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ON THE AUTOMORPHISMS OF CLUSTER ALGEBRAS

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Abstract. Let $\mathcal{A}_n(S)$ be a coefficient free cluster algebra over a field K. A cluster automorphism is an element of $Aut._K K(t_1, t_2 \cdots, t_n)$ which leaves the set of all cluster variables, χ_S, invariant. The group of all such automorphisms is studied in terms of the orbits of the symmetric group action on the set of all seeds $\mathcal{S}$ and the cluster pattern.

Introduction

Cluster algebras are introduced by S. Fomin, and A. Zelevinsky in [7 ,8, 2, 9]. The original motivation for this theory is to create an algebraic framework to study total positivity and canonical basis in semisimple algebraic groups. It is inspired by the discovery of connection between total positivity and canonical basis, due to G Lusztig, [11]

Great progress has been made in the theory, however we deviate from the original motivations.

A cluster algebra of rank n, $\mathcal{A}_n(S)$, is a commutative subalgebra of an ambient field, generated by a distinguished set of generators called cluster variables. The cluster variables are grouped in overlapping sets (clusters) of transcendence basis of the ambient field. Any two clusters can be obtained from each other by applying some sequence of mutations. We call the pair (χ_S, S_C), where χ_S is the set of all cluster variables and S_C denotes the class of all clusters, is called the cluster structure of $\mathcal{A}_n(S)$.

In this work, we introduce the cluster automorphisms of $\mathcal{A}_n(S)$, which are the field automorphisms which leave χ_S invariant. It turned out to be, under certain conditions, leaving χ_S invariant is equivalent to leaving the cluster structure invariant, (theorem 3.3). The group of all such automorphisms is called the cluster group of $\mathcal{A}_n(S)$ and is denoted by $C_n(S)$.

Also, we study the action of the symmetric group on the set $\mathcal{S}$, of all seeds of the field $F = K(t_1, t_2, \ldots, t_n)$. We show that in the simply-laced cluster algebras, the orbits of such actions are subsets of the orbits of the mutations group action on $\mathcal{S}$, theorem 2.4. However, the simply-laced hypothesis is necessary (example 2.5).

Every two seeds and a permutation group element define a field automorphism, we call such automorphism an exchange automorphism. The subgroup of $Aut._K F$, generated by the set of all exchange automorphisms, is called the exchange group of $\mathcal{A}_n(S)$, and is denoted by $\widetilde{\mathbf{m}}_n(S)$.

The main result of this article is providing a description for the intersection of the cluster group $C_n(S)$ and the exchange group $\widetilde{\mathbf{m}}_n(S)$ for any coefficient free cluster algebra satisfying the Fomin-Zelevinsky positivity conjecture, in terms of the orbits of the symmetric group action on S and the cluster pattern data (theorem 3.3).

Through out the paper, K is a field and $F = K(t_1, t_2 \dots t_n)$ is the field of rational functions in n independent (commutative) variables over k. We always denote (b_{ij}) for the square matrix B, (c_{ij}) for C, etc., and $[1, n] = \{1, 2, \dots, n\}$.

1. Preliminaries

Most of the material of this section is quoted from [2] [7] [8] [9] [10] and [16].

Definitions 1.1.
(1)A pre-seed of rank n in F over K is a pair (X, B) where $X = (x_1, x_2, \dots, x_n) \in F^n$, such that the $\{x_1, x_2, \dots, x_n\}$ is a transcendence basis of F over K and $B = (b_{ij})$ is an $n \times n$, sign-skew-symmetric integral matrix. Sign-skew-symmetric means, $b_{ij} = b_{ji} = 0$ or $b_{ij} b_{ji} < 0, \forall i, j \in [1, n]$.
(2)The diagram of a sign-skew-symmetric matrix $B = (b_{ij})$ is the weighted directed graph, $\Gamma(B)$, with set of vertices $[1, n]$, such that there is an edge from i to j if and only if $b_{ij} > 0$, and this edge is assigned the weight $|b_{ij} b_{ji}|$.
(3)A pre-seed (X, B) is called connected if $\Gamma(B)$ consists of exactly one connected component.

Definition 1.2 (Seed mutation). For each fixed $k \in \{1, \dots, n\}$, and each given pre-seed (X, B) we define a new pair $\mu_k(X, B) = (X^*, B^*)$ by setting $X^* = (x_1^*, \dots, x_n^*)$ with

$$x_i^* = \begin{cases} x_i & \text{if } i \neq k, \\ \frac{\prod_{b_{ji}>0} x_j^{b_{ji}} + \prod_{b_{ji}<0} x_j^{-b_{ji}}}{x_i} & \text{if } i = k. \end{cases} \tag{1.1}$$

and $B_k^* = (b_{ij}^*)$ with

$$b_{ij}^* = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}, \\ b_{ij} + \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2}, & \text{otherwise.} \end{cases} \tag{1.2}$$

The operation μ_k is called a mutation in $k-$direction.

Remark 1.3. If the pre-seed (X, B) is connected, then the mutation of (X, B) in any direction is still connected. Furthermore, one can see that $\mu_k^2 = 1$ for all $k \in [1, n]$, and $\{x_1, \dots, x_{i-1}, x_i^*, x_{i+1}, \cdots, x_n\}$ is always a transcendence basis of F over K for all $i \in [1, n]$. However, B^* need not be sign-skew-symmetric in general. So $(\mu_k(X), \mu_k(B))$ is not a pre-seed in general.

Definition 1.4. A connected pre-seed (X, B) is called a *seed* if we always obtain a pre-seed after applying seed mutations to (X, B) in all possible directions and all possible sequences, i.e., the matrix $\mu_{i_1} \mu_{i_2} \dots \mu_{i_q}(B)$ is again a sign skew symmetric matrix for all possible choices of $i_1, i_2 \dots i_q \in \{1, 2, \dots n\}$. In this case $\mu_{i_1} \mu_{i_2} \dots \mu_{i_q}((X, B))$ is called mutation-equivalent to (X, B). It is clear that mutation-equivalence is an equivalence relation and pre-seed (X, B) is a seed if and only if any one (that all) pre-seed that is mutation-equivalent to (X, B) is a seed.

Definition 1.5. (Distinguished seeds) A seed $p = (X, B)$ is called a distinguished seed if it satisfies the following two conditions

$$b_{ij}b_{ik} \geq 0, \quad \forall \quad i,j,k \in [1,n], \tag{1.3}$$

and the second condition is *Cartan counterpart* $A(B) = (a_{ij})$ of B, is of finite type as a Cartan matrix.
The type of p is the same as the Cartan-Killing type of $A(B)$. Where $A(B) = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 2, & \text{if } i = k, \\ -|b_{ij}|, & \text{if } i \neq k. \end{cases} \tag{1.4}$$

Remark 1.6. A connected pre-seed (X, B), with B skew-symmetrizable matrix, is always a seed. skew-symmetrizable means there exist a diagonal positive integral matrix $D = (d_i), i \in [1, n]$, such that DB is skew-symmetric.

Let $\mathcal{S}$ be the set of all seeds in F. Fix $p = (X, B) \in \mathcal{S}$. Let S denote the mutation equivalence class of p, and $S_C = \{Y; (Y, A) \in S\}$. Elements of S_C are called clusters and components of any cluster are called cluster variables. We call the pair, $(\mathcal{X}_S, S_C)$ the *cluster structure*, where $\mathcal{X}_S$ is the set of all cluster variables in S.

Definition 1.7. (Cluster algebra) Let $\mathcal{X}_S$ be the set of all cluster variables in S i.e. the union of all clusters in S_C. The cluster algebra $\mathcal{A}_n(S)$ of rank n, associated to the initial seed $p = (X, B)$ (of rank n), is defined to be the $\mathbb{Z}-$subalgebra of F generated by $\mathcal{X}_S$ i.e.

$$\mathcal{A}_n(S) := \mathbb{Z}[\mathcal{X}_S] \subset F \tag{1.5}$$

Definition 1.8. (**Cluster pattern of** $\mathcal{A}_n(S)$ [8]). The cluster pattern $\mathbb{T}_n(S)$ of the cluster algebra $\mathcal{A}_n(S)$ is an regular $n-$ary tree whose edges are labeled by the numbers $1, 2, \ldots, n$ such that the n edges emanating from each vertex receive different labels. The vertices are assigned to be the elements of S (the seeds) such that the endpoints of any edge are obtained from each other by seed mutation in the direction of the edge label.

