l-groups is equivalent to a full subcategory of the category of all superrigid directed cr-qsrings, as follows :

Proposition 4.8. *The forgetful functor, sending a directed cr-qsring R to the associated Abelian l-group $\Lambda = (E^+, \bullet, +, \vee)$, induces by restriction an equivalence from the full subcategory of those superrigid directed cr-qsrings R which satisfy the supplementary condition*

$$\forall x, y \in R, e^\bullet(x + y) = e^\bullet(x + v(y)) \vee e^\bullet(y + v(x))$$

to the category of Abelian l-groups.

Proof. First notice that in any directed cr-qsring R, for all $x, y \in R$, $x + v(y) \leq x + y$ since $v(y) \leq y$, therefore $(x + v(y)) \vee (y + v(x)) \neq \infty$, and $(x + v(y)) \vee (y + v(x)) \leq x + y$, in particular,

$$e^\bullet((x + v(y)) \vee (y + v(x))) = e^\bullet(x + v(y)) \vee e^\bullet(y + v(x)) \leq e^\bullet(x + y),$$

where the elements $e^\bullet(x + v(y)) \leq e^\bullet(x)$ and $e^\bullet(y + v(x)) \leq e^\bullet(y)$ are orthogonal, i.e. $e^\bullet(x + v(y)) \bullet e^\bullet(y + v(x)) = \varepsilon$.

Assuming that the inequality above becomes an identity for all $x, y \in R$, the following hold.

(i) $e^\bullet(x + x) = e^\bullet(x + v(x)) = e^\bullet(v(x)) = \varepsilon$, so $2x := x + x \in E^+$, and hence $-x = x$ for all $x \in R$.

(ii) For all $x, y \in R$, $x + x \wedge y = (x + x) \wedge (x + y) = e^+(x) \wedge (x + y) = v(x + y) = e^+(x \wedge y)$, and $x \wedge (x + y) = x + v(x + x + y) = x + e^+(x) + v(y) = x + v(y)$.

(iii) The map $\psi : R \longrightarrow \Lambda \times \Lambda_+, x \mapsto (v(x), e^+(e^\bullet(x)))$ is injective, identifying the semilattice $(R, \wedge)$ with a subsemilattice of the semilattice with support $\Lambda \times \Lambda_+$ defined by

$$(\gamma, \delta) \wedge (\gamma', \delta') := (\gamma + \gamma', \frac{\gamma \bullet \delta + \gamma' \bullet \delta'}{(\gamma + \gamma' \bullet \delta') \vee (\gamma' + \gamma \bullet \delta)}),$$

with the associated partial order

$$(\gamma, \delta) \leq (\gamma', \delta') \iff \gamma = \gamma' + \gamma \bullet \delta, \gamma \bullet \delta \leq \gamma' \bullet \delta' \iff \gamma = \gamma' + \gamma \bullet \delta, \delta \leq \delta'.$$

Notice that $\psi(E^+) = \{(\gamma, \varepsilon) \mid \gamma \in \Lambda\}$, and $\psi(E^\bullet)$ is identified with a convex subset of Λ_+.

Assuming in addition that R is superrigid, we have to show that ψ maps isomorphically R onto the directed cr-qsring $\mathcal{R}_\Lambda$ with support $\{(\gamma, \delta) \in \Lambda \times \Lambda_+ \mid \mid \gamma \mid + \delta = \varepsilon\}$, and the operations

$$(\gamma, \delta) + (\gamma', \delta') := (\frac{\gamma \bullet \delta + \gamma' \bullet \delta'}{\mid \frac{\gamma}{\gamma'} \mid + (\frac{\gamma}{\gamma'})_+ \bullet \delta + (\frac{\gamma}{\gamma'})_- \bullet \delta'}, \mid \frac{\gamma}{\gamma'} \mid + (\frac{\gamma}{\gamma'})_+ \bullet \delta + (\frac{\gamma}{\gamma'})_- \bullet \delta'),$$

$$(\gamma, \delta) \bullet (\gamma', \delta') := (\gamma \bullet \gamma', \delta + \delta'), -(\gamma, \delta) := (\gamma, \delta), (\gamma, \delta)^{-1} := (\gamma^{-1}, \delta), \epsilon := (\varepsilon, \varepsilon).$$

One checks easily that the operations as defined above make $\mathcal{R}_\Lambda$ a superrigid directed cr-qsring, generated by the union $E^+(\mathcal{R}_\Lambda) \cup E^\bullet(\mathcal{R}_\Lambda) = \{(\gamma, \varepsilon) \mid \gamma \in \Lambda\} \cup \{(\varepsilon, \delta) \mid \delta \in \Lambda_+\}$. More precisely, for any pair $(\gamma, \delta) \in \mathcal{R}_\Lambda$, we obtain $(\gamma_+, \delta), (\gamma_-, \delta) \in \mathcal{R}_\Lambda$, and

$$(\gamma, \delta) = (\gamma_+, \delta) \bullet (\gamma_-, \delta)^{-1} = ((\frac{\delta}{\gamma_+}, \varepsilon) + (\varepsilon, \gamma_+ \bullet \delta))^{-1} \bullet ((\frac{\delta}{\gamma_-}, \varepsilon) + (\varepsilon, \gamma_- \bullet \delta)).$$

To get the desired isomorphism $R \cong \mathcal{R}_\Lambda$, it remains to show that for all $x \in R$, the elements $\gamma := v(x)$ and $\delta := e^+(e^\bullet(x))$ of Λ are orthogonal, i.e. $\mid \gamma \mid + \delta = \varepsilon$. Since R is superrigid by assumption, there exists uniquely $y \in E^\bullet$ such that $\psi(y) = (\varepsilon, \gamma_+ \bullet \delta)$. It follows that $\psi(x \wedge y) = (\gamma_-^{-1}, (\frac{\delta}{\gamma_-})_+)$, and $\psi(x + x \wedge y) = ((\frac{\delta}{\mid \gamma \mid})_+ \bullet \gamma_-^{-1}, (\frac{\mid \gamma \mid + \delta}{\gamma_-})_+)$. Since $x + x \wedge y \in E^+$ by the property (ii) above, it follows that $(\frac{\mid \gamma \mid + \delta}{\gamma_-})_+ = \varepsilon$, i.e. $\mid \gamma \mid + \delta \leq \gamma_-$. Applying the same argument to the element $x^{-1} = \psi^{-1}((\gamma^{-1}, \delta))$, we get $\mid \gamma \mid + \delta \leq \gamma_+$, and hence $\mid \gamma \mid + \delta = \varepsilon$, as required. To end the proof, it remains to note that any morphism $\varphi : \Lambda \longrightarrow \Lambda'$ of Abelian l-groups extends uniquely to a morphism $\widetilde{\varphi} : \mathcal{R}_\Lambda \longrightarrow \mathcal{R}_{\Lambda'}, (\gamma, \delta) \mapsto (\varphi(\gamma), \varphi(\delta))$ of directed cr-qsrings. □

Remark 4.9. Using the superrigid directed cr-qsrings $\mathcal{R}_\Lambda$ as defined in Proposition 4.8., we can provide examples of directed cr-qsrings R with elements $x, y \in R$ satisfying $x - y \in E^+$ but $x \vee y = \infty$ (see Remarks 3.7.(1)). Let $\alpha \in \Lambda_+$. Then the subset $\{(\gamma, \delta) \in \mathcal{R}_\Lambda \mid \delta \leq \alpha\}$ is obviously a substructure of $\mathcal{R}_\Lambda$. Now, assuming that $\alpha \neq \varepsilon$, notice that the following are equivalent.

(1) The subset $S_\alpha := \{(\gamma, \delta) \in \mathcal{R}_\Lambda \mid \delta < \alpha\}$ is a substructure of $\mathcal{R}_\Lambda$.

(2) α is not a proper disjoint join in Λ_+, i.e. $\delta + \delta' \neq \varepsilon$ whenever $\alpha = \delta \vee \delta'$ with $\delta, \delta' \in \Lambda_+ \setminus \{\varepsilon\}$; equivalently, $\{\gamma \in \Lambda \mid \mid \gamma \mid = \alpha\} = \{\alpha, \alpha^{-1}\}$.

Indeed, (1) $\Longrightarrow$ (2) since, assuming that $\alpha = \delta \vee \delta'$ with $\delta, \delta' \neq \varepsilon, \delta + \delta' = \varepsilon$, it follows that $(\delta', \delta), (\delta, \delta') \in S_\alpha$ but $(\delta', \delta) + (\delta, \delta') = (\varepsilon, \alpha) \notin S_\alpha$. Conversely, (2) $\Longrightarrow$ (1) since, assuming that S_α is not a substructure of $\mathcal{R}_\Lambda$, it follows that there exist $(\gamma, \delta), (\gamma', \delta') \in S_\alpha$ such that $(\gamma, \delta) + (\gamma', \delta') \notin S_\alpha$, i.e. $\alpha = \rho \vee \rho'$ is a proper disjoint join, where $\rho = (\frac{\gamma}{\gamma'})_+ \bullet \delta + (\frac{\gamma}{\gamma'})_- \leq \delta < \alpha$ and $\rho' = (\frac{\gamma}{\gamma'})_- \bullet \delta' + (\frac{\gamma}{\gamma'})_+ \leq \delta' < \alpha$.

Assuming that α is not a proper disjoint join, so $R := S_\alpha$ is a directed cr-qsring as a substructure of $\mathcal{R}_\Lambda$, and assuming in addition that $\alpha = \sigma \vee \tau$ with $\varepsilon < \sigma, \tau < \alpha$, it follows that $x := (\varepsilon, \sigma), y := (\varepsilon, \tau) \in E^\bullet(R)$, in particular, $x - y = x + y = (\sigma + \tau, \varepsilon) \in E^+(R) = E^+(\mathcal{R}_\Lambda)$, but the elements x and y are not incident in R, though they are

incident in $\mathcal{R}_\Lambda$, with $x \vee y = (\varepsilon, \alpha)$. For instance, taking Λ the free Abelian l-group [26] with two generators ζ and η, and $R := S_\alpha$, where $\alpha := |\zeta| \bullet |\eta|$, the elements $x := (\varepsilon, |\zeta| \bullet \eta_+)$, $y := (\varepsilon, |\zeta| \bullet \eta_-) \in E^\bullet(R)$ satisfy the required conditions. For other examples see Remarks 5.3.(6) and 5.6.(5).

4.3 Locally linear directed commutative regular quasi-semirings

As an immediate consequence of definitions and Lemmas 4.6. and 4.3.(4), we obtain various characterizations for subdirectly irreducible directed cr-qsrings as follows..

Corollary 4.10. *The following are equivalent for a directed cr-qsring R.*

(1) *R is subdirectly irreducible.*

(2) *The Abelian l-group Λ is totally ordered.*

(3) *The poset $(R, \leq)$ is an order-tree, i.e. there exists the meet $x \wedge y$ for any pair $(x, y) \in R \times R$, and the poset $x \downarrow$ is totally ordered for all $x \in R$.*

(4) *For all $x, y \in R$ satisfying $x - y \in E^+$, either $x \leq y$ or $y \leq x$.*

(5) *For all $x, y, z \in R$, either $x \wedge y = y \wedge z$ or $x \wedge y = z \wedge x$ or $z \wedge x = y \wedge z$. In particular, the median $m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ of the triple (x, y, z) is well defined, and moreover $m(x, y, z) \in \{x \wedge y, y \wedge z, z \wedge x\}$ is the single element of the intersection $[x, y] \cap [y, z] \cap [z, x]$.*

(6) *For all $x, y \in R$, $[x, y] = [x \wedge y, x] \cup [x \wedge y, y]$.*

(7) *For all $x, y \in R$, $[x, y] = [x, z] \cup [z, y]$ provided $z \in [x, y]$.*

(8) *Any cell of R has at most two ends.*

Definition 4.11. *A directed cr-qsring R is called* locally linear *(abbreviated an lcr-qsring) if R satisfies the equivalent conditions of* Corollary 4.10.

By Corollary 4.10., any lcr-qsring has an underlying structure of Λ-tree [22], [1]. A natural extension of this arboreal structure to a larger class of directed cr-qsrings will be discussed in Section 5.

Remarks 4.12. (1) The totally ordered Abelian groups are identified with those lcr-qsrings R for which $E^\bullet = \{\varepsilon\}$. The simplest lcr-qsring R for which $E^\bullet$ is not a singleton is obtained by adding to the totally ordered multiplicative group $\Lambda = E^+ = \{\gamma^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$, with $\gamma > \varepsilon := \gamma^0$ the smallest positive element, a single element δ satisfying $\delta = -\delta = \delta^2 = e^\bullet(\delta), \delta + \delta = e^+(\delta) = \gamma, \delta + \gamma^n = \delta$ for $n \geq 1$ and γ^n otherwise, and $\delta \bullet \gamma^n = \gamma^n$, in particular, $v(\delta) = \varepsilon$. Thus $R = E^+ \cup \{\delta\}$, and $E^\bullet = \{\varepsilon, \delta\}$. More generally, the lcr-qsrings R, with $\Lambda = E^+ \cong \mathbb{Z}$ as above and $|E^\bullet| = 2$ are up to isomorphism classified by the pairs (F, l) consisting of a field F and a natural number l, the simplest case above corresponding to the pair $(\mathbb{F}_2, 0)$, where $\mathbb{F}_2$ is the field with two elements. Notice that all lcr-qsrings of this type are rigid but not superrigid. For any pair (F, l) as above, we define a lcr-qsring $L = L(F, l)$ as follows. For $l = 0$, put

$L = F^* \sqcup \Lambda = F \sqcup (\Lambda \setminus \{\gamma\})$, identifying the neutral element $0 \in F$ with the generator γ of Λ, and extend the operations on the field F and the ordered multiplicative group Λ by setting for all $x \in F^*, n \in \mathbb{Z}$, $x + \gamma^n = x$ if $n \geq 1$, $x + \gamma^n = \gamma^n$ if $n \leq 0$, and $x \bullet \gamma^n = \gamma^n$. For $l \geq 1$, put $L = G \sqcup \Lambda$, where $G := \mathbb{Z} \times F^*$ has the natural structure of Abelian group under the law $(n, x) \bullet (m, y) = (n + m, xy)$, with the neutral element $\delta := (0, 1)$, and $(n, x)^{-1} = (-n, x^{-1})$. In addition, define $-(n, x) = (n, -x)$,

$$(n, x) + (m, y) = \begin{cases} (n, x + y) & \text{if} \quad n = m, x + y \neq 0 \\ \gamma^{ln+1} & \text{if} \quad n = m, x + y = 0 \\ (n, x) & \text{if} \quad n < m, \end{cases}$$

$$(n, x) + \gamma^m = \begin{cases} (n, x) & \text{if} \quad m > ln \\ \gamma^m & \text{if} \quad m \leq ln, \end{cases}$$

and $(n, x) \bullet \gamma^m = \gamma^{ln+m}$. In particular, $e^+((n, x)) = \gamma^{ln+1}, v((n, x)) = \gamma^{ln}, (n, x) \wedge \gamma^m = \gamma^{\min(nl, m)}, d((n, x), \gamma^m) = \gamma^{|nl-m|+1}$, and $(n, x) \wedge (m, y) = \gamma^{\min(n,m)l}, d((n, x), (m, y)) = \gamma^{|n-m|l+2}$ provided $(n, x) \neq (m, y)$.

