arXiv:1011.0896v2 [math.RT] 7 Nov 2010

# ON SINGULAR LOCALIZATION OF $\mathfrak{g}$-MODULES.

ERIK BACKELIN AND KOBI KREMNIZER

Abstract. We prove a singular version of Beilinson-Bernstein localization for a complex semi-simple Lie algebra following the ideas from the positive characteristic case done in [BMR06].

## 1. Introduction

Let $\mathfrak{g}$ be a reductive complex Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and let $\mathcal{B}$ be the flag manifold of $\mathfrak{g}$. Let $\lambda \in \mathfrak{h}^*$ be regular and dominant and let $\chi_\lambda$ be the corresponding central character. Let $\mathcal{D}^\lambda_\mathcal{B}$ be the sheaf of $\lambda$-twisted differential operators on $\mathcal{B}$. The celebrated localization theorem of Beilinson and Bernstein, [BB81], states that the global section functor gives an equivalence between the category $\mathcal{D}^\lambda_\mathcal{B}$-mod and the category of $\mathrm{U}(\mathfrak{g})^\lambda := \mathrm{U}(\mathfrak{g}) / \operatorname{Ker} \chi_\lambda$-modules.

The problem of localization at a singular central character remained unsolved for a long time. It was known that the global section functor on $\mathcal{D}^\lambda_\mathcal{B}$-mod was a quotient functor whose kernel could be described rather explicitly, see [Kas93]. Thus $\mathrm{U}(\mathfrak{g})^\lambda$-mod is equivalent to a quotient category of $\mathcal{D}^\lambda_\mathcal{B}$-mod. The main drawback with this description, from the viewpoint of representation theory, is that it doesn't lead to a fully satisfactory $\mathcal{D}$-module description of the translation functors. (See [BG99] for the picture.)

A solution to the problem of singular localization was quite recently given in positive characteristic by [BMR06]. We sketch their basic construction here (it works in any characteristic):

Let $G$ be a reductive algebraic group such that $Lie\, G = \mathfrak{g}$. Instead of $\mathcal{B}$ consider a parabolic flag manifold $\mathcal{P} = G/P$, where $P \subset G$ is a parabolic subgroup whose parabolic roots coincide with the singular roots of $\lambda$. Replace the sheaf $\mathcal{D}^\lambda_\mathcal{B}$ by a sheaf $\mathcal{D}^\lambda_\mathcal{P} := \pi_*(\mathcal{D}_{G/U_P})^{L_P}$ modulo a certain ideal defined by $\lambda$. Here $L_P$ is the Levi factor and $U_P$ is the unipotent radical of $P$ and $\pi : G/U_P \to \mathcal{P}$ is the projection. The $L_P$-invariants are taken with respect to the right $L_P$-action on $G/U_P$. The sheaf $\pi_*(\mathcal{D}_{G/U_P})^{L_P}$ is locally isomorphic to $\mathcal{D}_\mathcal{P} \otimes \mathrm{U}(\mathfrak{l}_P)$, where $\mathfrak{l}_P = Lie\, L_P$. When $P = B$ we have $\mathcal{D}^\lambda_\mathcal{P} = \mathcal{D}^\lambda_\mathcal{B}$ and when $P = G$ we arrive at a tautological solution: $\mathcal{D}^\lambda_\mathcal{P} = \mathrm{U}(\mathfrak{g})^\lambda$ tensored with the sheaf of differential operators on a point $= \mathrm{U}(\mathfrak{g})^\lambda$.

What we do in this note is essentially to use this construction to prove a singular localization theorem in characteristic zero, see theorem 5.1 and theorem 5.2. This is probably well-known to the experts but we couldn't find it in the literature. Our proof is very similar to the original proof of [BB81], although the path to get there is slightly more ragged. (Technically speaking, in localization theory Beilinson and Bernstein introduced the method of tensoring a $\mathcal{D}$-module with a trivial vector bundle and then consider a filtration of this bundle with $G$-equivariant subquotient bundles. On $\mathcal{B}$ these subquotients can be taken to be line bundles, but on $\mathcal{P}$ one must use more general and less easily controlled vector bundles - because irreducible representations of $P$ are in general not one-dimensional.)

---

2000 *Mathematics Subject Classification.* Primary 17B10, 14-XX.

In the context of localization theory, positive characteristic is the most difficult, but the localization theorem can on the other hand then only hold at the level of derived categories. The result presented here gives an equivalence on the level of abelian categories just like Beilinson and Bernstein's theorem.

In subsequent research we will address the issue of singular localization for quantum groups at generic $q$ and at roots of unity. In these cases it is essential to use the language of equivariant sheaves on $G$, because one can quantize $\mathcal{O}(G)$ and $\mathcal{O}(P)$ and hence categories of $\mathcal{O}(P)$-comodules and $\mathcal{O}(G)$-modules, but a quantized flag variety does not exist as a "space". Regular localization for quantum groups was done in [BK06] (generic case, proof similar to that of [BB81]) and in [BK08] (root of unit case, proof resembling that of [BMR02]). Partly for this reason we have taken thorough care to justify our equivariant definitions and to give equivariant proofs for the results of this paper. For the convenience of the reader we have given parallel descriptions in the sometimes more geometrically intuitive non-equivariant language.

As an application we find that a block of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ will correspond to certain bi-equivariant $\mathcal{D}$-modules on $G$. If one reverses the reading order of these equivariance conditions and pushes forward to $\mathcal{B}$ it follows that even a singular block of category $\mathcal{O}$ is equivalent to a category of $\mathcal{D}$-modules on $\mathcal{B}$. See section 6.1. These modules are not holonomic however, unless $\lambda$ is regular. We hope that we in the future will be able to give a topological interpretation of them as some type of constructible sheaves, e.g., to use the equivariance to shrink the De Rham complex of such a $\mathcal{D}$-module to a constructible size.

We deduce that a singular block in category $\mathcal{O}$ is equivalent to a certain (non-standard) parabolic subcategory of a regular block in the category which is obtained from $\mathcal{O}$ by relaxing the defining semi-simplicity requirement for the action of $\mathfrak{h}$ to local finiteness and instead require true central character, proposition 6.1.

We also describe the functors on the $\mathcal{D}$-module side that correspond to translation functors on representations, generalizing some results of [BG99], see section 6.2.

## 2. Preliminaries

Here we fix notations and collect mostly well-known results that will be used in the paper.

2.1. **Notations.** We work over $\mathbb{C}$. Let $X$ be an algebraic variety, let $\mathcal{O}_X$ be the sheaf of regular functions on $X$ and $\mathcal{O}(X)$ its global sections. Denote by $\mathcal{O}_X$-mod the category of quasi-coherent sheaves on $X$. Let $\Gamma := \Gamma_X : \mathcal{O}_X\text{-mod} \to \mathcal{O}(X)\text{-mod}$ be the global section functor.

If $Y$ is another variety and there exists an obvious projection map $X \to Y$ we shall denote it by $\pi^Y_X$.

For $\mathcal{A}$ a sheaf of algebras on $X$ such that $\mathcal{O}_X \subseteq \mathcal{A}$ we abbreviate an $\mathcal{A}$-module for a sheaf of $\mathcal{A}$-modules that is quasi-coherent over $\mathcal{O}_X$. We denote by $\mathcal{A}$-mod the category of $\mathcal{A}$-modules. More generally, we will encounter categories such as ($\mathcal{A}$, *additional data*)-mod that consists of $\mathcal{A}$-modules with some *additional data*. We will then denote by ($\mathcal{A}$, *additional data*)-mod$_c$ its full subcategory of objects that are locally finitely generated over $\mathcal{A}$.

Assume that an algebraic group $L$ acts on $X$. Let $M \in \mathcal{O}_X$-mod be $L$-equivariant. In particular, $L$ acts on local sections of $M$ over $L$-invariant open subsets of $X$. There is the sheaf $(\pi^{X/L}_{X*}M)^L$ of $L$-invariant local sections in the direct image $\pi^{X/L}_{X*}M$. We can also think of $M^L$ as a sheaf on the set $X$ with the topology of $L$-invariant Zariski open subsets of $X$. We shall refer to $M^L$ as the sheaf of $L$-invariant local sections in $M$.

Unless stated otherwise, $\otimes = \otimes_{\mathbb{C}}$.

2.2. **Root data.** Let $G$ be a reductive complex linear algebraic group, $B \subset G$ a Borel subgroup and $T \subset B$ a maximal tori. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be their respective Lie algebras. For any parabolic subgroup $P$ of $G$ containing $B$, we denote by $U_P$ its unipotent radical and by $L_P$ its Levi subgroup and by $\mathfrak{u}_P$ and $\mathfrak{l}_P$ their Lie algebras. We denote by $\mathcal{B} = G/B$ the flag variety and by $\mathcal{P} = G/P$ the parabolic flag variety corresponding to $P$.

Let $\Lambda$ be the lattice of integral weights and let $\Lambda_r$ be the root lattice. Let $\Lambda_+$ and $\Lambda_{r+}$ be the positive weights and the positive linear combinations of the simple roots, respectively.

Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$. Let $\Delta$ be the simple roots and let $\Delta_P = \{\alpha \in \Delta : \mathfrak{g}^{-\alpha} \subset P\}$ be the subset of $P$-parabolic roots. Let $\mathcal{W}_P$ be the subgroup of $\mathcal{W}$ generated by simple reflections $s_\alpha$, for $\alpha \in \Delta_P$. Note that $\mathfrak{h}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{l}_P$. Denote by $S(\mathfrak{h})^{\mathcal{W}_P}$ the $\mathcal{W}_P$-invariants in $S(\mathfrak{h})$ with respect to the $\bullet$-action (here $w \bullet \lambda := w(\lambda + \rho) - \rho$, for $\lambda \in \mathfrak{h}^*$, $w \in \mathcal{W}$, $\rho$ is the half sum of the positive roots ). We have the Harish-Chandra homomorphism $S(\mathfrak{h})^{\mathcal{W}_P} \cong \mathcal{Z}(\mathfrak{l}_P)$ (as special cases $\mathcal{W} = \mathcal{W}_G$ and $S(\mathfrak{h})^{\mathcal{W}} \cong \mathcal{Z}(\mathfrak{g})$). Let $\lambda \in \mathfrak{h}^*$. Put $\Delta_\lambda = \{\alpha \in \Delta; \lambda(H_\alpha) = -1\}$ , where $H_\alpha \in \mathfrak{h}$ is the coroot corresponding to $\alpha$. Let $\chi_{\mathfrak{l}_P,\lambda} : \mathcal{Z}(\mathfrak{l}_P) \to \mathbb{C}$ be the character such that $\operatorname{Ker} \chi_{\mathfrak{l}_P,\lambda}$ annihilates the Verma module $M_\lambda$ with highest weight $\lambda$. Thus, $\chi_{\mathfrak{l}_P,\lambda} = \chi_{\mathfrak{l}_P,\mu} \iff \mu \in \mathcal{W}_P \bullet \lambda$. We write $\chi_\lambda = \chi_{\mathfrak{g},\lambda}$.