One can see, the cluster pattern of $\mathcal{A}_n(S)$ can be completely determined by any seed in S.

Definition 1.9. A cluster algebra, $\mathcal{A}_n(S)$, is called of finite type, if S is a finite set. Equivalently if $\mathcal{X}_S$ is finite.

The details for the following two theorems are available in [8], [16] and [7].

Theorem 1.10. ***(Finite type classification)*** *For a cluster algebra $\mathcal{A}_n(S)$, the following are equivalent:*

- *$\mathcal{A}_n(S)$ is of finite type;*
- *for every seed (X, B) in S, the entries of the matrix $B = (b_{ij})$ satisfy the inequalities $|b_{ij}b_{ji}| \leq 3$, for all $i, j \in [1, n]$;*
- *S contains a distinguished seed.*

Theorem 1.11. *Every finite type Cartan matrix corresponds to one and only one, up to field automorphism, finite type cluster algebra. Furthermore, a cluster algebra $\mathcal{A}_n(S)$ is of finite type if and only if S contains a distinguished seed and cluster type of $\mathcal{A}_n(S)$ is the same as the Cartan-Killing type of the Cartan counter part the distinguished seed.*

Remark 1.12. If there is a seed (X, B) in S such that, there is a linear ordering of $\{1, 2, \cdots n\}$, where $b_{ij} \geq 0$ for all $i < j$, then the seed (X, B) is called acyclic seed, and $\mathcal{A}_n(\mathcal{S})$ is called acyclic cluster algebra, and in this case we have;

$$\mathcal{A}_n(S) = \mathbb{Z}[x_k, x'_k; k \in [1, n]], \tag{1.6}$$

and $\mathcal{A}_n(S)$ is finitely generated as an algebra.

Theorem 1.13. ***(Laurent Phenomenon and Fomin-Zelevinsky Positivity Conjecture).*** *The cluster algebra $\mathcal{A}_n(S)$ is contained in the integral ring of Laurent polynomials $\mathbb{Z}[X^{\pm}]$, for any cluster $X \in S_C$, i.e.*

$$\mathcal{A}_n(S) \subset \mathbb{Z}[X^{\pm}] = \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, \ldots, x_n^{\pm}] \tag{1.7}$$

for any cluster $X = (x_1, x_2, \ldots, x_n) \in S_C$.
More precisely, every non zero element $y \in \mathcal{A}_n(S)$, can be uniquely written as

$$y = \frac{P(x_1, x_2, \ldots, x_n)}{x_1^{\alpha_1} \ldots x_n^{\alpha_n}}, \tag{1.8}$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, and $P(x_1, x_2, \ldots, x_n)$ is an element of the ring of polynomials $\mathbb{Z}[x_1, x_2, \ldots, x_n]$, is not divisible by any cluster variable x_i.
It is further conjectured that in the case of y is a cluster variable, the polynomials $P(x_1, x_2, \ldots, x_n)$ have nonnegative integer coefficients(known as Fomin-Zelevinsky positivity conjecture).
The conjecture has been proved in many cases including classical type cluster algebras [8], rank two affine cluster algebras as in [15], acyclic cluster algebra [3], cluster algebras arising from spaces [14], and more.

2. Cluster groups

Definition 2.1. Let $Aut_K(F)$ denote the automorphism group of F over K. An automorphism $\phi \in Aut_K(F)$ is called a cluster isomorphism from a cluster algebra $\mathcal{A}_n(S)$ onto a cluster algebra $\mathcal{B}_n(S')$ over F, if ϕ sends every cluster variable in $\mathcal{A}_n(S)$ onto a cluster variable in $\mathcal{A}_n(S')$.

In particular, ϕ is called cluster automorphism of $\mathcal{A}_n(S)$, if it leaves $\mathcal{X}_S$ invariant. (Where $\mathcal{X}_S$ is the set of all cluster variables of $\mathcal{A}_n(S)$).

The subgroup of $Aut_K(F)$ of all cluster automorphisms of a cluster algebra $\mathcal{A}_n(S)$ is called *the cluster group* of $\mathcal{A}_n(S)$ and is denoted by $C_n(S)$.

Remarks 2.2.
(1) One can see that any cluster automorphism of a cluster algebra $\mathcal{A}_n(S)$ is an algebra automorphism of $\mathcal{A}_n(S)$ over K, i.e. $C_n(S)$ is a subgroup of $Aut_K\mathcal{A}_n(S)$, where $Aut_K\mathcal{A}_n(S)$ denotes the automorphism group of the algebra $\mathcal{A}_n(S)$ over K.
(2) If $\psi : \mathcal{A}_n(S) \to \mathcal{B}_n(S')$ is a cluster isomorphism, then ψ induces a group isomorphism between $C_n(S)$, and $C_n(S')$. To see this last fact, we define the group isomorphism by $\pi : C_n(S) \to C_n(S')$, given by $\phi \mapsto \varphi$, where $\varphi(y) = \psi(\phi(\psi^{-1}(y)))$ for y a cluster variable in $\mathcal{B}_n(S')$. A routine check shows that π is a group isomorphism.

Let Σ_n be the symmetric group on n letters. One can see that $\Sigma_n \subset Aut_K(F)$ as follows: let $T = (t_1, t_2, \ldots, t_n)$, where $\{t_1, t_2, \ldots, t_n\}$ is a transcendence basis of

F over K, $\sigma \in \Sigma_n$, and $f = f(t_1, t_2, \dots t_n) \in F$. We have;

$$\sigma_T(f) := f(t_{\sigma(1)}, t_{\sigma(2)}, \dots t_{\sigma(n)}). \tag{2.1}$$

In the following we introduce an action of the symmetric group Σ_n on $\mathcal{S}$.

Definition 2.3. Let $X = (x_1, x_2, \dots x_n)$ be a fixed cluster, and let $\sigma \in \Sigma_n$. For any seed $p = (Y, B) \in \mathcal{S}$, where $Y = (y_1, y_2 \dots y_n)$, and $B = (b_{ij})$. The Laurent Phenomenon (Theorem 1.13) guarantees that $y_i = y_i(x_1, x_2, \dots x_n) \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}, \dots x_n^{\pm}]$ (i.e y_i is a Laurent polynomial in $\{x_1, x_2, \dots x_n\}$, for each $i \in [1, n]$). We define $\sigma_X(p)$, as follows;

$$\sigma_X(p) := (\sigma_X(Y), \sigma(B)), \tag{2.2}$$

where $\sigma_X(Y) = (\sigma_X(y_1), \sigma_X(y_2), \dots, \sigma_X(y_n))$, $\sigma(B) := (b_{\sigma(i)\sigma(j)})$ and $\sigma_X(y_i)$, for $i \in [1, n]$, is as defined in (2.1). We write $\sigma(p)$ instead of $\sigma_X(p)$ if there is no chance of confusion.

Before stating the next theorem, we need to develop some notations.
For a seed $p = (Y, B)$, *the neighborhood of a cluster variable* y_i is defined to be the subset of $\{y_1, y_2, \cdots, y_n\}$ of all cluster variables y_j, where $b_{ij} \neq 0$. We denote this $N_p(y_i)$. For every connected integral skew-symmetric matrix, $B = (b_{ij})$, it is convenient to assign a quiver. We define Q_B as $Q_B = (Q_{1B}, Q_{2B})$, where Q_{1B} denotes the vertices and Q_{2B} denotes the arrows. The vertices $Q_{1B} = \{1, \dots, n\}$ and there is b_{ij} arrows from i to j if and only if $b_{ij} > 0$.
The mutation operation of the matrix B can be translated to the associated quiver. Let $\mu_k(Q_B)$ denote the mutation at k of Q_B. First, all the arrows incident to k in Q_B are reversed in $\mu_k(Q_B)$. Second, for each pair of sets of arrows, one set is of the incoming arrows say from j to k, and the other is of the outgoing arrows from k say to i, we add number of arrows from j to i equals to the number of the product of the cardinal numbers of both sets. Last step, is canceling all two-cycles between j and i.

