A SYSTEM OF THIRD-ORDER DIFFERENTIAL OPERATORS CONFORMALLY INVARIANT UNDER $\mathfrak{so}(8, \mathbb{C})$

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Abstract. In earlier work, Barchini, Kable, and Zierau constructed a number of conformally invariant systems of differential operators associated to Heisenberg parabolic subalgebras in simple Lie algebras. The construction was systematic, but the existence of such a system was left open in several anomalous cases. Here, a conformally invariant system is shown to exist in the most interesting of these remaining cases. The construction may also be interpreted as giving an explicit homomorphism between generalized Verma modules for the Lie algebra of type D_4.

1. Introduction

Conformally invariant systems of differential operators on a manifold M on which a Lie algebra $\mathfrak{g}$ acts by first order differential operators were studied by Barchini, Kable, and Zierau in [1] and [2]. Loosely speaking, a conformally invariant system is a list of differential operators $D_1, \ldots, D_m$ that satisfies the bracket identity

$$[\Pi(X), D_j] = \sum_i C_{ij}(X) D_i,$$

where $\Pi(X)$ is the differential operator corresponding to $X \in \mathfrak{g}$ and $C_{ij}(X)$ are smooth functions on M. We shall give the definition of conformally invariant systems more precisely in Section 2. While a general theory of conformally invariant systems is developed in [2], examples of such systems of differential operators associated to the Heisenberg parabolic subalgebras of any complex simple Lie algebras are constructed in [1]. The purpose of this paper is to answer a question, left open in [1], concerning the existence of a certain conformally invariant system of third-order differential operators. This is done by constructing the required system. This result may be interpreted as giving an explicit homomorphism between two generalized Verma modules, one of which is non-scalar. The problem of constructing and classifying homomorphisms between scalar generalized Verma modules has received a lot of attention; for recent work, see, for example, [5]. Much less is known about maps between generalized Verma modules that are not necessarily scalar.

In order to explain our main results in this paper, we briefly review the results of [1] here. To begin with, let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be the parabolic subalgebra of Heisenberg type; that is, $\mathfrak{n}$ is a two-step nilpotent algebra with one-dimensional center. We denote by γ the highest root of $\mathfrak{g}$. For each root α let $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ be a corresponding $\mathfrak{sl}(2)$-triple, normalized as in Section 2 of [1]. Then $\mathrm{ad}(H_\gamma)$ on $\mathfrak{g}$ has eigenvalues -2, -1, 0, 1, 2, and the corresponding eigenspace decomposition of $\mathfrak{g}$ is denoted by

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus V^- \oplus \mathfrak{l} \oplus V^+ \oplus \mathfrak{z}(\mathfrak{n}).$$

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Let $\mathbb{D}[\mathfrak{n}]$ be the Weyl algebra of $\mathfrak{n}$. Then each system constructed in [1] derives from a $\mathbb{C}$-linear map $\Omega_k : \mathfrak{g}(2-k) \to \mathbb{D}[\mathfrak{n}]$ with $1 \leq k \leq 4$ and $\mathfrak{g}(2-k)$ the $2-k$ eigenspace of $\mathrm{ad}(H_\gamma)$. Let $\Pi_s : \mathfrak{g} \to \mathbb{D}[\mathfrak{n}]$ be the Lie algebra homomorphism constructed in Section 4 in [1]. Here s is a complex parameter. We say that the Ω_k system has special value s_0 when the system is conformally invariant for Π_{s_0}.

In [1] the special values of s are determined for the Ω_k systems with $k = 1, 2, 4$ for all complex simple Lie algebras, but only exceptional cases are considered for the Ω_3 system. A table in Section 8.10 in [1] lists the special values of s. The reader may want to notice that the entries in the columns for the systems Ω_2^{big} and Ω_2^{small} for types B_r and C_r should be transposed. Theorem 21 in [2] then shows that the Ω_3 system does not exist for A_r with $r \geq 3$, B_r with $r \geq 3$, and D_r with $r \geq 5$. There remain two open cases, namely, the Ω_3 system for type A_2 and the Ω_3 system for type D_4. The aim of this paper is to show that the Ω_3 system does exist for type D_4 (see Theorem 3.13). In order to achieve the result we use several facts from both [1] and [2]. By using these facts, we significantly reduce the amount of computation to show the existence of the system. In the other remaining case, for the algebra of type A_2, the Heisenberg parabolic subalgebra coincides with the Borel subalgebra, and the existence of the Ω_3 system(s) follows from the standard reducibility result for Verma modules (see for instance [3, Theorem 7.6.24]).

There are two differences between our conventions here and those used in [1]. One is that we choose the parabolic $Q_0 = L_0N_0$ for the real flag manifold, while the opposite parabolic $\bar{Q}_0 = L_0\bar{N}_0$ is chosen in [1]. Because of this, our special values of s are of the form $s = -s_0$, where s_0 are the special values shown in Section 8.10 in [1]. The other is that we identify $(V^+)^*$ with V^- by using the Killing form, while $(V^+)^*$ in [1] is identified with V^+ by using the non-degenerate alternating form $\langle \cdot, \cdot \rangle$ on V^+ defined by $[X_1, X_2] = \langle X_1, X_2 \rangle X_\gamma$ for $X_1, X_2 \in V^+$. Because of this difference the right action R, which will be defined in Section 2, will play the role played by Ω_1 in [1]. In addition to these notational differences, there are also some methodological differences between [1] and what we do here. These stem from the fact that we make systematic use of the results of [2] to streamline the process of proving conformal invariance.