Let $\lambda \in \mathfrak{h}^*$. We say that

- $\lambda$ is *$P$-dominant* if $\lambda(H_\alpha) \notin \{-2, -3, -4, \ldots\}$, for $\alpha \in \Delta_P$; $\lambda$ is dominant if it is $G$-dominant.
- $\lambda$ is *$P$-regular* if $\Delta_\lambda \subseteq \Delta_P$. $\lambda$ is regular if it is $B$-regular, that is if $w \bullet \lambda = \lambda \implies w = e$, for $w \in \mathcal{W}$.
- $\lambda$ is a *$P$-character*[1] if it extends to a character of $P$; thus $\lambda$ is a $P$-character iff $\lambda$ is integral and $\lambda|_{\Delta_P} = 0$.

Suppose now that $\lambda \in \mathfrak{h}^*$ is integral and $P$-dominant. Then there is an irreducible finite dimensional $P$-representation $V_P(\lambda)$ with highest weight $\lambda$. Note that $V_{L_P}(\lambda) := V_P(\lambda)$ is an irreducible representation for $L_P$. Of course, $\dim V_P(\lambda) = 1 \iff \lambda$ is a $P$-character.

The following is well-known:

**Lemma 2.1.** *Let $\lambda \in \mathfrak{h}^*$. Then $\lambda$ is dominant iff for all $\mu \in \Lambda_{r+} \setminus \{0\}$ we have $\chi_{\lambda+\mu} \neq \chi_\lambda$*

We also have

**Lemma 2.2.** *Let $\lambda \in \mathfrak{h}^*$ be $P$-regular and dominant. Let $\mu$ be a $P$-character and let $V$ be the finite dimensional irreducible representation of $\mathfrak{g}$ with extremal weight $\mu$. Then for any weight $\psi$ of $V$, $\psi \neq \mu$, we have $\chi_{\lambda+\mu} \neq \chi_{\lambda+\psi}$.*

*Proof.* This is well-known for $P = B$. We reduce to that case as follows: Let $\mathfrak{g}'$ be the semi-simple Lie subalgebra of $\mathfrak{g}$ generated by $X_{\alpha\pm}$, $\alpha \in \Delta \setminus \Delta_P$. Let $\mathfrak{h}' := \mathfrak{g}' \cap \mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}'$. The inclusion $\mathfrak{h}' \hookrightarrow \mathfrak{h}$ gives the projection $p : \mathfrak{h}^* \to \mathfrak{h}'^*$. Consider the restriction $V|_{\mathfrak{g}'}$ of $V$ to $\mathfrak{g}'$ and let $V'$ denote the irreducible $\mathfrak{g}'$-module with highest weight $p(\mu)$; $V'$ is a direct summand in $V|_{\mathfrak{g}'}$. Let $\Lambda(V)$ denote the set of weights of $V$. Then $p(\Lambda(V)) = \Lambda'(V|_{\mathfrak{g}'})$, the weights of $V|_{\mathfrak{g}'}$. By the assumption that $\mu$ is a $P$-character, it follows that $p(\Lambda(V))$ is contained in the convex hull $\overline{\Lambda'(V')}$ of $\Lambda'(V')$. Since $p(\lambda)$ is regular and dominant it is well known that $p(\lambda) + p(\mu) \notin \mathcal{W}'(p(\lambda) + \phi')$ for $\phi' \in \Lambda(V')$. But then it follows that $p(\lambda) + p(\mu) \notin \mathcal{W}'(p(\lambda) + \phi')$ for $\phi' \in \overline{\Lambda(V')}$. Now $\mathcal{W}' = p(\mathcal{W})$, so it follows that $\lambda + \mu \notin \mathcal{W}(\lambda + \phi)$, for $\phi \in \Lambda(V)$. $\square$

---

[1] [BMR06] use the terminology $P$-weights for what we call $P$-characters.

2.3. **Twisted Harish-Chandra modules.** See [Dix77] for generalities on Harish-Chandra-modules. Let $\lambda \in \mathfrak{h}^*$ and consider a parabolic subgroup $P \subset G$. Let $\chi_{\mathfrak{l}_P,\lambda} : Z(\mathfrak{l}_P) \to \mathbb{C}$ be the corresponding map given by the Harish-Chandra homomorphism.

There is the category of $\chi_{\mathfrak{l}_P,\lambda}$-twisted Harish-Chandra $(\mathfrak{g}, P)$-modules, which we shall denote by $(\mathrm{U}(\mathfrak{g}), P, \chi_{\mathfrak{l}_P,\lambda})$-mod or by $(\mathfrak{g}, P, \chi_{\mathfrak{l}_P,\lambda})$-mod. An object $M$ of this category is a (say left) $\mathrm{U}(\mathfrak{g})$-module, denote by $\epsilon$ the action map $\mathrm{U}(\mathfrak{g}) \to End(M)$, equipped with an algebraic (say right) action of $P$, denoted $\tau : P \to Aut(M)$. We require that the actions $\tau$ and $\epsilon$ commutes, that $d\tau|\mathfrak{u}_P = \epsilon|\mathfrak{u}_P$ and that $(\epsilon(z) - \chi_{\mathfrak{l}_P,\lambda}(z))m = 0$, for $m \in M^{L_P}$, $z \in \mathcal{Z}(\mathfrak{l}_P)$. Here, $M^{L_P}$ denotes the subspace of $P$-invariants in $M$.

We also have the category of $\widehat{\chi_{\mathfrak{l}_P,\lambda}}$-twisted Harish-Chandra $(\mathfrak{g}, P)$-modules, denoted by $(\mathrm{U}(\mathfrak{g}), P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-mod. In this case the compatibility conditions read: An object $M$ of this category is equipped with actions $\epsilon$ and $\tau$ as before satisfying $d\tau|\mathfrak{u}_P = \epsilon|\mathfrak{u}_P$ and $\epsilon(z) - \chi_{\mathfrak{l}_P,\lambda}(z)$ is locally nilpotent on $M^{L_P}$, for $z \in \mathcal{Z}(\mathfrak{l}_P)$.

Note that in the case $P = B$ we have $\mathfrak{l}_B = \mathfrak{h}$ and $\chi_{\mathfrak{h},\lambda} = \lambda$; thus we denote the above categories by $(\mathrm{U}(\mathfrak{g}), B, \lambda)$-mod and $(\mathrm{U}(\mathfrak{g}), B, \widehat{\lambda})$-mod, respectively, in this case.

We remark that for any $P$-character $\mu$ we have canonical equivalences

$$(\mathrm{U}(\mathfrak{g}), P, \chi_{\mathfrak{l}_P,\lambda})\text{-mod} \cong (\mathrm{U}(\mathfrak{g}), P, \chi_{\mathfrak{l}_P,\lambda+\mu})\text{-mod } and$$

$$(\mathrm{U}(\mathfrak{g}), P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})\text{-mod} \cong (\mathrm{U}(\mathfrak{g}), P, \widehat{\chi_{\mathfrak{l}_P,\lambda+\mu}})\text{-mod}$$

realized by twisting (tensoring) the $P$-actions with the one-dimensional $P$-representation $V_P(\mu)$.

If $\nu \in \mathfrak{h}^*$ is another weight we similarly have the categories $(\mathrm{U}(\mathfrak{g})^\nu, P, \chi_{\mathfrak{l}_P,\lambda})$-mod and $(\mathrm{U}(\mathfrak{g}), P, \chi_{\mathfrak{l}_P,\lambda})\text{-mod}^{\widehat{\nu}}$ obtained by replacing left $\mathrm{U}(\mathfrak{g})$-module in the definition by left $\mathrm{U}(\mathfrak{g})$-module with central character, respectively, generalized central character, $\chi_\nu$. The same sorts of equivalences as above hold also for these categories.

In this article we shall encounter various sheaf-versions of Harish-Chandra modules. It seemed most convenient to define them as they naturally occur. Equivalences analogous to the above will apply to the sheaf-versions as well.

2.4. **Equivariant $\mathcal{O}$-modules and induction.** See [Jan83] for details on this material.

Let $L$ be a linear algebraic group and $K$ a closed algebraic subgroup. For $X$ an algebraic variety equipped with a right (or left) action of $L$ we denote by $(\mathcal{O}_X, L)$-mod the category of $L$-equivariant sheaves of (quasi-coherent) $\mathcal{O}_X$-modules. If the $L$-action is free and the quotient is nice we have the equivalence

$$\pi^{X/L}_{X*}(\ )^L : (\mathcal{O}_X, L)\text{-mod} \to \mathcal{O}_{X/L}\text{-mod} : \pi^{X/L*}_X.$$

Since $L$ is affine, we have Serre's equivalence $\mathcal{O}_L$-mod $\to \mathcal{O}(L)$-mod, $M \mapsto \Gamma_L(M)$, for $M \in \mathcal{O}_L$-mod.

We denote by $\Gamma_{(L,K)}$ the global section functor on $(\mathcal{O}_L, K)$-mod that corresponds to the global section functor $\Gamma_{L/K}$ on $\mathcal{O}_{L/K}$-mod under the equivalence $(\mathcal{O}_L, K)\text{-mod} \cong \mathcal{O}_{L/K}$-mod. Then $\Gamma_{(L,K)}(M) = \Gamma_L(M)^K$, for $M \in (\mathcal{O}_L, K)$-mod. (Note that $\mathcal{O}(L/K) = \mathcal{O}(L)^K$.)

Let $Rep(L)$ denote the category of algebraic representations of $L$. We have $\mathcal{O}(L) \in Rep(L)$, via $(gf)(x) = f(g^{-1}x)$, for $g, x \in L$ and $f \in \mathcal{O}(L)$. We shall also consider the left $K$-action on $\mathcal{O}(L)$ given by $(kf)(x) = f(xk)$, for $k \in K, x \in L$ and $f \in \mathcal{O}(L)$. These actions commute.

For $V \in Rep(K)$ we consider the diagonal left $K$-action on $\widetilde{V} := \mathcal{O}(L) \otimes V$. The left $L$-action on $\mathcal{O}(L)$ defines a left $L$-action on $\widetilde{V}$ that commutes with the $K$-action and the multiplication map $\mathcal{O}(L) \otimes \widetilde{V} \to \widetilde{V}$ is $L$- and $K$-linear. Thus $\widetilde{V}$ belongs to the category $(L, \mathcal{O}(L), K)$-mod of $L$-$K$ bi-equivaraint $\mathcal{O}(L)$-modules. This gives the functor

$$p^* : Rep(K) \to (L, \mathcal{O}(L), K)\text{-mod},\ V \mapsto \widetilde{V}$$

(induced bundle of a representation, $p$ symbolizes projection from $L$ to $pt/K$.)

