REPRESENTATIONS OF FINITE POLYADIC GROUPS

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Abstract. We prove that there is a one-one correspondence between sets of irreducible representations of a polyadic group and its Post's cover. Using this correspondence, we generalize some well-known properties of irreducible characters in finite groups to the case of polyadic groups.

1. Introduction

A non-empty set G together with an n-ary operation $f : G^n \to G$ is called an *n-ary group* or a *polyadic group*, if the operation f is associative and for all $x_0, x_1, \ldots, x_n \in G$ and fixed $i \in \{1, \ldots, n\}$ there exists a unique element $z \in G$ such that

$$f(x_1^{i-1}, z, x_{i+1}^n) = x_0.$$

In the binary case (i.e., for $n = 2$), a polyadic group is just usual group.

In this paper, a sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we write $\overset{(t)}{x}$. In this convention $f(x_1, \ldots, x_n) = f(x_1^n)$ and hence the associativity of f can be formulated as

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

where $1 \leq i, j \leq n$, and $x_1, \ldots, x_{2n-1} \in G$.

The idea of investigations of such polyadic group goes back to E. Kasner's lecture [3] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper concerning the theory of n-ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [1]). In this paper Dörnte observed that any n-ary system (G, f) of the form $f(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n \circ b$, where $(G, \circ)$ is a group and b is its fixed element belonging to the center of $(G, \circ)$, is an n-ary group. Such n-ary group is called *b-derived* from the group $(G, \circ)$, and we will denote it by $der_b^n(G, \circ)$. In the case when b is the identity of $(G, \circ)$, we say that such n-ary group is *reducible* to the group $(G, \circ)$ or *derived* from $(G, \circ)$ and we denote it by $der^n(G, \circ)$. For every $n > 2$ there are n-ary groups which are not derived from any group. An n-ary group (G, f) is

Date: November 4, 2010.
MSC(2010): 20N15
Keywords: Polyadic Groups; Representations; Character degrees; Post's cover.

derived from some group if and only if it contains an element e (called an *n-ary identity*) such that

$$f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}) = x$$

holds for all $x \in G$ and for all $i = 1, \ldots, n$.

From the definition of an n-ary group (G, f) we can directly see that for every $x \in G$ there exists only one $z \in G$ satisfying the equation

$$f(\overset{(n-1)}{x}, z) = x.$$

This element is called *skew* to x and is denoted by $\overline{x}$. In a ternary group $(n = 3)$ derived from a binary group $(G, \circ)$, the skew element coincides with the inverse element in $(G, \circ)$. Thus, in some sense, the skew element is a generalization of the inverse element in binary groups.

Nevertheless, the concept of skew elements plays a crucial role in the theory of n-ary groups. Namely, as Dörnte proved, the following theorem is true.

Theorem 1.1. *In any n-ary group (G, f) the following identities*

$$f(\overset{(i-2)}{x}, \overline{x}, \overset{(n-i)}{x}, y) = f(y, \overset{(n-j)}{x}, \overline{x}, \overset{(j-2)}{x}) = y,$$

$$f(\overset{(k-1)}{x}, \overline{x}, \overset{(n-k)}{x}) = x$$

hold for all $x, y \in G$, $2 \leqslant i, j \leqslant n$ and $1 \leqslant k \leqslant n$.

Suppose (G, f) is an n-ary group. A map $\Lambda : G \to GL_m(\mathbb{C})$ with the property

$$\Lambda(f(x_1^n)) = \Lambda(x_1)\Lambda(x_2)\ldots\Lambda(x_n)$$

is a *representation* of G. The function

$$\chi(x) = Tr\ \Lambda(x)$$

is called the corresponding *character* of Λ. The number m is the degree of representation. Note that, Λ is a representation of (G, f), iff it is an n-ary homomorphism $G \to der^n(GL_m(\mathbb{C}))$.