Theorem 2.4. *Let $p = (X, B)$ be a simply-laced seed in F, i.e. $b_{ij} \in \{0, -1, 1\}$. Then for any $\sigma \in \Sigma_n$, $\sigma_X(p)$ is mutation-equivalent to p.*

Proof. Let $i, j \in \{1, 2, \dots, n\}$, such that $x_j \in N_p(x_i)$. Then $\sigma_{ij}(X, B)$ is mutation-equivalent to (X, B), where σ_{ij} stands for a transposition element of Σ_n, that sends every $k \in \{1, 2, \dots, n\}$ to itself except i and j. We have

$$\sigma_{ij}(X, B) = \mu_j\mu_i\mu_j\mu_i\mu_j((X, B)) = \mu_i\mu_j\mu_i\mu_j\mu_i((X, B)). \tag{2.3}$$

Now, since the symmetric group is generated by transpositions, and remarking that all seeds are connected, we are left to show the identity (2.3).
Sketch of proof of identities (2.3): Note that, each simply-laced sign skew symmetric matrix must be skew symmetric. So, if it is connected it associates to a (connected) quiver, which reduces to proving (2.3) on (X, Q_B) instead.
In the following we provide a proof for the identity (2.3) in some cases as examples. The proof of all other cases follow similarly:

(i) **Seeds of A_n type**
We provide a proof for A_3-type, a general A_n-type case is quite similar,

$$
\begin{aligned}
((x_1,x_2,x_3),\cdot_1\to\cdot_2\to\cdot_3) &\overset{\mu_1}{\Rightarrow} \Big(\Big(\frac{x_2+1}{x_1},x_2,x_3\Big),\cdot_1\leftarrow\cdot_2\to\cdot_3\Big)\\
&\overset{\mu_2}{\Rightarrow} \Big(\Big(\frac{x_2+1}{x_1},\frac{x_2x_3+x_3+x_1}{x_1x_2},x_3\Big),\cdot_1\to\cdot_2\leftarrow\cdot_3\Big)\\
&\overset{\mu_1}{\Rightarrow} \Big(\Big(\frac{x_3+x_1}{x_2},\frac{x_2x_3+x_3+x_1}{x_1x_2},x_3\Big),\cdot_1\leftarrow\cdot_2\leftarrow\cdot_3\Big)
\end{aligned}
$$

$$
\overset{\mu_2}{\Rightarrow} \Big(\Big(\frac{x_3+x_1}{x_2},x_1,x_3\Big),\ \begin{tikzcd} \cdot_1 \arrow[d] & \cdot_3 \arrow[l] \\ \cdot_2 \arrow[ur] & \end{tikzcd}\Big)
$$

$$
\overset{\mu_1}{\Rightarrow} ((x_2,x_1,x_3),\cdot_2\to\cdot_1\to\cdot_3)
$$

(ii) **For the exchange inside the n-cycles**
We prove it for A_3-type, a general n-cycle is quite similar:

$$
\Big((x_1,x_2,x_3),\ \begin{tikzcd} \cdot_1 \arrow[r] \arrow[d] & \cdot_3 \\ \cdot_2 \arrow[ur] & \end{tikzcd}\Big) \overset{\mu_1}{\Rightarrow} \Big(\Big(\frac{x_2x_3+1}{x_1},x_2,x_3\Big),\ \begin{tikzcd} \cdot_1 & \cdot_3 \arrow[l] \\ \cdot_2 \arrow[u] \arrow[ur] & \end{tikzcd}\Big)
$$

$$
\overset{\mu_2}{\Rightarrow} \Big(\Big(\frac{x_2x_3+1}{x_1},\frac{x_3(x_2x_3+1)+x_1}{x_1x_2},x_3\Big),\ \begin{tikzcd} \cdot_2 & \cdot_1 \arrow[l] \\ \cdot_3 \arrow[u] \arrow[ur] & \end{tikzcd}\Big)
$$

$$
\overset{\mu_1}{\Rightarrow} \Big(\Big(\frac{x_3+x_1}{x_2},\frac{x_3(x_2x_3+1)+x_1}{x_1x_2},x_3\Big),\ \begin{tikzcd} \cdot_2 \arrow[d] & \cdot_3 \arrow[l, "2"'] \\ \cdot_1 \arrow[ur] & \end{tikzcd}\Big)
$$

$$
\overset{\mu_2}{\Rightarrow} \Big(\Big(\frac{x_3+x_1}{x_2},x_1,x_3\Big),\ \begin{tikzcd} \cdot_2 \arrow[r, "2"] & \cdot_3 \arrow[dl] \\ \cdot_1 \arrow[u] & \end{tikzcd}\Big)
$$

$$
\overset{\mu_1}{\Rightarrow} \Big((x_2,x_1,x_3),\ \begin{tikzcd} \cdot_2 \arrow[r] \arrow[d] & \cdot_3 \\ \cdot_1 \arrow[ur] & \end{tikzcd}\Big).
$$

(remark that; the number 2 written over the arrows from 3 to 2 and from 2 to 3 in third and fourth steps respectively, refers to double arrows).
(iii) **Exchange of external vertex with adjacent one which is a vertex in an n-cycle**
We provide calculations for $n = 4$ case.

$$\left((x_1,x_2,x_3,x_4),\ \begin{tikzcd} \cdot_1 \arrow[r] & \cdot_2 \arrow[dl] \arrow[d] \\ \cdot_3 & \cdot_4 \arrow[l] \end{tikzcd}\right) \overset{\mu_1}{\Rightarrow} \left(\left(\frac{x_2+1}{x_1},x_2,x_3,x_4\right),\ \begin{tikzcd} \cdot_1 & \cdot_2 \arrow[l] \arrow[dl] \arrow[d] \\ \cdot_3 & \cdot_4 \arrow[l] \end{tikzcd}\right)$$

$$\overset{\mu_2}{\Rightarrow} \left(\left(\frac{x_2+1}{x_1},\frac{(x_2+1)x_3x_4+x_1}{x_1x_2},x_3,x_4\right),\ \begin{tikzcd} \cdot_1 \arrow[r] & \cdot_2 \\ \cdot_3 \arrow[ur] & \cdot_4 \arrow[l] \arrow[u] \end{tikzcd}\right)$$

$$\overset{\mu_1}{\Rightarrow} \left(\left(\frac{x_2x_4+x_1}{x_2},\frac{(x_2+1)x_3x_4+x_1}{x_1x_2},x_3,x_4\right),\ \begin{tikzcd} \cdot_1 & \cdot_2 \arrow[l] \\ \cdot_3 \arrow[ur] & \cdot_4 \arrow[l] \arrow[u] \end{tikzcd}\right)$$

$$\overset{\mu_2}{\Rightarrow} \left(\left(\frac{x_2x_4+x_1}{x_2},x_1,x_3,x_4\right),\ \begin{tikzcd} \cdot_1 \arrow[r] & \cdot_2 \arrow[dl] \arrow[d] \\ \cdot_3 \arrow[u] & \cdot_4 \arrow[l] \arrow[ul] \end{tikzcd}\right)$$

$$\overset{\mu_1}{\Rightarrow} \left((x_2,x_1,x_3,x_4),\ \begin{tikzcd} \cdot_2 \arrow[r] & \cdot_1 \arrow[dl] \arrow[d] \\ \cdot_3 & \cdot_4 \arrow[l] \end{tikzcd}\right).$$

Connected cycles and different quivers shapes are similar. □

For non simply-laced type seeds, the above result is not necessarily true, we provide the following counter example.

Example 2.5. Consider the seed (X, B), where

$$B = (b_{ij}) = \begin{pmatrix} 0 & +2 & 0 \\ -2 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In the following, we show that there is no sequence of mutations $\mu_{i_1}\mu_{i_2}\dots\mu_{i_k}$, such that $\sigma_{12}(B) = \mu_{i_1}\mu_{i_2}\dots\mu_{i_k}(B)$, where

$$\sigma_{12}(B) = \begin{pmatrix} 0 & -2 & +1 \\ +2 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

If we could show that, there is no sequence of mutations that sends the entry b_{23} to zero, we will be done. We do this by showing that every sequence of mutations sends b_{23} to an odd number. First we show by induction on the length of the sequence of mutations that, any sequence of mutations sends b_{13} and b_{12} to even numbers.