Let $Ind_K^L V := \widetilde{V}^K \in Rep(L)$.

We have the factorization $Ind_K^L = (\ )^K \circ p^*$. One can show that $R(\ )^K \circ p^* \cong RInd_K^L$ where $R(\ )^K$ and $RInd_K^L$ are computed in suitable derived categories. An important formula is the *tensor identity*

$$RInd_K^L(V \otimes W) \cong RInd_K^L(V) \otimes W,\ for\ V \in Rep(K), W \in Rep(L) \tag{2.1}$$

(In particular $RInd_K^L(W) \cong W$, for $W \in Rep(L)$.)

## 3. Parabolic Springer Resolutions

In order to treat sheaves of extended differential operators on parabolic flag varieties in the next section we will here gather information about their associated graded objects. This is encoded in the geometry of parabolic Grothendieck-Springer resolutions.

### 3.1. **Parabolic Flag Varieties.**

Let $\mathcal{P} = G/P$ be the variety of all parabolics of type $P$; it is equipped with a natural left $G$-action. There is a bijection between representations of $P$ and $G$-equivariant vector bundles on $\mathcal{P}$; a representation $V$ of $P$ correspond to the induced bundle $G \times_P V$ on $\mathcal{P}$. We denote by $\mathcal{O}(V) := \mathcal{O}_{\mathcal{P}}(V)$ the corresponding locally free sheaf on $\mathcal{P}$ which hence has a left $G$-equivariant structure.

Let $\lambda \in \mathfrak{h}^*$ be a $P$-character and write $\mathcal{O}(\lambda) := \mathcal{O}(V_P(\lambda))$ for the line-bundle corresponding to the one-dimensional $P$-representation $V_P(\lambda)$. We have $Pic(\mathcal{P}) = Pic_G(\mathcal{P}) \cong$ group of $P$-characters, (but note that not all vector bundles on $\mathcal{P}$ are $G$-equivariant). The ample line bundles $\mathcal{O}(-\mu)$ are given by $P$-characters $\mu$ such that $\mu(H_\alpha) > 0$ for all $\alpha \in \Delta \setminus \Delta_P$.

Next we define the parabolic Grothendieck resolution:

**Definition 3.1.** $\widetilde{\mathfrak{g}}_{\mathcal{P}} = \{(P', x) : P' \in \mathcal{P}, x \in \mathfrak{g}^*, x|_{\mathfrak{u}_{P'}} = 0\}$

Note that $\widetilde{\mathfrak{g}}_{\mathcal{P}} = G \times_P (\mathfrak{g}/\mathfrak{u}_P)^*$. We have a commutative square:

$$\begin{tikzcd} \widetilde{\mathfrak{g}}_{\mathcal{P}} \arrow[r] \arrow[d] & \mathfrak{l}_P^*/L_P = \mathfrak{h}^*/\mathcal{W}_P \arrow[d] \\ \mathfrak{g}^* \arrow[r] & \mathfrak{h}^*/\mathcal{W} \end{tikzcd} \tag{3.1}$$

where the top map sends $(P', x)$ to $x|\mathfrak{l}_{P'}/L_{P'} \in \mathfrak{l}_{P'}^*/L_{P'} \cong \mathfrak{l}_P^*/L_P$. Note that the isomorphism $\mathfrak{l}_{P'}^*/L_{P'} \cong \mathfrak{l}_P^*/L_P$ is canonical. [2]

---

[2] We can call $\mathfrak{l}_P^*/L_P$ the universal coadjoint quotient of the Levi Lie subalgebra.

This induces a map:

$$\pi_{\mathcal{P}} : \widetilde{\mathfrak{g}}_{\mathcal{P}} \to \mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*/\mathcal{W}_P \tag{3.2}$$

**Lemma 3.2.** $R\pi_{\mathcal{P}*}\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{P}}} = \mathcal{O}_{\mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*/\mathcal{W}_P}$

*Proof.* We shall reduce to the well-known case of the ordinary Grothendieck resolution for $\mathcal{P} = \mathcal{B}$. It states that

$$R\pi_{\mathcal{B}*}\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}} = \mathcal{O}_{\mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*} \tag{3.3}$$

Translating this to the equivariant language it reads:

$$RInd_B^G(S(\mathfrak{g}/\mathfrak{n})) = S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h}) \tag{3.4}$$

where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. The reason for this is that since $\mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*$ is affine the equality 3.3 is after taking global sections equivalent to the equality $R\Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}}) = \mathcal{O}(\mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*) = S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h})$ of $G$-modules. Since, the bundle projection $p : \widetilde{\mathfrak{g}}_{\mathcal{B}} \to G/B$ with fiber $(\mathfrak{g}/n)^*$ is affine, $p_*$ is exact and hence $R\Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}}) = R\Gamma(p_*(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}}))$.

Now, under the identification $\mathcal{O}_(G/B)$-mod $= (\mathcal{O}_G, B)$-mod we have that $p_*(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}})$ corresponds to $S(\mathfrak{g}/\mathfrak{n}) \otimes \mathcal{O}(G)$ so its derived global sections are given by $RInd_B^G(S(\mathfrak{g}/\mathfrak{n}))$ as stated. This proves 3.4.

By a similar argument, the statement of the lemma is equivalent to proving that

$$RInd_P^G(S(\mathfrak{g}/\mathfrak{b}_P)) = S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h})^{\mathcal{W}_P} \tag{3.5}$$

We know that

$$S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h}) = RInd_B^G S(\mathfrak{g}/\mathfrak{n}) = RInd_P^G \circ RInd_B^P(S(\mathfrak{g}/\mathfrak{n})) \tag{3.6}$$

We have a decomposition $\mathfrak{g} = \overline{\mathfrak{u}}_P \oplus \mathfrak{l}_P \oplus \mathfrak{u}_P$, where $\overline{\mathfrak{u}}_P$ is the image of $\mathfrak{u}_P$ under the Chevalley involution of $\mathfrak{g}$; thus $\mathfrak{g}/\mathfrak{n} = \mathfrak{l}_P/(\mathfrak{l}_P \cap \mathfrak{n}) \oplus \overline{\mathfrak{u}}_P$. Thus

$$RInd_B^P(S(\mathfrak{g}/\mathfrak{n})) = R\operatorname{Ind}_B^P(S(\mathfrak{l}_P/\mathfrak{l}_P \cap \mathfrak{n}) \otimes S(\overline{\mathfrak{u}}_P)) = R\operatorname{Ind}_B^P(S(\mathfrak{l}_P/\mathfrak{l}_P \cap \mathfrak{n})) \otimes S(\overline{\mathfrak{u}}_P) \tag{3.7}$$

where the last equality is the projection formula for induction (see 2.1) which applies since $S(\overline{\mathfrak{u}}_P)$ is a $P$-module. We have

$$R\operatorname{Ind}_B^P(S(\mathfrak{l}_P/\mathfrak{l}_P \cap \mathfrak{n})) = R\operatorname{Ind}_{L_P \cap B}^{L_P}(S(\mathfrak{l}_P/\mathfrak{l}_P \cap \mathfrak{n})) \tag{3.8}$$

of $P$-modules where the right hand side becomes a $P$-module by transporting the $U_P$ action from the left hand side. By 3.4 applied to $G$ replaced by $L_P$ we get that 3.8 equals $S(\mathfrak{l}_P) \otimes_{S(\mathfrak{h})^{\mathcal{W}_P}} S(\mathfrak{h})$. Thus the right hand side of 3.7 equals $S(\mathfrak{g}/\mathfrak{u}_P) \otimes_{S(\mathfrak{h})^{\mathcal{W}_P}} S(\mathfrak{h})$. Thus by 3.6 we have

$$S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h}) = RInd_P^G(S(\mathfrak{g}/\mathfrak{u}_P) \otimes_{S(\mathfrak{h})^{\mathcal{W}_P}} S(\mathfrak{h})) = RInd_P^G(S(\mathfrak{g}/\mathfrak{u}_P)) \otimes_{S(\mathfrak{h})^{\mathcal{W}_P}} S(\mathfrak{h})$$

Since $S(\mathfrak{h})$ is faithfully flat over $S(\mathfrak{h})^{\mathcal{W}_P}$ this implies 3.5. $\square$

Let $P \subset Q$ be two parabolic subgroups. The projection $\pi_{\mathcal{P}}^{\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ induces a map $\widetilde{\pi}_{\mathcal{P}}^{\mathcal{Q}} : \widetilde{\mathfrak{g}}_{\mathcal{P}} \to \widetilde{\mathfrak{g}}_{\mathcal{Q}}$ that fits into the following commutative square:

$$\begin{tikzcd}
\widetilde{\mathfrak{g}}_{\mathcal{P}} \arrow[r] \arrow[d, "\widetilde{\pi}^{\mathcal{Q}}_{\mathcal{P}}"] & \mathfrak{l}_P^*/L_P = \mathfrak{h}^*/\mathcal{W}_P \arrow[d] \\
\widetilde{\mathfrak{g}}_{\mathcal{Q}} \arrow[r] & \mathfrak{l}_Q^*/L_Q = \mathfrak{h}^*/\mathcal{W}_Q
\end{tikzcd} \tag{3.9}$$

With similar arguments as in the proof of lemma 3.2 one can prove

**Lemma 3.3.** $R\widetilde{\pi}^{\mathcal{Q}}_{\mathcal{P}*}\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{P}}} = \mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{Q}} \times_{\mathfrak{h}^*/\mathcal{W}_Q} \mathfrak{h}^*/\mathcal{W}_P}$

We observe that $\widetilde{\mathfrak{g}}_{\mathcal{P}}$ is an $L_P$-torsor over $T^*\mathcal{P}$. We put

**Definition 3.4.** $\widetilde{\mathfrak{g}}^{\lambda}_{\mathcal{P}} = \widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{h}^*/\mathcal{W}_P} \lambda$, for $\lambda \in \mathfrak{h}^*$.

We would like to view $\widetilde{\mathfrak{g}}^{\lambda}_{\mathcal{P}}$ as the classical Hamiltonian of $T^*(G/U_P)$ with respect to the (right) $L_P$-action. We have a moment map $\mu : T^*(G/U_P) \to \mathfrak{l}_P^*$. Recall that we can take the Hamiltonian reduction with respect to any subset of $\mathfrak{l}_P^*$ stable under the coadjoint action. Let $\mathcal{N}_\lambda \subset \mathfrak{l}_P^*$ be the preimage of $\lambda/\mathcal{W}_P \in \mathfrak{h}^*/\mathcal{W}_P \cong \mathfrak{l}_P^*/L_P$ under the quotient map. Then

$$T^*(G/U_P)//_{\mathcal{N}_\lambda} L_P = \mu^{-1}(\mathcal{N}_\lambda)/L_P = \widetilde{\mathfrak{g}}^{\lambda}_{\mathcal{P}}. \tag{3.10}$$

Note that we could also reduce with respect to $\lambda \in (\mathfrak{l}_P^*)^{L_P}$ in which case we would get twisted cotangent bundles.