In [2], a joint paper with W. Dudek, we studied representation theory of polyadic group, but representations we dealt with in that paper were considered to have non-empty kernels. In this paper, we study representations of polyadic groups without that assumption, i.e. the representations we deal with in this paper, may have empty kernels, as well. We will prove that there is a one-one correspondence between the sets of irreducible representations of polyadic groups and their *Post cover*. Using this correspondence, we will generalize some well-known properties of irreducible characters of finite groups to finite polyadic groups.

2. Generalities

Suppose (G, f) is an n-ary group and $a \in G$ is any fixed element. Let

$$G_a^* = \{(x, i) : \ x \in G, \ i \in \mathbb{Z}_{n-1}\}.$$

We define a binary operation of G_a^* by

$$(x, i) \cdot (y, j) = (f_*(x, \overset{(i)}{a}, y, \overset{(j)}{a}, \bar{a}, \overset{(n-2-i*j)}{a}), i * j),$$

where $i*j \equiv i+j+1 \ (mod\ n-1)$, and f_* indicates that f is used one or twice, depending on the value of $i*j$. The set G_a^* together with this operation is an ordinary group (see [4]), and we call it *Post's cover* of (G, f). The element $(\overline{a}, n-2)$ is the identity of the group G_a^*. The inverse element has the form

$$(x, i)^{-1} = (f_*(\overline{a}, \overset{(n-2-i)}{a}, \overline{x}, \overset{(n-3)}{x}, \overline{a}, \overset{(i+1)}{a}), k),$$

where $k = (n-3-i)(\mathrm{mod}\,(n-1))$. The following theorem is known as *Post's coset theorem*, and it is proved in [4] and also in[5].

Theorem 2.1. *Suppose*

$$H = \{(x, n-2) : \ x \in G\}.$$

Then $H \trianglelefteq G_a^*$ *and* $G_a^*/H \cong \mathbb{Z}_{n-1}$. *Further, we can identify* G *with the subset*

$$\{(x, 0) : \ x \in G\},$$

and in under this identification, G *is a coset of* H *and we have* $f(x_1^n) = x_1 x_2 \cdots x_n$.

It is proved that (see [4]), Post's cover G_a^* does not depend on a, i.e. for any $a, b \in G$, we have $G_a^* \cong G_b^*$.

Proposition 2.2. *Suppose* A *is an ordinary group and* $a \in A$. *Then for any* $n \geq 2$, *we have*

$$(der^n(A))_a^* \cong A \times \mathbb{Z}_{n-1}.$$

Proof. It is enough to suppose $a = e$, the identity element of A. We have

$$(der^n(A))_e^* = \{(x, i) : \ x \in A, i \in \mathbb{Z}_{n-1}\},$$

and also

$$\begin{aligned} (x, i) \cdot (y, j) &= (xy, i * j) \\ &= (xy, i + j + 1) \\ &= (x, i)(y, j)(e, 1). \end{aligned}$$

This shows that

$$(der^n(A))_e^* = der^2_{(e,1)}(A \times \mathbb{Z}_{n-1}).$$

Now, define a map $\varphi : der^2_{(e,1)}(A \times \mathbb{Z}_{n-1}) \to A \times \mathbb{Z}_{n-1}$, by $\varphi(x, i) = (x, i+1)$. It is easy to check that φ is an isomorphism.

As a result, we see that if $a \in GL_m(\mathbb{C})$, then

$$(der^n(GL_m(\mathbb{C})))^*_a \cong GL_m(\mathbb{C}) \times \mathbb{Z}_{n-1}.$$

Now, let (G, f) be an n-ary group and suppose $\Lambda : G \to der^n(GL_m(\mathbb{C}))$ is any representation. Let $a \in G$ be fixed. We define a new map

$$\Lambda^*_a : G^*_a \to (der^n(GL_m(\mathbb{C})))^*_{\Lambda(a)}$$

by the rule

$$\Lambda^*_a(x, i) = (\Lambda(x), i).$$

Lemma 2.3. *Λ^*_a is a group homomorphism.*

Proof. Let $B = \Lambda(a)$. Note that we have $\Lambda(\bar{a}) = \Lambda(a)^{2-n} = \bar{B}$. Now, for any $x, y \in G$ and $i, j \in \mathbb{Z}_{n-1}$, we have

