arXiv:1011.0966v1 [math.PR] 3 Nov 2010

A SPATIAL VERSION OF THE ITÔ-STRATONOVICH CORRECTION

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ABSTRACT. We consider a class of stochastic PDEs of Burgers type in spatial dimension 1, driven by space-time white noise. Even though it is well-known that these equations are well-posed, it turns out that if one performs a spatial discretisation of the nonlinearity in the "wrong" way, then the sequence approximate equations does converge to a limit, but this limit exhibits an additional correction term.

This correction term is proportional to the local quadratic cross-variation (in space!) of the gradient of the conserved quantity with the solution itself. This can be understood as a consequence of the fact that for any fixed time, the law of the solution is locally equivalent to Wiener measure, where space plays the role of time. In this sense, the correction term is similar to the usual Itô-Stratonovich correction term that arises when one considers different temporal discretisations of stochastic ODEs.

1. INTRODUCTION

In this work, we give a rigorous analysis of the behaviour of stochastic Burgers equations in one spatial dimension under various approximation schemes. It was recently argued in [HV10] that if the approximation scheme fails to satisfy a certain symmetry condition, then one expects the approximations to converge to a modified equation, with the appearance of an additional correction term in the limit. This correction term is somewhat similar to the Itô-Stratonovich correction that appears in the study of SDEs when one compares centred and one-sided approximations. The present article provides a rigorous justification of the results observed in [HV10], at least in the case where the nonlinearity of the equation is of gradient type, and therefore the limiting equation is well-posed in a classical sense.

More precisely, we will consider in this work stochastic PDEs of the form

$$\partial_t u = \nu \partial_x^2 u + F(u) + \nabla G(u) \partial_x u + \xi \,, \tag{1.1}$$

where $u = u(x,t)$ is an $\mathbf{R}^n$-valued function, with $x \in [0, 2\pi]$, $t \geq 0$. In this equation, $\nu > 0$ is a positive constant, the functions $F, G : \mathbf{R}^n \to \mathbf{R}^n$ are assumed to be $\mathcal{C}^3$, and the stochastic forcing term ξ consists of independent space-time white noises in each component of $\mathbf{R}^n$. For the sake of simplicity, we endow this equation with periodic boundary conditions, but we do not expect this to alter our results significantly.

Equation (1.1) is locally well-posed [Gyö98], provided that we rewrite the nonlinearity as $\partial_x G(u)$ and consider solutions either in the weak or the mild form [DPZ92]. (Note that our noise term is *not* the gradient of space-time white noise, as in [BG97]. Therefore, our solutions are actually continuous functions.) The aim

Date: November 4, 2010.

2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 35K55, 60H30, 60H35.

Key words and phrases. Itô-Stratonovich correction, stochastic Burgers equation, spatial discretisations, Wiener chaos.

of this article is to show that this well-posedness is much less stable than one may imagine at first. Indeed, if we set

$$D_\varepsilon^+ u(x) = \frac{u(x+\varepsilon) - u(x)}{\varepsilon} ,$$

and consider the family u_ε of solutions to the approximate equation

$$\partial_t u_\varepsilon = \nu \partial_x^2 u_\varepsilon + F(u_\varepsilon) + \nabla G(u_\varepsilon) D_\varepsilon^+ u_\varepsilon + \xi ,$$

then our main result, Theorem 1.6 below, implies that $u_\varepsilon \Rightarrow \bar{u}$, where $\bar{u}$ is the solution to (1.1), but with F replaced by

$$\bar{F}(u) = F(u) - \frac{1}{4\nu} \Delta G(u) , \tag{1.2}$$

where Δ is the usual Laplacian on $\mathbf{R}^n$.

Remark 1.1. An explicit calculation allows to check that the local quadratic variation (in space!) of u for the solution to (1.1) is precisely given by $1/(2\nu)$. Therefore, one can interpret the correction term appearing in (1.2) as precisely being equal to $-\frac{1}{2}$ times the quadratic covariation between u and $\nabla G(u)$. Recall that is exactly the correction term that appears when one switches between Itô and Stratonovich integral in the usual setting of stochastic calculus.

Remark 1.2. This correction term is a purely stochastic effect and is completely unrelated to the fact that our discretisation scheme is not an upwind scheme (see [CIR52, MRtTB05]). In the absence of noise, we would still have the regularising property from the non-vanishing viscosity, so that pretty much any "reasonable" numerical scheme would converge to the correct solution.