## 4. Extended differential operators on $\mathcal{P}$

In this section we construct the sheaf of extended differential operators on a parabolic flag variety and describe its global sections.

4.1. **Torsors.** Let $X$ be an algebraic variety equipped with a free right action of a linear algebraic group $L$ and let $p : X \to X/L$ be the projection. We assume that $X$, locally in the Zariski topology, is of the form $Y \times L$, for some variety $Y$, and $p$ is first projection. Such $X$ is called an $L$-torsor. We get induced right $L$-actions on the sheaf $\mathcal{D}_X$ of regular differential operators on $X$ and on the direct image sheaf $p_*(\mathcal{D}_X)$. Denote by $\widetilde{\mathcal{D}}_{X/L} := p_*(\mathcal{D}_X)^L$ the sheaf on $X/L$ of $L$-invariant local sections of $p_*(\mathcal{D}_X)$.

Let $\mathfrak{l} = \mathit{Lie}\, L$. The infinitesimal $L$-action gives an algebra map $\tilde{\epsilon} : U(\mathfrak{l}) \to p_*\mathcal{D}_X$, which is injective since the $L$-action is free. It follows from the definition of differentiating a group action that $[\tilde{\epsilon}(U(\mathfrak{l})), \widetilde{\mathcal{D}}_{X/L}] = 0$.

Notice that $\tilde{\epsilon}(\mathrm{U}(\mathfrak{l})) \nsubseteq \widetilde{\mathcal{D}}_{X/L}$, unless $L$ is abelian, but $\tilde{\epsilon}(\mathcal{Z}(\mathfrak{l})) \subseteq \widetilde{\mathcal{D}}_{X/L}$. We denote by $\epsilon : \mathcal{Z}(\mathfrak{l}) \to \widetilde{\mathcal{D}}_{X/L}$ the restriction of $\tilde{\epsilon}$ to $\mathcal{Z}(\mathfrak{l})$. By the discussion above it is a central embedding.

Now, using that $p$ is locally trivial we can give a local description of $\widetilde{\mathcal{D}}_{X/L}$. Let $Y \times L$ be a Zariski open subset of $X$ over which $p$ is trivial. Then $\mathcal{D}_X|_{Y\times L} = \mathcal{D}_Y \otimes \mathcal{D}_L$ and $\widetilde{\mathcal{D}}_{X/L}|_Y = \mathcal{D}_Y \otimes \mathrm{U}(\mathfrak{l})$, where $\mathrm{U}(\mathfrak{l})$ is identified with the algebra of right $L$-invariant differential operators $\mathcal{D}_L^L$ on $L$.

Note that $\tilde{\epsilon}(\mathrm{U}(\mathfrak{l}))|_{Y\times L} = 1 \otimes {}^{L}\mathcal{D}_L$ is the algebra of left $L$-invariant differential operators on $Y \times L$, with respect to the natural left $L$-action on $Y \times L$, that are constant along $Y$. Since $\mathcal{Z}({}^{L}\mathcal{D}_L) = \mathcal{Z}(\mathcal{D}_L^L)$ we get that $\epsilon$ is locally given by the embedding

$$\mathcal{Z}(\mathfrak{l}) \hookrightarrow \mathrm{U}(\mathfrak{l}) \cong 1 \otimes \mathrm{U}(\mathfrak{l}) \hookrightarrow \mathcal{D}_Y \otimes \mathrm{U}(\mathfrak{l}).$$

This implies that $\epsilon(\mathcal{Z}(\mathfrak{l})) = \mathcal{Z}(\widetilde{\mathcal{D}}_{X/L})$.

Denote by $(\mathcal{D}_X, L)$-mod the category of weakly equivariant $(\mathcal{D}_X, L)$-modules. In order to simplify the description of this category we assume henceforth that $X$ is quasi-affine. Its object $M$ is then a left $\mathcal{D}_X$-module equipped with an algebraic right action $\rho = \{\rho_U\}$, where $\rho_U : L \to \mathrm{Aut}_{\mathbb{C}_U}(M(U))^{\mathrm{op}}$ are homomorphism compatible with the restriction maps in $M$, for each Zariski-open $L$-invariant subset $U$ of $X$. We require that $\mathcal{D}_X \otimes M \to M$ is $L$-linear (over $L$-invariant open sets) with respect to the diagonal $L$-action on a tensor. (For a general $X$, $\rho$ would have to be replaced by a given isomorphism $pr^*M \cong act^*M$ satisfying a cocycle condition, where $pr$ and $act : X \times L \to X$ are projection and the action map, respectively.)

Denote by $(\mathcal{D}_X, L, \mathfrak{l})$-mod the category of strongly equivariant $(\mathcal{D}_L, L)$-modules. Its object $(M, \rho)$ is a weakly equivariant $(\mathcal{D}_X, L)$-module such that $d\rho(x)m = \epsilon(x)m$ for $x \in \mathfrak{l}$ and $m \in M$.

For $M \in (\mathcal{D}_X, L)$-mod we consider the sheaf $(p_*M)^L$ of $L$-invariant local sections in $p_*M$; it has a natural $\widetilde{\mathcal{D}}_{X/L}$-module structure. Thus we get a functor $p_*$ whose right adjoint is $p^*$ (the pullback in the category of $\mathcal{O}$-modules with its natural equivariant structure). The following is standard (see [BB93]):

**Lemma 4.1.**

$$i)\ p_*(\ )^L : (\mathcal{D}_X, L)\textit{-mod} \leftrightarrows \widetilde{\mathcal{D}}_{X/L}\textit{-mod} : p^* \ \textit{and}$$

$$ii)\ p_*(\ )^L : (\mathcal{D}_X, L, \mathfrak{l})\textit{-mod} \leftrightarrows \mathcal{D}_{X/L}\textit{-mod} : p^*$$

*are mutually inverse equivalences of categories.*

4.2. **Definition of extended differential operators.** On $G/U_P$ we shall always consider the right $L_P$-action $(\overline{g}, h) \mapsto \overline{gh}$, for $g \in G$ and $h \in L_P$. Thus, $G/U_P$ is an $L_P$-torsor. We put

**Definition 4.2.** $\widetilde{\mathcal{D}}_{\mathcal{P}} = \pi^{\mathcal{P}}_{G/U_P*}(\mathcal{D}_{G/U_P})^{L_P}$.

By the results of the previous section we have that locally on $\mathcal{P}$, $\widetilde{\mathcal{D}}_{\mathcal{P}} \cong \mathcal{D}_{G/P} \otimes \mathrm{U}(\mathfrak{l}_P)$, and we have the algebra homomorphisms $\epsilon : \mathcal{Z}(\mathfrak{l}_P) \to \widetilde{\mathcal{D}}_{\mathcal{P}}$.

For $\lambda \in \mathfrak{h}^*$ we define:

**Definition 4.3.** $\mathcal{D}^{\lambda}_{\mathcal{P}} = \widetilde{\mathcal{D}}_{\mathcal{P}} \otimes_{\epsilon(\mathcal{Z}(\mathfrak{l}_P))} \mathbb{C}_{\lambda}$.

4.3. **Equivariant description.** The categories $\widetilde{\mathcal{D}}_{\mathcal{P}}$-mod, $\mathcal{D}^{\lambda}_{\mathcal{P}}$-mod and $\widetilde{\mathcal{D}}_{\mathcal{P}}$-mod$^{\widehat{\lambda}}$ can be described equivariantly on $G$ and on $G/U_P$. It is better to work on $G$. We start with $G/U_P$ as an intermediate step.

By lemma 4.1 we have mutually inverse equivalences

$$\pi^{\mathcal{P}}_{G/U_P*}(\ )^{L_P} : (\mathcal{D}_{G/U_P}, L_P)\text{-mod} \leftrightarrows \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod} : \pi^{\mathcal{P}*}_{G/U_P} \tag{4.1}$$

Transporting conditions from the right-hand side to the left-hand side we see that $\mathcal{D}^{\lambda}_{\mathcal{P}}$-mod is equivalent to the full subcategory $(\mathcal{D}_{G/U_P}, L_P, \chi_{\mathfrak{l}_P,\lambda})$-mod of $(\mathcal{D}_{G/U_P}, L_P)$-mod whose object $M$ satisfy $\mathrm{Ker}\,\epsilon(\mathcal{Z}(\mathfrak{l}_P)) \cdot M^{L_P} = 0$. Similarly, $\widetilde{\mathcal{D}}_{\mathcal{P}}$-mod$^{\widehat{\lambda}}$ is equivalent to the full subcategory

$(\mathcal{D}_{G/U_P}, L_P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-mod of $(\mathcal{D}_{G/U_P}, L_P)$-mod whose object $M$ satisfies that $\operatorname{Ker}\epsilon(\mathcal{Z}(\mathfrak{l}_P))$ is locally nilpotent on $M^{L_P}$.

Now we pass to $G$. Let us introduce some notations:

Denote by $\mu_l$ and $\mu_r$ the actions of left and right multiplication of $G$ on itself, respectively. The infinitesimal actions of $\mu_l$ and $\mu_r$ give algebra embeddings $\epsilon_l$ and $\epsilon_r : \mathrm{U}(\mathfrak{g}) \to \mathcal{D}_G$. We have that $\epsilon_l(\mathrm{U}(\mathfrak{g})) = \mathcal{D}_G^G$ consists of *right* invariant differential operators on $G$ and $\epsilon_r(\mathrm{U}(\mathfrak{g})) = {}^G\mathcal{D}_G$ consists of *left* invariant differential operators on $G$, $\mathcal{Z}(\mathfrak{g}) = \epsilon_l(\mathrm{U}(\mathfrak{g})) \cap \epsilon_r(\mathrm{U}(\mathfrak{g}))$ and $\epsilon_l|_{\mathcal{Z}(\mathfrak{g})} = \epsilon_r|_{\mathcal{Z}(\mathfrak{g})}$.

The actions $\mu_l$ and $\mu_r$ induce left and right actions of $G$ on $\mathcal{D}_G$ denoted by the same symbols, respectively.