Conversely, let R be a lcr-qsring with $\Lambda = E^+ \cong \mathbb{Z}$ and $E^\bullet = \{\varepsilon, \delta\}$, so $R = E^+ \sqcup G$, where $G := \{x \in R \mid e^\bullet(x) = \delta\}$. Thus $(G, \bullet, ^{-1})$ is an Abelian group with the neutral element δ, and the qvaluation $v : R \longrightarrow \Lambda = E^+$ induces a morphism $\varphi : G \longrightarrow \mathbb{Z}$ such that $v(x) = \gamma^{\varphi(x)}$ for all $x \in G$. Put $G_0 := \text{Ker}(\varphi) = \{x \in G \mid v(x) = \varepsilon\}$, and let $l \in \mathbb{N}$ be such that $\varphi(G) = l\mathbb{Z}$, in particular, $G = G_0$ for $l = 0$. It follows easily that $x \in G_0 \Longrightarrow -x \in G_0$, $x + y \in G_0$ whenever $x, y \in G_0, x \neq -y$, and $e^+(x) = e^+(\delta) = \gamma$ for all $x \in G_0$, and hence $F := G_0 \sqcup \{\gamma\}$ becomes a field with γ as the neutral element for addition, and δ as the neutral element for multiplication. If $l = 0$ then R is canonically isomorphic with $L(F, 0)$ as defined above. If $l \geq 1$, for any $\pi \in G$ satisfying $\varphi(\pi) = l$, the map $G \longrightarrow \mathbb{Z} \times F^*, g \mapsto (\frac{\varphi(g)}{l}, \pi^{-\frac{\varphi(g)}{l}} \bullet g)$ extends to an isomorphism $R \cong L(F, l)$ which is the identity on Λ and sends the *local uniformizer* π to the pair $(1, \delta)$.

(2) The simplest superrigid lcr-qsrings are of the form $\mathcal{R}_\Lambda$ (see Proposition 4.8.), where Λ is a totally ordered Abelian group, so $\mathcal{R}_\Lambda = E^+ \cup E^\bullet = \Lambda \times \{\varepsilon\} \cup \{\varepsilon\} \times \Lambda_+$.

According to a classical result of Garrett Birkhoff, we obtain

Proposition 4.13.   (1) *Every directed cr-qsring has a subdirect representation for which all the factors are lcr-qsrings.*

(2) *The variety of all directed cr-qsrings is generated by the class of all lcr-qsrings.*

Any directed cr-qsring $(R, +, \bullet, -, ^{-1}, \varepsilon)$ is canonically embedded as a subalgebra of signature $(2, 2, 1, 1, 0)$ into the direct product $\widetilde{R} := \prod_P R_P$ of lcr-qsrings, where P ranges over the minimal prime convex submonoids of Λ_+, in 1-1 correspondence with the ultrafilters of the distributive lattice $(\Lambda_+, +, \vee)$ with the least element ε, as well as with the minimal prime convex l-subgroups of Λ, and $R_P := R/ \equiv_P$ is the quotient of R by the congruence $\equiv_P$ associated to P as in Lemma 4.6. The corresponding Abelian l-group of the product $\widetilde{R}$ is the direct product $\widetilde{\Lambda} := \prod_P \Lambda_P$ of the maximal totally ordered factors of Λ, in which the Abelian l-group Λ is canonically embedded as a subdirect product.

A basic consequence of Proposition 4.13. is provided by the statement (2) of the next result.

Corollary 4.14. *Let R be a directed cr-qsring.*

(1) *For all $x, y, z \in R$, $\frac{d(x,z) \bullet d(y,z)}{d(x,y)} = (x,y)_z^2$, where*

$$(x,y)_z := \frac{\widehat{\lambda}(x,y,z)}{d(x,y)} = \frac{e^+(z) \bullet e^+(x \wedge y)}{e^+(x \wedge z) \bullet e^+(y \wedge z)} \in \Lambda_+.$$

The following hold.

   (i) $(x,y)_z = \varepsilon \iff z \in [x, y]$;

   (ii) $(x,y)_z \geq d(x,z) \iff (x,y)_z = d(x,z) \iff x \in [y, z]$.

(2) *For all $x, y, z, u \in R$, $(x,y)_u + (x,z)_u \leq (y,z)_u$, i.e. the ternary map $R^3 \longrightarrow \Lambda_+, (x,y,z) \mapsto (x,y)_u + (x,z)_u$ is symmetric.*

(3) *For all $x, y \in R$, the map $d(x,-) : [x,y] \longrightarrow [\varepsilon, d(x,y)], z \mapsto d(x,z)$ is a Λ-isometry (not necessarily surjective), i.e. $d(z,u) = |\frac{d(x,z)}{d(x,u)}|$ for $z, u \in [x,y]$. In particular, the map $d(x,-) : [x,y] \longrightarrow [\varepsilon, d(x,y)]$ is a monomorphism of posets, where the partial order $\prec_x$ on $[x,y]$ is defined by $z \prec_x u \iff z \in [x,u]$.*

(4) *For all $x, y \in R$, the map $[x,y] \longrightarrow [x, x \wedge y] \times [x \wedge y, y], z \mapsto (z \wedge x, z \wedge y)$ is a monomorphism of posets with respect to the order $\prec_x$ on $[x,y]$ and the order $(a,b) \prec (a', b') \iff a' \leq a, b \leq b'$ on the product $[x, x \wedge y] \times [x \wedge y, y]$. Its image consists of those pairs (a,b) for which $a \vee b \neq \infty$. The composition of the injective monotone map above with the isomorphism of bounded distributive lattices*

$$[x, x \wedge y] \times [x \wedge y, y] \longrightarrow [\varepsilon, \lambda(x,y)] \times [\lambda(x,y), d(x,y)], (a,b) \mapsto (d(x,a), d(x,b))$$

equals the composition of the injective monotone map $d(x,-)$ from (3) with the monomorphism of bounded distributive lattices

$$[\varepsilon, d(x,y)] \longrightarrow [\varepsilon, \lambda(x,y)] \times [\lambda(x,y), d(x,y)], \gamma \mapsto (\gamma + \lambda(x,y), \gamma \vee \lambda(x,y)).$$

(5) *If $[x,y] = [z,u]$ then $x \wedge y = z \wedge u, \lambda(x,z) = \lambda(u,y), \lambda(x,u) = \lambda(z,y), d(x,z) = d(y,u), d(x,u) = d(y,z)$, and $d(x,y) = d(z,u)$. In particular, the diameter $d(x,y)$ is an invariant of the cell $C = [x,y]$.*

(6 *For all $x, y, z \in R$, the intersection $[x,y] \cap [y,z] \cap [z,x]$ has at most one element.*

Proof. The assertion (1) is immediate by Lemma 4.1.(3).
  (2) By (1) we have to show that

$$\mathcal{L} := e^+(x \wedge y) \bullet e^+(z \wedge u) + e^+(x \wedge z) \bullet e^+(y \wedge u) \leq \mathcal{R} := e^+(y \wedge z) \bullet e^+(x \wedge u)$$

for all $x, y, z, u \in R$. By Proposition 4.13., it suffices to check that the inequality $\mathcal{L} \leq \mathcal{R}$ holds in the product $\widetilde{R}$, i.e. it is true in every factor R_P. Consequently, we may assume

from the beginning that R is a lcr-qsring. Setting $a := x \wedge y \wedge z$, we distinguish by Corollary 4.10.(5) the following three cases.

(i) $a = x \wedge y = x \wedge z$: By Lemmas 3.4.(4, 5) and 3.6.(1, 2), we get

$$\mathcal{L} = e^+(a) \bullet (e^+(z \wedge u) + e^+(y \wedge u)) = e^+(a) \bullet e^+(y \wedge z \wedge u) = e^+(a \wedge u) \bullet e^+(a \vee (y \wedge z \wedge u)) \leq \mathcal{R},$$

as required.

(ii) $a = x \wedge y = y \wedge z$: Then, again by the lemmas above, we obtain

$$A := e^+(x \wedge y) \bullet e^+(z \wedge u) = e^+(a) \bullet e^+(z \wedge u) = e^+(a \wedge u) \bullet e^+(a \vee (z \wedge u))$$

and

$$\mathcal{R} = e^+(a) \bullet e^+(x \wedge u) = e^+(a \wedge u) \bullet e^+(a \vee (x \wedge u)),$$

therefore $\mathcal{L} \leq A \leq \mathcal{R}$ provided $z \wedge u \leq x \wedge u$. Otherwise, $x \wedge u = x \wedge z$ by Corollary 4.10.(5), and hence $A = e^+(a) \bullet e^+(z \wedge u), \mathcal{R} = e^+(a) \bullet e^+(x \wedge u)$, and $B := e^+(x \wedge z) \bullet e^+(y \wedge u) = e^+(a) \bullet e^+((x \wedge u) \vee (y \wedge u))$, therefore $\mathcal{L} = A + B = e^+(a) \bullet e^+((z \wedge u) \wedge ((x \wedge u) \vee ((y \wedge u)))) = \mathcal{R}$, as desired.

(iii) $a = x \wedge z = y \wedge z$: We proceed as in the case (ii).

(3) is a consequence of (2). Indeed, let $x, y, z, u \in R$ be such that $z, u \in [x, y]$, i.e. $(y, z)_x = d(x, z)$ and $(y, u)_x = d(x, u)$ by (1),(ii). The inequality $d(z, u) \geq | \frac{d(x,z)}{d(x,u)} |$ is obvious by the triangle inequality. On the other hand, it follows by (2) that

$$d(x, z) + d(x, u) = (y, z)_x + (y, u)_x \leq (z, u)_x,$$

therefore $d(z, u) \leq \frac{d(x,z) \bullet d(x,u)}{(d(x,z) + d(x,u))^2} = | \frac{d(x,z}{d(x,u)} |$ as desired.

The rest of the statements follow by straightforward computation.   □

5 Median and locally faithfully full directed commutative regular quasi-semirings

We are now ready to introduce an interesting class of directed cr-qsrings containing all locally linear directed cr-qsrings, as well as arbitrary direct products of them.

Lemma 5.1. *The following are equivalent for a directed cr-qsring R.*

(1) *For all $x, y, z \in R$, the elements $x \wedge y, y \wedge z$, and $z \wedge x$ are bounded above, and hence the element $m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ is well defined.*

(2) *Every finite family $(x_i)_{1 \leq i \leq n}, n \geq 1$, of pairwise incident elements of R is bounded above, and hence the join $\bigvee \{x_i \mid i = 1, \cdots, n\}$ is well defined.*

(3) *For all $x, y, z \in R$, the intersection $[x, y] \cap [y, z] \cap [z, x]$ consists of a single element.*

(4) *For all $x, y, z \in R$, the intersection $[x, y] \cap [y, z] \cap [z, x]$ is nonempty.*

(5) *For all $x, y, z \in R$, there exists $u \in R$ such that $[x, y] \cap [x, z] = [x, u]$ and $u \in [y, z]$.*

Proof. The implications $(2) \implies (1)$, $(3) \implies (4)$, $(4) \implies (1)$, and $(5) \implies (4)$ are obvious. $(1) \implies (2)$ follows by induction, $(4) \implies (3)$ by Corollary 4.12.(6), and $(1) \implies (5)$ with $u = m(x, y, z)$, by straightforward verification. $\square$

Definition 5.2. *A directed cr-qsring R satisfying the equivalent conditions from* Lemma 5.1. *is said to be* median.

Remarks 5.3. (1) Assuming that R is a median directed cr-qsring, it follows by Lemmas 4.3.(2) and 5.1.(1, 3) that $(R, m : R^3 \longrightarrow R)$ is a *median set (algebra)* [2], [7, Proposition 3.1.]. Moreover, since $m(x, y, z) \wedge u = m(x \wedge u, y \wedge u, z)$ for all $x, y, z, u \in R$, the algebra $(R, m, \wedge)$ of signature $(3, 2)$ is a *directed median set* [4]. By Proposition 3.1.(5) and Lemmas 3.4.(3) and 3.6.(3), for any $a \in R$, the translations $x \mapsto a + x$ and $x \mapsto a \bullet x$ are endomorphisms of the directed median set $(R, m, \wedge)$. By Lemma 4.4., Remark 4.5. and Lemma 5.1.(3), the map $x \mapsto -x$ is an involutive automorphism of $(R, m, \wedge)$, while the map $x \mapsto x^{-1}$ is an involutive automorphism of the median set (R, m), i.e. $m(x, y, z)^{-1} = m(x^{-1}, y^{-1}, z^{-1})$ for all $x, y, z \in R$, and also an isomorphism between the directed median sets $(R, m, \wedge)$ and $(R, m, \sqcap)$.