We now outline the remainder of this paper. In Section 2, we review the setting and results of Section 5 in [2], simultaneously specializing them to the situation considered here. It would be helpful for the reader to be familiar with [2], particularly the concepts of $\mathfrak{g}$-manifold and $\mathfrak{g}$-bundle, at this point; the definitions may be found on pp. 790-791 of [2]. In Section 3, we specialize further by taking $\mathfrak{g}$ to be of type D_4. We fix a suitable Chevalley basis and give the definition of the $\tilde{\Omega}_3$ system whose conformal invariance is to be established. A remark on notation might be helpful here. In [1], a system Ω_3' is initially defined. It is then shown to decompose as a sum of a leading term $\tilde{\Omega}_3$ and a correction term C_3. These two are recombined with different coefficients to give Ω_3, which is finally shown to be conformally invariant for exceptional algebras. For type D_4, it emerges that $\Omega_3 = \tilde{\Omega}_3$, so that the correction term C_3 is discarded completely. For this reason, we do not recapitulate the process. Rather, we simply introduce $\tilde{\Omega}_3$ and proceed to show that it is conformally invariant. This is done in Theorem 3.11, which is our main result.

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2. Conformally Invariant Systems

The purpose of this section is to introduce the notion of conformally invariant systems. Let G_0 be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$. Let Q_0 be a parabolic subgroup of G_0 and $Q_0 = L_0N_0$ a Levi decomposition of Q_0. By the Bruhat decomposition, the subset $\bar{N}_0Q_0$ of G_0 is open and dense in G_0, where $\bar{N}_0$ is the nilpotent subgroup of G_0 opposite to N_0. Let $\bar{\mathfrak{n}}$ and $\mathfrak{q}$ be the complexifications of the Lie algebras of $\bar{N}_0$ and Q_0, respectively; we have the direct sum $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{q}$. For $Y \in \mathfrak{g}$, write $Y = Y_{\bar{\mathfrak{n}}} + Y_{\mathfrak{q}}$ for the decomposition of Y in this direct sum. Similarly, write the Bruhat decomposition of $g \in \bar{N}_0Q_0$ as $g = \bar{\mathbf{n}}(g)\mathbf{q}(g)$ with $\bar{\mathbf{n}}(g) \in \bar{N}_0$ and $\mathbf{q}(g) \in Q_0$. Note that for $Y \in \mathfrak{g}_0$ we have

$$Y_{\bar{\mathfrak{n}}} = \frac{d}{dt}\bar{\mathbf{n}}(\exp(tY))\big|_{t=0},$$

and a similar equality holds for $Y_{\mathfrak{q}}$.

We consider the homogeneous space G_0/Q_0. Let $\mathbb{C}_{\chi^{-s}}$ be the one-dimensional representation of L_0 with character χ^{-s}. The representation χ^{-s} is extended to a representation of Q_0 by making it trivial on N_0. For any manifold M, denote by $C^\infty(M, \mathbb{C}_{\chi^{-s}})$ the smooth functions from M to $\mathbb{C}_{\chi^{-s}}$. The group G_0 acts on the space

$$C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) = \{F \in C^\infty(G_0, \mathbb{C}_{\chi^{-s}}) \mid F(gq) = \chi^{-s}(q^{-1})F(g) \text{ for all } q \in Q_0 \text{ and } g \in G_0\}$$

by left translation, and the action Π_s of $\mathfrak{g}$ on $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ arising from this action is given by

$$(\Pi_s(Y)\bullet F)(g) = \frac{d}{dt}F(\exp(-tY)g)\big|_{t=0}$$

for $Y \in \mathfrak{g}_0$. Here the dot $\bullet$ denotes the action of $\Pi_s(Y)$. This action is extended $\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We use the same symbols for the extended actions.

The restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ is an injection whose image is dense for the smooth topology. We may define the action of $\mathcal{U}(\mathfrak{g})$ on the image of the restriction map by $\Pi_s(u)\bullet f = \big(\Pi_s(u)\bullet F\big)|_{\bar{N}_0}$ for $u \in \mathcal{U}(\mathfrak{g})$ and $F \in C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ with $f = F|_{\bar{N}_0}$. Define a right action R of $\mathcal{U}(\bar{\mathfrak{n}})$ on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$\big(R(X)\bullet f\big)(\bar{n}) = \frac{d}{dt}f\big(\bar{n}\exp(tX)\big)\big|_{t=0}$$

for $X \in \bar{\mathfrak{n}}_0$ and $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. A direct computation shows that

$$\big(\Pi_s(Y)\bullet f\big)(\bar{n}) = -sd\chi\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)f(\bar{n}) - \big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet f\big)(\bar{n}) \tag{2.1}$$

for $Y \in \mathfrak{g}$ and f in the image of the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. This equation implies that the representation Π_s extends to a representation of $\mathcal{U}(\mathfrak{g})$ on the whole space $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. Note that for all $Y \in \mathfrak{g}$, the linear map $\Pi_s(Y)$ is in $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}}) \oplus \mathfrak{X}(\bar{N}_0)$, where

$\mathfrak{X}(\bar{N}_0)$ is the space of smooth vector fields on $\bar{N}_0$. This property of $\Pi_s(Y)$ makes $\bar{N}_0$ a $\mathfrak{g}$-*manifold* in the sense of [2, page 790].