Let $(\mathcal{D}_G, P, \mathfrak{u}_P)$-mod be the category whose object $M$ satisfies

1. $M$ is a left $\mathcal{D}_G$-module
2. $M$ has a right algebraic $P$-action $\rho$ such that $\mathcal{D}_G \otimes M \to M$ is $P$-linear, with respect to the right $P$-action $\mu_r|_P$ on $\mathcal{D}_G$ and the diagonal $P$-action on a tensor.
3. $d\rho|_{\mathfrak{u}_P} = \epsilon_r|_{\mathfrak{u}_P}$ on $M$.

By 4.1 and lemma 4.1 *ii)* (applied to $X = G$ and $L = U_P$) we have an equivalence

$$\pi^{\mathcal{P}}_{G*}(\ )^P : (\mathcal{D}_G, P, \mathfrak{u}_P)\text{-mod} \leftrightarrows \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod} : \pi^{\mathcal{P}*}_G \tag{4.2}$$

Let $\lambda \in \mathfrak{h}^*$. Let $(\mathcal{D}_G, P, \mathfrak{u}_P, \chi_{\mathfrak{l}_P,\lambda})$-mod be the full subcategory of $(\mathcal{D}_G, P, \mathfrak{u}_P)$-mod whose object $M$ in addition to $(1) - (3)$ satisfies

(4) $\epsilon_r(z) - \chi_\lambda^{L_P}(z) = 0$ on $\Gamma_G(M)^{L_P}$, for $z \in \mathcal{Z}(\mathfrak{l}_P)$,

where we remind that $\Gamma_G(M)$ denotes the $\mathcal{O}(G)$-module corresponding to $M \in \mathcal{O}_G$-mod. In condition (4) we could have replaced $\Gamma_G(M)^{L_P}$ by $M^{L_P}$, which we remind is a sheaf on the set $G$ with the topology of $L_P$-invariant Zariski-open subsets of $G$. Lemma 4.4 *ii*) would then tautologically hold. Normally we don't bather to distinguish between $M$ and $\Gamma_G(M)$, but here we liked to emphasize that it is enough to consider *global* $L_P$-invariants, because in future research on singular localization of quantum groups we will want to altogether avoid sheaves of local invariants.

**Lemma 4.4.** *i) Let $M \in (\mathcal{O}_G, L_P)$-mod. Then $M = \mathcal{O}_G \cdot \Gamma_G(M)^{L_P}$. ii) There is an equivalence $(\mathcal{D}_G, P, \mathfrak{u}_P, \chi_{\mathfrak{l}_P,\lambda})$-mod $\cong \mathcal{D}^\lambda_{\mathcal{P}}$-mod induced from the equivalence 4.2.*

*Proof. i)* Since $G$ and its closed subgroup $L_P$ are reductive $G/L_P$ is an affine variety (see [Mat60]). Let $p : G \to G/L_P$ be the projection and $M \in (\mathcal{O}_G, L_P)$-mod. Then we have, since $\pi^{G/L_P}_{G*}(M)^{L_P} \in \mathcal{O}_{G/L_P}$-mod and $G/L_P$ is affine, that

$$\pi^{G/L_P}_{G*}(M)^{L_P} = \mathcal{O}_{G/L_P} \cdot \Gamma_{G/L_P}(\pi^{G/L_P}_{G*}(M)^{L_P}) \tag{4.3}$$

Note that $\Gamma_G(M)^{L_P} = \Gamma_{G/L_P}(\pi^{G/L_P}_{G*}(M)^{L_P})$. Thus, on $G$, 4.3 reads that

$$M^{L_P} = (\mathcal{O}_G)^{L_P} \cdot \Gamma_G(M)^{L_P}$$

holds over any $L_P$-invariant open subset of $G$. Since $G$ locally is of the form $Z \times L_P$ and $M$ is $L_P$-equivariant, we trivially have $M = \mathcal{O}_G \cdot M^{L_P}$. Thus, $M = \mathcal{O}_G \cdot \Gamma_G(M)^{L_P}$.

*ii*) We have the embeddings $\epsilon : \mathcal{Z}(\mathfrak{l}_P) \to \widetilde{\mathcal{D}}_{\mathcal{P}}$ and $\epsilon_r|_{\mathcal{Z}(\mathfrak{l}_p)} : \mathcal{Z}(\mathfrak{l}_P) \to \mathcal{D}_G$. We have $\widetilde{\mathcal{D}}_{\mathcal{P}} = \pi^{\mathcal{P}}_{G*}(\mathcal{D}_G)^{L_P}/J$ where $J$ is the ideal generated by $d\rho(x) - \epsilon_r(x)$, for $x \in \mathfrak{u}_P$. Note that the map $\mathcal{Z}(\mathfrak{l}_P) \xrightarrow{\epsilon} \widetilde{\mathcal{D}}_{\mathcal{P}}$ coincides with the composition $\mathcal{Z}(\mathfrak{l}_P) \xrightarrow{\epsilon_r} \pi^{\mathcal{P}}_{G*}(\mathcal{D}_G)^{L_P} \to \pi^{\mathcal{P}}_{G*}(\mathcal{D}_G)^{L_P}/J = \widetilde{\mathcal{D}}_{\mathcal{P}}$.

Let $M \in (\mathcal{D}_G, P, \mathfrak{u}_P)$-mod. Then we have

$$\pi^{\mathcal{P}}_{G*}(M)^{L_P} \in \mathcal{D}^{\lambda}_{\mathcal{P}}\text{-mod} \iff \text{local sections of } \pi^{\mathcal{P}}_{G*}(M)^{L_P} \text{ are annihilated by } \epsilon(\operatorname{Ker}\chi_{\mathfrak{l}_P,\lambda})$$

$$\iff \text{local sections of } M^{L_P} \text{ are annihilated by } \epsilon_r(\operatorname{Ker}\chi_{\mathfrak{l}_P,\lambda}).$$

Since $M$ has an underlying object in $(\mathcal{O}_G, L_P)$-mod we have by *i)* that $M = \mathcal{O}_G \cdot \Gamma_G(M)^{L_P}$, so that $M^{L_P} = \mathcal{O}_G^{L_P} \cdot \Gamma_G(M)^{L_P}$. Since, for $z \in \epsilon_r(\mathcal{Z}(\mathfrak{l}_P))$, $v \in M^{L_P}$ and $f \in \mathcal{O}_G^{L_P}$, we have $zfm = fzm$, it follows that the last condition is equivalent to

$$\Gamma_G(M)^{L_P} \text{ is annihilated by } \epsilon_r(\operatorname{Ker}\chi_{\mathfrak{l}_P,\lambda})$$

and this is exactly the condition of (4). □

Similarly, there is the category $(\mathcal{D}_G, P, \mathfrak{u}_P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-mod that is equivalent to $\widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}}$. An object $M$ of this category satisfies $(1) - (3)$ above and in addition

$$(\widehat{4}) \qquad \epsilon_r(z) - \chi^{L_P}_{\lambda}(z) \text{ is locally niloptent on } \Gamma_G(M)^{L_P}, \text{ for } z \in \mathcal{Z}(\mathfrak{l}_P).$$

We have omitted the proof of this fact which is essentially the same as that of lemma 4.4.

*Remark* 4.5. When $L_P = T$ (i.e., when $P = B$), condition (4) can be written as

$$(\epsilon_r(z) - \chi_\lambda(z))m = d\rho(z)m, \text{ for } m \in M, z \in \mathfrak{h} \tag{4.4}$$

This is so, because by (4), 4.4 holds for $m \in M^{L_P}$ and $z \in \mathfrak{h}$. But then it follows from Leibniz's rule that 4.4 holds for all $m$ of the form $m = fm'$, for $f \in \mathcal{O}_G$ and $m' \in M^{L_P}$, i.e., it holds for all $m \in M$, $z \in \mathfrak{h}$. Traditionally, this is how such equivariance conditions are written down (see [BB93]).

For $P \neq B$, (4) can not be written in the form 4.4. To understand this, note that (4) merely gives that $(\epsilon_r(z) - \chi_\lambda(z))m = d\rho(z)m$ $(= 0)$ for $z \in \mathcal{Z}(\mathfrak{l}_P)$ and $m \in M^{L_P}$. Since $\mathcal{Z}(\mathfrak{l}_P)$ is not generated by a Lie subalgebra of $\mathfrak{g}$, we can not apply Leibniz's rule to extend the equality of the actions $\epsilon_r(z) - \chi_\lambda(z)$ and $d\rho(z)$ to the whole of $M$. Actually, for $P = G$ the actions coincide only on $M^{L_P}$. There is of course a general relation between these actions on the whole of $M$, but it is difficult to give an explicit formula for it. Since $G/L_P$ is affine we might as well work with $M^{L_P}$ (also in the quantum case).

Similar remarks apply to $(\widehat{4})$.

**Example 4.6.** For the reader's convenience (and later use) let us analyze the simplest case when $P = G$. Since $\mathfrak{u}_G = 0$ we write $(\mathcal{D}_G, G, \chi_\lambda)\text{-mod} := (\mathcal{D}_G, G, \mathfrak{u}_G, \chi_\lambda)$-mod.

Then, from the equivalence $\mathbb{C}\text{-mod} \cong (\mathcal{O}_G, G)$-mod, $V \mapsto \mathcal{O}_G \otimes V$, it follows "by hand" that for any $\lambda \in \mathfrak{h}^*$ there is the equivalence $\mathrm{U}(\mathfrak{g})^{\lambda}\text{-mod} \cong (\mathcal{D}_G, G, \chi_\lambda)$-mod given by

$$V \mapsto \mathcal{O}_G \otimes V$$

We get $(\mathcal{O}_G \otimes V)^G = V$ and $V$ is a left module for $\epsilon_l(\mathrm{U}(\mathfrak{g}))$ with central character $\chi_\lambda$.

This by hand-description is the same as the conditions $(1) - (4)$. In fact, that $V$ as a left module for $\epsilon_l(\mathrm{U}(\mathfrak{g}))$ has central character $\chi_\lambda$ arises from (4) as follows: $V$ is a left module for $\epsilon_r(\mathcal{Z}(\mathfrak{g}))$ and this $\mathcal{Z}(\mathfrak{g})$-action differs by $\chi_\lambda$ from the $\mathcal{Z}(\mathfrak{g})$-action on $V$ that is obtained from differentiating the (trivial) $G$-action on $V$ and restrict it to the center of $\mathrm{U}(\mathfrak{g})$. Moreover, $\epsilon_r(\mathcal{Z}(\mathfrak{g})) = \epsilon_l(\mathcal{Z}(\mathfrak{g}))$.

A similar description holds for $\chi_\lambda$ replaced by $\widehat{\chi_\lambda}$.