(2) With the same assumption as above, for all $x, y \in R$, the poset $([x, y], \prec_x)$, as defined in Corollary 4.14.(3), is a bounded distributive lattice with the meet operation $(z, u) \in [x, y] \mapsto m(z, x, u)$, and the join operation $(z, u) \mapsto m(z, y, u)$. The Λ-isometry $d(x, -) : [x, y] \longrightarrow [\varepsilon, d(x, y)], z \mapsto d(x, z)$ is a monomorphism of bounded distributive lattices.

(3) In a median directed cr-qsring R, the element $(x, y)_z \in \Lambda_+$ as defined in Corollary 4.14.(1) is exactly the distance $d(m(x, y, z), z)$, and hence the inequality from Corollary 4.14.(2) can be alternatively proved as follows. Setting $a := m(x, y, u), b := m(x, z, u), c := m(y, z, u)$, we obtain

$$[u, a] \cap [u, b] = [u, m(a, b, u)] = [u, m(c, x, u)] = [u, x] \cap [u, c].$$

Since $a, b \in [u, x], m(a, b, u) \in [u, c]$, and the map $d(u, -) : [u, x] \longrightarrow [\varepsilon, d(u, x)]$ is a morphism of bounded distributive lattices, it follows that

$$(x, y)_u + (x, z)_u = d(u, a) \wedge d(u, b) = d(u, m(a, b, u)) \leq d(u, c) = (y, z)_u,$$

as desired.

(4) Extending the signature $(+, \bullet, -, ^{-1}, \varepsilon)$ with a ternary operation standing for the median m, it follows by Lemma 5.1.(4) that the class of all median directed cr-qsrings is a variety defined by the finitely many equational axioms for directed cr-qsrings extended with axioms expressing the fact that the ternary map m is symmetric and $m(x, y, z) \in [x, y]$ for all x, y, z. By Corollary 4.10., its subdirectly irreducible members are the lcr-qsrings, identified with those directed median cr-qsrings which satisfy the suplementary very restrictive universal axiom

$$m(x, y, z) \in \{x \wedge y, y \wedge z, z \wedge x\}.$$

(5) The directed median cr-qsrings are the models of the inductive theory T_m in the first order language L with signature $(+, \bullet, -, ^{-1}, \varepsilon)$, obtained from the universal

theory T of directed cr-qsrings by adding the $\forall\exists$-sentence

$$\forall x, y, z \,\exists u \,(x \wedge y \leq u) \bigwedge (y \wedge z \leq u) \bigwedge (z \wedge x \leq u).$$

By Proposition 4.13., $T = (T_m)_\forall$, the universal theory of T_m, i.e. the directed cr-qsrings form the class of L-substructures of the models of T_m.

(6) For any Abelian l-group Λ, the superrigid directed cr-qsring $\mathcal{R}_\Lambda$ as defined in Proposition 4.8. is median. As shown in Remark 4.9., for any $\alpha \in \Lambda_+ \setminus \{\varepsilon\}$, the necessary and sufficient condition for the subset $S_\alpha := \{(\gamma, \delta) \in \mathcal{R}_\Lambda \mid \delta < \alpha\}$ to be a substructure of $\mathcal{R}_\Lambda$ in the signature $(+, \bullet, -, {}^{-1}, \varepsilon)$ is that α is not a proper disjoint join in Λ_+. The necessary and sufficient condition for such a substructure S_α to be also closed under the median operation is that the cardinality of any subset of $[\varepsilon, \alpha] \setminus \{\varepsilon, \alpha\}$ consisting of pairwise disjoint elements is at most 2. Indeed, assuming that there exist $\varepsilon < \sigma_i < \alpha, i = 1, 2, 3$, such that $\sigma_i + \sigma_j = \varepsilon$ for $i \neq j$, it follows that $m(x_1, x_2, x_3) = (\varepsilon, \alpha) \in \mathcal{R}_\Lambda \setminus S_\alpha$, where $x_i := (\varepsilon, \alpha \sigma_i^{-1}), i = 1, 2, 3$. Conversely, assuming that S_α is not median, there exist $x_i := (\gamma_i, \delta_i) \in S_\alpha, i = 1, 2, 3$, such that $x := m(x_1, x_2, x_3) \in \mathcal{R}_\Lambda \setminus S_\alpha$, therefore

$$\alpha = e^+(e^\bullet(x)) = \bigvee_{1 \leq i < j \leq 3} e^+(e^\bullet(x_i \wedge x_j)) \leq m(\delta_1, \delta_2, \delta_3) \leq \alpha.$$

Consequently, $\delta_i \vee \delta_j = \alpha$ for $1 \leq i < j \leq 3$, so $\{\alpha \delta_i^{-1} \mid i = 1, 2, 3\}$ is a subset of $[\varepsilon, \alpha] \setminus \{\varepsilon, \alpha\}$ consisting of three pairwise disjoint elements as desired.

The lexicographic extensions of totally ordered Abelian groups by Abelian l-groups provide examples of such substructures S_α which are not median. For instance, take Λ the free Abelian (multiplicative) group with 4 generators $\sigma_1, \sigma_2, \sigma_3$ and α. With respect to the partial order

$$\sigma_1^{n_1} \sigma_2^{n_2} \sigma_3^{n_3} \alpha^n \leq \sigma_1^{m_1} \sigma_2^{m_2} \sigma_3^{m_3} \alpha^m \iff \text{eiher } n < m \text{ or } n = m \text{ and } n_i \leq m_i, i = 1, 2, 3,$$

Λ becomes an Abelian l-group with

$$\Lambda_+ = \{\sigma_1^{n_1} \sigma_2^{n_2} \sigma_3^{n_3} \alpha^n \mid \text{either } n > 0 \text{ or } n = 0 \text{ and } n_i \geq 0, i = 1, 2, 3\}.$$

As α is not a proper disjoint join in Λ_+, S_α is a substructure of $\mathcal{R}_\Lambda$ in the signature $(+, \bullet, -, {}^{-1}, \varepsilon)$ but S_α is not closed under the median operation since $m(x_1, x_2, x_3) = (\varepsilon, \alpha) \in \mathcal{R}_\Lambda \setminus S_\alpha$, where $x_i := (\varepsilon, \alpha \sigma_i^{-1}) \in S_\alpha, i = 1, 2, 3$.

Another interesting class of directed cr-qsrings, which turns out to be a proper subclass of the class of all directed median cr-qsrings, is introduced by the next statement.

Lemma 5.4. *The following are equivalent for a directed cr-qsring R.*

(1) *For all $x, y \in R$, $x \vee y \neq \infty \iff x - y \in E^+$.*

(2) *For all $x, y \in R$, there exists a (unique) element $z \in R$ such that $d(x, z) = d(y, x \wedge y)$ and $d(y, z) = d(x, x \wedge y)$, in particular, $z \in [x, y]$.*

(3) *The Λ-metric space (R,d) is* locally full, *i.e. for all $x,y \in R$, the map*
$d(x,-) : [x,y] \longrightarrow [\varepsilon, d(x,y)], z \mapsto d(x,z)$ *is surjective.*

(4) *The Λ-metric space (R,d) is* locally faithfully full, *i.e. for all $x,y \in R$, the map*
$d(x,-) : [x,y] \longrightarrow [\varepsilon, d(x,y)]$ *is bijective.*

Proof. The implications $(4) \implies (3)$ and $(3) \implies (2)$ are obvious, while $(3) \implies (4)$ follows by Corollary 4.14.(3).

$(1) \implies (3)$: For $x,y \in R$, let $\gamma \in [\varepsilon, d(x,y)]$. We have to find an element $z \in [x,y]$ such that $d(x,z) = \gamma$. By Lemma 4.3.(4), the elements

$$z_1 := x + \frac{e^+(x)}{\gamma + \lambda(x,y)} = x + \frac{e^+(x) \bullet e^+(x \wedge y)}{e^+(x) + (\gamma \bullet e^+(x \wedge y))}$$

and

$$z_2 := y + \frac{e^+(y) \bullet (\gamma \vee \lambda(x,y))}{d(x,y)} = y + \frac{\gamma \bullet e^+(x \wedge y)^2}{e^+(x) + (\gamma \bullet e^+(x \wedge y))}$$

are uniquely determined by the conditions $z_1 \in [x, x \wedge y], d(x,z_1) = \gamma + d(x, x \wedge y)$ and $z_2 \in [x \wedge y, y], d(x,z_2) = \gamma \vee d(x, x \wedge y)$ respectively. As $z_1 - z_2 = x - y + e^+(x \wedge y) = v(x-y) \in E^+$, it follows by assumption that $z := z_1 \vee z_2 \neq \infty$. The element z has the required property thanks to Corollary 4.14.(4).

$(2) \implies (1)$: Let $x,y \in R$ be such that $x - y \in E^+$, i.e. $e^+(x \wedge y) = e^+(x) + e^+(y)$. We have to show that $x \vee y \neq \infty$. By assumption there exists $z \in [x,y]$ such that $d(x,z) = d(y, x \wedge y)$. As $x \wedge y \leq z = (x \wedge z) \vee (y \wedge z)$, it follows that

$$d(x,z) := \frac{e^+(x) \bullet e^+(z)}{e^+(x \wedge z)^2} = \frac{e^+(x) \bullet (e^+(x \wedge z) \vee e^+(y \wedge z))}{e^+(x \wedge z)^2} = \frac{e^+(x) \bullet e^+(y \wedge z)}{e^+(x \wedge z) \bullet e^+(x \wedge y)} =$$

$$d(y, x \wedge y) := \frac{e^+(y)}{e^+(x \wedge y)},$$

therefore $\dfrac{e^+(y \wedge z)}{e^+(x \wedge z)} = \dfrac{e^+(y)}{e^+(x)}$. Consequently,

$$\left(\frac{e^+(y \wedge z)}{e^+(x \wedge z)}\right)_+ := \frac{e^+(y \wedge z)}{e^+(x \wedge z) + e^+(y \wedge z)} = \frac{e^+(y \wedge z)}{e^+(x \wedge y)} =$$

$$\left(\frac{e^+(y)}{e^+(x)}\right)_+ := \frac{e^+(y)}{e^+(x) + e^+(y)} = \frac{e^+(y)}{e^+(x \wedge y)},$$

and hence $e^+(y \wedge z) = e^+(y)$, and similarly, $e^+(x \wedge z) = e^+(x)$. Thus $x \wedge z = x, y \wedge z = y$, and hence $z = x \vee y \neq \infty$ as desired. $\square$

Definition 5.5. *A directed cr-qsring R satisfying the equivalent conditions from* Lemma 5.4. *is said to be* locally faithfully full *(abbreviated* lff*).*

Remarks 5.6. (1) Every lff directed cr-qsring R is median. Indeed, for arbitrary $x, y, z \in R$, the element $(x, y)_z \in \Lambda_+$ as defined in Corollary 4.14.(1) is bounded above by the distance $d(x, z)$, and hence, by Lemma 5.4.(4), there exists uniquely $u \in [x, z]$ such that $d(z, u) = (x, y)_z$. According to Remarks 5.3.(3), we obtain $u = m(x, y, z)$, the median of the triple (x, y, z).

(2) In a lff directed cr-qsring R, for all $x, y \in R$, the map $d(x, -) : [x, y] \longrightarrow [\varepsilon, d(x, y)]$ is an *isomorphism* of Λ-metric spaces as well as an *isomorphism* of bounded distributive lattices.

(3) Extending the signature $(+, \bullet, -, ^{-1}, \varepsilon)$ with a binary operation $\vee'$, the class of all lff directed cr-qsrings becomes a variety defined by the finitely many equational axioms for directed cr-qsrings extended with the following two equational axioms

(i) $x \vee' y = y \vee' x$

and

(ii) $e^+(x) \bullet (x - y) \bullet e^+(x \vee' y) = e^+(y) \bullet (x - (x \vee' y))^2$,

defining $x \vee' y$ as the unique element z satisfying the conditions $d(x, z) = d(y, x \wedge y)$ and $d(y, z) = d(x, x \wedge y)$ from Lemma 5.4.(2). Notice that, as shown in the proof of the implication $(2) \Longrightarrow (1)$ of Lemma 5.4., $x \vee' y = x \vee y \Longleftrightarrow x - y \in E^+$.

By Corollary 4.10., the subdirectly irreducible members of the variety of all lff directed cr-qsrings are the lcr-qsrings, identified with those lff directed cr-qsrings which satisfy the supplementary very restrictive universal axiom

$$(x \vee' y \le x) \bigvee (x \vee' y \le y).$$

(4) The lff directed cr-qsrings are the models of the inductive theory T_{lff} in the first order language L with signature $(+. \bullet, -, ^{-1}, \varepsilon)$, obtained from the universal theory T of directed cr-qsrings by adding the $\forall \exists$-sentence

$$\forall x, y \, \exists z \, (2(x - y) \ne x - y) \bigvee ((x \le z) \bigwedge (y \le z)).$$

By Proposition 4.13., $T = (T_{lff})_\forall$, the universal theory of T_{lff}, i.e. the directed cr-qsrings form the class of L-substructures of the models of T_{lff}.

