Conservation of vacuum in an interferometer

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We show that the total quantum efficiency of multiple optical modes cannot increase under linear-optical processing. That is, the sum of quantum efficiencies of any K output modes of a linear-optical scheme cannot exceed the sum of quantum efficiencies in the K input modes with the highest efficiencies. This result, which includes our earlier work (D. W. Berry and A. I. Lvovsky, arXiv:1004.2245) as a special case for $K = 1$, imposes severe limitations on quantum processing using linear optics. In particular, it rules out catalytic improvement of single-photon sources, in which a single available photon source of high quantum efficiency would permit generating multiple high-efficiency single photons in separate optical modes.

A leading approach to quantum information processing is via linear-optical quantum computing (LOQC), first proposed in 2001 by Knill, Laflamme and Milburn [1]. Major progress has been made towards implementation of LOQC, both on the theoretical and experimental fronts. Modifications have been proposed that greatly reduce the overhead costs [2], a quantum error correction protocol has been introduced [3, 4], and experimental implementation of primary gates has been demonstrated [5].

In spite of this progress, practical linear-optical quantum computing is still out of reach. Many of the difficulties arise because the single-photon sources required for LOQC, as well as LOQC circuits themselves, suffer from losses. Although a certain degree of tolerance to losses does exist in some LOQC schemes [4], the efficiency of existing single-photon sources [6] as well as the quality of individual circuit elements and waveguides are far below the required minima.

Under these circumstances it appears beneficial to develop a procedure that would eliminate the effect of losses, perhaps at a cost of introducing extra resources. It could be used, for example, to convert the outputs of N imperfect single-photon sources into $K < N$ single-photon sources of improved quantum efficiency. Such a procedure would be straightforward to implement if nonlinear-optical interactions with single photons were readily available. Indeed, with nonlinear optics one could set up a quantum nondemolition measurement of the photon number observable, followed by postselection on the desired measurement result. Unfortunately, however, implementation of such interactions at the present level of technology is barely feasible [7] and, perhaps more importantly, inherently lossy, which defeats the purpose of efficiency correction.

In this paper, we are investigating the question of whether improvement of quantum efficiency is possible under linear-optical processing. Under this processing we understand arbitrary interferometric transformations and conditioning on results of arbitrary *destructive* measurements on some of the optical modes involved. The efforts to construct such a scheme began in 2004, mostly ending with various no-go results [8–11]. Recently [12], we have proven the most general no-go result so far, showing that the quantum efficiency in any of the optical states obtained through linear processing cannot exceed the quantum efficiency of the best available input.

However, Ref. [12] is limited to a single output mode. Therefore it does not rule out the possibility of "catalytic" efficiency improvement, i.e. a scheme in which a single or a few available photon sources of high quantum efficiency would permit generating multiple high-efficiency single photons in separate optical modes. In the present work, we prove a more general result. We show that the total efficiency of any K output modes cannot exceed the total efficiency of the K highest efficiency modes available at the input. That is, any loss that has occurred at the input can neither be removed nor redistributed so as to improve the efficiency in some of the modes at the expense of lower-efficiency modes.

We start with the general definition of the efficiency of a quantum optical state $\hat\rho$ [12]

$$E(\hat\rho) := \inf\{p \mid \exists \hat\rho_0 \geq 0 \ : \ \mathcal{E}_p(\hat\rho_0) = \hat\rho\}, \tag{1}$$

where $\mathcal{E}_p$ represents a loss channel with transmissivity p. In other words, $E(\hat\rho)$ is the lowest possible transmissivity of a loss channel such that state $\hat\rho$ can be obtained from another state $\hat\rho_0$ by transmitting it through that channel. For example, the efficiency of a coherent state is 0, and

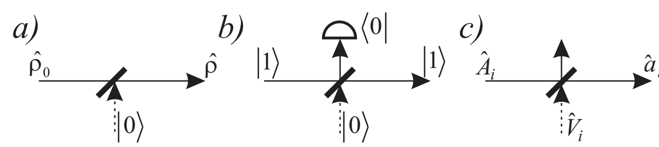

FIG. 1: The beam splitter model of absorption. a) Application to the definition of the generalized efficiency. b) The model is invalid if the transmitted state is conditional upon a measurement in the reflected channel. In this example, conditioning on the detection of vacuum results in a high-efficiency single-photon state emerging in the transmitted channel, independent of the beam splitter transmissivity. c) Illustration for Eq. (12).

the efficiency of any coherent superposition of a finite number of Fock states is 1. The efficiency of any light state that has propagated through an attenuator with transmissivity p cannot exceed p.