4.4. **Global sections.** Notice that the left $G$-action on $G/U_P$, $(g,\overline{g'})\mapsto\overline{gg'}$, commutes with the right $L_P$-action and therefor induces a homomorphism $\mathrm{U}(\mathfrak{g})\to\widetilde{\mathcal{D}}_{\mathcal{P}}$, that commute with the map $\epsilon:\mathcal{Z}(\mathfrak{l}_P)\to\widetilde{\mathcal{D}}_{\mathcal{P}}$. We also get a homomorphism $\mathrm{U}(\mathfrak{g})\to\mathcal{D}^{\lambda}_{\mathcal{P}}$.

Consider the sheaf of algebras $\mathcal{O}_{\mathcal{P}}\otimes\mathrm{U}(\mathfrak{g})$ on $\mathcal{P}$ with multiplication determined by those in $\mathcal{O}_{\mathcal{P}}$ and in $\mathrm{U}(\mathfrak{g})$ and by the requirement that $[A,f]=\epsilon(A)(f)$ for $A\in\mathfrak{g}$ and $f\in\mathcal{O}_{\mathcal{P}}$. Then we have a surjective algebra homomorphism $\eta:\mathcal{O}_{\mathcal{P}}\otimes\mathrm{U}(\mathfrak{g})\to\widetilde{\mathcal{D}}_{\mathcal{P}}$. Its kernel is the ideal generated by $\xi\in\mathcal{O}_{\mathcal{P}}\otimes\mathfrak{u}_P$, $\xi(x)\in\mathfrak{p}_x$, for $x\in\mathcal{P}$ and $\mathfrak{p}_x\subseteq\mathfrak{g}$ the corresponding parabolic subalgebra.

Hence, to define a $\widetilde{\mathcal{D}}_{\mathcal{P}}$-module structure on an $\mathcal{O}_{\mathcal{P}}$-module $M$ is the same thing as defining a $\mathrm{U}(\mathfrak{g})$-module structure on $M$ such that $\operatorname{Ker}\eta$ vanishes on $M$ and $A(fm)=f(Am)+\epsilon(A)(f)m$, for $A\in\mathfrak{g}$, $f\in\mathcal{O}_{\mathcal{P}}$ and $m\in M$.

Let $\mu\in\mathfrak{h}^*$ be integral and $P$-dominant. Recall that $V_P(\mu)$ denotes the corresponding irreducible representation of $P$ with highest weight $\mu$ and $\mathcal{O}(V_P(\mu))$ the corresponding left $G$-equivariant locally free sheaf on $\mathcal{P}$.

Let $M\in\widetilde{\mathcal{D}}_{\mathcal{P}}$-mod. We shall show that the $\mathcal{O}_{\mathcal{P}}$-module $M\otimes_{\mathcal{O}_{\mathcal{P}}}\mathcal{O}(V_P(\mu))$ is naturally a $\widetilde{\mathcal{D}}_{\mathcal{P}}$-module. We proceed as follows:

The $G$-action on $\mathcal{O}(V_P(\mu))$ differentiates to a left $\mathfrak{g}$-action on it, which extends to a $\mathfrak{g}$-action on $M\otimes_{\mathcal{O}_{\mathcal{P}}}\mathcal{O}(V_P(\mu))$ by Leibniz's rule. Since $V_P(\mu)$ is an irreducible $P$-module we have that $U_P$ acts trivially on it (recall $V_P(\mu)=V_{L_P}(\mu)$). Hence, $\mathfrak{u}_P$ acts trivially $\mathcal{O}(V_P(\mu))$ and from this it now follows that the compatibilities for being a $\widetilde{\mathcal{D}}_{\mathcal{P}}$-module are satisfied by $M\otimes_{\mathcal{O}_{\mathcal{P}}}\mathcal{O}(V_P(\mu))$.

Assume that $M\in\widetilde{\mathcal{D}}_{\mathcal{P}}$-mod. In the equivariant language on $G$ we see that $M$ and $M\otimes_{\mathcal{O}_{\mathcal{P}}}\mathcal{O}(V_P(\mu))$ correspond to $\pi^{\mathcal{P}*}_G M$ and $M_{V_P(\mu)}:=\pi^{\mathcal{P}*}_G M\otimes V_P(\mu)\in(\mathcal{D}_G,L_P,\mathfrak{u}_P)$-mod, respectively. Here, the left $\mathfrak{g}$-action and the left $\mathcal{O}_G$-action on $M_{V_P(\mu)}$ are given by the actions on the first tensor. Again, it is the fact that $U_P$ acts trivially on $V_P(\mu)$ that shows that $M_{V_P(\mu)}$ is an object of $(\mathcal{D}_G,L_P,\mathfrak{u}_P)$-mod.

**Lemma 4.7.** *Let $\lambda\in\mathfrak{h}^*$, $M\in\mathcal{D}^{\lambda}_{\mathcal{P}}$-mod and $\mu\in\mathfrak{h}^*$ be integral and $P$-dominant. Then $M\otimes_{\mathcal{O}_{\mathcal{P}}}\mathcal{O}(V_P(\mu))\in\oplus_{\nu\in\Lambda(V_P(\mu))}\widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda+\nu}}$, where $\Lambda(V_P(\mu))$ denotes the set of weights of $V_P(\mu)$.*

*Proof.* In equivariant translation we have $\pi^{\mathcal{P}*}_G M\in(\mathcal{D}_G,P,\mathfrak{u}_P,\chi_{\mathfrak{l}_P,\lambda})$-mod and want to prove that

$$M_{V_P(\mu)}:=\pi^{\mathcal{P}*}_G M\otimes V_P(\mu)\in\oplus_{\nu\in\Lambda(V_P(\mu))}(\mathcal{D}_G,P,\mathfrak{u}_P,\widehat{\lambda+\nu})\text{-mod}.$$

Consider the forgetful functor

$$for:(\mathcal{D}_G,P,\mathfrak{u}_P,\chi_{\mathfrak{l}_P,\lambda})\text{-mod}\longrightarrow(\mathcal{D}_{L_P},L_P,\chi_{\mathfrak{l}_P,\lambda})\text{-mod}$$

By example 4.6 with $G$ replaced by $L_P$ we find a a left $\epsilon_l(\mathrm{U}(\mathfrak{l}_P))$-module $W$ with central character $\chi_{\mathfrak{l}_P,\lambda}$ equipped with a trivial $L_P$-action such that $\pi^{\mathcal{P}*}_G M=\mathcal{O}_{L_P}\otimes W$. We get

$$for(M_{V_P(\mu)})=for(\pi^{\mathcal{P}*}_G M\otimes V_P(\mu))=\mathcal{O}_{L_P}\otimes W\otimes V_P(\mu)$$

By the Peter-Weyl theorem $\mathcal{O}_{L_P}=\oplus_{\phi}V_P(\phi)\otimes V^*_P(\phi)$ as an $L_P$-bimodule, where $V^*_P(\phi)$ is the dual representation of $V_P(\phi)$ and $\phi$ runs over all integral $P$-dominant weights. Thus we have the following equalities of $\epsilon_r(\mathrm{U}(\mathfrak{l}_P))$-modules:

$$(M_{V_P(\mu)})^{L_P}=(\mathcal{O}_{L_P}\otimes W\otimes V_P(\mu))^{L_P}=(\mathcal{O}_{L_P}\otimes V_P(\mu))^{L_P}\otimes W=$$

$$\oplus_{\phi}V_P(\phi)\otimes(V^*_P(\phi)\otimes V_P(\mu))^{L_P}\otimes W=V_P(\mu)\otimes W$$

where the last equality holds since $(V_P^*(\phi)\otimes V_P(\mu))^{L_P}$ is isomorphic to the trivial representation of $L_P$, if $\mu=\phi$, and 0 else, by Schur's lemma. It is known, see [BerGel81], that $V_P(\mu)\otimes W$ is a direct sum of $\epsilon_r(\mathrm{U}(\mathfrak{l}_P))$-submodules with generalized central characters $\widehat{\lambda+\nu}$, for $\nu\in\Lambda(V_P(\mu))$. Such a decomposition will give the prescribed decomposition of $M_{V_P(\mu)}$. ☐

Let $\mu$ be a $P$-character. Then $\mathcal{O}(-\mu)=\mathcal{O}(V_P(\mu))$ is a line-bundle. We set $M(-\mu):=M\otimes_{\mathcal{O}}\mathcal{O}(V_P(\mu))\in\mathcal{D}_{\mathcal{P}}^{\lambda+\mu}$-mod. We have

**Theorem 4.1.** *i)* $R\pi^{\mathcal{P}}_{\mathcal{B}*}\widetilde{\mathcal{D}}_{\mathcal{B}}=\widetilde{\mathcal{D}}_{\mathcal{P}}\otimes_{\mathcal{Z}(\mathfrak{l}_{\mathcal{P}})}S(\mathfrak{h})$, *ii)* $R\pi^{\mathcal{Q}}_{\mathcal{P}*}\widetilde{\mathcal{D}}_{\mathcal{P}}=\widetilde{\mathcal{D}}_{\mathcal{Q}}\otimes_{\mathcal{Z}(\mathfrak{l}_{\mathcal{Q}})}S(\mathfrak{h})^{\mathcal{W}_P}$, *iii)* $R\Gamma(\widetilde{\mathcal{D}}_{\mathcal{P}})=U(\mathfrak{g})\otimes_{\mathcal{Z}(\mathfrak{g})}S(\mathfrak{h})^{\mathcal{W}_P}$ *and iv)* $R\Gamma(\mathcal{D}^\lambda_{\mathcal{P}})=\mathrm{U}(\mathfrak{g})^\lambda$ .

*Proof.* By lemma 3.2 and lemma 3.3 the associated graded maps *i)* and *ii)* are isomorphisms; hence *i)* and *ii)* are also isomorphisms. *iii)* and *iv)* follows. ☐

In both cases the global section functor $\Gamma:\mathcal{D}^\lambda_{\mathcal{P}}\text{-mod}\to\mathrm{U}(\mathfrak{g})^\lambda\text{-mod}$ and $\Gamma:\widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}}\to\mathcal{U}(\mathfrak{g})\text{-mod}^{\widehat{\lambda}}$, respectively, has a left adjoint denoted by $\mathcal{L}$, which we call the localization functor. In the first case it is given by

$$\mathcal{L}=\mathcal{D}^\lambda_{\mathcal{P}}\otimes_{\mathrm{U}(\mathfrak{g})^\lambda}(\ ):\mathrm{U}(\mathfrak{g})^\lambda\text{-mod}\to\mathcal{D}^\lambda_{\mathcal{P}}\text{-mod}$$

and in the second case it is given by

$$\mathcal{L}=\varprojlim_n\mathcal{D}_{\mathcal{P}}/(\operatorname{Ker}\chi_\lambda)^n\otimes_{\mathrm{U}(\mathfrak{g})}(\ ):\mathcal{U}(\mathfrak{g})\text{-mod}^{\widehat{\lambda}}\to\widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}}.$$

## 5. Singular Localization

Here we prove the singular version of Beilinson-Bernstein localization.