(5) For any Abelian l-group Λ, the superrigid directed cr-qsring $\mathcal{R}_\Lambda$ as defined in Proposition 4.8. is lff, in particular median, where $(\gamma, \delta) \vee (\gamma', \delta') = (\gamma \vee \gamma', \delta \vee \delta')$ for all $(\gamma, \delta), (\gamma', \delta') \in \mathcal{R}_\Lambda$ satisfying $(\gamma, \delta) + (\gamma', \delta') \in E^+(\mathcal{R}_\Lambda)$, i.e. $\gamma + \gamma' \bullet \delta' = \gamma' + \gamma \bullet \delta$. For all $\alpha \in \Lambda_+ \setminus \{\varepsilon\}$, the necessary and sufficient condition for the subset $S_\alpha := \{(\gamma, \delta) \in \mathcal{R}_\Lambda \mid \delta < \alpha\}$ to be lff, i.e. a substructure of $\mathcal{R}_\Lambda$ in the extended signature $(+, \bullet, \vee', -, ^{-1}, \varepsilon)$ is that $\sigma + \tau \ne \varepsilon$ for all $\varepsilon < \sigma, \tau < \alpha$, in particular, α is not a proper disjoint join in Λ_+. Indeed, assuming that there exists a pair of disjoint elements (σ, τ) in $[\varepsilon, \alpha] \setminus \{\varepsilon, \alpha\}$, it follows that $x \vee y = (\varepsilon, \alpha) \in \mathcal{R}_\Lambda \setminus S_\alpha$, where $x := (\varepsilon, \alpha\sigma^{-1}), y = (\varepsilon, \alpha\tau^{-1}) \in S_\alpha$. The converse is obvious.

As an example of a median substructure S_α which is not lff, take Λ the free Abelian (multiplicative) group with three generators σ_1, σ_2 and α, becoming an l-group under the partial order

$$\sigma^{n_1}\sigma_2^{n_2}\alpha^n \le \sigma_1^{m_1}\sigma_2^{m_2}\alpha^m \Longleftrightarrow \text{either } n < m \text{ or } n = m \text{ and } n_i \le m_i, i = 1, 2.$$

In this case, S_α is a median substructure of $\mathcal{R}_\Lambda$ according to Remarks 5.3.(6), but S_α is not lff since $x_1 \vee x_2 = (\varepsilon, \alpha) \in \mathcal{R}_\Lambda \setminus S_\alpha$, where $x_i := (\varepsilon, \alpha\sigma_i^{-1}) \in S_\alpha, i = 1, 2$.

6 l-valuations and Prüfer extensions associated to super-rigid directed commutative regular quasi-semirings

In this section we associate a l-valued commutative ring to a superrigid directed cr-qsring, and we discuss the case when the associated commutative ring is a Prüfer extension of its l-valuation subring.

6.1 The l-valued commutative ring associated to a superrigid directed commutative regular quasi-semiring

For a given directed cr-qsring R, let $R_\alpha := \{x \in R \,|\, e^+(x) = \alpha\}$ for any $\alpha \in \Lambda := E^+(R)$. The family $(R_\alpha, +)_{\alpha \in \Lambda}$ is an inverse system of Abelian groups with the connecting morphisms $\pi_{\alpha\beta} : R_\beta \longrightarrow R_\alpha, x \mapsto x + \alpha$ for $\alpha \leq \beta$. We denote by $B := B(R) = \varprojlim_{\alpha \in \Lambda} R_\alpha \cong \varprojlim_{\alpha \in \Lambda_+} R_\alpha$, the inverse limit of the inverse system above. Thus the Abelian group B consists of all maps $\varphi : \Lambda_+ \longrightarrow R$ satisfying $e^+(\varphi(\alpha)) = \alpha$ for $\alpha \in \Lambda_+$, and $\varphi(\alpha) \leq \varphi(\beta)$ for $\alpha \leq \beta$. The operations $+, -$ are defined pointwise, while the neutral element 0 is the embedding $\Lambda_+ \longrightarrow R, \alpha \mapsto \alpha$. The injective map $\varphi \in B \mapsto \widetilde{\varphi} : \Lambda \longrightarrow R$, where $\widetilde{\varphi}(\alpha) := \varphi(\alpha_+) + \alpha$ for $\alpha \in \Lambda$, so $\widetilde{\varphi}|_{\Lambda_+} = \varphi$, maps B onto the Abelian group consisting of the sections $\psi : \Lambda \longrightarrow R$ of the surjective map $e^+ : R \longrightarrow \Lambda, x \mapsto e^+(x)$, satisfying $\psi(\alpha \wedge \beta) = \psi(\alpha) \wedge \psi(\beta), \psi(\alpha \vee \beta) = \psi(\alpha) \vee \psi(\beta)$ for all $\alpha, \beta \in \Lambda$.

The Abelian group $(B, +, -, 0)$ becomes a commutative ring (not necessarily unital) with respect to the multiplication $B \times B \longrightarrow B, (\varphi, \psi) \mapsto \varphi \cdot \psi$, defined by

$$(\varphi \cdot \psi)(\alpha) := \varphi(\alpha \bullet v(\psi(\varepsilon))^{-1}) \bullet \psi(\alpha \bullet v(\varphi(\varepsilon))^{-1}) + \alpha =$$

$$\lim_{\gamma \to \infty} (\varphi(\gamma) \bullet \psi(\gamma) + \alpha), \text{ for } \alpha \in \Lambda_+.$$

The ring B is unital if and only if the directed cr-qsring R is superrigid, and in this case the unit $1 \in B$ is the injective map $\Lambda_+ \longrightarrow R$, sending $\alpha \in \Lambda_+$ to the unique element $1_\alpha \in E^\bullet(R)$ satisfying $e^+(1_\alpha) = \alpha$. We assume in the following that R is superrigid.

For every $\varphi \in B$, the map $v \circ \varphi : \Lambda_+ \longrightarrow \Lambda$ belongs to the commutative l-monoid $\widehat{\Lambda}$ as defined in 1.1, and the map $w : B \longrightarrow \widehat{\Lambda}, \varphi \mapsto w(\varphi) := v \circ \varphi$ is a l-valuation on the commutative ring B, with $\mathrm{supp}(w) = \{0\}$, so the ring B is reduced, and $A := A_w = w^{-1}(\widehat{\Lambda}_+) = \{\varphi \in B \,|\, \varphi(\varepsilon) = \varepsilon\}$, the l-valuation ring of w. Call the map $w : B \longrightarrow \widehat{\Lambda}$ the *l-valuation associated to the superrigid directed cr-qsring R*.

For every $\alpha \in \Lambda_+$, the subgroup $T_\alpha := \{x \in R_\alpha \,|\, \varepsilon \leq x\} = \{x \in R_\alpha \,|\, v(x) \in \Lambda_+\}$ of the Abelian group $(R_\alpha, +)$ becomes a commutative ring with multiplication $x \cdot y := x \bullet y + \alpha$ for $x, y \in T_\alpha$, and unit 1_α. For $\alpha \leq \beta$, the connecting map $\pi_{\alpha\beta} : R_\beta \longrightarrow R_\alpha$ induces by restriction a ring morphism $T_\beta \longrightarrow T_\alpha$, and $A = \varprojlim_{\alpha \in \Lambda_+} T_\alpha \cong \varprojlim_{\alpha \in \Lambda_+} A/I_\alpha$,

where $I_\alpha := \mathrm{Ker}\,(A \longrightarrow T_\alpha, \varphi \mapsto \varphi(\alpha)) = \{\varphi \in B \,|\, w(\varphi) \geq \alpha\}$ is an ideal of $A = I_\varepsilon$. On the other hand, for every $\alpha \in \Lambda_+$, R_α becomes an A-module under the action $\varphi \cdot x := \varphi(\alpha \bullet v(x)_-) \bullet x + \alpha$ for $\varphi \in A, x \in R_\alpha$, in particular, $\varphi \cdot x = \varphi(\alpha) \cdot x$ for $x \in T_\alpha$. Thus the canonical map $\pi_\alpha : B \longrightarrow R_\alpha, \varphi \mapsto \varphi(\alpha)$, for $\alpha \in \Lambda_+$, is a morphism of A-modules, and $B \cong \varprojlim_{\alpha \in \Lambda_+} R_\alpha \cong \varprojlim_{\alpha \in \Lambda_+} B/I_\alpha$ as A-modules.

Lemma 6.1. *Let R be a superrigid directed cr-qsring R, $w : B \longrightarrow \widehat{\Lambda}$ its associated l-valuation, and A the l-valuation ring. Then $w^{-1}(\Lambda) = B^*$, and $w^{-1}(\varepsilon) = A^*$, so $w(B) \cap \Lambda \cong B^*/A^*$. In particular, B is a field and A is a valuation ring of B whenever Λ is totally ordered, i.e. R is a superrigid lcr-qsring.*

Proof. We have to show that $w^{-1}(\Lambda) \subseteq B^*$. Let $\varphi \in B$ with $w(\varphi) = \gamma \in \Lambda$. Define the map $\psi : \Lambda_+ \longrightarrow R$ by $\psi(\alpha) = \varphi(\alpha \bullet \gamma_+^2)^{-1} + \alpha$ for $\alpha \in \Lambda_+$. As $\alpha \bullet \gamma_+^2 \geq \gamma_+$ for all $\alpha \in \Lambda_+$, it follows that

$$e^+(\psi(\alpha)) = \frac{\alpha \bullet \gamma_+^2}{v(\varphi(\alpha \bullet \gamma_+^2))^2} + \alpha = \frac{\alpha \bullet \gamma_+^2}{\gamma^2} + \alpha = \alpha \bullet \gamma_-^2 + \alpha = \alpha.$$

To conclude that $\psi \in B$, it suffices to show that $\varphi(\alpha \bullet \gamma_+^2)^{-1} \leq \varphi(\beta \bullet \gamma_+^2)^{-1}$ for $\alpha \leq \beta$. By Lemma 3.5.(4), it follows that

$$\varphi(\alpha \bullet \gamma_+^2)^{-1} \wedge \varphi(\beta \bullet \gamma_+^2)^{-1} = \varphi(\beta \bullet \gamma_+^2)^{-1} \bullet e^\bullet(\varphi(\alpha \bullet \gamma_+^2)) \leq \varphi(\alpha \bullet \gamma_+^2)^{-1}.$$

Applying e^+ to the both members of the last inequality, we obtain by Lemma 2.6.(4) the same value $\alpha \bullet \gamma_-^2$, so the inequality becomes an equality as desired.

Finally note that $w(\psi) = \gamma^{-1}$ and

$$(\varphi\psi)(\alpha) = \varphi(\alpha \bullet \gamma_+) \bullet \psi(\alpha \bullet \gamma_-) + \alpha = \varphi(\alpha \bullet \gamma_+) \bullet (\varphi(\alpha \bullet |\gamma| \bullet \gamma_+)^{-1} + \alpha \bullet \gamma_-) + \alpha = 1_\alpha$$

for all $\alpha \in \Lambda_+$, and hence $\varphi = \psi^{-1} \in B^*$, as required. $\hfill\square$

Lemma 6.2. *With the same data as in Lemma 6.1, let $E^\bullet(B) := \{\varphi \in B \,|\, \varphi^2 = \varphi\}$ be the boolean algebra of idempotent elements of the commutative ring B. Then the l-valuation $w : B \longrightarrow \widehat{\Lambda}$ induces by restriction an anti-isomorphism of boolean algebras $E^\bullet(B) \longrightarrow \partial\widehat{\Lambda}_+$; we denote its inverse by η.*

Proof. The inclusion $w(E^\bullet(B)) \subseteq E^\bullet(\widehat{\Lambda}) = \partial\widehat{\Lambda}_+$ is obvious, in particular $E^\bullet(B) \subseteq A$. First let us show that the map $w|_{E^\bullet(B)} : E^\bullet(B) \longrightarrow \partial\widehat{\Lambda}_+$ is injective. Let $\varphi \in E^\bullet(B)$, i.e. $\varphi(\alpha)^2 + \alpha = \varphi(\alpha)$, in particular $v(\varphi(\alpha)) \vee \dfrac{\alpha}{v(\varphi(\alpha))} = \alpha$, for all $\alpha \in \Lambda_+$. By multiplication with $\varphi(\alpha)^{-1}$, it follows that

$$\varphi(\alpha) + \frac{\alpha}{v(\varphi(\alpha))} \leq \varphi(\alpha)^{-1} \bullet (\varphi(\alpha)^2 + \alpha) = e^\bullet(\varphi(\alpha)) = 1_{\frac{\alpha}{v(\varphi(\alpha))}}.$$

The inequality above becomes an identity since $e^+(\varphi(\alpha) + \dfrac{\alpha}{v(\varphi(\alpha))}) = \dfrac{\alpha}{v(\varphi(\alpha))} = e^+(1_{\frac{\alpha}{v(\varphi(\alpha))}})$, and hence $1_{\frac{\alpha}{v(\varphi(\alpha))}} \leq \varphi(\alpha)$. As $v(\varphi(\alpha)) \leq \varphi(\alpha)$ too, it follows that

$v(\varphi(\alpha)) \vee 1_{\frac{\alpha}{v(\varphi(\alpha))}} \leq \varphi(\alpha)$. Applying e^+ to the both members of the inequality above, we get the same value α, therefore $\varphi(\alpha) = v(\varphi(\alpha)) \vee 1_{\frac{\alpha}{v(\varphi(\alpha))}} = w(\varphi)(\alpha) \vee 1_{\frac{\alpha}{w(\varphi)(\alpha)}}$, so $\varphi \in E^\bullet(B)$ is completely determined by $w(\varphi)$, i.e. the map $w|_{E^\bullet(B)} : E^\bullet(B) \longrightarrow \partial\widehat{\Lambda}_+$ is injective.