Let $\mathcal{L}_{-s}$ be the trivial bundle of $\bar{N}_0$ with fiber $\mathbb{C}_{\chi^{-s}}$. Then the space of smooth sections of $\mathcal{L}_{-s}$ is identified with $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. An operator $D : C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ is said to be a *differential operator* if it is of the form

$$D = \sum_{|\alpha| \leq k} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha},$$

where $a_\alpha \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$, $k \in \mathbf{Z}_{\geq 0}$, and multi-index notation is being used.

Denote the space of differential operators by $\mathbb{D}(\mathcal{L}_{-s})$. The elements of $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ may be regarded as differential operators by identifying them with the multiplication operator they induce. A computation shows that in $\mathbb{D}(\mathcal{L}_{-s})$,

$$\big([\Pi_s(Y), f]\big)(\bar{n}) = -\big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big) \bullet f\big)(\bar{n})$$

for $Y \in \mathfrak{g}$ and $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. This verifies that Π_s gives $\mathcal{L}_{-s}$ the structure of a $\mathfrak{g}$-*bundle* in the sense of [2, page 791].

Definition 2.2. *Let Π_s and $\mathcal{L}_{-s}$ be as above. A* conformally invariant system *on $\mathcal{L}_{-s}$ with respect to Π_s is a list of differential operators $D_1, \ldots, D_m \in \mathbb{D}(\mathcal{L}_{-s})$ so that the following two conditions are satisfied:*

- (C1) *The list $D_1, \ldots, D_m$ is linearly independent at each point of $\bar{N}_0$.*
- (C2) *For each $Y \in \mathfrak{g}$ there is an $m \times m$ matrix $C(Y)$ of smooth functions on $\bar{N}_0$ so that, in $\mathbb{D}(\mathcal{L}_{-s})$,*

$$[\Pi_s(Y), D_j] = \sum_i C_{ij}(Y) D_i.$$

The map $C : \mathfrak{g} \to M_{m\times m}(C^\infty(\bar{N}_0))$ is called the *structure operator.*

Now we define

$$\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}} = \{D \in \mathbb{D}(\mathcal{L}_{-s}) \mid [\Pi_s(X), D] = 0 \text{ for all } X \in \bar{\mathfrak{n}}\}.$$

Proposition 2.3. [2, Proposition 13] *Let $D_1, \ldots, D_m$ be a list of operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. Suppose that the list is linearly independent at e and that there is a map $b : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{C})$ such that*

$$\big([\Pi_s(Y), D_i] \bullet f\big)(e) = \sum_{j=1}^m b(Y)_{ji} (D_j \bullet f)(e)$$

for all $Y \in \mathfrak{g}$, $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$, and $1 \leq i \leq m$. Then $D_1, \ldots, D_m$ is a conformally invariant system on $\mathcal{L}_{-s}$. The structure operator of the system is given by $C(Y)(\bar{n}) = b(\mathrm{Ad}(\bar{n}^{-1})Y)$ for all $\bar{n} \in \bar{N}_0$ and $Y \in \mathfrak{g}$.

As shown on p.802 in [2] the differential operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ can be described in terms of elements of the generalized Verma module

$$\mathcal{M}(\mathbb{C}_{sd\chi}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{sd\chi},$$

where $\mathbb{C}_{sd\chi}$ is the $\mathfrak{q}$-module derived from the Q_0-representation $(\chi^s, \mathbb{C})$. By identifying $\mathcal{M}(\mathbb{C}_{sd\chi})$ as $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{sd\chi}$, the map $\mathcal{M}(\mathbb{C}_{sd\chi}) \to \mathcal{U}(\bar{\mathfrak{n}})$ given by $u \otimes 1 \mapsto u$ is an isomorphism. The composition

$$\mathcal{M}(\mathbb{C}_{sd\chi}) \to \mathcal{U}(\bar{\mathfrak{n}}) \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}} \tag{2.4}$$

is then a vector-space isomorphism, where the map $\mathcal{U}(\bar{\mathfrak{n}}) \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is given by $u \mapsto R(u)$.

Suppose that $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ and $l \in L_0$. Then we define an action of L_0 on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$(l \cdot f)(\bar{n}) = \chi^{-s}(l) f(l^{-1}\bar{n}l).$$

This action agrees with the action of L_0 by left translation on the image of the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. In terms of this action we define an action of L_0 on $\mathbb{D}(\mathcal{L}_{-s})$ by

$$(l \cdot D)\bullet f = l \cdot \big(D\bullet(l^{-1} \cdot f)\big).$$

One can check that we have $l \cdot R(u) = R(\mathrm{Ad}(l)u)$ for $l \in L_0$ and $u \in \mathcal{U}(\bar{\mathfrak{n}})$; in particular this L_0-action stabilizes the subspace $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. Also L_0 acts on $\mathcal{M}(\mathbb{C}_{sd\chi})$ by $l \cdot (u \otimes z) = \mathrm{Ad}(l)u \otimes z$, and with these actions, the isomorphism (2.4) is L_0-equivariant. For $D \in \mathbb{D}(\mathcal{L}_{-s})$, we denote by $D_{\bar{n}}$ the linear functional $f \mapsto (D\bullet f)(\bar{n})$ for $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. The following result is the specialization of Theorem 15 in [2] to the present situation.