In our treatment, we model the loss channel by a beam splitter, through which the initial state $\hat{\rho}_0$ propagates in order to generate $\hat{\rho}$ [Fig. 1(a)], with vacuum entering the beam splitter's other input port. In order for the model to be valid, the transmitted state must not be conditioned on any measurement performed in the reflected channel of this beam splitter [Fig. 1(b)].

We consider a general method for processing optical modes, as shown in Fig. 2. There are N input modes with annihilation operators $\hat{a}_j$, in a tensor product of N single-mode states with efficiencies p_j. These modes are passed through a general interferometer which performs a unitary operation U on the mode operators. We retain K of the output modes, and the remaining $N-K$ modes are subjected to a generalized destructive quantum measurement. We consider postselection on a particular result of this measurement, and determine the efficiencies in the remaining output modes. Below, we show that the sum of these efficiencies cannot exceed the total of the K largest values of p_j.

First we note that we can obtain each of the interferometer input states by combining some initial state, in a mode with annihilation operator $\hat{b}_j$, with vacuum on a beam splitter with transmissivity p_j (Fig. 2):

$$\hat{a}_j = \sqrt{p_j}\hat{b}_j + \sqrt{1-p_j}\hat{v}_j. \quad (2)$$

The action of the interferometer can then be written as

$$\hat{a}'_i = \sum_{j=1}^{N} U_{ij}\hat{a}_j = \sum_{j=1}^{N} U_{ij}\sqrt{p_j}\hat{b}_j + \sum_{j=1}^{N} U_{ij}\sqrt{1-p_j}\hat{v}_j. \quad (3)$$

We see that each vacuum mode contributes to each of the output modes, including those that are subjected to measurements. It is thus difficult to evaluate which vacuum contribution leads to an efficiency loss, and which is "compromised" by conditional measurements. We address this issue by performing an RQ decomposition on the matrix $U_{ij}\sqrt{1-p_j}$ such that

$$U_{ij}\sqrt{1-p_j} = \sum_{k=1}^{N} R_{ik}Q_{kj}, \quad (4)$$

where Q is unitary and R is an upper triangular matrix, so $R_{ik}=0$ for $k<i$. Then we get

$$\hat{a}'_i = \sum_{k=1}^{N} U_{ik}\sqrt{p_k}\hat{b}_k + \sum_{k=1}^{N} R_{ik}\hat{v}'_k, \quad (5)$$

where

$$\hat{v}'_k = \sum_{j=1}^{N} Q_{kj}\hat{v}_j \quad (6)$$

are obtained by transforming modes $\hat{v}_j$ in a fictitious interferometer Q. Because all the $\hat{v}_j$ are in vacuum states, so are the $\hat{v}'_k$.

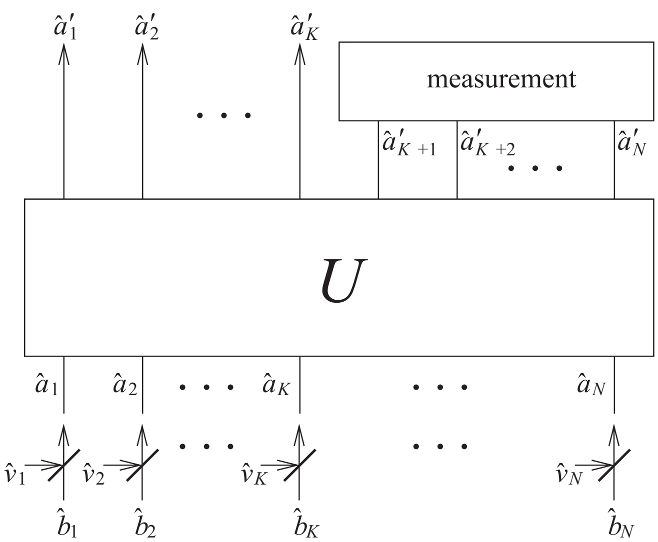

FIG. 2: A general setup for processing photon sources. The inputs are in a tensor product of N single-mode states with generalized efficiencies p_j. The modes pass through a general interferometer, and all but K of the output modes are detected via a measurement. The remaining K modes can be conditioned on a particular measurement result (they need not be in a product state). Each interferometer input mode $\hat{a}_j$, which has efficiency p_j, can be obtained by a beam splitter combining mode $\hat{b}_j$ of generalized efficiency 1 and mode $\hat{v}_j$ containing a vacuum state.