$$\begin{aligned}
\Lambda^*_a((x,i)\cdot(y,j)) &= \Lambda^*(f_*(x, \overset{(i)}{a}, y, \overset{(j)}{a}, \bar{a}, \overset{(n-2-i*j)}{a}), i*j)\\
&= (\Lambda(x)\Lambda(a)^i\Lambda(y)\Lambda(a)^j\Lambda(\bar{a})\Lambda(a)^{n-2-i*j}, i*j)\\
&= (\Lambda(x)\Lambda(a)^i\Lambda(y)\Lambda(a)^j\Lambda(a)^{2-n}\Lambda(a)^{n-2-i*j}, i*j)\\
&= (\Lambda(x)\Lambda(a)^i\Lambda(y)\Lambda(a)^{j-i*j}, i*j).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Lambda^*_a(x,i)\cdot\Lambda^*_a(y,j) &= (\Lambda(x), i)\cdot(\Lambda(y), j)\\
&= (\Lambda(x)B^i\Lambda(y)B^j\bar{B}B^{n-2-i*j}, i*j)\\
&= (\Lambda(x)B^i\Lambda(y)B^jB^{2-n}B^{n-2-i*j}, i*j)\\
&= (\Lambda(x)B^i\Lambda(y)B^{j-i*j}, i*j)\\
&= (\Lambda(x)\Lambda(a)^i\Lambda(y)\Lambda(a)^{j-i*j}, i*j).
\end{aligned}$$

This shows that Λ^*_a is a group homomorphism.

Note that we have an isomorphism

$$q : (der^n(GL_m(\mathbb{C})))^*_{\Lambda(a)} \to (der^n(GL_m(\mathbb{C})))^*_I,$$

where I is the identity matrix. It is easy to see that

$$q(X, i) = (X\Lambda(a)^i, i),$$

for any $X \in GL_m(\mathbb{C})$. As we saw in the previous section, we have also an isomorphism

$$\varphi : (der^n(GL_m(\mathbb{C})))^*_I \to GL_m(\mathbb{C}) \times \mathbb{Z}_{n-1},$$

such that $\varphi(X, i) = (X, i+1)$. Now, let $\pi : GL_m(\mathbb{C}) \times \mathbb{Z}_{n-1} \to GL_m(\mathbb{C})$ be the projection. Combining all of these maps, we obtain

$$\Lambda^* = \pi\varphi q\Lambda^*_a : G^*_a \to GL_m(\mathbb{C}),$$

which is an ordinary representation of G_a^*. Note that

$$\begin{aligned}\Lambda^*(x,i) &= \pi\varphi q\Lambda_a^*(x,i)\\ &= \pi\varphi q(\Lambda(x),i)\\ &= \pi\varphi(\Lambda(x)\Lambda(a)^i,i)\\ &= \pi(\Lambda(x)\Lambda(a)^i,i+1)\\ &= \Lambda(x)\Lambda(a)^i.\end{aligned}$$

Conversely, suppose $\Gamma : G_a^* \to GL_m(\mathbb{C})$ is an ordinary representation of G_a^*. Since $G \subseteq G_a^*$, so by restriction we obtain a representation Γ_G for (G,f).

Lemma 2.4. *The maps $\Lambda \mapsto \Lambda^*$ and $\Gamma \mapsto \Gamma_G$ are inverse to each other.*

Proof. We have

$$\begin{aligned}(\Lambda^*)_G(x,0) &= \Lambda^*(x,0)\\ &= \Lambda(x)\Lambda(a)^0\\ &= \Lambda(x).\end{aligned}$$

On the other hand

$$\begin{aligned}(\Gamma_G)^*(x,i) &= \Gamma(x,0)\Gamma(a,0)^i\\ &= \Gamma(x,0)\Gamma(a,i-1)\\ &= \Gamma((x,0)\cdot(a,i-1))\\ &= \Gamma(f_*(x,\overset{(0)}{a},a,\overset{(i-1)}{a},\bar{a},\overset{(n-2-0*(i-1))}{a}),0*(i-1))\\ &= \Gamma(f_*(x,\overset{(i)}{a},\bar{a},\overset{(n-i-2)}{a}),i)\\ &= \Gamma(x,i).\end{aligned}$$

So, the maps are inverse to each other.