For single element sequences: one can see that, only μ_2 and μ_3 may change b_{13} and b_{12} respectively: that is μ_2 and μ_3 send b_{13} and b_{12} to 2 respectively.

Now, assume that every sequence of mutations of length k sends b_{13} and b_{12} to an even number, and let $\mu_{i_{k+1}}\mu_{i_k}\dots\mu_{i_1}$ be a sequence of length $k+1$. So if

$$\mu_{i_k}\dots\mu_{i_1}((b_{ij})) = (b'_{ij}). \tag{2.4}$$

Then $b'_{23} = 2d$ for some integer number d. Now we have

$$\begin{aligned}\mu_{i_{k+1}}(b'_{13}) &= b'_{13} + \frac{b'_{12}|b'_{23}| + b'_{23}|b'_{12}|}{2} \\ &= b'_{13} + d|b'_{23}| + |d|b'_{23} \\ &= b'_{13} + d\begin{cases}\pm 2b_{23}, & \text{if } b_{23}d > 0, \\ 0, & \text{if } b_{23}d < 0.\end{cases}\end{aligned}$$

Since b'_{13} is an even number then $\mu_{i_{k+1}}(b'_{13})$ must be an even too. This shows that any sequence of mutations will send b_{13} to an even number. In a similar way one can show that any sequence of mutation sends b_{12} to an even number.
Secondly, we show that every sequence of mutations sends $|b_{23}|$ to an odd number. We show this by induction on the number of occurrences of μ_1 in the sequence. Note that any sequence not containing μ_1 will not change $|b_{23}|$.
Sequences contains only one copy of μ_1: Without loss of generality, let $\mu_{i_1}\mu_{i_2}\dots\mu_{i_k}$ be a sequence of mutations such that $\mu_{i_k} = \mu_1$, and $\mu_{i_j} \neq \mu_1$, $\forall j \in [1, k]$. This becomes clear by considering that the possible change in $|b_{23}|$ appears only after applying μ_1, and there is no change in $|b_{23}|$ due to μ_2 or μ_3. Then using same notation as in (2.4), we have

$$b'_{23} = \pm 1 + \frac{b'_{21}|b'_{13}| + |b'_{21}|b'_{13}}{2}. \tag{2.5}$$

However b'_{21} and b'_{13} are both even numbers, so $\frac{b'_{21}|b'_{13}|+|b'_{21}|b'_{13}}{2}$ must be even, and b'_{23} is an odd number.
Sequences contains more than one copy of μ_1: Assume that any sequence of mutations, with μ_1 repeated $k-$ times sends b_{23} to an odd number.
Let $\mu_{i_t}\mu_{i_2}\dots\mu_{i_1}$ be a sequence of mutations containing μ_1, $k+1-$times, then we can assume that $\mu_{i_t} = \mu_1$. Let

$$\mu_{i_t}\dots\mu_{i_1}((b_{ij})) = (b''_{ij}), \quad \text{and} \quad \mu_{i_{t-1}}\dots\mu_{i_1}((b_{ij})) = (b'_{ij}), \tag{2.6}$$

then one can see that b'_{23} is an odd number and b'_{12} and b'_{13} are both even numbers. Then,

$$b''_{23} = b'_{23} + \frac{b'_{21}|b'_{13}| + b'_{13}|b'_{21}|}{2}, \tag{2.7}$$

a sum of an odd and even numbers, b''_{23} is an odd number. $\square$

Definition (2.3) gives rise to an equivalence relation on $\mathcal{S}$, as we will see in the following definition.

Definition 2.6. Let $B = (b_{ij})$ and $B' = (b'_{ij})$ be any two sign skew symmetric integral matrices, and σ be an element of $\sum_n$. Then we say that B and B' are σ-similar if $B' = \epsilon\sigma(B)$, where $\epsilon \in \{-1, +1\}$.

Now we define an equivalence relation $\sim$ on $\mathcal{S}$. Let $p = (X, B)$ and $p' = (Y, B')$ be two seeds. Then

$$p \sim p' \ \text{ if and only if } \ B \ \text{ and } \ B' \ \text{ are } \ \sigma-\text{similar } \text{ for some permutation } \ \sigma. \tag{2.8}$$

This yields an equivalence relation on $\mathcal{S}$ with the equivalence class of p denoted by $[p]$ and the equivalence class of B is denoted by $\langle B\rangle$. (Note, $\sim$ defines an equivalence relation on the set of all sign skew-symmetric integral $n \times n$ matrices, for all non-negative integer n)

Lemma 2.7. *Let $p = (X, B)$ and $p' = (Y, B')$ be any two seeds, and $\sigma \in \sum_n$. So, if $T_{pp',\sigma} \in Aut._K(F)$, given by; $x_i \mapsto y_{\sigma(i)}$. Then, we have;*

(2.9)
$$B \text{ and } B' \text{ are } \sigma-\text{similar if and only if } T_{pp',\sigma}(\mu_i(x_i)) = \mu_{\sigma(i)}(y_{\sigma(i)}), \quad \forall i \in [1, n].$$

In particular, $p \sim p'$ if and only if for some permutation σ, $T_{pp',\sigma}$ sends $\mu_i(X)$ to $\mu_{\sigma(i)}(Y)$.

Proof. $\Rightarrow$) Assume that B and B' are two σ- similar seeds. Then $B' = \epsilon(\sigma(B))$. So, if $B = (b_{ij})$ and $B' = (b'_{ij})$, then $b'_{ij} = \epsilon b_{\sigma(i)\sigma(j)}$, where $\epsilon \in \{+1, -1\}$. Then we have;

$$\begin{aligned}
T_{pp',\sigma}(\mu_i(x_i)) &= T_{pp',\sigma}\left(\frac{\prod_{b_{ji}>0} x_j^{b_{ji}} + \prod_{b_{ji}<0} x_j^{-b_{ji}}}{x_i}\right) \\
&= \frac{\prod_{b_{ji}>0} y_{\sigma(j)}^{b_{ji}} + \prod_{b_{ji}<0} y_{\sigma(j)}^{-b_{ji}}}{y_{\sigma(i)}} \\
&= \frac{\prod_{b_{ji}>0} y_{\sigma(j)}^{\epsilon b'_{\sigma(j)\sigma(i)}} + \prod_{b_{ji}<0} y_{\sigma(j)}^{-\epsilon b'_{\sigma(j)\sigma(i)}}}{y_{\sigma(i)}} \\
&= \begin{cases} \frac{\prod_{b'_{\sigma(j)\sigma(i)}>0} y_{\sigma(j)}^{b'_{\sigma(j)\sigma(i)}} + \prod_{b'_{\sigma(j)\sigma(i)}<0} y_{\sigma(j)}^{-b'_{\sigma(j)\sigma(i)}}}{y_{\sigma(i)}}, & \text{if } \epsilon = 1, \\ \frac{\prod_{b'_{ji}<0} y_{\sigma(j)}^{-b'_{\sigma(j)\sigma(i)}} + \prod_{b'_{\sigma(j)\sigma(i)}>0} y_{\sigma(i)}^{b'_{\sigma(j)\sigma(i)}}}{y_{\sigma(i)}}, & \text{if } \epsilon = -1. \end{cases} \\
&= \mu_{\sigma(i)}(y_{\sigma(i)})
\end{aligned}$$