If D_ε^+ is replaced by D_ε^-, defined by $D_\varepsilon^- u(x) = \big(u(x) - u(x-\varepsilon)\big)/\varepsilon$, then a similar result is true, but the sign in front of the correction term in (1.2) changes. We will actually consider a much more general class of approximations to (1.1), where we also allow both the linear operator ∂_x^2 and the noise term ξ to be replaced by approximate versions that are still translation-invariant, but modified at the lengthscale ε.

1.1. **Statement of the main result.** For $\varepsilon > 0$ we consider approximating stochastic PDEs of the type

$$\partial_t u_\varepsilon = \nu \Delta_\varepsilon u_\varepsilon + F(u_\varepsilon) + \nabla G(u_\varepsilon) D_\varepsilon u_\varepsilon + \xi_\varepsilon .$$

Since our system is invariant under spatial translations, it seems natural to restrict ourselves to a class of approximations that enjoys the same property. Throughout this article, we will therefore use approximate differential operators Δ_ε and D_ε, as well as an approximate space-time white noise ξ_ε given by their Fourier transforms:

$$\widehat{\Delta_\varepsilon u}(k) = -k^2 f(\varepsilon|k|)\widehat{u}(k) , \quad \widehat{D_\varepsilon u}(k) = ikg(\varepsilon k)\widehat{u}(k) , \quad \widehat{\xi_\varepsilon}(k) = h(\varepsilon|k|)\widehat{\xi}(k) .$$

We will make the following standing assumptions on these objects.

Assumption 1.3. *The function $f : [0,\infty) \to [0,+\infty]$ is twice differentiable at 0 with $f(0) = 1$ and $f'(0) = 0$. Furthermore, there exists $q \in (0,1]$ such that $f(k) \geq q$ for all $k > 0$.*

If $f(k) = +\infty$ for some values of k, we use the convention $\exp(-t\infty) = 0$ for every $t > 0$. In this case, the semigroup generated by Δ_ε is not strongly continuous, but this is of no consequence for our analysis.

Assumption 1.4. *There exists a signed Borel measure μ such that*

$$\int_{\mathbf{R}} e^{ikx}\,\mu(dx) = ik\,g(k)\ ,$$

and such that

$$\mu(\mathbf{R}) = 0\ ,\quad |\mu|(\mathbf{R}) < \infty\ ,\quad \int_{\mathbf{R}} x\,\mu(dx) = 1\ ,\quad \int_{\mathbf{R}} |x|^4\,|\mu|(dx) < \infty\ . \tag{1.3}$$

In particular, we have $(D_\varepsilon u)(x) := \frac{1}{\varepsilon}\int_{\mathbf{R}} u(x+\varepsilon y)\,\mu(dy)$.

Assumption 1.5. *The function h is bounded and such that h^2/f is of bounded variation. Furthermore, h is twice differentiable at the origin with $h(0) = 1$ and $h'(0) = 0$.*

Let $\bar{u}$ be the solution to the equation

$$\partial_t \bar{u} = \nu\partial_x^2 \bar{u} + \overline{F}(\bar{u}) + \nabla G(\bar{u})\partial_x \bar{u} + \xi\ , \tag{1.4}$$

In this equation,

$$\overline{F} := (F - \Lambda\Delta G)$$

and $\Lambda \in \mathbf{R}$ is a correction constant given by

$$\Lambda \stackrel{\text{def}}{=} \frac{1}{2\pi\nu}\int_{\mathbf{R}_+}\int_{\mathbf{R}} \frac{(1-\cos(yt))h^2(t)}{t^2 f(t)}\,\mu(dy)\,dt\ . \tag{1.5}$$

Note that a straightforward calculation shows that Λ is indeed well-defined, as a consequence of the fact that h^2/f is bounded by assumption and that $|\mu|$ has a finite second moment.

Before we state our main result, note that the equation (1.4) is locally well-posed in L^∞, see [BCF91, BCJL94, DPDT94, Gyö98, Hai09]. As a consequence, it has a well-defined blow-up time τ_* (possibly infinite) such that, almost surely, $\lim_{t\to\tau_*}\|\bar{u}(t)\|_{L^\infty} = +\infty$ on the event $\{\tau_* < \infty\}$. With this notation, we are now ready to state the main result of this paper.