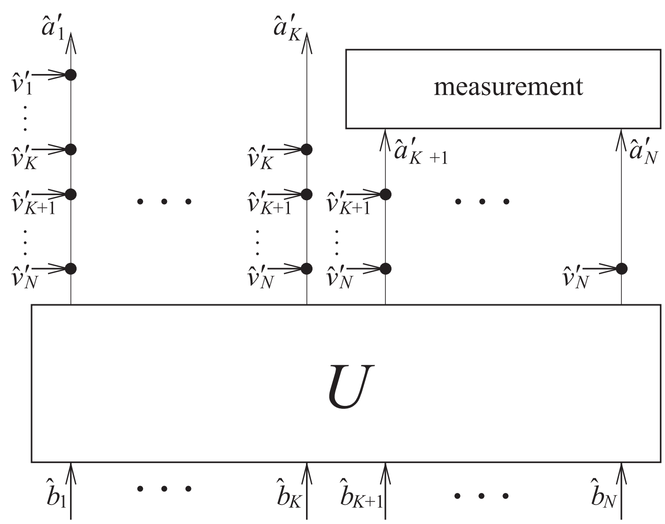

FIG. 3: The processing scheme modified in accordance with Eq. (5). Vacuum inputs enter *after* the interferometer and contribute to the signal according to the upper triangular matrix R_{ik}. Note that the summations in Eq. (5) (displayed as black dots), are not unitary, and hence cannot be represented by beam splitters.

In the following, we show that the efficiency of output mode $\hat{a}'_i$, with $1 \le i \le K$, is upper bounded by

$$E(\hat{a}'_i) \le 1 - \sum_{k=1}^{K} |R_{ik}|^2. \tag{7}$$

In this sum, we exclude the contribution of modes $\hat{v}'_{K+1}, \ldots, \hat{v}'_N$ because they contribute to the set of output modes $M := \{\hat{a}'_{K+1}, \ldots, \hat{a}'_N\}$ that are measured (Fig. 3). Because the output is conditioned on the measurement result, the effect of the latter vacuum modes on the output efficiency cannot be generally quantified.

To formalize this intuition, we fix a specific value of i for which Eq. (7) is to be proven, and define

$$\varrho := \sum_{k=1}^{K} |R_{ik}|^2. \tag{8}$$

There are two special cases that need to be considered before we proceed further. If $\varrho = 0$, the sum in Eq. (7) is zero. Because the efficiency cannot exceed 1, the inequality (7) is trivial. The other special case is that $\varrho = 1$. In that case, the only contribution to $\hat{a}'_i$ is vacuum modes. This is because $\hat{a}'_i$ is a unitary combination of other modes, so Eq. (5) gives

$$\sum_{k=1}^{N} |U_{ik}|^2 p_k + \sum_{k=1}^{N} |R_{ik}|^2 = 1. \tag{9}$$

If $\varrho = 1$, then that implies $\forall k\; U_{ik}\sqrt{p_k} = 0$; hence the first term in Eq. (3) is zero, and so is the efficiency. Equation (7) is again trivially satisfied.

Now that we have eliminated these special cases, we can define, for $\varrho \in (0,1)$,

$$\hat{A}_i := \frac{1}{\sqrt{1-\varrho}} \left(\sum_{k=1}^{N} U_{ik}\sqrt{p_k}\hat{b}_k + \sum_{k=K+1}^{N} R_{ik}\hat{v}'_k \right), \tag{10}$$

$$\hat{V}_i := \frac{1}{\sqrt{\varrho}} \sum_{k=1}^{K} R_{ik}\hat{v}'_k. \tag{11}$$

The operator $\hat{A}_i$ represents those terms in Eq. (5) that correspond to the input signal modes, as well as the last $N-K$ vacuum modes. The operator $\hat{V}_i$ represents the first K vacuum modes. The choice of ϱ ensures that these operators are normalized, in the sense that $[\hat{A}_i, \hat{A}_i^\dagger] = [\hat{V}_i, \hat{V}_i^\dagger] = 1$.