On the other hand, we define a map $\eta : \partial\widehat{\Lambda}_+ \longrightarrow E^\bullet(B)$ as follows. Let $\theta \in \partial\widehat{\Lambda}_+, \alpha \in \Lambda_+$. As $e^+(1_{\frac{\alpha}{\theta(\alpha)}}) + \theta(\alpha) = \varepsilon$, so $(\frac{\alpha}{\theta(\alpha)^2})_+ = \frac{\alpha}{\theta(\alpha)}, (\frac{\alpha}{\theta(\alpha)^2})_- = \theta(\alpha)$, it follows that $\eta(\theta)(\alpha) := 1^{-1}_{\frac{\alpha}{\theta(\alpha)^2}} = \theta(\alpha) \vee 1_{\frac{\alpha}{\theta(\alpha)}}$, where $1_{\frac{\alpha}{\theta(\alpha)^2}} := 1_{\frac{\alpha}{\theta(\alpha)}} + \frac{\alpha}{\theta(\alpha)^2}$(cf. 4.2). One checks easily that the map $\eta(\theta) : \Lambda_+ \longrightarrow R$ belongs to $E^\bullet(B)$, so we have obtained a map $\eta : \partial\widehat{\Lambda}_+ \longrightarrow E^\bullet(B)$ such that $w \circ \eta = 1_{\partial\widehat{\Lambda}_+}$. Consequently, the map $w|_{E^\bullet(B)} : E^\bullet(B) \longrightarrow \partial\widehat{\Lambda}_+$ is bijective and η is its inverse. Moreover it is an anti-isomorphism of boolean algebras since $w(\varphi \cdot \psi) = w(\varphi) \bullet w(\psi) = w(\varphi) \vee w(\psi)$ for $\varphi, \psi \in E^\bullet(B)$, and

$$w(1 - \varphi)(\alpha) = v(1_\alpha - \varphi(\alpha)) = v(1_\alpha - (1_{\frac{\alpha}{v(\varphi(\alpha))}} \vee v(\varphi(\alpha)))) =$$

$$e^+(1_\alpha \wedge (1_{\frac{\alpha}{v(\varphi(\alpha))}} \vee v(\varphi(\alpha)))) = e^+(1_{\frac{\alpha}{v(\varphi(\alpha))}}) = \frac{\alpha}{v(\varphi(\alpha))} = (\neg w(\varphi))(\alpha)$$

for $\varphi \in E^\bullet(B), \alpha \in \Lambda_+$. $\qquad\qquad\square$

Proposition 6.3. *Let R be a superrigid directed cr-qsring. With the notation above, for any $\theta \in \partial\widehat{\Lambda}_+$, let $R^\theta := \{x \in R \mid \widetilde{\theta}(e^+(x)) = \widetilde{\theta}(v(x)) = \varepsilon\} = \{x \in R \mid d(x, \varepsilon) \in \mathrm{Ker}\,(\widetilde{\theta})_+\}$; in particular, $R^\varepsilon = R, R^\omega = \{\varepsilon\}$. Then the following assertions hold.*

(1) *R^θ is a substructure of R with $E^+(R^\theta) = \mathrm{Ker}\,(\widetilde{\theta}), E^\bullet(R^\theta) = \{1_\alpha \mid \alpha \in \mathrm{Ker}\,(\widetilde{\theta})_+\}$. Call R^θ the* superrigid directed cr-qsring induced by θ, *and denote by $B^\theta := B(R^\theta)$ and $w^\theta : B^\theta \longrightarrow \widehat{\mathrm{Ker}\,(\widetilde{\theta})}$ the commutative ring and the l-valuation associated as above to R^θ.*

(2) *$(w^\theta)^{-1}(\mathrm{Ker}\,(\widetilde{\theta})) = (B^\theta)^*$ and $(w^\theta)^{-1}(\varepsilon) = (A^\theta)^*$, where $A^\theta \subseteq B^\theta$ denotes the l-valuation ring of w^θ.*

(3) *$w^{-1}(\widehat{\Lambda}\theta) = B\eta(\theta) := \{\varphi \cdot \eta(\theta) \mid \varphi \in B\} = \{\varphi \in B \mid \varphi \cdot \eta(\theta) = \varphi\}$.*

(4) *The restriction map $\varphi \mapsto \varphi|_{\mathrm{Ker}\,(\widetilde{\theta})_+}$ yields a ring isomorphism $B\eta(\theta) \longrightarrow B^\theta$, and $w^\theta(\varphi|_{\mathrm{Ker}\,(\widetilde{\theta})_+}) = w(\varphi)|_{\mathrm{Ker}\,(\widetilde{\theta})_+}$ for all $\varphi \in B\eta(\theta)$. In particular, $w^{-1}(\Lambda\theta) = (B\eta(\theta))^* = B^*\eta(\theta) \cong (B^\theta)^*$ and $w^{-1}(\theta) = (A\eta(\theta))^* = A^*\eta(\theta) \cong (A^\theta)^*$.*

Proof. As $d(x, \varepsilon) = \frac{e^+(x)}{e^+(x \wedge \varepsilon)^2} = |v(x)| \bullet e^+(e^\bullet(x))$ for all $x \in R$, and $\mathrm{Ker}\,(\widetilde{\theta})$ is a convex l-subgroup of Λ, it follows that $d(x, \varepsilon) \in \mathrm{Ker}\,(\widetilde{\theta})_+ \iff \widetilde{\theta}(e^+(x)) = \widetilde{\theta}(v(x)) = \varepsilon$.

(1) We have to show that R^θ is closed under the operations $+, -, \bullet, ^{-1}$. As $d(-x, \varepsilon) = d(x^{-1}, \varepsilon) = d(x, \varepsilon)$ for all $x \in R$ by Lemma 4.4.(2), it follows that R^θ is closed under the unary operations $-$ and $^{-1}$. Let $x, y \in R^\theta$. Applying the endomorphism $\widetilde{\theta}$ to the relations $e^+(x + y) = e^+(x) \wedge e^+(y)$ and $v(x) \wedge v(y) \leq v(x + y) \leq e^+(x + y)$, we obtain $x + y \in R^\theta$, while $x \bullet y \in R^\theta$ follows by applying $\widetilde{\theta}$ to the identities $v(x \bullet y) = v(x) \bullet v(y)$ and $e^+(x \bullet y) = e^+(x) \bullet v(y) \wedge v(x) \bullet e^+(y)$. Since $d(\alpha, \varepsilon) = |\alpha|$ for $\alpha \in \Lambda = E^+$, and

$d(1_\alpha, \varepsilon) = \alpha$ for $\alpha \in \Lambda_+$, it follows that $E^+(R^\theta) = \text{Ker}\,(\widetilde{\theta})$ and $E^\bullet(R^\theta) = \{1_\alpha \,|\, \alpha \in \text{Ker}\,(\widetilde{\theta})_+)$, and hence the directed cr-qsring R^θ is superrigid.

(2) Apply Lemma 6.1 to the superrigid directed cr-qsring R^θ.

(3) Let $\varphi \in B$. As $\text{supp}(w) = \{0\}$, it follows by Lemma 1.10.(1), (i) $\iff$ (ii), and Lemma 6.2 that

$$\varphi \in w^{-1}(\widehat{\Lambda}\theta) \iff w(\varphi \cdot (1 - \eta(\theta))) = w(\varphi) \cdot \neg\theta = \omega \iff \varphi \cdot (1 - \eta(\theta)) = 0 \iff \varphi \in B\eta(\theta).$$

(4) Let $\varphi \in B\eta(\theta), \alpha \in \text{Ker}\,(\widetilde{\theta})_+$. Since $e^+(\varphi(\alpha)) = \alpha \in \text{Ker}\,(\widetilde{\theta})$ and $v(\varphi(\alpha)) = w(\varphi)(\alpha) \in \text{Ker}\,(\widetilde{\theta})$ by Lemma 1.10.(1), (i) $\iff$ (iii), it follows that $\varphi(\alpha) \in R^\theta$, therefore $\varphi|_{\text{Ker}\,(\widetilde{\theta})_+} \in B^\theta$. The map $B\eta(\theta) \longrightarrow B^\theta, \varphi \mapsto \varphi|_{\text{Ker}\,(\widetilde{\theta})_+}$ is obviously a ring morphism. The morphism above is injective. Indeed, assuming that $\varphi|_{\text{Ker}\,(\widetilde{\theta})_+} = 0$, i.e. $\varphi(\alpha) = \alpha$ for all $\alpha \in \text{Ker}\,(\widetilde{\theta})_+$, it follows by Lemma 1.10.(1), (i) $\iff$ (vi), that

$$w(\varphi)(\alpha) = \theta(\alpha) \cdot w(\varphi)\left(\frac{\alpha}{\theta(\alpha)}\right) = \theta(\alpha) \cdot v\left(\varphi\left(\frac{\alpha}{\theta(\alpha)}\right)\right) = \theta(\alpha) \cdot \frac{\alpha}{\theta(\alpha)} = \alpha$$

for all $\alpha \in \Lambda_+$, so $w(\varphi) = \omega$ and hence $\varphi = 0$ as required. To show that the morphism above is surjective, let $\psi \in B^\theta$, and consider for any $\alpha \in \Lambda_+$ the pair

$$\left(\psi\left(\frac{\alpha}{\theta(\alpha)}\right) \in R^\theta \subseteq R, \theta(\alpha) \bullet v(\psi(\varepsilon)) \in \Lambda = E^+\right).$$

Since

$$e^+\left(\psi\left(\frac{\alpha}{\theta(\alpha)}\right)\right) + \theta(\alpha) \bullet v(\psi(\varepsilon)) = \frac{\alpha}{\theta(\alpha)} + \theta(\alpha) \bullet v(\psi(\varepsilon)) = v(\psi(\varepsilon)) \le v\left(\psi\left(\frac{\alpha}{\theta(\alpha)}\right)\right),$$

it follows by Lemma 3.6.(7) that the element $\varphi(\alpha) := \psi\left(\frac{\alpha}{\theta(\alpha)}\right) \vee \theta(\alpha) \bullet v(\psi(\varepsilon)) \in R$ is well defined. As $e^+(\varphi(\alpha)) = \frac{\alpha}{\theta(\alpha)} \vee \theta(\alpha) \bullet v(\psi(\varepsilon)) = \alpha$ and $\alpha \le \beta \implies \varphi(\alpha) \le \varphi(\beta)$, we deduce that the map $\varphi : \Lambda_+ \longrightarrow R$ is an element of the ring B such that $\varphi|_{\text{Ker}\,(\widetilde{\theta})_+} = \psi$. Moreover it follows by (3) and Lemma 1.10.(1), (i) $\iff$ (v), that $\varphi \in B\eta(\theta)$ as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 6.4. The correspondence $R \mapsto (B, w : B \longrightarrow \widehat{\Lambda})$ extends to a covariant functor defined on the category of superrigid directed cr-qsrings with morphisms $f : R \longrightarrow R'$ satisfying the condition that the Abelian l-group $\Lambda' := E^+(R')$ is the convex hull of the image $f(\Lambda) \subseteq \Lambda'$. For such a morphism $f : R \longrightarrow R'$, define the ring morphism $B(f) : B(R) \longrightarrow B(R')$ by $B(f)(\varphi)(\alpha') :=$

$$\lim_{\alpha \to \infty}(f(\varphi(\alpha)) + \alpha') = f(\varphi(\alpha)) + \alpha' \text{ for some (for all) } \alpha \in \Lambda_+ \text{ such that } f(\alpha) \ge \alpha',$$

for $\varphi \in B(R), \alpha' \in \Lambda'_+$. On the other hand, let $\widehat{f} : \widehat{\Lambda} \longrightarrow \widehat{\Lambda'}$ be the morphism of commutative l-monoids induced by the morphism of Abelian l-groups $f|_\Lambda : \Lambda \longrightarrow \Lambda'$ (cf. Lemma 1.2.(1)). It follows that the pair $(B(f), \widehat{f}) : (B(R), w) \longrightarrow (B(R'), w')$ is a morphism of l-valued commutative rings, i.e. $w' \circ B(f) = \widehat{f} \circ w$.

Remark 6.5. There exist nontrivial superrigid directed cr-qsrings R with the associated l-valuation $w : B \longrightarrow \widehat{\Lambda}$ such that $A_w = B$. For instance, let $R = \mathcal{R}_\Lambda$ as defined in the proof of Proposition 4.8, where Λ is a nontrivial Abelian l-group. It follows that the associated l-valuation $w : B \longrightarrow \widehat{\Lambda}$ is injective with image $w(B) = \partial\widehat{\Lambda}_+$, identifying the ring $B = A_w$ with the underlying boolean ring of the boolean algebra with support $\partial\widehat{\Lambda}_+$ and opposite order. In particular, $B = \mathbb{F}_2$, the field with 2 elements, provided Λ is totally ordered.

6.2 Prüfer commutative regular quasi-semirings

Let R be a superrigid directed cr-qsring, and $B := B(R)$, $w : B \longrightarrow \widehat{\Lambda}$, $A := A_w$, its associated commutative ring, l-valuation and l-valuation subring respectively. Let $M := M(A, B)$ be the commutative sl-monoid of all f.g. A-submodules of B, associated to the ring extension $A \subseteq B$ (cf. 1.2). Let $\widetilde{M} := \{I \in M \mid IB = B\} = \{I \in M \mid \exists J \in M, A \subseteq IJ\}$ be the monoid of B-regular f.g. A-submodules of B, and M^* be the subgroup of invertible A-submodules of B. As shown in 1.3, the l-valuation w induces a canonical morphism of sl-monoids $\widehat{w} : M \longrightarrow \widehat{\Lambda}$, defined by $\widehat{w}(I) := \bigwedge_{1 \leq i \leq n} w(\varphi_i)$, where $\{\varphi_i \mid i = 1, \cdots, n\}$ is an arbitrary system of generators of the f.g. A-submodule I of B. Recall that $M^* \subseteq \widetilde{M} \subseteq \widehat{w}^{-1}(\Lambda)$, the morphism $\widehat{w}$ induces by restriction a monomorphism of ordered groups $M^* \longrightarrow \Lambda$, and $I_{\widehat{w}(\mathbf{a})} := \{\varphi \in B \mid w(\varphi) \geq \widehat{w}(\mathbf{a})\} = \mathbf{a}$ for all $\mathbf{a} \in M^*$.