Theorem 1.6. *Let $\kappa > 0$. There exists a sequence of stopping times τ_ε satisfying $\lim_{\varepsilon\to 0}\tau_\varepsilon = \tau_*$ in probability, and such that*

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big(\sup_{t\le\tau_\varepsilon}\|u_\varepsilon(t) - \bar{u}(t)\|_{L^\infty} > \varepsilon^{\frac{1}{8}-\kappa}\Big) = 0\ .$$

Remark 1.7. The statement of Theorem 1.6 is slightly incomplete since we do not specify how we choose the initial conditions. In order to avoid further technical complications, we will actually consider sequences of initial conditions that have the property that the initial condition for u_ε "behaves like" the solution $u_\varepsilon(t)$ for positive times. This statement will be made more precise in Theorem 2.2 below.

Before we proceed, we list some of the most common examples of discretisations that do fit our framework. For $a, b \ge 0$ with $a + b > 0$, it is natural to discretise the derivative operator by choosing

$$\mu := \frac{\delta_a - \delta_{-b}}{a+b}\ .$$

This is also the discretisation that was considered in [HV10]. As far as the discretisations of the noise and the Laplacian are concerned, there are at least three natural choices.

No discretisation. This is the case $f = h = 1$ where only the nonlinearity is discretised. With this choice, one can check that the correction factor is given by $\Lambda = \frac{1}{4\nu}\frac{a-b}{a+b}$.

Finite difference discretisation. In this case, we divide the interval $[0, 2\pi]$ into N equally sized intervals. For convenience we assume that N is odd and we set

$$\big(\Delta_\varepsilon u\big)(x) = \frac{1}{\varepsilon^2}\big(u(x+\varepsilon) + u(x-\varepsilon) - 2u(x)\big)\ , \quad \varepsilon = \frac{2\pi}{N}\ .$$

We furthermore identify a function u with the trigonometric polynomial of degree $(N-1)/2$ agreeing with u at the gridpoints. This corresponds to the choice

$$f(k) = \begin{cases} \frac{4}{k^2}\sin^2(k/2), & k \in [0,\pi) \\ +\infty, & k \in [\pi,\infty) \end{cases}\ , \qquad h = \mathbf{1}_{[0,\pi)}\ ,$$

The natural choice for the discretisation of the derivative operator in this case is to choose a and b to be integers, so that discretisation takes place on the gridpoints. With this choice, it can be shown that the correction factor is identical to that obtained in the previous case. Note however that this is *not* the case if the discretisation of the derivative operator is not adapted to the gridsize.

Galerkin discretisation. In this case, we approximate Δ and ξ by only keeping those Fourier modes that appear in the approximation by trigonometric polynomials. This corresponds to the choice

$$f(k) = \begin{cases} 1, & k \in [0,\pi) \\ +\infty, & k \in [\pi,\infty) \end{cases}\ , \qquad h = \mathbf{1}_{[0,\pi)}\ .$$

The correction factor Λ is then given by

$$\Lambda = \frac{\cos(\pi a) + \pi a\,\mathrm{Si}(\pi a) - \cos(\pi b) - \pi b\,\mathrm{Si}(\pi b)}{2\pi^2\nu(a+b)}\ ,$$

where $\mathrm{Si}\,t = \int_0^t \frac{\sin x}{x}\,dx$.

The rest of this paper is structured as follows. In Section 2 we introduce notation, we give a refined formulation of the main result and present an outline of the proof of the main result (Theorem 2.2). In Section 3 we prove several useful bounds on the approximating semigroups and the approximations of the gradient. Section 4 is devoted to several estimates for stochastic convolutions, the most crucial one being Proposition 4.6, which is responsible for the correction term appearing in the limiting equation. Most of the work is performed in Section 5, where convergence of various approximating equations is proved.

Acknowledgements. We are grateful to Hendrik Weber, Jochen Voß and Andrew Stuart for numerous discussions on this and related problems. Part of this work has been carried out while JM was visiting the Courant Institute and the University of Warwick. He thanks both institutions for their kind hospitality and support. MH would like to acknowledge financial support provided by the EPSRC through grants EP/E002269/1 and EP/D071593/1, the Royal Society through a Wolfson Research Merit Award, and the Leverhulme Trust through a Philip Leverhulme prize. JM was supported by Rubicon grant 680-50-0901 of the Netherlands Organisation for Scientific Research (NWO).