Theorem 2.5. *Suppose that F is a finite-dimensional $\mathfrak{q}$-submodule of the generalized Verma module $\mathcal{M}(\mathbb{C}_{sd\chi})$. Let $f_1, \ldots, f_k$ be a basis of F and define constants $a_{ri}(Y)$ by*

$$Y f_i = \sum_{r=1}^{k} a_{ri}(Y) f_r$$

for $1 \le i \le k$ and $Y \in \mathfrak{q}$. Let $D_1, \ldots, D_k \in \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ correspond to the elements $f_1, \ldots, f_k \in F$. Then

$$[\Pi_s(Y), D_i]_{\bar{n}} = \sum_{r=1}^{k} a_{ri}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)(D_r)_{\bar{n}} - sd\chi\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)(D_i)_{\bar{n}}$$

for all $Y \in \mathfrak{g}$, $1 \le i \le k$, and $\bar{n} \in \bar{N}_0$.

3. The Ω_3 System on $\mathfrak{so}(8, \mathbb{C})$

In this section, we specialize to the situation where G_0 is a real form of the group $SO(8, \mathbb{C})$ that contains a real parabolic subgroup of Heisenberg type. In this setting, we construct a system of differential operators on the bundle $\mathcal{L}_1$ and show that it is conformally invariant. We first introduce some notation.

Let $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Fix Δ^+ a positive system and denote by S the corresponding set of simple roots. We denote the highest root by γ. Let $B_{\mathfrak{g}}$ denote a positive multiple of the Killing form on $\mathfrak{g}$ and denote by $(\cdot, \cdot)$ the corresponding inner product induced on $\mathfrak{h}^*$. The normalization of $B_{\mathfrak{g}}$ will be specified below. Let us write $||\alpha||^2 = (\alpha, \alpha)$ for any $\alpha \in \Delta$. For $\alpha \in \Delta$, we let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ corresponding to α. For any $\mathrm{ad}(\mathfrak{h})$-invariant subspace $V \subset \mathfrak{g}$, we denote by $\Delta(V)$ the set of roots α so that $\mathfrak{g}_\alpha \subset V$.

It is known that we can choose $X_\alpha \in \mathfrak{g}_\alpha$ and $H_\alpha \in \mathfrak{h}$ for each $\alpha \in \Delta$ in such a way that the following conditions hold. The reader may want to note that our normalizations are special cases of those used in [1].

(C1) For each $\alpha \in \Delta^+$, $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ is an $\mathfrak{sl}(2)$-triple. In particular,

$$[X_\alpha, X_{-\alpha}] = H_\alpha.$$

(C2) For each $\alpha, \beta \in \Delta$, $[H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta$.

(C3) For $\alpha \in \Delta$ we have $B_{\mathfrak{g}}(X_\alpha, X_{-\alpha}) = 1$; in particular, $(\alpha, \alpha) = 2$.

(C4) For $\alpha, \beta \in \Delta$ we have $\beta(H_\alpha) = (\beta, \alpha)$.

Let $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type; that is, the parabolic subalgebra corresponding to the subset $\{\alpha \in S \mid (\alpha, \gamma) = 0\}$. Denote by $\mathfrak{l}$ the Levi factor of $\mathfrak{q}$ and by $\mathfrak{n}$ the nilpotent radical of $\mathfrak{q}$. Then the action of $\mathrm{ad}(H_\gamma)$ on $\mathfrak{g}$ has eigenvalues -2, -1, 0, 1, 2, and the corresponding eigenvalue decomposition of $\mathfrak{g}$ is denoted by

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus V^- \oplus \mathfrak{l} \oplus V^+ \oplus \mathfrak{z}(\mathfrak{n}).$$

Note that V^+ and V^- are irreducible $\mathfrak{l}$-modules, since the Heisenberg parabolic $\mathfrak{q}$ is maximal (see [4, Exercise 5, page 638] for instance).

Let $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ be the deleted Dynkin diagram associated to the Heisenberg parabolic $\mathfrak{q}$; that is, the subdiagram of the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$ obtained by deleting the node corresponding to the simple root that is not orthogonal to γ, and the edges that involve it.

As on p.789 in [1] the operator Ω_2 is given in terms of R by

$$\Omega_2(Z) = -\frac{1}{2} \sum_{\alpha, \beta \in \Delta(V^+)} N_{\beta, \beta'} M_{\alpha, \beta'}(Z) R(X_{-\alpha}) R(X_{-\beta})$$

for $Z \in \mathfrak{l}$. It follows from Theorem 5.2 of [1] and the data tabulated in Section 8.10 of [1] that each Ω_2 system associated to a singleton component of $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ is conformally invariant on the line bundle $\mathcal{L}_1$. The reader may want to note here that the special values of our Ω_2 system are of the form $-s_0$ with s_0 the special values of the Ω_2 system given in [1], because the parabolic $\mathfrak{q}$ is chosen in this paper, while the opposite parabolic $\bar{\mathfrak{q}}$ is chosen in [1]. One can also check that we have $\Omega_2(\mathrm{Ad}(l)Z) = \chi(l) l \cdot \Omega_2(Z)$ for all $l \in L_0$. Note that this is different from the $\mathrm{Ad}(l)$ transformation law that appears in [1], for the same reason. We extend the $\mathbb{C}$-linear maps $d\chi$, R, and Ω_2 to be left $C^\infty(\bar{N}_0)$-linear so that certain relationships can be expressed more easily.