The next lemma permits us to characterize those superrigid directed cr-qsrings R for which the associated ring extension $A \subseteq B$ is Prüfer, i.e. $M^* = \widetilde{M}$, so w is a Prüfer l-valuation. Call them *Prüfer cr-qsrings*.

Lemma 6.6. *Let R be a superrigid directed cr-qsring, and let $\varphi, \psi \in B := B(R)$. Then the following assertions are equivalent.*

(1) *The elements $\psi, \varphi\psi$ and $\varphi(1 - \varphi\psi)$ belong to the l-valuation ring $A := A_w$.*

(2) $\psi(\gamma^2) = \varphi(\gamma)^{-1} + \gamma^2$, *i.e.* $\psi(\gamma^2) \leq \varphi(\gamma)^{-1}$, *where* $\gamma := w(\varphi)_- = v(\varphi(\varepsilon))^{-1}$.

Proof. (1) $\Longrightarrow$ (2) : By assumption, $\psi(\varepsilon) = (\varphi \cdot \psi)(\varepsilon) = \varepsilon$, and $\varphi(\varepsilon) = (\varphi^2 \cdot \psi)(\varepsilon)$. Consequently, we obtain $(\varphi \cdot \psi)(\varepsilon) = \varphi(\varepsilon) \bullet \psi(\gamma) + \varepsilon = \varepsilon$, therefore $\varphi(\varepsilon) \bullet \psi(\gamma) = \varepsilon$ since $e^+(\varphi(\varepsilon) \bullet \psi(\gamma)) \leq v(\varphi(\varepsilon)) \bullet \gamma = \varepsilon$. It follows that $v(\psi(\gamma)) = v(\varphi(\varepsilon))^{-1} = \gamma = e^+(\psi(\gamma))$, so $\psi(\gamma) = \gamma \in E^+$. Further we obtain $(\varphi \cdot \psi)(\gamma) = \varphi(\gamma) \bullet \psi(\gamma^2) + \gamma$, and hence $(\varphi \cdot \psi)(\gamma) = \varphi(\gamma) \bullet \psi(\gamma^2)$ since $v(\varphi(\gamma)) = v(\varphi(\varepsilon)) = \gamma^{-1}$ implies $e^+(\varphi(\gamma) \bullet \psi(\gamma^2)) \leq v(\varphi(\gamma)) \bullet \gamma^2 = \gamma$. Consequently,

$$\varphi(\varepsilon) = (\varphi^2 \cdot \psi)(\varepsilon) = \varphi(\varepsilon) \bullet (\varphi \cdot \psi)(\gamma) + \varepsilon = \varphi(\varepsilon) \bullet \varphi(\gamma) \bullet \psi(\gamma^2)$$

since $e^+(\varphi(\varepsilon) \bullet (\varphi \cdot \psi)(\gamma)) \leq v(\varphi(\varepsilon)) \bullet \gamma = \varepsilon$. As

$$e^+(e^\bullet(\psi(\gamma^2)) = \frac{\gamma^2}{v(\psi(\gamma^2))} \leq \frac{\gamma^2}{v(\psi(\gamma))} = \gamma = e^+(e^\bullet(\varphi(\varepsilon))) \leq e^+(e^\bullet(\varphi(\gamma))),$$

we obtain the desired inequality by multiplying with $\varphi(\varepsilon)^{-1} \bullet \varphi(\gamma)^{-1}$ the both terms of the identity above

$$\varphi(\gamma)^{-1} \geq e^\bullet(\varphi(\varepsilon)) \bullet \varphi(\gamma)^{-1} = e^\bullet(\varphi(\varepsilon)) \bullet e^\bullet(\varphi(\gamma)) \bullet \psi(\gamma^2) = \psi(\gamma^2).$$

(2) $\Longrightarrow$ (1) : We obtain $w(\psi) \geq v(\psi(\gamma^2)) = v(\varphi(\gamma)^{-1}) + \gamma^2 = \gamma + \gamma^2 = \gamma \geq \varepsilon$, $w(\varphi\psi) = w(\varphi) \cdot w(\psi) \geq v(\varphi(\varepsilon)) \bullet \gamma = \varepsilon$, therefore $\psi, \varphi\psi \in A$. Further we obtain

$$(\varphi\psi)(\gamma) = \varphi(\gamma) \bullet \psi(\gamma^2) + \gamma = \varphi(\gamma) \bullet \psi(\gamma^2) = \varphi(\gamma) \bullet (\varphi(\gamma)^{-1} + \gamma^2) \geq$$

$$e^\bullet(\varphi(\gamma)) + \varphi(\gamma)) \bullet \gamma^2 = 1_{\gamma^2} + \gamma = 1_\gamma,$$

and hence $w(1 - \varphi\psi)) \geq v(1_\gamma - (\varphi\psi)(\gamma)) \geq v(1_\gamma - 1_\gamma) = \gamma$. Thus $w(\varphi(1 - \varphi\psi)) = w(\varphi)w(1 - \varphi\psi) \geq \gamma^{-1} \bullet \gamma = \varepsilon$, therefore $\varphi(1 - \varphi\psi) \in A$ as desired. $\qquad\square$

Corollary 6.7. *Let R be a superrigid directed cr-qsring, with the associated commutative ring extension $A \subseteq B$. The necessary and sufficient condition for R to be a Prüfer cr-qsring is that for all $\varphi \in B$ there exists $\psi \in B$ such that $\psi(\gamma^2) \leq \varphi(\gamma)^{-1}$, where $\gamma := w(\varphi)_- = v(\varphi(\varepsilon))^{-1}$.*

Proof. The statement is a consequence of Lemma 6.6 and 1.4. (P 2). $\qquad\square$

Notice that, by constrast with Prüfer ring extensions, the property of a superrigid directed cr-qsring to be Prüfer is not an elementary property.

A particular class of Prüfer cr-qsrings is obtained by considering the following completeness property for directed cr-qsrings.

Definition 6.8. *A directed cr-qsring R, with $\Lambda = E^+(R)$, is said to be* spherically complete *(for short,* complete*) if for every element $x \in R$, the following equivalent conditions are satisfied.*

(1) *There exists a map $\varphi : \Lambda \longrightarrow R$ such that $e^+(\varphi(\alpha)) = \alpha$ for all $\alpha \in \Lambda$, $\varphi(\alpha) \leq \varphi(\beta)$ provided $\alpha \leq \beta$, and $\varphi(e^+(x)) = x$.*

(2) *There exists a coherent family of balls with center x and arbitrary radia, i.e. a map $\psi : \Lambda \longrightarrow R$ such that $d(\psi(\alpha), x) = |\alpha|$ for all $\alpha \in \Lambda$, in particular, $\psi(\varepsilon) = x$, and $\psi(\alpha) \leq \psi(\beta)$ provided $\alpha \leq \beta$.*

[For φ satisfying (1), define $\psi(\alpha) = \varphi(\alpha \bullet e^+(x))$ for $\alpha \in \Lambda$. Conversely, for ψ satisfying (2), define $\varphi(\alpha) = \psi(\frac{\alpha}{e^+(x)}) = \psi((\frac{\alpha}{e^+(x)})_+) + \alpha$ for $\alpha \in \Lambda$.]

Corollary 6.9. *A superrigid directed cr-qsring R is Prüfer whenever it is complete.*

Proof. Let $A \subseteq B$ be the ring extension associated to R. Assuming that R is complete, it follows that for all $\alpha \in \Lambda := E^+(R)$, the canonical map $\pi_\alpha : B \longrightarrow R_\alpha, \varphi \mapsto \widetilde{\varphi}(\alpha) := \varphi(\alpha_+) + \alpha$ is onto. In particular, for every $\varphi \in B$, with $\gamma := w(\varphi)_- = v(\varphi(\varepsilon))^{-1}$, there is $\psi \in B$ such that $\psi(\gamma^3) = \pi_{\gamma^3}(\psi) = \varphi(\gamma)^{-1} \in R_{\gamma^3}$, and hence $\psi(\gamma^2) \leq \psi(\gamma^3) = \varphi(\gamma)^{-1}$. Consequently, R is a Prüfer cr-qsring by Corollary 6.7. $\qquad\square$

6.3 Prüfer-Manis commutative regular quasi-semirings

Let R be a Prüfer cr-qsring. According to 1.4, the canonical embedding $M^* \longrightarrow \widehat{M^*}$ of the Abelian l-group M^* into its commutative l-monoid extension $\widehat{M^*}$, as defined in 1.1, extends to a morphism of sl-monoids $\widehat{\mathfrak{w}} : M \longrightarrow \widehat{M^*}$, defined by $\widehat{\mathfrak{w}}(I)(\mathbf{a}) := I + \mathbf{a}$ for $I \in M, \mathbf{a} \in M^*_+$, while the map $\mathfrak{w} : B \longrightarrow \widehat{M^*}, \varphi \mapsto \widehat{\mathfrak{w}}(A\varphi)$, induced by $\widehat{\mathfrak{w}}$, is the Prüfer-Manis l-valuation associated to the Prüfer ring extension $A \subseteq B$. Notice that $A_{\mathfrak{w}} = A$, $\mathrm{supp}(\mathfrak{w}) = \cap_{\mathbf{a} \in M^*_+} \mathbf{a}$ is the conductor of A in B, and the reduced ring extension $A/\mathrm{supp}(\mathfrak{w}) \subseteq B/\mathrm{supp}(\mathfrak{w})$ is Prüfer. The relation between the l-valuations $w : B \longrightarrow \widehat{\Lambda}$ and $\mathfrak{w} : B \longrightarrow \widehat{M^*}$, as well as that between the morphisms $\widehat{w} : M \longrightarrow \widehat{\Lambda}$ and $\widehat{\mathfrak{w}} : M \longrightarrow \widehat{M^*}$, are described by Lemma 1.20. In particular, $M^* \cong \widehat{w}(M^*) = \widehat{w}(M)^* = \widehat{w}(M) \cap \Gamma$, where Γ is the convex hull of $\widehat{w}(M)^*$ in Λ. Using Lemma 1.20.(3), we introduce the following class of Prüfer cr-qsrings.

Definition 6.10. *A Prüfer cr-qsring R is called* Prüfer-Manis *if $\Lambda = E^+(R)$ is the convex hull of $\widehat{w}(M)^* \cong M^*$, in particular, $\widehat{w}(M) \cap \Lambda = \widehat{w}(M)^*$ and w is equivalent with the Prüfer-Manis l-valuation $\mathfrak{w}$ associated to the Prüfer extension $A \subseteq B$. It is called* Prüfer-Manis in a strong sense *if $M^* \cong \widehat{w}(M)^* = \Lambda$.*

Lemma 6.11. *Let R be a superrigid directed cr-qsring. If R is complete and $M^* \cong \widehat{w}(M)^* = \Lambda$ then R is Prüfer-Manis in a strong sense and lff, in particular, median.*

Proof. By Lemma 6.9, R is a Prüfer cr-qsring, and hence Prüfer-Manis in a strong sense cf. Definition 6.10.

To show that R is lff, let $x, y \in R$ be such that $x - y \in \Lambda = E^+(R)$. According to Lemma 5.4 and Definition 5.5, we have to show that $x \vee y \neq \infty$, i.e. $x \leq z, y \leq z$ for some $z \in R$. Put $e^+(x) = \alpha, e^+(y) = \beta$, so $x - y = v(x - y) = e^+(x - y) = \alpha + \beta = \alpha \wedge \beta$ by assumption. Let $\gamma := \alpha_+ \vee \beta_+$. By completeness, it follows that there exist $\varphi, \psi \in B := B(R)$ such that $\widetilde{\varphi}(\alpha) := \varphi(\gamma) + \alpha = x, \widetilde{\psi}(\beta) := \psi(\gamma) + \beta = y$, i.e. $x \leq \varphi(\gamma), y \leq \psi(\gamma)$. Consequently, $(\varphi - \psi)(\gamma) = \varphi(\gamma) - \psi(\gamma) \geq x - y = \alpha \wedge \beta$, therefore $w(\varphi - \psi) \geq w(\varphi - \psi)(\gamma) \geq \alpha \wedge \beta$, i.e. $\varphi - \psi \in I_{\alpha \wedge \beta}$. On the other hand, since $\widehat{w}(M)^* = \Lambda$ by assumption, it follows that $I_\alpha, I_\beta \in M^*$, and $I_{\alpha \wedge \beta} = I_\alpha + I_\beta \in M^*$. Thus $\varphi - \psi = \rho + \mu$ for some $\rho \in I_\alpha, \mu \in I_\beta$. Setting $\theta := \varphi - \rho = \psi + \mu$, we deduce that $\theta(\gamma) = \varphi(\gamma) - \rho(\gamma) \geq x, \theta(\gamma) = \psi(\gamma) + \mu(\gamma) \geq y$, and hence $x \vee y \neq \infty$ as desired.

As R is lff, it is median by Remarks 5.6.(1). $\qquad\square$

Denote by $\mathfrak{R}$ the category of Prüfer-Manis cr-qsrings, with morphisms $f : R \longrightarrow R'$ satisfying the condition from Remark 6.4 : $\Lambda' := E^+(R')$ is the convex hull of $f(\Lambda)$. The correspondence above $R \mapsto (A \subseteq B)$ extends as in Remark 6.4 to a covariant functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$, where $\mathfrak{P}$ denotes the category of Prüfer extensions with morphisms $g : (A \subseteq B) \longrightarrow (A' \subseteq B')$ satisfying the supplementary natural condition : the Abelian l-group $M^*_{B'/A'}$ of invertible A'-submodules of B' is the convex hull of the image $\{A'g(\mathbf{a}) \,|\, \mathbf{a} \in M^*_{B/A}\}$ of $M^*_{B/A}$ through the morphism g.