2. Proof of the main result

In order to shorten notations, we introduce the semigroups S and S_ε, defined as rescaled versions of the heat semigroup and its approximation:

$$S(t) \stackrel{\text{def}}{=} e^{-t(1-\nu\partial_x^2)} \;, \qquad S_\varepsilon(t) \stackrel{\text{def}}{=} e^{-t(1-\nu\Delta_\varepsilon)} \;,$$

where we define S_ε by Fourier analysis, making use of the convention $e^{-\infty} = 0$. Since we will always work with the mild formulation, it will be convenient to have a notation for the convolution (in time) of a function with one of the semigroups. We will henceforth write

$$(S * w)(t) \stackrel{\text{def}}{=} \int_0^t S(t-s)w(s)\,ds \;.$$

Let $(W(t))_{t\in\mathbf{R}}$ be a two-sided cylindrical Wiener process on $\mathcal{H} \stackrel{\text{def}}{=} L^2([0,2\pi],\mathbf{R}^n)$ (see [DPZ92, Hai09] for precise definitions) and let Q_ε be the bounded operator on $\mathcal{H}$ defined as a Fourier multiplier by

$$\widehat{Q_\varepsilon u}(k) = h(\varepsilon|k|)\widehat{u}(k) \;.$$

(We assume that it acts independently on each component.) Finally, we define the $\mathcal{H}$-valued processes ψ and $\widetilde{\psi}$ by

$$\psi(t) = \int_{-\infty}^t S(t-s)\,dW(s) \;, \qquad \widetilde{\psi}(t) = \int_{-\infty}^t S_\varepsilon(t-s)Q_\varepsilon\,dW(s) \;,$$

so that, in the notations of the previous section, they are the *stationary* solutions to the linear equations

$$\partial_t\psi = (\nu\partial_x^2 - 1)\psi + \xi \;, \qquad \partial_t\widetilde{\psi} = (\nu\Delta_\varepsilon - 1)\widetilde{\psi} + \xi_\varepsilon \;.$$

With all of these notations at hand, we can rewrite the equations for $\bar{u}$ and u_ε in the mild form as

$$\bar{u}(t) = S(t)v_0 + \psi(t) + S * \big(\overline{F}(\bar{u}) + \nabla G(\bar{u})\partial_x\bar{u}\big)(t) \;, \tag{2.1}$$

$$u_\varepsilon(t) = S_\varepsilon(t)v_0 + \widetilde{\psi}(t) + S_\varepsilon * \big(F(u_\varepsilon) + \nabla G(u_\varepsilon)D_\varepsilon u_\varepsilon\big)(t) \;. \tag{2.2}$$

Remark 2.1. Note that we have used here a common initial condition v_0 for the difference $\bar{u}-\psi$ and $u_\varepsilon - \widetilde{\psi}$. As a consequence, the two equations *do not* start with the same initial condition! However, as $\varepsilon \to 0$, the initial condition of u_ε converges to that of $\bar{u}$. The reason for not starting with the same initial condition is mostly of technical nature.

It will be convenient to define for any $0 < \gamma < \chi$,

$$\psi_\gamma \stackrel{\text{def}}{=} (I - \Pi_{\varepsilon^{-\gamma}})\psi \;, \qquad \psi^\gamma \stackrel{\text{def}}{=} \Pi_{\varepsilon^{-\gamma}}\psi \;, \qquad \psi_\gamma^\chi \stackrel{\text{def}}{=} (\Pi_{\varepsilon^{-\chi}} - \Pi_{\varepsilon^{-\gamma}})\psi \;.$$

The expressions $\widetilde{\psi}_\gamma$, $\widetilde{\psi}^\gamma$, and $\widetilde{\psi}_\gamma^\chi$ are defined analogously. Here Π_N denotes the projection onto the low-frequency components of the Fourier expansion, defined by $\Pi_N e_n \stackrel{\text{def}}{=} \mathbf{1}_{|n|\le N}e_n$, where $e_n(x) = (2\pi)^{-\frac{1}{2}}e^{inx}$.

We set

$$\bar{v} \stackrel{\text{def}}{=} \bar{u} - \psi \;, \qquad \widetilde{v} \stackrel{\text{def}}{=} u_\varepsilon - \widetilde{\psi} \;.$$

In the proof it will be convenient to work with the functions $\bar{v}^\gamma$ and $\widetilde{v}^\gamma$ defined by

$$\bar{v}^\gamma \stackrel{\text{def}}{=} \bar{v} + \psi^\gamma = \bar{u} - \psi_\gamma \;, \qquad \widetilde{v}^\gamma \stackrel{\text{def}}{=} \widetilde{v} + \widetilde{\psi}^\gamma = u_\varepsilon - \widetilde{\psi}_\gamma \;.$$