**Theorem 5.1.** *Let $\lambda$ be dominant and $P$-regular then $\Gamma:\mathcal{D}^\lambda_{\mathcal{P}}$-mod $\to\mathrm{U}(\mathfrak{g})^\lambda$-mod is an equivalence of categories.*

*Proof.* Essentially taken from [BB81]. Since $\Gamma(\mathcal{D}^\lambda_{\mathcal{P}})=\mathrm{U}(\mathfrak{g})^\lambda$, which is a generator of the target category, the theorem will follow from the following two claims:

*a)* Let $\lambda$ be dominant. Then $\Gamma:\mathcal{D}^\lambda_{\mathcal{P}}\text{-mod}\to\mathrm{U}(\mathfrak{g})^\lambda\text{-mod}$ is exact.

*b)* Let $\lambda$ be dominant and $P$-regular and $M\in\mathcal{D}^\lambda_{\mathcal{P}}$-mod, then if $\Gamma(M)=0$ it follows that $M=0$.

Let $V$ be a finite dimensional irreducible $G$-module and let

$$0=V_{-1}\subset V_0\subset\ldots\subset V_n=V$$

be a filtration of $V$ by $P$-submodules, such that $V_i/V_{i-1}\cong V_P(\mu_i)$ is an irreducible $P$-module.

Assume first that the highest weight $\mu_0$ of $V$ is a $P$-character. Thus $M\otimes\mathcal{O}(V_0)=M(-\mu_0)$ and we get an embedding $M(-\mu_0)\hookrightarrow M\otimes\mathcal{O}(V)$, which twists to the embedding $M\hookrightarrow M(\mu_0)\otimes\mathcal{O}(V)\cong M(\mu_0)^{\dim V}$. Now, by lemmas 2.1, 4.7 and theorem 4.1 iii) we get that this inclusion splits on derived global sections, so $R\Gamma(M)$ is a direct summand of $R\Gamma(M(\mu_0))^{\dim V}$. Now, for $\mu_0$ big enough and if $M$ is $\mathcal{O}$-coherent we have $R^{>0}\Gamma(M(\mu_0))=0$ (since $\mathcal{O}(\mu_0)$ is very ample). Hence, $R^{>0}\Gamma(M)=0$ in this case. A general $M$ is the union of coherent submodules and by a standard limit-argument it follows that $R^{>0}\Gamma(M)=0$. This proves *a)*.

Now, for *b)* we assume instead that the lowest weight $\mu_n$ of $V$ is a $P$-character. Then we have a surjection $M^{\dim V}\cong M\otimes\mathcal{O}(V)\to M(-\mu_n)$. Applying global sections and using lemmas 2.2, 4.7 and theorem 4.1 iv) we get that $\Gamma(M(-\mu_n))$ is a direct summand of $\Gamma(M)^{\dim V}$. For $\mu_n$ small enough we get that $\Gamma(M(-\mu_n))\neq 0$. Hence, $\Gamma(M)\neq 0$. This proves *b)*. ☐

**Theorem 5.2.** *Let $\lambda$ be dominant and $P$-regular then $\Gamma : \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-}mod^{\widehat{\lambda}} \to \mathcal{U}(\mathfrak{g})\text{-}mod^{\widehat{\lambda}}$ is an equivalence of categories.*

*Proof.* This follows from theorem 5.1 and a simple devissage. ☐

## 6. Category $\mathcal{O}$

We shall relate singular category $\mathcal{O}$ to a certain (non-standard) parabolic category $\mathcal{O}$ and discuss translation functors in the context of singular localization.

6.1. **Equivariant $\mathcal{D}$-modules and singular and parabolic category $\mathcal{O}$.** We want to describe blocks in the Bernstein-Gelfand-Gefand category $\mathcal{O}$ of finitely generated $\mathrm{U}(\mathfrak{g})$-modules which are locally finite over $\mathrm{U}(\mathfrak{n})$ and semi-simple over $\mathfrak{h}$. Let $\mathcal{O}_\lambda, \mathcal{O}_{\widehat{\lambda}} \subset \mathcal{O}$ be the subcategories of modules with central character, respectively, generalized central character, $\chi_\lambda$. We can assume that $\lambda$ is dominant since category $\mathcal{O}_\lambda$ only depends on $\chi_\lambda$. Pick any regular dominant $\lambda' \in \mathfrak{h}^*$ such that $\lambda - \lambda'$ is integral. Note that $\mathcal{O}_\lambda$ coincides with the category $(\mathrm{U}(\mathfrak{g})^\lambda, B, \lambda')\text{-mod}_c$ and $\mathcal{O}_{\widehat{\lambda}} \subset \mathcal{O}$ coincides with the category of $(\mathrm{U}(\mathfrak{g}), B, \lambda')\text{-mod}_c^{\widehat{\lambda}}$.

Let $P$ be a parabolic such that $\lambda$ is $P$-regular. The equivalence in theorem 5.2 thus induces an equivalence between $\mathcal{O}_{\widehat{\lambda}}$ and a category that we denote by $(\widetilde{\mathcal{D}}_{\mathcal{P}}, B, \lambda')\text{-mod}_c^{\widehat{\lambda}}$. In the equivariant description on $G$ we see that an object $M$ of $(\widetilde{\mathcal{D}}_{\mathcal{P}}, B, \lambda')\text{-mod}^{\widehat{\lambda}}$ will satisfy $(1), (2), (3), (\widehat{4})$ from section 4.2 and in addition there is a left $B$-action $\tau : B \to \mathrm{Aut}(M)$ such that

$$d\tau(x) - \epsilon_l(x) - \lambda'(x) = 0 \text{ on } M, \text{ for } x \in \mathfrak{b}. \tag{5}$$

By reading the defining conditions of $(\widetilde{\mathcal{D}}_{\mathcal{P}}, B, \lambda')\text{-mod}^{\widehat{\lambda}}$ in a different order we see that $(\widetilde{\mathcal{D}}_{\mathcal{P}}, B, \lambda')\text{-mod}^{\widehat{\lambda}}$ is equivalent to the category $(\widetilde{\mathcal{D}}_{\mathcal{B}}^{\lambda'}, P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-mod. Since $\lambda'$ is dominant and regular we get from Beilinson-Bernstein localization that $(\widetilde{\mathcal{D}}_{\mathcal{B}}^{\lambda'}, P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-mod is equivalent to the category $(\mathrm{U}(\mathfrak{g})^{\lambda'}, P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-mod, see section 2.3. Summarizing we get

**Proposition 6.1.** *$\mathcal{O}_{\widehat{\lambda}}$ is equivalent to $(\mathrm{U}(\mathfrak{g})^{\lambda'}, P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-$mod_c$.*

In the case when $\lambda$ is regular, we have that $\mathcal{O}_{\widehat{\lambda}}$ is equivalent to $(\mathrm{U}(\mathfrak{g})^{\lambda'}, B, \widehat{\lambda})$-mod which equals the category of $\mathfrak{g}$-representations which are locally finite over $\mathrm{U}(\mathfrak{b})$ and admit central character $\chi_{\lambda'}$. This was first proved in [Soe86]. In general it gives an equivalence between singular category $\mathcal{O}$ and a (non-standard) version of a parabolic category $\mathcal{O}$. It is not the parabolic-singular Koszul duality of [BGS96].

*Remark* 6.2. The $\mathcal{D}$-module category $(\widetilde{\mathcal{D}}_{\mathcal{B}}^{\lambda'}, P, \widehat{\chi_{\mathfrak{l}_P,\lambda}})$-mod corresponding to a singular block in category $\mathcal{O}$ will not consist of holonomic modules. For instance, if $\chi = -\rho$ (totally singular case, so we must take $P = G$) and $\lambda' = 0$, we have that category $\mathcal{O}_{\widehat{-\rho}}$ will consist of direct sums of copies of the simple Verma module $M_{-\rho}$. Corresponding to $M_{-\rho}$ is the non-holonomic $\mathcal{D}$-module $\mathcal{D}_{\mathcal{B}}$.

6.2. **Translation functors.** Let $\lambda, \mu \in \mathfrak{h}^*$ satisfy $\lambda - \mu$ is integral. Then there is the translation functor

$$T_\lambda^\mu = T_{\mathfrak{g},\lambda}^\mu : \mathrm{U}(\mathfrak{g})\text{-mod}^{\widehat{\lambda}} \to \mathrm{U}(\mathfrak{g})\text{-mod}^{\widehat{\mu}},\ M \mapsto pr_\mu(M \otimes E)$$

where $E$ is an irreducible finite dimensional representation of $\mathfrak{g}$ with extremal weight $\lambda - \mu$ and $pr_\mu = pr_{\mathcal{Z}(\mathfrak{g}),\mu} : \mathrm{U}(\mathfrak{g})\text{-mod}^{\mathcal{Z}(g)-fin} \to \mathrm{U}(\mathfrak{g})\text{-mod}^{\widehat{\mu}}$ is projection onto generalized central character. Here, $\mathrm{U}(\mathfrak{g})\text{-mod}^{\mathcal{Z}(g)-fin}$ stands for $\mathrm{U}(\mathfrak{g})$-modules that are locally finite over $\mathcal{Z}(g)$. See [BerGel81] for further information about translation functors.

We shall give a $\mathcal{D}$-module interpretation of these functors. Define for any parabolic subgroup $P \subset G$ a geometric translation functor

$$\mathbb{T}^{\mu}_{P,\lambda} : \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}} \to \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\mu}},\ M \mapsto pr_{\mathcal{Z}(\mathfrak{l}_P),\mu}(M \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(E'))$$

for $M \in (\widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod})^{\widehat{\lambda}}$, where $E'$ is an irreducible $P$-representation with highest weight in $\mathcal{W}_P(\mu-\lambda)$. Here, $pr_{\mathcal{Z}(\mathfrak{l}_P),\mu} : \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\mathcal{Z}(\mathfrak{l}_P)-fin} \to \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\mu}}$ denotes the projection onto generalized $\mathfrak{l}_P$ central character $\mu$. This description makes sense both in the equivariant and the non-equivariant description of the category $\widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}}$.

Note that if $\mu-\lambda$ is a $P$-character then $\mathcal{O}_{\mathcal{P}}(E') = \mathcal{O}_{\mathcal{P}}(\mu-\lambda)$ and in this case $\mathbb{T}^{\mu}_{\lambda} = (\ ) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(\mu-\lambda)$ is an equivalence with inverse given by $\mathbb{T}^{\lambda}_{\nu} = (\ ) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(\lambda-\mu)$. In particular, for $P = B$ we have $\mathbb{T}^{\mu}_{\lambda} = (\ ) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(\mu-\lambda)$ for any $\mu$ and $\lambda$.