$\Leftarrow$) Suppose that p and p' are not σ-similar, then $B' \neq \pm\sigma(B)$, i.e. $(b'_{ij}) \neq \pm(b_{\sigma(i)\sigma(j)})$. Then $b'_{ij_0} \neq \pm b_{\sigma(i)\sigma(j_0)}$, for some $j_0 \in [1, n]$. Now, we have;

$$\begin{aligned}
T_{pp',\sigma}(\mu_i(x_i)) &= T_{pp',\sigma}\left(\frac{\prod_{b_{ij>0}} x_j^{b_{ij}} + \prod_{b_{ik}<0} x_k^{-b_{ik}}}{x_i}\right) \\
&= T_{pp',\sigma}\left(\frac{\prod_{b_{ij>0, j\neq j_0}} x_j^{b_{ij}} \cdot x_{j_0}^{b_{ij_0}} + \prod_{b_{ik}<0} x_k^{-b_{ik}}}{x_i}\right) \\
&= \frac{\prod_{b_{ij>0, j\neq j_0}} y_{\sigma(j)}^{b_{ij}} \cdot y_{\sigma(j_0)}^{b_{ij_0}} + \prod_{b_{ik}<0} y_{\sigma(k)}^{-b_{ik}}}{y_{\sigma(i)}} \\
&= \begin{cases} \frac{\prod_{b_{ij>0, j\neq j_0}} y_{\sigma(j)}^{b_{ij}} \cdot y_{\sigma(j_0)}^{b_{ij_0}} + \prod_{b_{ik}<0} y_{\sigma(k)}^{-b_{ik}}}{y_{\sigma(i)}}, & \text{if } b_{ij_0} > 0, \\ \frac{\prod_{b_{ij>0,}} y_{\sigma(j)}^{b_{ij}} + \prod_{b_{ik}<0, j\neq j_0} y_{\sigma(k)}^{-b_{ik}} \cdot y_{\sigma(j_0)}^{b_{ij_0}}}{y_{\sigma(i)}}, & \text{if } b_{ij_0} < 0 \end{cases} \\
&\neq \mu_{\sigma(i)}(y_{\sigma(i)}).
\end{aligned}$$

Last line is due to the fact that, $y_{\sigma(j_0)}^{b_{ij_0}}$ appears in $\mu_{\sigma(i)}(y_{\sigma(i)})$ with a different exponent. The last part of the statement is straightforward. $\square$

Theorem 2.8. *Let $p = (X, B)$ and $p' = (Y, B')$ be any two σ-similar seeds. Then for any sequence of mutations $\mu_{i_k}, \mu_{i_{k-1}}, \dots, \mu_{i_1}$, the following are true:*
(1) $\mu_{i_k}\mu_{i_{k-1}} \dots \mu_{i_1}(X, B)$ and $\mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})} \dots \mu_{\sigma(i_1)}(Y, B')$ are σ- similar,
(2) $T_{pp',\sigma}(\mu_{i_k}\mu_{i_{k-1}} \dots \mu_{i_1}(X)) = \mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})} \dots \mu_{\sigma(i_1)}(Y)$, where $T_{pp',\sigma}$ is as defined in lemma 2.7.

Proof.
(1) This part follows as a simple corollary of the identity

$$\sigma(\mu_k(B)) = \mu_{\sigma(k)}(\sigma(B)), \quad \forall k \in [1, n]. \tag{2.10}$$

To prove that; let $\mu_k(B) = (b^*_{ij})$, and $\mu_{\sigma(k)}(\sigma(B)) = (b'^*_{\sigma(i)\sigma(j)})$. One can see that; we obtain the matrix $\sigma(B)$ from B, by relocating the entries of B using σ, i.e. the (i, j) entry b_{ij} of B moves to the position $\sigma(i) - \sigma(j)$ in $\sigma(B)$. So, we have $b_{\sigma(i)\sigma(j)} = b_{ij}$, $\forall i, j \in [1, n]$. Therefore, if $\sigma(\mu_k(B)) = (b^*_{\sigma(i)\sigma(j)})$, then, $b^*_{\sigma(i)\sigma(j)} = b^*_{ij}$, $\forall i, j \in [1, n]$. Now, let's look at the matrix $\mu_{\sigma(k)}(\sigma(B)) = (b'^*_{\sigma(i)\sigma(j)})$. We have $\forall i, j \in [1, n]$;

$$\begin{aligned} b'^*_{\sigma(i)\sigma(j)} &= \begin{cases} -b_{\sigma(i)\sigma(j)}, & \text{if } \sigma(k) \in \{i, j\}, \\ b_{\sigma(i)\sigma(j)} + \frac{b_{\sigma(i)\sigma(k)}|b_{\sigma(k)\sigma(j)}| + |b_{\sigma(i)\sigma(k)}|b_{\sigma(k)\sigma(j)}}{2}, & \text{otherwise} \end{cases} \\ &= \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}, \\ b_{ij} + \frac{b_{ik}|b_{kj}| + |b_{ik}|b_{kj}}{2}, & \text{otherwise} \end{cases} \\ &= b^*_{ij}. \end{aligned}$$

Remark that; $k \in \{i, j\}$ if and only if $\sigma(k) \in \{\sigma(i), \sigma(j)\}$. Therefore, $b'^*_{\sigma(i)\sigma(j)} = b^*_{\sigma(i)\sigma(j)}, \forall i, j \in [1, n]$, and this finishes the proof of (2.10). Changing the sign in (2.10) is immaterial to the identity, so we may rewrite it as

$$\epsilon\sigma(\mu_k(B)) = \mu_{\sigma(k)}(\epsilon\sigma(B)), \quad \forall \;\; k \in [1, n], \text{where} \;\; \epsilon \in \{-1, +1\}. \tag{2.11}$$

Which is equivalent to saying, if B and B' are σ-similar, then for any $k \in [1, n]$, $\mu_k(B)$, and $\mu_{\sigma(k)}(B')$ are also σ-similar. An induction process generalizes this fact to an arbitrary sequence of mutations, $\mu_{i_k}, \mu_{i_{k-1}}, \dots, \mu_{i_1}$, that is $\mu_{i_k}\mu_{i_{k-1}} \dots \mu_{i_1}(X, B)$ and $\mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})} \dots \mu_{\sigma(i_1)}(Y, B')$ are also σ-similar. Hence (1) is proved.

(2) Let $\mu_{i_k}\mu_{i_{k-1}} \dots \mu_{i_1}$ be a sequence of mutations, $p_{i_1 i_k} = \mu_{i_k}\mu_{i_{k-1}} \dots \mu_{i_1}(p)$, and $p'_{\sigma(i_1)\sigma(i_k)} = \mu_{\sigma(i_k)}\mu_{\sigma(i_{k-1})} \dots \mu_{\sigma(i_1)}(p')$, for $j \in [1, k]$. Part (1) tells us that $p_{i_1 i_{k-1}}$ and $p'_{\sigma(i_1)\sigma(i_{k-1})}$ are σ-similar. Then lemma 2.7 implies that

$$T_{p_{i_1 i_k} p'_{\sigma(i_1)\sigma(i_k)},\sigma}(\mu_{i_k}(\mu_{i_{k-1}} \dots \mu_{i_1}(X))) = \mu_{i_{\sigma(k)}}(\mu_{i_{\sigma(k-1)}} \dots \mu_{\sigma(i_1)}(Y)). \tag{2.12}$$

So, it remains to show that

$$T_{p_{i_1 i_k} p'_{\sigma(i_1)\sigma(i_k)},\sigma}(\mu_{i_k}(\mu_{i_{k-1}} \dots \mu_{i_1}(X))) = T_{pp',\sigma}(\mu_{i_k}(\mu_{i_{k-1}} \dots \mu_{i_1}(X)). \tag{2.13}$$

To get to this, let $q = (Z, D)$, and $q' = (T, C)$ be any two σ-similar seeds, and let $q_1 = \mu_i(Z, D) = (Z', D')$, $q'_1 = \mu_{\sigma(i)}(T, C) = (T', C')$. Where $Z = (z_1, z_2, \dots, z_n)$, and $T = (t_1, t_2, \dots, t_n)$. Next we show that

$$T_{q_1 q'_1, \sigma}(\mu_k \mu_i(Z)) = T_{qq', \sigma}(\mu_k \mu_i(Z)). \tag{2.14}$$