Moreover we obtain

Lemma 6.12. *The functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$ takes values in the full subcategory $\mathfrak{CP}$ of $\mathfrak{P}$ consisting of the* complete *Prüfer extensions, i.e. those Prüfer extensions $A \subseteq B$*

for which the canonical morphisms $A \longrightarrow \varprojlim_{\mathbf{a} \in (M^*_{B/A})_+} A/\mathbf{a}$ *and* $B \longrightarrow \varprojlim_{\mathbf{a} \in (M^*_{B/A})_+} B/\mathbf{a}$ *are*

isomorphisms.

Proof. Let R be a Prüfer-Manis cr-qsring, with $\mathcal{B}(R) = (A \subseteq B)$, its associated Prüfer ring extension, and $M := M_{B/A}$ the *sl*-monoid of f.g. A-submodules of B. Since $\Lambda := E^+(R)$ is the convex hull of $\widehat{w}(M^*) = \widehat{w}(M) \cap \Lambda \cong M^*$, it follows that $A \cong \varprojlim_{\mathbf{a} \in M^*_+} A/I_{\widehat{w}(\mathbf{a})}$, $B \cong \varprojlim_{\mathbf{a} \in M^*_+} B/I_{\widehat{w}(\mathbf{a})}$, where $I_\alpha := \{\varphi \in A \,|\, w(\varphi) \geq \alpha\}$ for $\alpha \in \Lambda_+$. It remains to recall that $I_{\widehat{w}(\mathbf{a})} = \mathbf{a}$ for all $\mathbf{a} \in M^*_+$. □

7 Directed commutative regular quasi-semirings associated to Prüfer extensions

Extending [6], we construct in this section a covariant functor from the category of Prüfer extensions to the category of superrigid directed cr-qsrings, whose image turns out to be equivalent with the category of complete Prüfer extensions.

Let $A \subseteq B$ be a Prüfer ring extension, and $\Lambda := M^*_{B/A}$ be the multiplicative Abelian *l*-group of the invertible A-submodules of B, with A as neutral element, and $\Lambda_+ := \{\alpha \in \Lambda \,|\, \alpha \subseteq A\}$, the *l*-monoid of nonnegative elements of Λ. As shown in 1.4, the map $w : B \longrightarrow \widehat{\Lambda}$, defined by $w(x)(\alpha) = \alpha + Ax$ for $x \in B, \alpha \in \Lambda_+$, is the Prüfer-Manis *l*-valuation associated to the Prüfer ring extension $A \subseteq B$.

Let $R = R_{B/A} := \sqcup_{\alpha \in \Lambda} B/\alpha$ be the disjoint union of the factor A-modules B/α for α ranging over Λ. Notice that R is a singleton if and only if $A = B$.

The injective map $\Lambda \longrightarrow R, \alpha \mapsto 0 \bmod \alpha$ identifies Λ with a subset of R, while the ring structure of B induces on the residue set R an algebra $(R, +, \bullet, -, ^{-1}, \varepsilon)$ of signature $(2, 2, 1, 1, 0)$ as follows.

For $\mathbf{x} := x \bmod \alpha$, $\mathbf{y} := y \bmod \beta \in R$, we define

$$\mathbf{x} + \mathbf{y} := (x + y) \bmod (\alpha + \beta),$$

$$-\mathbf{x} := (-x) \bmod \alpha,$$

and

$$\mathbf{x} \bullet \mathbf{y} := xy \bmod \gamma,$$

with

$$\gamma = \alpha\beta + x\beta + y\alpha = \alpha v(\mathbf{y}) + \beta v(\mathbf{x}),$$

where $v(\mathbf{x}) := w(x)(\alpha) = \alpha + Ax \in \Lambda$.

We see at once that the operations above are well defined, making $(R, +, -)$ a regular commutative semigroup with $E^+ = (\Lambda, +)$ as its semilattice of idempotents, and $(R, \bullet)$ a commutative semigroup with $E^\bullet = \{1 \bmod \alpha \,|\, \alpha \in \Lambda_+\}$ as its semilattice of idempotents, canonically isomorphic through the map $1 \bmod \alpha \mapsto \alpha$ with the semilattice $(\Lambda_+, +)$. Put $\varepsilon := 0 \bmod A = 1 \bmod A$, and notice that $E^+ \cap E^\bullet = \{\varepsilon\}$. Moreover the commutative semigroup $(R, \bullet)$ is regular with the quasi-inverse $\mathbf{x}^{-1}$ of any element $\mathbf{x} = x \bmod \alpha$ defined as follows. Since $v(\mathbf{x}) = \alpha + Ax \in \Lambda$, we obtain $A = v(\mathbf{x})^{-1}(\alpha + Ax) =$

$v(\mathbf{x})^{-1}\alpha + v(\mathbf{x})^{-1}Ax$, therefore there exists $z \in v(\mathbf{x})^{-1}$ such that $xz \equiv 1 \bmod v(\mathbf{x})^{-1}\alpha$. One checks easily that the element

$$\mathbf{x}^{-1} := z \bmod \alpha v(\mathbf{x})^{-2}$$

is well defined and is the required quasi-inverse of $\mathbf{x}$.

With the notation from Section 2, we get $e^{+}(\mathbf{x}) = \alpha, e^{\bullet}(\mathbf{x}) = 1 \bmod \alpha v(\mathbf{x})^{-1}$ for $\mathbf{x} = x \bmod \alpha \in R$. The canonical partial order $\leq$ on R is given by

$$x \bmod \alpha \leq y \bmod \beta \Longleftrightarrow \beta \subseteq \alpha \text{ and } x \equiv y \bmod \alpha.$$

The axioms from Definition 2.3 are easily verified, so R, with the operations as defined above, is a cr-qsring. Moreover, as $\varepsilon \leq \mathbf{x}$ for all $\mathbf{x} \in E^{\bullet}$, the equivalent conditions of Proposition 3.1. are satisfied, and hence the cr-qsring R is directed, with the meet-semilattice operation with respect to the partial order $\leq$ given by

$$x \bmod \alpha \wedge y \bmod \beta = x \bmod (\alpha + \beta + A(x - y)).$$

In addition, R is superrigid and, according to Section 4, a Λ-metric space with the Λ-valued distance map $d : R \times R \longrightarrow \Lambda_{+}$ defined by

$$d(x \bmod \alpha, y \bmod \beta) = \alpha\beta(\alpha + \beta + A(x - y))^{-2}.$$

Proceeding as in 6.1, we associate to the superrigid directed cr-qsring R above the commutative ring $\widehat{B} := B(R)$ with carrier $\varprojlim_{\alpha \in \Lambda_{+}} B/\alpha$ and the l-valuation $\mathbf{w} : \widehat{B} \longrightarrow \widehat{\Lambda}$, whose l-valuation subring is $\widehat{A} := \varprojlim_{\alpha \in \Lambda_{+}} A/\alpha$. Composing $\mathbf{w}$ with the canonical ring morphism $B \longrightarrow \widehat{B}$, we obtain the Prüfer-Manis l-valuation $w : B \longrightarrow \widehat{\Lambda}$ associated to the Prüfer ring extension $A \subseteq B$, in particular, $\mathcal{M}_{\mathbf{w}}^{*} = \mathcal{M}_{w}^{*} = \Lambda$ and $\mathbf{w}$ is a Manis l-valuation.

Lemma 7.1. *Let $A \subseteq B$ be a Prüfer commutative ring extension, $\Lambda := M_{B/A}^{*}$ the Abelian l-group of invertible A-submodules of B, and $R := R_{B/A}$ the associated super-rigid directed cr-qsring. Then R is complete and Prüfer-Manis in a strong sense.*

In particular, R is lff, and hence median, with the median operation $m : R^{3} \longrightarrow R$ defined by

$$m(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \vee_{1 \leq i < j \leq 3}(\mathbf{x}_i \wedge \mathbf{x}_j) = x \bmod \alpha \text{ for } \mathbf{x}_i = x_i \bmod \alpha_i, i = 1, 2, 3,$$

where

$$\alpha = \cap_{1 \leq i < j \leq 3}(\alpha_i + \alpha_j + A(x_i - x_j)),$$

and $x \bmod \alpha$ is uniquely determined by the conditions

$$x \equiv x_i \equiv x_j \bmod (\alpha_i + \alpha_j + A(x_i - x_j)), 1 \leq i < j \leq 3.$$

Proof. To check that the directed cr-qsring R is complete, let $\mathbf{x} = x \bmod \gamma \in R$. For any $y \in B$ satisfying $y \equiv x \bmod \gamma$, the map $\varphi_y : \Lambda \longrightarrow R$, defined by $\varphi_y(\alpha) := y \bmod \alpha$, satisfies the condition (1) from Definition 6.8. as desired. In particular, by Corollary 6.9, R is a Prüfer cr-qsring, i.e. the commutative ring extension $\widehat{A} \subseteq \widehat{B}$ associated to R is Prüfer. Moreover R is Prüfer-Manis in a strong sense (cf. Definition 6.10.) since $\Lambda = \widehat{w}(M_{B/A})^* \subseteq \widehat{\mathbf{w}}(M_{\widehat{B}/\widehat{A}})^* \subseteq \Lambda$, and hence $M_{\widehat{B}/\widehat{A}}^* \cong \widehat{\mathbf{w}}(M_{\widehat{B}/\widehat{A}})^* = \Lambda$ as required. By Lemma 6.11, R is lff, and hence median. $\qquad\square$

With the notation from 6.3, the correspondence above $(A \subseteq B) \mapsto R_{B/A}$ extends to a covariant functor $\mathcal{R} : \mathfrak{P} \longrightarrow \mathfrak{R}$. By Lemma 7.1, the functor $\mathcal{R}$ takes values in the full subcategory $\mathfrak{CR}$ of $\mathfrak{R}$ consisting of those superrigid directed cr-qsrings which are complete and Prüfer-Manis in a strong sense.

The relationship of adjunction between $\mathcal{R}$ and the functor $\mathcal{B}$ as defined in 6.3. is described by the next statement.

Theorem 7.2. (1) *The covariant functor $\mathcal{R} : \mathfrak{P} \longrightarrow \mathfrak{R}$ is the left adjoint of the covariant functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$.*

(2) *The endofunctor $\mathcal{B} \circ \mathcal{R} : \mathfrak{P} \longrightarrow \mathfrak{P}$, factorizing through $\mathfrak{CP}$, sends a Prüfer extension $A \subseteq B$ to its completion $\widehat{A} := \varprojlim_{\alpha \in (M_{B/A}^*)_+} A/\alpha \subseteq \widehat{B} := \varprojlim_{\alpha \in (M_{B/A}^*)_+} B/\alpha$.*

(3) *By restriction, $\mathcal{R}$ and $\mathcal{B}$ yield inverse equivalences of the categories $\mathfrak{CP}$ and $\mathfrak{CR}$.*

Proof. We define two natural transformations $\mathfrak{r} : \mathcal{R} \circ \mathcal{B} \longrightarrow 1_{\mathfrak{R}}, \mathfrak{p} : 1_{\mathfrak{P}} \longrightarrow \mathcal{B} \circ \mathcal{R}$ as follows. For each R in $\mathfrak{R}$, $\mathcal{R}(\mathcal{B}(R))$ is an object of $\mathfrak{CR}$, canonically identified with a subobject of R in $\mathfrak{R}$. Let $\mathfrak{r}_R : \mathcal{R}(\mathcal{B}(R)) \longrightarrow R$ denote this embedding, and notice that $\mathfrak{r}_R$ is an isomorphism if and only if R is in $\mathfrak{CR}$. On the other hand, for each $(A \subseteq B)$ in $\mathfrak{P}$, $\mathcal{B}(\mathcal{R}(A \subseteq B)) = (\widehat{A} \subseteq \widehat{B})$ is an object of $\mathfrak{CP}$, the completion of the Prüfer extension $A \subseteq B$. Let $\mathfrak{p}_{A \subseteq B} : (A \subseteq B) \longrightarrow (\widehat{A} \subseteq \widehat{B})$ be the canonical completion morphism in $\mathfrak{P}$, and notice that $\mathfrak{p}_{A \subseteq B}$ is an isomorphism if and only if $(A \subseteq B)$ is in $\mathfrak{CP}$. It follows easily that the families of morphisms $\mathfrak{r}_R$ and $\mathfrak{p}_{A \subseteq B}$, for $R, (A \subseteq B)$ ranging over the objects of the categories $\mathfrak{R}$ and $\mathfrak{P}$ respectively, are natural transformations. Moreover it follows that $\mathcal{R}\mathfrak{p} = (\mathfrak{r}\mathcal{R})^{-1} : \mathcal{R} \longrightarrow \mathcal{R} \circ \mathcal{B} \circ \mathcal{R}$ and $\mathfrak{p}\mathcal{B} = (\mathcal{B}\mathfrak{r})^{-1} : \mathcal{B} \longrightarrow \mathcal{B} \circ \mathcal{R} \circ \mathcal{B}$ are natural isomorphisms, so, in particular, the *counit-unit equations* $1_{\mathcal{R}} = \mathfrak{r}\mathcal{R} \circ \mathcal{R}\mathfrak{p}, 1_{\mathcal{B}} = \mathcal{B}\mathfrak{r} \circ \mathfrak{p}\mathcal{B}$ are satisfied. We conclude that $(\mathcal{R}, \mathcal{B})$ form an *adjoint pair* with *counit* $\mathfrak{r}$ and *unit* $\mathfrak{p}$, inducing by restriction inverse equivalences of the full subcategories $\mathfrak{CR}$ and $\mathfrak{CP}$.