In the rest of this paper our line bundle is assumed to be $\mathcal{L}_1$ and for simplicity we denote Π_1 by Π. Now we define an operator $\tilde{\Omega}_3$ on $C^\infty(\bar{N}_0, \mathbb{C}_\chi)$ by

$$\tilde{\Omega}_3(Y) = \sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\Omega_2\big([X_\epsilon, Y]\big)$$

for $Y \in V^-$.

Lemma 3.1. *Let $W_1, \ldots, W_m$ be a basis for V^+ and $W_1^*, \ldots, W_m^*$ be the $B_{\mathfrak{g}}$-dual basis of V^-. Then*

$$\tilde{\Omega}_3(Y) = \sum_{i=1}^{m} R(W_i^*)\Omega_2\big([W_i, Y]\big).$$

Proof. Suppose that $\Delta(V^+) = \{\epsilon_1, \ldots, \epsilon_m\}$. Each W_i then may be expressed by

$$W_i = \sum_{j=1}^{m} a_{ij} X_{\epsilon_j}$$

for $a_{ij} \in \mathbb{C}$. Let $[a_{ij}]$ be the change of basis matrix and set $[b_{ij}] = [a_{ij}]^{-1}$. Then define

$$W_i^* = \sum_{k=1}^{m} b_{ki} X_{-\epsilon_k}$$

for $i = 1, \ldots, m$. Since $B_{\mathfrak{g}}(X_{\epsilon_i}, X_{-\epsilon_j}) = \delta_{ij}$ with δ_{ij} the Kronecker delta, it follows that

$$B_{\mathfrak{g}}(W_i, W_j^*) = \delta_{ij}.$$

Thus $\{W_1^*, \ldots, W_m^*\}$ is the dual basis of $\{W_1, \ldots, W_m\}$. Note that we have $\sum_{i=1}^{m} b_{ki}a_{ij} = \delta_{kj}$ since $[b_{ij}][a_{ij}] = I$. Then a direct computation shows that

$$\begin{aligned}\sum_{i=1}^{m} R(W_i^*)\Omega_2([W_i, Y]) &= \sum_{j,k=1}^{m} \big(\sum_{i=1}^{m} b_{ki}a_{ij}\big) R(X_{-\epsilon_k})\Omega_2([X_{\epsilon_j}, Y]) \\ &= \sum_{j=1}^{m} R(X_{-\epsilon_j})\Omega_2([X_{\epsilon_j}, Y]).\end{aligned}$$

This completes the proof. □

Lemma 3.2. *For all $l \in L_0$, $Z \in \mathfrak{l}$, and $Y \in V^-$, we have*

$$\tilde{\Omega}_3(\mathrm{Ad}(l)Y) = \chi(l)l \cdot \tilde{\Omega}_3(Y) \tag{3.3}$$

and

$$[\Pi(Z), \tilde{\Omega}_3(Y)] = \tilde{\Omega}_3\big([Z, Y]\big) - d\chi(Z)\tilde{\Omega}_3(Y).$$

Proof. Recall that $l \cdot R(u) = R(\mathrm{Ad}(l)u)$ for $l \in L_0$ and $u \in \mathcal{U}(\bar{\mathfrak{n}})$. Since we have $\Omega_2(\mathrm{Ad}(l)W) = \chi(l)l \cdot \Omega_2(W)$ for $l \in L_0$ and $W \in \mathfrak{l}$, it follows that

$$\chi(l)l \cdot \tilde{\Omega}_3(Y) = \sum_{\epsilon\in\Delta(V^+)} R(\mathrm{Ad}(l)X_{-\epsilon})\Omega_2\big([\mathrm{Ad}(l)X_\epsilon, \mathrm{Ad}(l)Y]\big). \tag{3.4}$$

By Lemma 3.1, the value of $\tilde{\Omega}_3(Y)$ is independent from a choice of a basis for V^+. Therefore the right hand side of (3.4) is equal to the sum $\sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\Omega_2\big([X_\epsilon, \mathrm{Ad}(l)Y]\big)$, which is $\tilde{\Omega}_3(\mathrm{Ad}(l)Y)$. The second equality is obtained by differentiating the first. □