Let z_j be a cluster variable in Z, then for $j \neq i$, both of $T_{q_1 q'_1, \sigma}$, and $T_{qq', \sigma}$ leave z_j unchanged. Now, let $j = i$, and we have

$$\begin{aligned} T_{q_1 q'_1, \sigma}(\mu_i(z_i)) &= T_{q_1 q'_1, \sigma}\left(\frac{\prod_{d_{ij}>0} z_j^{d_{ij}} + \prod_{d_{ik}<0} z_k^{-d_{ik}}}{z_i}\right) \\ &= \frac{\prod_{d_{ij}>0} t_{\sigma(j)}^{d_{ij}} + \prod_{d_{ik}<0} t_{\sigma(k)}^{-d_{ik}}}{T_{p_1 p'_1, \sigma}(z_i)}. \end{aligned}$$

However,

$$\begin{aligned} T_{q_1 q'_1, \sigma}(\mu_i(z_i)) &= \mu_{\sigma(i)}(t_{\sigma(i)}) \\ &= \frac{\prod_{c_{ij}>0} t_{\sigma(j)}^{c_{ij}} + \prod_{c_{ik}<0} t_{\sigma(k)}^{-c_{ik}}}{t_{\sigma(i)}} \\ &= \frac{\prod_{d_{ij}>0} t_{\sigma(j)}^{\epsilon d_{ij}} + \prod_{d_{ik}<0} t_{\sigma(k)}^{-\epsilon d_{ik}}}{t_{\sigma(i)}}. \end{aligned}$$

Hence, $T_{p_1 p'_1, \sigma}(z_i) = t_{\sigma(i)}$. This shows that $T_{q_1 q'_1, \sigma}$, and $T_{qq', \sigma}$ have the same action on every cluster variable in Z, and since cluster variables from the $\mu_k \mu_i(Z)$ are integral laurent polynomials of cluster variables from Z, this gives (2.14).

For equation (2.13) we use induction on the length of the mutation sequence. Assume that equation (2.13) is true for any sequence of mutation of length less than or equal $k - 1$. Now we have;

$$\begin{aligned} T_{p_{i_1 i_k} p'_{i_1 i_k}, \sigma}(\mu_{i_k} \mu_{i_{k-1}} \dots \mu_{i_1}(X)) &= T_{p_{i_1 i_{k-2}} p'_{i_1 i_{k-2}}, \sigma}(\mu_{i_k} \mu_{i_{k-1}} \dots \mu_{i_1}(X)) \\ &= T_{pp', \sigma}(\mu_{i_k} \mu_{i_{k-1}} \dots \mu_{i_1}(X)), \end{aligned}$$

where the first equality is by identity (2.14), and the second, by the induction hypotheses.

□

Theorem 2.9. *Let $\mathcal{A}_n(S)$ be a cluster algebra, and (X, B) be a self σ-similar seed in S for some $\sigma \in \Sigma_n$. Then, σ_X is a cluster automorphism.*

Proof. Let $y \in \mathcal{X}_S$. Then there exists a seed (X, B) such that for a cluster variable x_i in X, we have $y = \mu_{i_1} \mu_{i_2} \dots \mu_{i_k}(x_i)$. So for some sequence of mutations $\mu_{i_1} \mu_{i_2} \dots \mu_{i_k}$, we apply this sequence of mutation to X.

We are left to show,

$$\sigma_X(y) = \mu_{\sigma(i_1)} \mu_{\sigma(i_2)} \cdots \mu_{\sigma(i_k)}(x_{\sigma(i)}). \tag{2.15}$$

From part (1) Theorem 2.8, $\mu_{i_2}\mu_{i_3}\dots\mu_{i_t}((X,B))$ and $\mu_{\sigma(i_2)}\mu_{\sigma(i_3)}\cdots\mu_{\sigma(i_t)}((X,B))$, are σ-similar $\forall t\in[1,q]$. Part (2) of the same theorem implies that equation (2.15) is correct.

$\square$

Example 2.10. Let $\mathcal{A}_n(S)$ be a cluster algebra of A_n-type. Then, $C_n(S)$ is not trivial.

To see that, fix an initial seed (X,B) such that $B=(b_{ij})$ where either $b_{ij}\geq 0,\forall i>j$ or $b_{ij}\leq 0,\forall i>j$, such seed is always exist in any cluster algebra of A_n-type . The following permutation is a cluster automorphism with symmetric group action defined, with respect to the initial cluster X;

$$\tau_X=\begin{cases}(1\ \ 2k+1)_X(2\ \ 2k)_X\dots(k\ \ k+2)_X, & \text{if } n=2k+1,\\ (1\ \ 2k)_X(2\ \ 2k-1)_X\dots(k\ \ k+1)_X, & \text{if } n=2k.\end{cases} \tag{2.16}$$

Now one can see that (X,B) is self σ-similar.

3. Exchange groups

Every path in the cluster pattern defines a field automorphism, which we codify in the following definition. In this section, we study the intersection of the group generated by all such automorphisms and the cluster group.

Definitions 3.1. Let $p=(X,B)$, and $p'=(Y,B')$ be any two vertices in the exchange graph $g_n(S)$ of the cluster algebra $\mathcal{A}_n(S)$. For any $\sigma\in\sum_n$, The field automorphism $T_{pp',\sigma}:F\to F$ induced by $x_i\mapsto y_{\sigma(i)}$ is called an exchange automorphism.

The subgroup of $Aut._K(F)$ generated by set of all exchange automorphisms is called the exchange group of $\mathcal{A}_n(S)$, and is denoted by $\widetilde{\mathbf{m}}_n(S)$.

Remark 3.2. let $\mathcal{A}_n(S)$ be a simply laced cluster algebra, and fix an initial seed $p=(X,B)$. Then, every symmetric group element can be seen as a field automorphism, (as in the paragraph proceeding definition (2.3)), taking $T=X$. From Theorem 2.4, every symmetric group element (in the above sense) corresponds to a path in the exchange graph of $\mathcal{A}_n(S)$. So, the symmetric group elements can be seen as exchange automorphisms.

Before we state the main results we sharpen the notations of the neighbors and monomials of the cluster variables. Let x_{i_0} be a cluster variable in $p=(X,B)$, i.e.$X=(x_1,\dots,x_{i_0-1},x_{i_0},x_{i_0+1},\dots,x_n)$. The set of neighbors of x_{i_0} at the seed p, denoted by $N_p(x_{i_0})$ and defined as, $N_p(x_{i_0}):=N_{p,+}(x_{i_0})\bigcup N_{p,-}(x_{i_0})$, where $N_{p,+}(x_{i_0})=\{x_i;b_{i_0i}>0\}$ and $N_{p,-}(x_{i_0})=\{x_i;b_{i_0i}<0\}$. The positive and negative monomials of the cluster variable x_{i_0} at the seed p are denoted by $m_{p,+}(x_{i_0})$, and $m_{p,-}(x_{i_0})$ respectively. Where, $m_{p,+}(x_{i_0})=\prod_{x_i\in N_{p,+}(x_{i_0})}x_i^{b_{ii_0}}$, and $m_{p,-}(x_{i_0})=\prod_{x_i\in N_{p,-}(x_{i_0})}x_i^{-b_{ii_0}}$. We denote $f_{p,x_{i_0}}=m_{p,+}(x_{i_0})+m_{p,-}(x_{i_0})$, one can see that $f_{p,x_{i_0}}$ is not divisible by $x_i,\forall i\in[1,n]$.
The following theorem provides a description for $C_n(S)$, through $\widetilde{\mathbf{m}}_n(S)$ and the equivalent classes of $\sim$. In the proof of the theorem, we assume that the positivity conjecture is true, Theorem 1.13. However, a proof without the positivity conjecture can be written in rank two and finite type cluster algebras.