Hence $\hat{A}_i$ and $\hat{V}_i$ represent mutually orthogonal optical modes that can be combined to give $\hat{a}'_i$ via

$$\hat{a}'_i = \sqrt{1-\varrho}\hat{A}_i + \sqrt{\varrho}\hat{V}_i. \tag{12}$$

This corresponds to a beam splitter transformation, as shown in Fig. 1(c). The intensity transmissivity $1-\varrho$ of this beam splitter equals the right-hand side of (7), so to prove this inequality it only remains to prove that the reflected output of the beam splitter is not measured.

Because $\{\hat{a}'_1, \ldots, \hat{a}'_N\}$ correspond to the output modes of the initial interferometer (Fig. 2), they form an orthonormal set, and M (the measured modes) and $\hat{a}'_i$ must be orthogonal. The only vacuum modes that contribute to M are $\hat{v}'_{K+1}, \ldots, \hat{v}'_N$, so $\hat{V}_i$ is also orthogonal to M. Because $\hat{A}_i$ can be expressed as a linear combination of $\hat{a}'_i$ and $\hat{V}_i$, it must be orthogonal to M as well. Hence, Eq. (12) corresponds to a beam splitter transformation where none of the modes are measured, and Eq. (7) must hold.

While the beam splitter transformation (12) is not physically present in the Fig. 2 scheme, it is possible to rearrange this scheme into an equivalent one where $\hat{a}'_i$ is explicitly obtained by a beam splitter. This rearrangement is presented in the Appendix.

Now that inequality (7) is proven, we can bound the total efficiency in output modes $\hat{a}'_1, \ldots, \hat{a}'_K$ according to

$$\begin{aligned}
\sum_{i=1}^{K} E(\hat{a}'_i) &\le K - \sum_{i=1}^{K}\sum_{k=1}^{K} |R_{ik}|^2 \\
&= K - \sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{\ell,j=1}^{K} U_{i\ell}\sqrt{1-p_\ell}V^*_{k\ell}U^*_{ij}\sqrt{1-p_j}V_{kj} \\
&= K - \sum_{k=1}^{K}\sum_{\ell,j=1}^{N} \delta_{\ell j}\sqrt{1-p_\ell}V^*_{k\ell}\sqrt{1-p_j}V_{kj} \\
&= \sum_{k=1}^{K}\sum_{j=1}^{N} p_j|V_{kj}|^2 = \sum_{k=1}^{K} q_k,
\end{aligned} \tag{13}$$

where we defined

$$q_k := \sum_{j=1}^{N} p_j |V_{kj}|^2. \tag{14}$$

Because V is unitary, $|V_{ij}|^2$ is a doubly stochastic matrix. It then follows [13] that $\vec{p}$ majorizes $\vec{q}$, i.e. $\sum_{k=1}^{K} q_k \le \sum_{k=1}^{K} p_k$ as long as both $\vec{p}$ and $\vec{q}$ are sorted in non-increasing order. Hence this total efficiency of the output cannot be any larger than the total of the K largest values of p_j.

A few comments are in order. First, the requirement that the input be in a tensor product state is important. This is because the total efficiency of multiple single-mode states, dealt with in the present paper, can be different from the efficiency of a state taken as a whole. For example, consider the two-mode entangled state $(|10\rangle + |01\rangle)/\sqrt{2}$. This state cannot be produced by attenuating any other state, i.e. it has efficiency one when analyzed as a single entity. After passing this state through a symmetric beam splitter, one obtains $|10\rangle$, which has unit efficiency in one of the output modes. On the other hand, each input mode considered separately is in the mixed state $(|1\rangle\langle 1| + |0\rangle\langle 0|)/2$, from which one may incorrectly conclude that the highest single-mode efficiency that can be obtained is 1/2.