Proposition 3.5. *We have*

$$[\Pi(X), R(Y)]_{\bar{n}} = R\big([\mathrm{Ad}(\bar{n}^{-1})X, Y]_{V^-}\big)_{\bar{n}} - d\chi\big([\mathrm{Ad}(\bar{n}^{-1})X, Y]_{\mathfrak{l}}\big)$$

for all $X \in \mathfrak{g}$*,* $Y \in V^-$*, and* $\bar{n} \in \bar{N}_0$*.*

Proof. Let F be the subspace of $\mathcal{M}(\mathbb{C}_{-d\chi})$ spanned by $X_{-\alpha} \otimes 1$ and $1 \otimes 1$ with $\alpha \in \Delta(V^+)$. A direct computation shows that F is a $\mathfrak{q}$-submodule of $\mathcal{M}(\mathbb{C}_{-d\chi})$ and that for $Z \in \mathfrak{l}$ and $U \in \mathfrak{n}$ we have

$$Z(X_{-\alpha} \otimes 1) = [Z, X_{-\alpha}] \otimes 1 - d\chi(Z) X_{-\alpha} \otimes 1$$

and

$$U(X_{-\alpha} \otimes 1) = -d\chi([U, X_{-\alpha}]_{\mathfrak{l}}) 1 \otimes 1.$$

Then it follows from Theorem 2.5 that if $X \in \mathfrak{g}$ and $\big(\mathrm{Ad}(\bar{n}^{-1})X\big)_{\mathfrak{q}} = Z + U$ with $Z \in \mathfrak{l}$ and $U \in \mathfrak{n}$ then for $Y \in V^-$,

$$[\Pi(X), R(Y)]_{\bar{n}} = R\big([Z, Y]\big)_{\bar{n}} - d\chi\big([U, Y]\big).$$

Since $[Z, Y] = [\mathrm{Ad}(\bar{n}^{-1})X, Y]_{V^-}$ and $[U, X_{-\alpha}]_{\mathfrak{l}} = [\mathrm{Ad}(\bar{n}^{-1})X, Y]_{\mathfrak{l}}$, this completes the proof. □

Let $\omega_2(X)$ denote the element in $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-d\chi}$ that corresponds to $\Omega_2(X)$ under R.

Lemma 3.6. *For* $W, Z \in \mathfrak{l}$*, we have*

$$\omega_2\big([Z, W]\big) = Z\omega_2(W) + 2d\chi(Z)\omega_2(W).$$

Proof. Since $\Omega_2(\mathrm{Ad}(l)W) = \chi(l) l \cdot \Omega_2(W)$ for $l \in L_0$, we have $\omega_2(\mathrm{Ad}(l)W) = \chi(l)\mathrm{Ad}(l)\omega_2(W)$ by Lemma 18 in [2]. Then the formula is obtained by replacing l by $\exp(tZ)$ with $Z \in \mathfrak{l}_0$, differentiating, and setting at $t = 0$. □

Proposition 3.7. *We have*

$$[\Pi(X), \Omega_2(W)]_{\bar{n}} = \Omega_2\big([\mathrm{Ad}(\bar{n}^{-1})X, W]_{\mathfrak{l}}\big)_{\bar{n}} - d\chi\big((\mathrm{Ad}(\bar{n}^{-1})X)_{\mathfrak{l}}\big)\Omega_2(W)_{\bar{n}}$$

for all $X \in \mathfrak{g}$*,* $W \in \mathfrak{l}$*, and* $\bar{n} \in \bar{N}_0$*.*

Proof. Recall that the Ω_2 system is conformally invariant on the line bundle $\mathcal{L}_1$. Therefore $F \equiv \mathrm{span}_{\mathbb{C}}\{\omega_2(W) \mid W \in \mathfrak{l}\}$ is a $\mathfrak{q}$-submodule of $\mathcal{M}(\mathbb{C}_{-d\chi})$. By applying Lemma 3.6 with $Z = H_\gamma$, we obtain $H_\gamma \omega_2(W) = -4\omega_2(W)$ for all $W \in \mathfrak{l}$. For $U \in V^+$ we have $H_\gamma U \omega_2(W) = -3U\omega_2(W)$, and $H_\gamma X_\gamma \omega_2(W) = -2X_\gamma \omega_2(W)$ for all $W \in \mathfrak{l}$. Therefore if $U \in \mathfrak{n}$ then $U\omega_2(W) = 0$ for all $W \in \mathfrak{l}$, because otherwise $U\omega_2(W)$ would have the wrong H_γ-eigenvalue to lie in F. Since Lemma 3.6 shows that

$$Z\omega_2(W) = \omega_2([Z, W]) - 2d\chi(Z)\omega_2(W)$$

for $Z, W \in \mathfrak{l}$, the proposed formula now follows from Theorem 2.5. □

Lemma 3.8. *For* $X \in V^+$ *and* $Y \in V^-$*, we have*

$$\sum_{\epsilon \in \Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big) = 2\Omega_2([X, Y]).$$

Proof. Since we have $||\epsilon||^2 = 2$ for all $\epsilon \in \Delta(V^+)$, it follows from Proposition 2.2 of [1] that

$$\sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big) = \frac{1}{2}\sum_{\mathcal{C}} p(D_4, \mathcal{C})\Omega_2\big(\mathrm{pr}_{\mathcal{C}}([X, Y])\big),$$

where $\mathcal{C}$ are the connected components of $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ as in [1] and $\mathrm{pr}_{\mathcal{C}}([X, Y])$ is the projection of $[X, Y]$ onto $\mathfrak{l}(\mathcal{C})$, the ideal of $[\mathfrak{l}, \mathfrak{l}]$ corresponding to $\mathcal{C}$. (See Section 2 of [1] for further discussion.) One can find in Section 8.4 of [1] that $p(D_4, \mathcal{C}) = 4$ for all the components $\mathcal{C}$. Then the fact that $\Omega_2(H_\gamma) = 0$ shows that we obtain

$$\sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big) = 2\Omega_2\big([X, Y]\big),$$

which is the proposed formula. □

Now with the above lemmas and propositions we are ready to show the following key theorem.