Theorem 3.3. *Let $\mathcal{A}_n(S)$ be a cluster algebra satisfies the positivity conjecture, (theorem 1.13). Let $p = (X, B)$ and $p' = (Y, B')$ be any two vertices in the exchange graph of $\mathcal{A}_n(S)$. Then, for the field automorphism $T_{pp',\sigma}$ (Definition 3.1.), the following are equivalent*

(1) *$T_{pp',\sigma}$ is a cluster automorphism,*

(2) *p and p' are σ- similar,*

(3) *$T_{pp',\sigma}$ permutes the clusters.*
Furthermore for any seed $q = (Z, D)$, we have;

$$T_{pp',\sigma}(Z) \in \{Y; \quad (Y, M) \in [q]\}. \tag{3.1}$$

Proof.
(1)⇒(2)
Let $T_{pp',\sigma}$ be a cluster automorphism. From Lemma 2.7, to show p and p' are σ-similar, it is enough to show that, $T_{pp',\sigma}(\mu_i(x_i)) = \mu_{\sigma(i)}(y_{\sigma(i)}), \forall i \in [1, n]$.
Let $z = T_{pp',\sigma}(\mu_i(x_i))$, and $\xi = \mu_{\sigma(i)}(y_{\sigma(i)})$ where $X = (x_1, x_2, \ldots, x_n)$, and $Y = (y_1, y_2, \ldots, y_n)$. Then, we have;

$$z = \frac{T_{pp',\sigma}(f_{p,x})}{y_{\sigma(i)}}, \qquad \text{and} \qquad \xi = \frac{f_{p',y_{\sigma(i)}}}{y_{\sigma(i)}}. \tag{3.2}$$

Both $T_{pp',\sigma}(f_{p,x})$ and $f_{p',y_{\sigma(i)}}$ are polynomials in the integer ring of polynomials $\mathbb{Z}[y_{\sigma(1)}, \cdots, y_{\sigma(i-1)}, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)}]$, and are not divisible by $y_{\sigma(j)}$, for all j in $[1, n]$.
Now, suppose that z is a cluster variable. Then, by Laurent phenomenon (theorem 1.13), z can be written uniquely as;

$$z = \frac{P(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(i-1)}, \xi, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)})}{y_{\sigma(1)}^{\alpha_1} \cdots y_{\sigma(i-1)}^{\alpha_{i-1}} \xi^{\alpha_i} y_{\sigma(i+1)}^{\alpha_{i+1}} \cdots y_{\sigma(n)}^{\alpha_n}}, \tag{3.3}$$

where $P(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(i-1)}, \xi, y_{\sigma(i+1)}, \ldots, y_{\sigma(n)})$ is a polynomial with integers coefficients, which is not divisible by any of the following cluster variables $y_{\sigma(1)}$,$y_{\sigma(2)}$, …, $y_{\sigma(i-1)}$, ξ,$y_{\sigma(i+1)}, \ldots, y_{\sigma(n-1)}$ and $y_{\sigma(n)}$, and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$.
Comparing z from (3.2) and (3.3), we have;

$$T_{pp',\sigma}(f_{p,x}) \ldots y_{\sigma(1)}^{\alpha_1} \ldots y_{\sigma(i-1)}^{\alpha_{i-1}} \ldots \xi^{\alpha_i} \ldots y_{\sigma(i+1)}^{\alpha_{i+1}} \ldots y_{\sigma(n)}^{\alpha_n} = P \cdot y_{\sigma(i)}. \tag{3.4}$$

Since, $f_{p,x}$ is not divisible by any cluster variable x_i, for any $i \in [1, n]$. Then $T_{pp',\sigma}(f_{p,x})$ is not divisible by $y_i, \forall i \in [1, n]$. More precisely $T_{pp',\sigma}(f_{p,x})$ is a sum of two monomials in variables from the cluster Y, with positive exponents. Therefore, $\alpha_j = 0$ for all $j \in [1, n] - \{i\}$, and $i = -1$. Hence, (3.4) can be simplified as

$$T_{pp',\sigma}(f_{p,x}) = P \cdot f_{p',y_{\sigma(i)}}. \tag{3.5}$$

Now we have that $f_{p',y_{\sigma(i)}}$ is also a sum of two monomials in variables from the cluster Y, with positive exponents. However, P is a polynomial with positive integers coefficients, and not divisible by any cluster variable from $Y' = \mu_{\sigma(i)}(Y)$. Then it must be either a sum of at least two monomials in variables from Y', or it

is a positive integer. Equation (3.5) says that, the first option for P is impossible because $T_{pp',\sigma}(f_{p,x})$ and $f_{p',y_{\sigma(i)}}$ sums of exactly two monomials, so P must be an integer. Because the coefficients of $T_{pp',\sigma}(f_{p,x})$ and $f_{p',y_{\sigma(i)}}$ are all ones, which must be exactly 1.
Hence,

$$T_{pp',\sigma}(f_{p,x}) = f_{p',y_{\sigma(i)}}. \tag{3.6}$$

Therefore,

$$T_{pp',\sigma}(\mu_i(x_i)) = \mu_{\sigma(i)}(y_{\sigma(i)}), \forall i \in [1,n]. \tag{3.7}$$

Now, lemma 2.7 implies p and p' are σ-similar.
(2)$\Rightarrow$ (3)

Let $Z = (z_1, z_2, \ldots, z_n)$ be the cluster of the seed (Z, D). Then there is a sequence of mutations $\mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}$ such that

$$(Z, D) = \mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}(X, B)$$

Since (X, B), and (Y, B') are σ-similar then, Theorem 2.8 part 1 implies that $\mu_{i_2}\ldots\mu_{i_k}(X, B)$ and $\mu_{\sigma(i_2)}\ldots\mu_{\sigma(i_k)}(Y, B')$ are σ-similar too.
But, from Theorem 2.8 part 2, we have,

$$T_{pp',\sigma}(\mu_{i_1}(\mu_{i_2}\ldots\mu_{i_k}(X))) = \mu_{\sigma(i_1)}\mu_{\sigma(i_2)}\ldots\mu_{\sigma(i_k)}(Y),$$

and since the right hand side is a cluster and the left hand side is only $T_{pp',\sigma}(Z)$, then $T_{pp',\sigma}$ sends Z to a cluster. So, $T_{pp',\sigma}(Z)$ permutes the clusters. For the belonging (3.1), is immediate from the above argument.

(3)$\Rightarrow$(1) Permuting the clusters implies leaving χ_S invariant, because every cluster variable is contained in some cluster. □

The following is a corollary of the proof of Theorem 3.3, and is actually the generalization of the statement of the same theorem, to the level of cluster isomorphism.

Corollary 3.4. *Let $\mathcal{A}_n(S)$, and $\mathcal{A}_n(S')$ be any two cluster algebras over F. If $p = (X, B) \in S$, $p' = (Y, B') \in S'$, and $\sigma \in \sum_n$. Then the following are equivalent*

(1) *the field automorphism $\phi_{pp',\sigma} : F \to F$, given by $x_i \mapsto y_{\sigma(i)}$ is a cluster isomorphism from $\mathcal{A}_n(S)$ onto $\mathcal{A}_n(S')$,*

(2) *p and p' are σ-similar,*

(3)*$\phi_{pp',\sigma}$ sends every cluster in S_C onto a cluster in S'_C.*
In particular, two cluster algebras are cluster isomorphic if and only if they contain two σ-similar seeds for some permutation σ.

Proof. Follow the proof of Theorem 3.3, mutatis mutandis. □

Corollary 3.5. *If $\mathcal{A}_n(S)$ is a cluster algebra of simply-laced type then $C_n(S) \neq 1$.*

Proof. This follow from Theorem 2.4 and theorem 3.3.
However, the converse is not necessarily true. Consider the the cluster algebra given in example 2.5, and the cluster automorphism is the transposition σ_{23}. A routine check shows that, σ_{23} corresponds to the sequence of mutations $\mu_2\mu_3\mu_2\mu_3\mu_2$. Then Theorem 2.4 implies the transposition σ_{23} is a cluster automorphism, while the cluster algebra is not simply-laced. □

Conjecture 3.6. The set of all cluster variables χ_S can be complectly determined by $[p]$ as follows;

$$\chi_S = \bigcup \{Y; \quad (Y,M) \in [p]\}, \tag{3.8}$$

In the following we calculate the cluster and exchange groups for some cluster algebras of low ranks.