Second, the measurement performed on $\hat{a}'_{K+1}, \ldots \hat{a}'_N$ is an arbitrary generalized destructive measurement. This includes the situation when no measurement is performed, and one or all of the modes are simply discarded. This means that the above bound on the total output efficiency is automatically valid for any $K' \leq K$ output modes. In particular, if $K' = 1$, we obtain the result of Ref. [12].

The result presented in this work establishes a major limitation in linear-optical quantum information processing. Once an incoherent vacuum contribution has been injected into a linear-optical circuit, it cannot be eliminated by any linear optical means. Not only cannot the total efficiency be increased, but also the efficiencies cannot be redistributed so as to further increase the efficiency of higher-efficiency modes at the expense of lower-efficiency modes. One consequence of this fact is the exclusion of any possibility for "catalytic" efficiency improvement, in which a single highly efficient optical source would permit simultaneous preparation of multiple high-efficiency optical states in separate optical modes.

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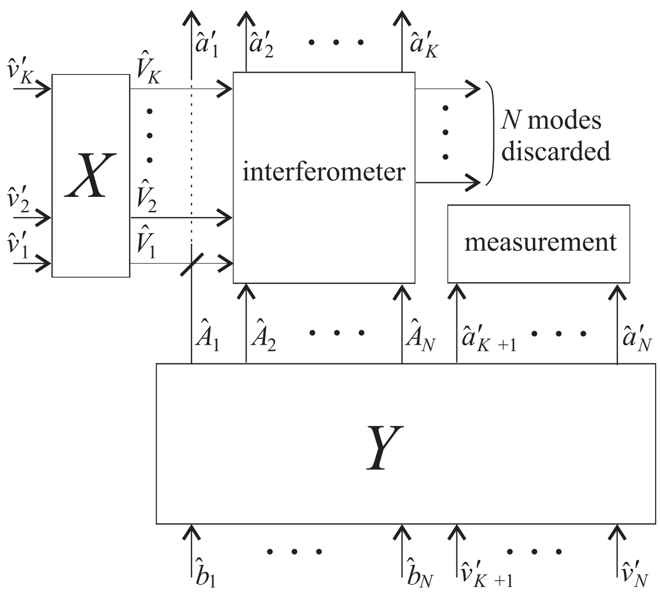

FIG. 4: The alternative representation of the processing scheme, illustrated for the example of $i = 1$. The output mode $\hat{a}'_i$ is now explicitly obtained from signal mode $\hat{A}_i$ and vacuum mode $\hat{V}_i$ by a simple beam splitter operation (12), so its efficiency can be upper bounded.

Appendix. We can obtain interferometric transformations that lead to Eqs. (10) and (11) as follows. Because $\hat{V}_i$ is normalized and a linear combination of $\{\hat{v}'_1, \ldots, \hat{v}'_K\}$, we can use the Gram-Schmidt procedure to construct an orthonormal set, $\{\hat{V}_1, \ldots, \hat{V}_K\}$. We then define X to be the unitary transforming between these two orthonormal sets. Similarly, $\hat{A}_i$ and $\{\hat{a}'_{K+1}, \ldots, \hat{a}'_N\}$ are orthonormal, and linear combinations of $\{\hat{b}_1, \ldots \hat{b}_N, \hat{v}'_{K+1}, \ldots, \hat{v}'_N\}$. We can therefore use the Gram-Schmidt procedure to complete the orthonormal set $\{\hat{A}_1, \ldots \hat{A}_N, \hat{a}'_{K+1}, \ldots, \hat{a}'_N\}$, and define Y to be the corresponding unitary. Because X and Y are unitary, we can obtain these transformations via an interferometer, as shown in Fig. 4.

Each output mode $\hat{a}'_1, \ldots, \hat{a}'_K$ can be written as a linear combination of states $\{\hat{A}_1, \ldots, \hat{A}_N, \hat{V}_1 \ldots, \hat{V}_K\}$, which can be interpreted as another interferometric transformation with some of its output modes discarded (Fig. 4). Because $\hat{a}'_i$ is a linear combination of $\hat{A}_i$ and $\hat{V}_i$ alone, it can be obtained via combining these at a beam splitter. Combining the remaining modes at a separate interferometer then produces the other $\hat{a}'_\ell$ and the discarded modes.

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