Let $Q \subset G$ be another parabolic subgroup with $P \subset Q$. We have

**Lemma 6.3.** *The diagram*

$$\begin{tikzcd}
\widetilde{\mathcal{D}}_{\mathcal{P}}\text{-}mod^{\widehat{\lambda}} \arrow[r, "\mathbb{T}^{\mu}_{P,\lambda}"] \arrow[d, "\pi^{\mathcal{Q}}_{\mathcal{P}*}"] & \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-}mod^{\widehat{\mu}} \arrow[d, "\pi^{\mathcal{Q}}_{\mathcal{P}*}"] \\
\widetilde{\mathcal{D}}_{\mathcal{Q}}\text{-}mod^{\widehat{\lambda}} \arrow[r, "\mathbb{T}^{\mu}_{Q,\lambda}"] & \widetilde{\mathcal{D}}_{\mathcal{Q}}\text{-}mod^{\widehat{\mu}}
\end{tikzcd}$$

*of exact functors commutes up to natural equivalence.*

In the case of $P = B$ and $Q = G$ this was proved in [BG99].

*Proof.* Let $V$ (resp., $V'$) be an irreducible finite dimensional representation for $Q$ (resp., for $P$) whose highest weight belongs to $\mathcal{W}_Q(\mu-\lambda)$ (resp., $\mathcal{W}_P(\mu-\lambda)$). Let $M \in \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}}$. Then, since $V$ is a $Q$-representation, we have $\mathcal{O}_{\mathcal{P}}(V) = \pi^{\mathcal{Q}*}_{\mathcal{P}}(\mathcal{O}_{\mathcal{Q}}(V))$ and therefore it follows from the projection formula that

$$\pi^{\mathcal{Q}}_{\mathcal{P}*}(\mathcal{O}_{\mathcal{P}}(V) \otimes_{\mathcal{O}_{\mathcal{P}}} M) = \mathcal{O}_{\mathcal{Q}}(V) \otimes_{\mathcal{O}_{\mathcal{Q}}} \pi^{\mathcal{Q}}_{\mathcal{P}*}(M).$$

Thus we get

$$\mathbb{T}^{\mu}_{Q,\lambda} \circ \pi^{\mathcal{Q}}_{\mathcal{P}*}(M) = pr_{\mathcal{Z}(\mathfrak{l}_Q),\mu}(\mathcal{O}_{\mathcal{Q}}(V) \otimes_{\mathcal{O}_{\mathcal{Q}}} \pi^{\mathcal{Q}}_{\mathcal{P}*}(M)) =$$

$$pr_{\mathcal{Z}(\mathfrak{l}_Q),\mu}(\pi^{\mathcal{Q}}_{\mathcal{P}*}(\mathcal{O}_{\mathcal{P}}(V) \otimes_{\mathcal{O}_{\mathcal{P}}} M)) = \pi^{\mathcal{Q}}_{\mathcal{P}*}(pr_{\mathcal{Z}(\mathfrak{l}_P),\mu}(\mathcal{O}_{\mathcal{P}}(V) \otimes_{\mathcal{O}_{\mathcal{P}}} M)) \overset{(*)}{=}$$

$$\pi^{\mathcal{Q}}_{\mathcal{P}*}(pr_{\mathcal{Z}(\mathfrak{l}_P),\mu}(\mathcal{O}_{\mathcal{P}}(V') \otimes_{\mathcal{O}_{\mathcal{P}}} M)) = \pi^{\mathcal{Q}}_{\mathcal{P}*} \circ \mathbb{T}^{\mu}_{P,\lambda}(M)$$

The equality $(*)$ follows from lemma 2.2 applied to the reductive Lie algebra $\mathfrak{l}_Q$ and its parabolic subalgebra $\mathfrak{l}_Q \cap \mathfrak{p}$ (compare with the proof of the localization theorem). ☐

Let us geometrically describe **translation to the wall**: In this case $\mu$ is more singular than $\lambda$, i.e., we assume that $\Delta_\lambda \subsetneq \Delta_\mu$ and $\lambda$ and $\mu$ are dominant. We choose the parabolic subgroups

$P \subset Q \subset G$ such that the parabolic roots of $P$ equal $\Delta_\lambda$ and the parabolic roots of $Q$ equal $\Delta_\mu$. It follows from lemma 6.3 that the diagram below commutes up to natural equivalence:

$$\begin{tikzcd}
\mathrm{U}(\mathfrak{g})\text{-mod}^{\widehat{\lambda}} \arrow[dd, "(4)\ T^{\mu}_{\lambda}"] & \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}} \arrow[l, "(1)\ \Gamma"'] \arrow[d, "(3)\ \pi^{\mathcal{Q}}_{\mathcal{P}*}"] \arrow[dr, "(2)\ \mathbb{T}^{\mu}_{P,\lambda}"] & \\
 & \widetilde{\mathcal{D}}_{\mathcal{Q}}\text{-mod}^{\widehat{\lambda}} \arrow[d, "(5)\ \mathbb{T}^{\mu}_{Q,\lambda}"] & \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\mu}} \arrow[dl, "(7)\ \pi^{\mathcal{Q}}_{\mathcal{P}*}"] \\
\mathrm{U}(\mathfrak{g})\text{-mod}^{\widehat{\mu}} & \widetilde{\mathcal{D}}_{\mathcal{Q}}\text{-mod}^{\widehat{\mu}} \arrow[l, "(6)\ \Gamma"'] &
\end{tikzcd} \tag{6.1}$$

Note that (1) and (6) are equivalences by the choices of $P$ and $Q$ and that $(2) = (\ )\otimes_{\mathcal{O}_{\mathcal{P}}}\mathcal{O}(\mu-\lambda)$ is an equivalence, since $\mu - \lambda$ is a $P$-character.

We see that (3) is an equivalence of categories because both the source and the target category are D-affine, since $\lambda$ is $P$- and $Q$-regular, and $\Gamma \circ \pi^{\mathcal{Q}}_{\mathcal{P}*} = \Gamma$. On the other hand, the functor (7) is not faithful, because $\mu$ is not $P$-regular. (5) is also not faithful. We remind that all functors involved are exact.

Let us now describe **translation out of the wall**: This is done by taking the diagram of adjoint functors in the diagram 6.1, so we keep assuming that $\lambda$, $\mu$, $P$ and $Q$ are as in 6.1. The left and right adjoint of $T^{\mu}_{\lambda}$ is $T^{\lambda}_{\mu}$, the translation out of the wall. The equivalences $(1), (2), (3)$ and $(6)$ of course have left and right adjoints that coincide. Also, the left and right adjoint of (5) coincide; it is given by $\mathbb{T}^{\lambda}_{\mathfrak{l}_Q,\mu}$. Finally (7) has the left adjoint $\pi^{\mathcal{Q}*}_{\mathcal{P}}$; thus, $\pi^{\mathcal{Q}*}_{\mathcal{P}}$ must also be the right adjoint of (7). Summing up we have:

$$\begin{tikzcd}
\mathrm{U}(\mathfrak{g})\text{-mod}^{\widehat{\lambda}} \arrow[r, "\mathcal{L}"] & \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\lambda}} & \\
 & \widetilde{\mathcal{D}}_{\mathcal{Q}}\text{-mod}^{\widehat{\lambda}} \arrow[u, "\pi^{\mathcal{Q}*}_{\mathcal{P}}"] & \widetilde{\mathcal{D}}_{\mathcal{P}}\text{-mod}^{\widehat{\mu}} \arrow[ul, "\mathbb{T}^{\lambda}_{P,\mu}"'] \\
\mathrm{U}(\mathfrak{g})\text{-mod}^{\widehat{\mu}} \arrow[uu, "T^{\lambda}_{\mu}"] \arrow[r, "\mathcal{L}"] & \widetilde{\mathcal{D}}_{\mathcal{Q}}\text{-mod}^{\widehat{\mu}} \arrow[u, "\mathbb{T}^{\lambda}_{Q,\mu}"] \arrow[ur, "\pi^{\mathcal{Q}*}_{\mathcal{P}}"'] &
\end{tikzcd} \tag{6.2}$$

*Remark* 6.4. Translation functors restrict to functors between blocks in category $\mathcal{O}$. Using the description of a singular block of category $\mathcal{O}$ as a category of $\mathcal{D}$-modules on $\mathcal{B}$ from the previous section we see that translation functors can be interpreted as functors between (twisted) $\mathcal{D}$-module categories on $\mathcal{B}$.

## REFERENCES

[BK06] E. Backelin, K. Kremnizer, *Quantum flag varieties, equivariant D-modules and localization of quantum groups*, Adv in Math, Volume 203, Issue 2, Page 408-429 (July 2006)

[BK08] E. Backelin, K. Kremnizer, *Localization of a quantum group at a root of unity*, J. Amer. Math. Soc. 21 (2008), 1001-1018.

[BB81] A. Beilinson and J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sc. Paris, 292 (Série I) (1981) 15-18.

[BB93] A. Beilinson and J. Bernstein, *Proof of Jantzen's conjecture*, Advances in Sovjet Mathematics, Volume 19, part 1 (1993) 1-50.

[BG99] A. Beilinson and V. Ginzburg, *Wall-crossing functors and $\mathcal{D}$-modules on flag manifolds*, Representation theory (www.ams.org/ert) 3, (1999) 1-31.

[BGS96] A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), 473–527.

[BerGel81] J. Bernstein and S.I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Comp. Math. 41 (1981), 245–285.

[BMR02] A. Bezrukavnikov, I. Mircovic and D. Rumynin, *Localization for a semi-simple Lie algebra in prime characteristic*, arXiv:math.RT/0205144.

[BMR06] A. Bezrukavnikov, I. Mircovic and D. Rumynin, *Singular localization and intertwining operators for Lie algebras in prime characteristic*, arXiv:math/0602075v2

[Dix77] J. Dixmier. Enveloping Algebras, North-Holland, Amsterdam/New York/Oxford (1977).

[Jan83] J. C. Jantzen, *Algebraic groups*, Springer-Verlag (1983).

[Kas93] M. Kashiwara, *$\mathcal{D}$-modules on flagmanifolds*, J. Amer. Math. Soc. 6 (1993), 905–1011.

[Mat60] Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J. 16 (1960), 205–218.

[Soe86] W. Soergel, *Équivalence de certain catégories de $\mathfrak{g}$-modules*, C.R. Acad Paris Sér 1 303 (1986), no. 15, 725-727.

ERIK BACKELIN, DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LOS ANDES, CARRERA 1 N. 18A - 10, BOGOTÁ, COLOMBIA

*E-mail address*: `erbackel@uniandes.edu.co`

KOBI KREMNIZER, MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, 2429 ST GILES' OXFORD OX1 3LB, UK

*E-mail address*: `kremnitzer@maths.ox.ac.uk`