Note that G is a generating set for G_a^* and hence, Λ is irreducible, iff Λ^* is irreducible. Hence, we proved;

Theorem 2.5. *Let (G,f) be an n-ary group. Then there is a one-one correspondence between the set of all irreducible representations of G and those of G_a^*, for any $a \in G$. This correspondence is the map $\Lambda \mapsto \Lambda^*$. Especially, the number of irreducible representations of any finite polyadic group is finite.*

3. Applications

In this section, we apply the correspondence just we obtained, to generalize some results of representations theory of finite groups to the case of finite polyadic groups. In this section (G,f) is a finite n-ary group. Let $a \in G$ be any fixed element.

Suppose Λ_1, ..., Λ_k are all non-equivalent irreducible representations of G with degrees

$$d_1, d_2, \ldots, d_k.$$

Then the set of irreducible representations of G_a^* is

$$\Lambda_1^*, \ldots, \Lambda_k^*$$

with the same set of degrees. Since for any i, the order of G_a^* is divisible by d_i, and since we have

$$\sum_{i=1}^{k} d_i^2 = |G_a^*|,$$

so we have;

Theorem 3.1. *The number $(n-1)|G|$ is divisible by all d_i, and also we have*

$$\sum_{i=1}^{k} d_i^2 = (n-1)|G|.$$

Denote by $\mathrm{Irr}(G,f)$, the set of all irreducible characters of G. We can generalize the orthogonality property of ordinary irreducible characters, for those elements of $\mathrm{Irr}(G,f)$, which have non-empty kernels.

Theorem 3.2. *Let $\chi, \psi \in Irr(G,f)$ and $p \in \ker\chi$, $q \in \ker\psi$, and $a \in G$. Then we have*

$$\frac{1}{(n-1)|G|}\sum_{i=0}^{n-2}\sum_{x\in G}\chi(f(x, \overset{(i)}{a}, \overset{(n-i-1)}{p}))\psi(f(x, \overset{(i)}{a}, \overset{(n-i-1)}{q}))^* = \delta_{\chi,\psi},$$

where $*$ denotes complex conjugation.

Proof. Let $\hat{\chi}$ and $\hat{\psi}$ be the corresponding characters of G_a^*. Suppose Λ is a representation of G, whose character is χ. Then

$$\begin{aligned}
\hat{\chi}(x,i) &= Tr\ \Lambda^*(x,i) \\
&= Tr\ \Lambda(x)\Lambda(a)^i \\
&= Tr\ \Lambda(x)\Lambda(a)^i\Lambda(p)^{n-i-1} \\
&= Tr\ \Lambda(f(x, \overset{(i)}{a}, \overset{(n-i-1)}{p})) \\
&= \chi(f(x, \overset{(i)}{a}, \overset{(n-i-1)}{p})).
\end{aligned}$$

Similarly, for ψ, we have

$$\hat{\psi}(x,i) = \psi(f(x, \overset{(i)}{a}, \overset{(n-i-1)}{q})).$$

Hence, we have

$$\begin{aligned}\delta_{\chi,\psi} &= \delta_{\hat{\chi},\hat{\psi}}\\ &= \frac{1}{|G_a^*|}\sum_{(x,i)\in G_a^*}\hat{\chi}(x,i)\hat{\psi}(x,i)^*\\ &= \frac{1}{(n-1)|G|}\sum_{i=0}^{n-2}\sum_{x\in G}\chi(f(x,\overset{(i)}{a},\overset{(n-i-1)}{p}))\psi(f(x,\overset{(i)}{a},\overset{(n-i-1)}{q}))^*.\end{aligned}$$

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