Example 3.7. **Cluster and exchange groups of rank 1** In this case, $F = K(t)$, and $Aut_K F$ is isomorphic to the projective linear group $PGL_2(K)$, and $S = \{(x,(0)),(\frac{1}{x},(0))\}$, hence $\mathcal{A}_1(S) = \mathbb{Z}[x^{\pm 1}]$. So the cluster group and the exchange groups associated to $\mathcal{A}_1(S)$ are the subgroup of $Aut._K(F)$ generated by the automorphism $T_1 : K(x) \to K(x)$ induced by $x \mapsto \frac{1}{x}$, and then;

$$\widetilde{m}(S) = C_1^1((x,(0))) = C_1^1((\frac{1}{x},(0))) = C_1(S) \cong \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle < PGL_2(K).$$

Example 3.8. **Cluster groups of rank 2,** $C_2(\mathcal{S})$
In this case $F = K(x_1,x_2)$, and applying the mutations on the initial seed $\theta = \{(x_1,x_2), \begin{pmatrix} 0 & m \\ -n & 0 \end{pmatrix}\}$, leads to the following recursive relation for the cluster variables of $\mathcal{A}(\mathcal{S})$

$$x_{t-1}x_{t+1} = \begin{cases} x_t^m + 1, & \text{if t is odd} \\ x_t^n + 1, & \text{otherwise.} \end{cases} \tag{3.9}$$

Thus, the cluster algebra $\mathcal{A}(\mathcal{S})(m.n)$ corresponding to θ is the subalgebra of $F = K(x_1,x_2)$ generated by $\{x_t; t \in \mathbb{Z}\}$, however, since θ is acyclic seed then $\mathcal{A}(\mathcal{S})(m.n) = \mathbb{Z}[x_0,x_1,x_2,x_3] \subset K(x_1,x_2)$.

Theorem 3.9. [11] *The sequence (3.9) of the cluster variables $\{x_t\}_{t\in\mathbb{Z}}$ in $\mathcal{A}(\mathcal{S})(m.n)$ is periodic if and only if $mn \le 3$, and is of period 5 (resp., 6, 8) if $mn = 1$ (resp., 2, 3).*

In the following, let $C_2(m,n)$ denote the cluster group associated to $\mathcal{A}(\mathcal{S})(m,n)$.

Lemma 3.10. *There is a cluster isomorphism between the cluster algebra $\mathcal{A}_2(S)(m,n)$ and $\mathcal{A}_2(S)(n,m)$.*

Proof. Fix (x_1,x_2) and (y_1,y_2) as initial clusters for $\mathcal{A}_2(S)(m,n)$ and $\mathcal{A}_2(S)(n,m)$ respectively. Consider the following cluster isomorphism

$$\sigma_{12} : \mathcal{A}_2(S)(m,n) \to \mathcal{A}_2(S)(n,m)$$

given by:

$$x_1 \mapsto y_2 \quad \text{and} \quad x_2 \mapsto y_1,$$

one can see that this automorphism induces a one to one correspondence between the sets of all clusters of $\mathcal{A}_2(S)(m,n)$ and $\mathcal{A}_2(S)(n,m)$.

□

Corollary 3.11. $C_2(m,n) \cong C_2(n,m)$.

Proof. From the previous lemma, and part (2) in remarks 3.2. □

Example 3.12. **The Cluster and exchange groups of** $\mathcal{A}(\mathcal{S})(1,1)$. In this case, we have exactly 5 cluster variables which, in terms of the initial cluster variables (x_1, x_2) are

$$\{x_1, x_2, \frac{x_1+1}{x_2}, \frac{x_2+1}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\},$$

and the following unordered pairs as clusters

$$(x_1,x_2),(x_1,\frac{x_1+1}{x_2}),(\frac{x_2+1}{x_1},x_2),(\frac{x_2+1}{x_1},\frac{1+x_1+x_2}{x_1x_2}),(\frac{1+x_1+x_2}{x_1x_2},\frac{x_1+1}{x_2}).$$

So, $C_2(1,1)$ is the subgroup of $Aut._K K(x_1,x_2)$ generated by the following involuting automorphisms T_1, and T_2 where, T_1 is induced by;

$$x_1 \mapsto \frac{x_2+1}{x_1}, \quad \text{and} \quad x_2 \mapsto x_2, \tag{3.10}$$

and T_2

$$x_1 \mapsto x_1 \quad \text{and} \quad x_2 \mapsto \frac{x_1+1}{x_2}, \tag{3.11}$$

To show, these are the only generators. Let's try one different choice, consider the automorphism η induced by, $x_1 \mapsto \frac{x_1+1}{x_2}$ then we have;

$$\eta((x_1,\frac{x_1+1}{x_2})) = (\eta(x_1),\eta(\frac{x_1+1}{x_2})) = (\frac{x_1+1}{x_2},\frac{x_1+x_2+1}{x_2^2})$$

but $(\frac{x_1+1}{x_2},\frac{x_1+x_2+1}{x_2^2})$ is not a cluster, so the automorphism η can not be a cluster automorphism. In a complete similar way we can argue all other possible choices. Therefore,

$$C_2(1,1) = \langle T_1, T_2\rangle < Aut._K K(x_1,x_2).$$

Also, we can see that

$$C_2(1,1) = \Sigma_2 = \widetilde{m}_2(S) = C_2^d(p), \quad \forall \ \text{ seed } \ p \in S, \quad \text{and} \ \forall \ \ d \in [1,5].$$

Remark 3.13. $C_2(1,1)$ is a Coexter group with the following presentation

$$C_2(1,1) = \{T_1,T_2 | T_1^2 = T_2^2 = 1, (T_1T_2)^{10} = 1\}.$$

Example 3.14. **The cluster group** $C_2(2,1)$.
We have exactly 6 different cluster variables, which are

$$\{x_1,x_2,\frac{x_2+1}{x_1},\frac{(x_2+1)^2+x_1^2}{x_1^2x_2},\frac{x_1^2+x_2+1}{x_1x_2},\frac{x_1^2+1}{x_1}\} \tag{3.12}$$

and the following unordered pairs as the set of clusters
$(x_1,x_2),(\frac{x_2+1}{x_1},x_2),(\frac{x_2+1}{x_1},\frac{(x_2+1)^2+x_1^2}{x_1^2x_2}),(\frac{x_1^2+x_2+1}{x_1x_2},\frac{(x_2+1)^2+x_1^2}{x_1^2x_2}),(\frac{x_1^2+x_2+1}{x_1x_2},\frac{x_1^2+1}{x_1})$. Notice that, the cluster variables x_1 and x_2 are not symmetrical as in $\mathcal{A}(\mathcal{S})(1,1)$, which implies that the symmetric group element σ_{12} is not a cluster automorphism i.e. is not an element of $C_2(2,1)$, and hence the generators are only T_1 as defined in (3.5), together with automorphism $T_2 \in Aut_K K(x_1,x_2)$ which is induced by

$$x_1 \mapsto x_1 \quad \text{and} \quad x_2 \mapsto \frac{x_1^2+1}{x_2}. \tag{3.13}$$

Then we have

(3.14)
$$C_2(2,1) = \widetilde{m}_2(S) = C_2^d(p) = \langle T_1, T_2 \rangle < Aut_K K(x_1, x_2), \ \ \forall \ \text{seed} \ \ p \in S, \ \ \text{and} \ \forall \ d.$$

Remark 3.15. $C_2(2,1)$ is a Coexter group with the following presentation
$$C_2(2,1) = \{T_1, T_2 | \ \ T_1^2 = T_2^2 = 1, (T_1T_2)^3 = 1\}.$$

Example 3.16. **The cluster group** $C_2(3,1)$.
We have exactly 8 different cluster variables, and in a similar way of $C_2(2,1)$, we have; $C_2(3,1)$ is generated by T_1 as defined in (3.5) and T_2, induced by

$$x_1 \mapsto x_1 \quad \text{and} \quad x_2 \mapsto \frac{x_1^3+1}{x_2}, \tag{3.15}$$

and we have
(3.16)
$$C_2(3,1) = \widetilde{m}_2(S) = C_2^d(p) = \langle T_1, T_2 \rangle < Aut_K K(x_1, x_2), \ \ \forall \ \text{seed} \ \ p \in S, \ \ \text{and} \ \forall \ d.$$

Remark 3.17. $C_2(3,1)$ is a Coexter group with the following presentation
$$C_2(2,1) = \{T_1, T_2 | \ \ T_1^2 = T_2^2 = 1, (T_1T_2)^4 = 1\}.$$

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