The associated *monad* in the category $\mathfrak{P}$ is the triple $(T, \mathfrak{p}, \mu)$, where the endofunctor $T : \mathfrak{P} \longrightarrow \mathfrak{P}$ is given by $T = \mathcal{B} \circ \mathcal{R}$, the *unit of the monad* is just the unit $\mathfrak{p} : 1_{\mathfrak{P}} \longrightarrow T$ of the adjunction, and the *multiplication* $\mu : T^2 = T \circ T \longrightarrow T$ is the natural isomorphism $\mathcal{B}\mathfrak{r}\mathcal{R}$. Dually, the endofunctor $\mathcal{R} \circ \mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{R}$ together with the natural transformation $\mathfrak{r} : \mathcal{R} \circ \mathcal{B} \longrightarrow 1_{\mathfrak{R}}$, as *counit*, and the natural isomorphism $\mathcal{R}\mathfrak{p}\mathcal{B} : \mathcal{R} \circ \mathcal{B} \longrightarrow (\mathcal{R} \circ \mathcal{B})^2$, as *comultiplication*, defines a *comonad* in the category $\mathfrak{R}$. $\qquad\square$

Remarks 7.3. (1) Let us denote by $\mathfrak{P}_0$ the full subcategory of $\mathfrak{P}$ consisting of those Prüfer extensions $A \subseteq B$ for which $M_{B/A}^*$ is nontrivial and totally ordered, i.e. $B \neq A$ and $B \setminus A$ is multiplicatively closed (cf. Corollary 1.22). The objects

of its full subcategory $\mathfrak{C}\mathfrak{P}_0 := \mathfrak{P}_0 \cap \mathfrak{C}\mathfrak{P}$ are the complete valued fields. On the other hand, we denote by $\mathfrak{R}_0$ the full subcategory of $\mathfrak{R}$ consisting of those R for which $\Lambda = E^+(R)$ is nontrivial and totally ordered. Let $\mathfrak{C}\mathfrak{R}_0 := \mathfrak{R}_0 \cap \mathfrak{C}\mathfrak{R}$. By restriction, we obtain an adjoint pair $(\mathcal{R}_0 : \mathfrak{P}_0 \longrightarrow \mathfrak{R}_0, \mathcal{B}_0 : \mathfrak{R}_0 \longrightarrow \mathfrak{P}_0)$, inducing inverse equivalences of $\mathfrak{C}\mathfrak{P}_0$ and $\mathfrak{C}\mathfrak{R}_0$.

(2) Given a family $(A_i \subseteq B_i)_{i \in I}$ of Prüfer extensions, let $A := \prod_{i \in I} A_i, B := \prod_{i \in I} B_i$, with the canonical projections $p_i : B \longrightarrow B_i$, satisfying $p_i(A) = A_i, i \in I$. By [19, I, Proposition 5.17.], the ring extension $A \subseteq B$ is Prüfer, so the projections

$p_i : (A \subseteq B) \longrightarrow (A_i \subseteq B_i), i \in I$, are morphisms in $\mathfrak{P}$. The commutative sl-monoid $M_{B/A}$ of f.g. A-submodules of B is identified through the canonical morphism $M_{B/A} \longrightarrow \prod_{i \in I} M_{B_i/A_i}, \alpha \mapsto (p_i(\alpha))_{i \in I}$ to the subdirect product consisting of those families $\alpha := (\alpha_i)_{i \in I} \in \prod_{i \in I} M_{B_i/A_i}$ for which there is a bound $n_\alpha \in \mathbb{N}$ such that the A_i-submodule α_i of B_i admits a system of at most n_α generators for all $i \in I$. By restriction, the Abelian l-group $\Lambda := M_{B/A}^*$ is identified with a subdirect product of the family of Abelian l-groups $\Lambda_i := M_{B_i/A_i}^*, i \in I$, consisting of those $\alpha := (\alpha_i)_{i \in I} \in \prod_{i \in I} \Lambda_i$ satisfying the boundedness condition above. In particular, $\Lambda = \prod_{i \in I} \Lambda_i$ whenever either the index set I is finite or there is a bound $n \in \mathbb{N}$ such that each invertible A_i-submodule of B_i admits a system of at most n generators for all $i \in I$.

By Lemma 1.3 and Remark 1.5, the commutative l-monoid $\widehat{\Lambda}$ is identified with the subdirect product of the family $(\widehat{\Lambda_i})_{i \in I}$ consisting of those $\varphi \in \prod_{i \in I} \widehat{\Lambda_i} \cong \widehat{\prod_{i \in I} \Lambda_i}$ for which $\varphi \wedge \alpha \in \Lambda$ for all $\alpha \in \Lambda$.

Let $R_i := \mathcal{B}(A_i \subseteq B_i), i \in I, R := \mathcal{B}(A \subseteq B)$. R is identified with the subdirect product of the family $(R_i)_{i \in I}$ consisting of those $\mathbf{x} \in \prod_{i \in I} R_i$ for which $e^+(\mathbf{x}) \in \Lambda$. Applying the functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$ to the objects $R_i, i \in I$, and R of the category $\mathfrak{C}\mathfrak{R}$, we obtain the completions $\mathcal{B}(R_i) := (\widehat{A_i} \subseteq \widehat{B_i}), i \in I, \mathcal{B}(R) := (\widehat{A} \subseteq \widehat{B})$ of the Prüfer extensions $A_i \subseteq B_i, i \in I$, and $A \subseteq B$ respectively. The complete Prüfer extension $\widehat{A} \subseteq \widehat{B}$ is identified with the subdirect product of the family $\widehat{A_i} \subseteq \widehat{B_i}, i \in I$, consisting of those elements $(x_i)_{i \in I} \in \prod_{i \in I} \widehat{B_i}$ for which $(w_i(x_i))_{i \in I} \in \widehat{\Lambda}$, where $w_i : \widehat{B_i} \longrightarrow \widehat{\Lambda_i}, i \in I$, are the Prüfer-Manis l-valuations associated to the complete Prüfer extensions $\widehat{A_i} \subseteq \widehat{B_i}, i \in I$. In particular, $R \cong \prod_{i \in I} R_i$, and $(\widehat{A} \subseteq \widehat{B}) \cong \prod_{i \in I}(\widehat{A_i} \subseteq \widehat{B_i})$ whenever either I is finite or there is $n \in \mathbb{N}$ such that for all $i \in I$, each $\alpha \in \Lambda_i$ admits a system of at most n generators.

Notice that, in general, the Prüfer extension $A \subseteq B$ together with the projections $p_i : (A \subseteq B) \longrightarrow (A_i \subseteq B_i), i \in I$, is not a direct product in the category $\mathfrak{P}$ since, given an arbitrary family $f_i : (C \subseteq D) \longrightarrow (A_i \subseteq B_i), i \in I$, of morphisms in $\mathfrak{P}$, the canonical morphism of Prüfer extensions $f : (C \subseteq D) \longrightarrow (A \subseteq B)$ is not necessarily a morphism in $\mathfrak{P}$. However f is in $\mathfrak{P}$ whenever the index set I is finite, and hence the category $\mathfrak{P}$ has finite products. Similarly, the category $\mathfrak{R}$ has finite products, and for each finite family $(R_i)_{i \in I}$ of objects of $\mathfrak{R}$, it follows by the continuity of the right adjoint functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$ that $\mathcal{B}(\prod_{i \in I} R_i) \cong \prod_{i \in I} \mathcal{B}(R_i)$. In particular, the equivalent full subcategories $\mathfrak{C}\mathfrak{P} \subseteq \mathfrak{P}$ and $\mathfrak{C}\mathfrak{R} \subseteq \mathfrak{R}$ are both closed under finite products.

(3) Assume that A is a Dedekind (in particular, Prüfer) domain with its quotient field $B \neq A$. Let us denote by $\mathcal{P} := \mathcal{P}(A)$ the nonempty set of the non-null prime (maximal) ideals of A. For every $\mathfrak{p} \in \mathcal{P}$, we denote by $w_{\mathfrak{p}} : B \longrightarrow \mathbb{Z} \cup \{\infty\}$ the corresponding discrete valuation with valuation ring $A_{\mathfrak{p}}$ and maximal ideal $\mathfrak{p} A_{\mathfrak{p}}$. With the notation above, we have $M := M_{B/A} = \Lambda \cup \{\{0\}\}$, where $\Lambda := M^* \cong \mathbb{Z}^{(\mathcal{P})}$ is the free Abelian multiplicative group generated by $\mathcal{P}$, with the canonical lattice operations. The corresponding commutative l-monoid extension $\widehat{\Lambda}$ of Λ is identified, as in Example 1.1, with the collection of those formal products $\prod_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}^{n_{\mathfrak{p}}}$ with $n_{\mathfrak{p}} \in \mathbb{Z} \cup \{\infty\}$ for which the set $\{\mathfrak{p} \mid n_{\mathfrak{p}} < 0\}$ is finite. The Prüfer-Manis l-valuation $w : B \longrightarrow \widehat{\Lambda}$ associated to the Prüfer extension $A \subseteq B$ sends any $0 \neq x \in B$ to $Ax = \prod_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}^{w_{\mathfrak{p}}(x)} \in \Lambda$, while $w(0) = \omega$, so $M \cong \mathcal{M}_w$. Let $R := \mathcal{R}(A \subseteq B), R_{\mathfrak{p}} := \mathcal{R}(A_{\mathfrak{p}} \subseteq B)$ for $\mathfrak{p} \in \mathcal{P}$, so $\Lambda_{\mathfrak{p}} := E^+(R_{\mathfrak{p}}) \cong (\mathbb{Z}, +), \Lambda = E^+(R) = \oplus_{\mathfrak{p} \in \mathcal{P}} \Lambda_{\mathfrak{p}}$. By the *approximation theorem in Dedekind domains* [9, VII, 4, Proposition 2] it follows that the canonical subdirect representation

$$R \longrightarrow \prod_{\mathfrak{p} \in \mathcal{P}} R_{\mathfrak{p}}, \ x \bmod \alpha \in R_\alpha := B/\alpha \mapsto (x \bmod \mathfrak{p}^{w_{\mathfrak{p}}(\alpha)})_{\mathfrak{p} \in \mathcal{P}},$$

for $\alpha := \prod_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}^{w_{\mathfrak{p}}(\alpha)} \in \Lambda$, identifies R with the substructure of the product $\prod_{\mathfrak{p} \in \mathcal{P}} R_{\mathfrak{p}}$ consisting of those families $\mathbf{x} := (\mathbf{x}_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{P}}$ for which $d(\mathbf{x}, \varepsilon := 0 \bmod A) := (d_{\mathfrak{p}}(\mathbf{x}_{\mathfrak{p}}, \varepsilon_{\mathfrak{p}} := 0 \bmod A_{\mathfrak{p}}))_{\mathfrak{p} \in \mathcal{P}} \in \Lambda$, i.e. the set $\{\mathfrak{p} \in \mathcal{P} \mid \mathbf{x}_{\mathfrak{p}} \neq \varepsilon_{\mathfrak{p}}\}$ is finite. In particular, $R \cong \prod_{\mathfrak{p} \in \mathcal{P}} R_{\mathfrak{p}} \Longleftrightarrow A$ is semilocal, i.e. $\mathcal{P}$ is finite. As median set, R is *simplicial*, i.e. each cell $[\mathbf{x}, \mathbf{y}] \subseteq R$ has finitely many elements. Moreover for each finite subset $X \subseteq R$ with $|X| \geq 2$, the convex closure $[X]$ of X in the median set R is isomorphic to a finite product of finite simplicial trees : $[X] = \prod_{\mathfrak{p} \in \mathcal{P}} [X_{\mathfrak{p}}] \cong \prod_{|X_{\mathfrak{p}}| \geq 2} [X_{\mathfrak{p}}]$, where $[X_{\mathfrak{p}}]$ is the subtree of the simplicial tree $R_{\mathfrak{p}}$ generated by the image $X_{\mathfrak{p}}$ of X trough the projection map $R \longrightarrow R_{\mathfrak{p}}$. Applying the functor $\mathcal{B} : \mathfrak{R} \longrightarrow \mathfrak{P}$ to the object $R = \mathcal{R}(A \subseteq B)$ of $\mathfrak{CR}$, we obtain the complete Prüfer extension $\mathcal{B}(R) = (\widehat{A} \subseteq \widehat{B})$ - the completion of the Prüfer extension $A \subseteq B$ - with $\widehat{A} = \prod_{\mathfrak{p} \in \mathcal{P}} \widehat{A_{\mathfrak{p}}}, \ \widehat{B} = \{(x_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{P}} \in \prod_{\mathfrak{p} \in \mathcal{P}} \widehat{B_{\mathfrak{p}}} \mid \{\mathfrak{p} \in \mathcal{P} \mid x_{\mathfrak{p}} \notin \widehat{A_{\mathfrak{p}}}\} \text{ is finite}\}$-the ring of *restricted adeles* of the Dedekind domain A, where $\widehat{B_{\mathfrak{p}}}(\widehat{A_{\mathfrak{p}}})$ denotes the completion of the field B (the domain A) with respect to the valuation $w_{\mathfrak{p}}$. The Prüfer-Manis l-valuation $\mathfrak{w} : \widehat{B} \longrightarrow \widehat{\Lambda}$ associated to the Prüfer extension $\widehat{A} \subseteq \widehat{B}$, sending any $x := (x_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{P}} \in \widehat{B}$ to the formal product $\prod_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}^{w_{\mathfrak{p}}(x_{\mathfrak{p}})} \in \widehat{\Lambda}$, is onto, inducing the isomorphisms

$$M_{\widehat{B}/\widehat{A}} = \{\widehat{A}x \mid x \in \widehat{B}\} \cong \mathcal{M}_{\mathfrak{w}} \cong \widehat{\Lambda}, \ M^*_{\widehat{B}/\widehat{A}} = \{\widehat{A}x \mid x \in \widehat{B}^*\} \cong \widehat{B}^*/\widehat{A}^* \cong \Lambda.$$

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