It follows from (2.1) and (2.2) that these functions satisfy the following equations:

$$\bar{v}^\gamma(t) = S(t)v_0 + \psi^\gamma(t) + S * \big(\overline{F}(\bar{v}^\gamma + \psi_\gamma) + \partial_x\big(G(\bar{v}^\gamma + \psi_\gamma)\big)\big)(t)\ , \tag{2.3}$$

$$\widetilde{v}^\gamma(t) = S_\varepsilon(t)v_0 + \widetilde{\psi}^\gamma(t) + S_\varepsilon * \big(F(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) + \nabla G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma)D_\varepsilon(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma)\big)(t). \tag{2.4}$$

For large parts of this article, it will be convenient to work in the fractional Sobolev space H^α for some $\alpha > \frac{1}{2}$, so that $H^\alpha \subset L^\infty$. Furthermore, we will need to use a high-frequency cut-off, which will smoothen out the solutions at a scale ε^χ for some $\chi > 1$. It turns out that a reasonable choice for these parameters is given by

$$\alpha = \tfrac{3}{4}\ , \qquad \gamma = \tfrac{1}{3}\ , \qquad \chi = \tfrac{3}{2}\ , \tag{2.5}$$

and we will fix these values from now on. With these notations at hand, the following theorem, which is essentially a more precise reformulation of Theorem 1.6, is a more precise statement of our main result. Here and in the rest of the paper we write $\|u\|_\beta$ to denote the norm of an element u in the fractional Sobolev space H^β for $\beta \in \mathbf{R}$.

Theorem 2.2. *Let $\kappa > 0$ be an arbitrary (small) exponent and let $v_0 \in H^\beta$ for all $\beta < \frac{3}{2}$. There exists a sequence of stopping times τ_ε satisfying $\tau_\varepsilon \to \tau_*$ in probability as $\varepsilon \to 0$, such that*

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{t\le\tau_\varepsilon} \|u_\varepsilon(t) - \bar{u}(t)\|_{L^\infty} > \varepsilon^{\frac{1}{8}-\kappa}\Big) = 0\ . \tag{2.6}$$

In fact, we have the bounds

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{t\le\tau_\varepsilon} \|\widetilde{v}^\gamma(t) - \bar{v}^\gamma(t)\|_\alpha > \varepsilon^{\frac{1}{8}-\kappa}\Big) = 0\ , \tag{2.7}$$

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{t\le\tau_\varepsilon} \|\widetilde{\psi}_\gamma(t) - \psi_\gamma(t)\|_{L^\infty} > \varepsilon^{\frac{1}{2}-\kappa}\Big) = 0\ . \tag{2.8}$$

Remark 2.3. We emphasise again that the initial conditions $\bar{u}(0)$ and $u_\varepsilon(0)$ are slightly different. In fact, one has $u_\varepsilon(0) = \bar{u}(0) + \widetilde{\psi}(0) - \psi(0)$.

Remark 2.4. The rate $\frac{1}{8}$ is not optimal. By adjusting the parameters α, γ and χ in an optimal way, and by sharpening some of the arguments in our proof, one could achieve a slightly better rate. However, we do not believe that any rate obtained in this way would reflect the true speed of convergence, so we keep with the values (2.5) that yield simple fractions.

Remark 2.5. From a technical point of view, the general methodology followed in this section and the subsequent sections is inspired from [Hai10], where a somewhat similar phenomenon was investigated. Besides the structural differences in the equations considered here and in [Hai10], the main technical difficulties that need to be overcome for the present work are the following:

(1) In [Hai10], it is possible to simply subtract the stochastic convolution ψ (or $\tilde{\psi}$) and work with the equation for the remainder. Here, we instead subtract only the highest Fourier modes of ψ. The reason for this choice is that it entails that $\bar{v}^\gamma \to \bar{u}$ as $\varepsilon \to 0$. This allows us to linearise the nonlinearity around $\bar{v}^\gamma$ in order to exhibit the desired correction term. As a consequence, our a priori regularity estimates are much worse than those in [Hai10] and our convergence rates are worse. The main reason why

we need this complication is that our approximate derivative D_ε does not satisfy the chain rule.

(2) All of our fixpoint arguments need to be performed in the fractional Sobolev space H^α, for some $\alpha > \frac{1}{2}$. This is in contrast to [Hai10] where some of the arguments could be performed first in L^∞, and then lifted to H^α by a standard bootstrapping argument. These bootstrapping arguments fail here, since the nonlinearity of our approximating equation contains an approximate derivative, which gives rise to correction terms which are not easy to control.