Theorem 3.9. *We have* $[\Pi(X), \tilde{\Omega}_3(Y)]_e = 0$ *for all* $X \in V^+$ *and all* $Y \in V^-$.

Proof. The commutator $[\Pi(X), \tilde{\Omega}_3(Y)]$ is a sum of two terms. One of them is given by

$$\begin{aligned}(3.10)\quad &\sum_{\epsilon\in\Delta(V^+)} [\Pi(X), R(X_{-\epsilon})]\Omega_2\big([X_\epsilon, Y]\big)\\ &= \sum_{\epsilon\in\Delta(V^+)} R\big([\mathrm{Ad}(\cdot^{-1})X, X_{-\epsilon}]_{V^-}\big)\Omega_2\big([X_\epsilon, Y]\big) - \sum_{\epsilon\in\Delta(V^+)} d\chi\big([\mathrm{Ad}(\cdot^{-1})X, X_{-\epsilon}]_{\mathfrak{l}}\big)\Omega_2\big([X_\epsilon, Y]\big),\end{aligned}$$

by Proposition 3.5. At e, the first term is zero, since $[X, X_{-\epsilon}]_{V^-} = 0$ for all $\epsilon \in \Delta(V^+)$. By writing out X as a linear combination of X_α with $\alpha \in \Delta(V^+)$, one can see that at the identity the second term in (3.10) evaluates to

$$-\sum_{\epsilon\in\Delta(V^+)} d\chi\big([X, X_{-\epsilon}]\big)\Omega_2\big([X_\epsilon, Y]\big)_e = -\Omega_2\big([X, Y]\big)_e$$

since $d\chi(H_\alpha) = 1$ for $\alpha \in \Delta(V^+)$. The other term is given by

$$\begin{aligned}(3.11)\quad &\sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\big[\Pi(X), \Omega_2([X_\epsilon, Y])\big]\\ &= \sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\Omega_2\big([\mathrm{Ad}(\cdot^{-1})X, [X_\epsilon, Y]]_{\mathfrak{l}}\big) - \sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})d\chi\big((\mathrm{Ad}(\cdot^{-1})X)_{\mathfrak{l}}\big)\Omega_2\big([X_\epsilon, Y]\big),\end{aligned}$$

by Proposition 3.7. To further evaluate this expression, we make use of a simple general observation. Namely, if D is a first order differential operator, ϕ and ψ are smooth functions, and $\phi(e) = 0$ then $D_e(\phi\psi) = D_e(\phi)\psi(e)$. Notice that $\bar{n} \mapsto \mathrm{ad}(\mathrm{Ad}(\bar{n}^{-1})X)$ is a smooth function on $\bar{N}_0$. It follows from the left $C^\infty(\bar{N}_0)$-linear extension of Ω_2 that the first term of the right hand side of (3.11) can be expressed as

$$\sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\big(\mathrm{ad}(\mathrm{Ad}(\cdot^{-1})X)_{\mathfrak{l}} \cdot \Omega_2\big([X_\epsilon, Y]\big)\big),$$

where $\mathrm{ad}(\mathrm{Ad}(\cdot^{-1})X)_{\mathfrak{l}}$ denotes the map $Z \mapsto [\mathrm{Ad}(\cdot^{-1})X, Z]_{\mathfrak{l}}$ for $Z \in \mathfrak{g}$. Since we have

$$\big(R(X_{-\epsilon})\bullet(\mathrm{Ad}(\cdot^{-1})X)\big)(e) = [X, X_{-\epsilon}],$$

$[X,[X_\epsilon,Y]]_{\mathfrak{l}}=0$, and $X_{\mathfrak{l}}=0$, the right hand side of (3.11) then evaluates at the identity to

$$\sum_{\epsilon\in\Delta(V^+)}\Omega_2\big([[X,X_{-\epsilon}],[X_\epsilon,Y]]\big)_e-\sum_{\epsilon\in\Delta(V^+)}d\chi\big([X,X_{-\epsilon}]\big)\Omega_2\big([X_\epsilon,Y]\big)_e,$$

which is equivalent to

$$\sum_{\epsilon\in\Delta(V^+)}\Omega_2\big([[X,X_{-\epsilon}],[X_\epsilon,Y]]\big)_e-\Omega_2\big([X,Y]\big)_e.$$

Therefore we obtain

$$[\Pi(X),\tilde{\Omega}_3(Y)]_e=\sum_{\epsilon\in\Delta(V^+)}\Omega_2\big([[X,X_{-\epsilon}],[X_\epsilon,Y]]\big)_e-2\Omega_2\big([X,Y]\big)_e.$$

Now it follows from Lemma 3.8 that $[\Pi(X),\tilde{\Omega}_3(Y)]_e=0$. □

Proposition 3.12. *For $Y\in V^-$, we have $[\Pi(X_\gamma),\tilde{\Omega}_3(Y)]_e=0$.*

Proof. Since $\mathfrak{z}(\mathfrak{n})=[V^+,V^+]$, it suffices to show that $[\Pi\big([X_1,X_2]\big),\tilde{\Omega}_3(Y)]_e=0$ for $X_1,X_2\in V^+$. Note that we have $\Pi\big([X_1,X_2]\big)=[\Pi(X_1),\Pi(X_2)]$, so it follows from the Jacobi identity that $[\Pi\big([X_1,X_2]\big),\tilde{\Omega}_3(Y)]$ may be expressed as a sum of two terms. The first is

$$[\Pi(X_1),[\Pi(X_2),\tilde{\Omega}_3(Y)]]=\Pi(X_1)[\Pi(X_2),\tilde{\Omega}_3(Y)]-[\Pi(X_2),\tilde{\Omega}_3(Y)]\Pi(X_1).$$

By (2.1), we have $\Pi(X)_e=0$ for all $X\in\mathfrak{n}$. Using this fact and Theorem 3.9, it is obtained that $[\Pi(X_1),[\Pi(X_2),\tilde{\Omega}_3(Y)]]_e=0$ since $(D_1D_2)_e=(D_1)_eD_2$ for $D_1,D_2\in\mathbb{D}(\mathcal{L}_1)$. The second term is

$$[\Pi(X_2),[\tilde{\Omega}_3(Y),\Pi(X_1)]]=\Pi(X_2)[\tilde{\Omega}_3(Y),\Pi(X_1)]-[\tilde{\Omega}_3(Y),\Pi(X_1)]\Pi(X_2).$$

It follows from the same argument for the first term that we have $[\Pi(X_2),[\tilde{\Omega}_3(Y),\Pi(X_1)]]_e=0$. This concludes that the proposition. □

Theorem 3.13. *Let $\mathfrak{g}$ be the complex simple Lie algebra of type D_4, and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then the $\tilde{\Omega}_3$ system is conformally invariant on the line bundle $\mathcal{L}_1$.*

Proof. For $Y\in V^-$, it follows from Lemma 3.2 that

$$[\Pi(Z),\tilde{\Omega}_3(Y)]_e=\tilde{\Omega}_3\big([Z,Y]\big)_e-d\chi(Z)\tilde{\Omega}_3(Y)_e$$

for all $Z\in\mathfrak{l}$. Also Theorem 3.9 and Proposition 3.12 show that $[\Pi(U),\tilde{\Omega}_3(Y)]=0$ for all $U\in\mathfrak{n}$. By the definition of $\tilde{\Omega}_3(Y)$, it is clear that $[\Pi(\bar{U}),\tilde{\Omega}_3(Y)]_e=0$ for all $\bar{U}\in\bar{\mathfrak{n}}$. Now by applying Proposition 2.3 we conclude that the $\tilde{\Omega}_3$ system is conformally invariant on $\mathcal{L}_1$. □

Let $\omega_3(Y)$ denote the element in $\mathcal{U}(\bar{\mathfrak{n}})\otimes\mathbb{C}_{-d\chi}$ that corresponds to $\tilde{\Omega}_3(Y)$ under R. Theorem 3.13 then implies that $E\equiv\mathrm{span}_{\mathbb{C}}\{\omega_3(Y)\mid Y\in V^-\}$ is a $\mathfrak{q}$-submodule of $\mathcal{M}(\mathbb{C}_{-d\chi})$. Note that it follows from (3.3) that we have $\omega_3(\mathrm{Ad}(l)Y)=\chi(l)\mathrm{Ad}(l)\omega_3(Y)$ for $l\in L_0$. By using the $\mathrm{Ad}(l)$ transformation law, one can check that a map $Y\otimes 1\mapsto\omega_3(Y)$ from $V^-\otimes\mathbb{C}_{-d\chi}$ to E is L_0-equivariant with the standard action of L_0 on $V^-\otimes\mathbb{C}_{-d\chi}$. In particular, E is an irreducible $\mathfrak{l}$-module, because V^- is $\mathfrak{l}$-irreducible. Since ω_3 has the same $\mathrm{Ad}(l)$ transformation law as ω_2, we have

$$\omega_3\big([Z,Y]\big)=Z\omega_3(Y)+2d\chi(Z)\omega_3(Y) \tag{3.14}$$

for $Y \in V^-$ and $Z \in \mathfrak{l}$. The same argument in the proof of Proposition 3.7 then shows that $\mathfrak{n}$ acts on E trivially. Hence, E is a leading $\mathfrak{l}$-type in the sense of [2, page 808].

Now there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism from a generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} E$ to $\mathcal{M}(\mathbb{C}_{-d\chi})$, that is given by

$$u \otimes \omega_3(Y) \mapsto u \cdot \omega_3(Y).$$

It follows from (3.14) that H_γ acts on E by -5, while it acts on $\mathbb{C}_{-d\chi}$ by -2; in particular, E is not equivalent to $\mathbb{C}_{-d\chi}$. We now conclude the following corollary.

Corollary 3.15. *Let $\mathfrak{g}$ be the complex simple Lie algebra of type D_4, and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then the generalized Verma module $\mathcal{M}(\mathbb{C}_{-d\chi})$ is reducible.*

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