(3) In one crucial step where a Gaussian concentration inequality is employed in [Hai10], it was necessary that the stochastic convolutions belong to H^α for some $\alpha > \frac{1}{2}$. This is the case in [Hai10] as a consequence of the extra regularising effect caused by a small fourth-order term present in the linear part. This additional regularising effect is not always present in the current work. We therefore perform another truncation in Fourier space, at very high frequencies. This is the purpose of the exponent χ.

Note also that Proposition 4.6 is the analogue of Proposition 4.1 in [Hai10]. One difference is that we have a much cleaner separation of the probabilistic and the analytical aspects of this result.

By a standard Picard fixed point argument (see, e.g. [Hai09]) it can be shown that (2.1) admits a unique mild solution $\bar{u}$ defined on a random time interval $[0, \tau_*]$. Moreover, the spatial regularity of ψ and $\bar{u}$ equals that of a Brownian path, in the sense that $\psi(t)$ and $\bar{u}(t)$ are continuous and belong to H^β for any $\beta < \frac{1}{2}$ and any $t > 0$, but not to $H^{\frac{1}{2}}$. We shall take advantage of the fact that the process $\bar{v}$ is much more regular. In fact, $\bar{v}(t) \in H^\beta$ almost surely for any $\beta < \frac{3}{2}$ and any $t > 0$, but one does not expect it to belong to $H^{\frac{3}{2}}$ in general. This follows immediately from the mild formulation (2.1) combined with a standard bootstrapping argument. It follows from these considerations that, for every fixed time horizon T, the stopping time

$$\tau_*^K := T \wedge \inf\{t \,:\, \|\bar{v}(t)\|_\alpha \vee \|\bar{u}(t)\|_{L^\infty} \geq K\}$$

converges in probability to $\tau_* \wedge T$ as $K \to \infty$.

It will be shown in Section 4 that a number of functionals of ψ and $\widetilde{\psi}$ scale in the following way:

$$\begin{aligned}
\|\widetilde{\psi}^\chi_\gamma(t)\|_{L^\infty} &\lesssim \varepsilon^{\frac{\gamma}{2}-\kappa}\,, & \|\widetilde{\psi}_\gamma(t)\|_{L^\infty} &\lesssim \varepsilon^{\frac{\gamma}{2}-\kappa}\,,\\
\|\psi_\gamma(t)\|_{L^\infty} &\lesssim \varepsilon^{\frac{\gamma}{2}-\kappa}\,, & \|\widetilde{\psi}_\chi(t)\|_{L^\infty} &\lesssim \varepsilon^{\frac{\chi}{2}-\kappa}\,,\\
\|\psi^\gamma(t)\|_\alpha &\lesssim \varepsilon^{-\gamma(\alpha-\frac{1}{2})-\kappa}\,, & \|\widetilde{\psi}^\chi_\gamma(t)\|_\alpha &\lesssim \varepsilon^{-\chi(\alpha-\frac{1}{2})-\kappa}\,,\\
\|\widetilde{\psi}^\gamma(t) - \psi^\gamma(t)\|_\alpha &\lesssim \varepsilon^{2-\gamma(\alpha+\frac{3}{2})-\kappa}\,, & \Theta_\varepsilon(\widetilde{\psi}_\gamma(t)) &\lesssim \varepsilon^{-1-\kappa}\,,\\
\Theta_\varepsilon(\widetilde{\psi}_\chi(t)) &\lesssim \varepsilon^{\chi-2-\kappa}\,, & \|\Lambda - \Xi_\varepsilon(\widetilde{\psi}^\chi_\gamma(t))\|_{-\alpha} &\lesssim \varepsilon^{\frac{1}{2}-\kappa}\,,
\end{aligned}$$

where the quantities Θ_ε and Ξ_ε are defined by

$$\Theta_\varepsilon(u) \stackrel{\text{def}}{=} \int_{\mathbf{R}} y^2 \|\widehat{D}_{\varepsilon y} u\|_{L^2}^2\, |\mu|(dy)\,, \quad \Xi_\varepsilon(u) \stackrel{\text{def}}{=} \int_{\mathbf{R}} \frac{\varepsilon y^2}{2} \widehat{D}_{\varepsilon y} u \otimes \widehat{D}_{\varepsilon y} u\, \mu(dy)\,.$$

Note that all of these relations are of the form $\mathbf{\Psi}_i^\varepsilon(t) \lesssim \varepsilon^{\alpha_i - \kappa}$ for some expression $\mathbf{\Psi}_i^\varepsilon$ depending on ε and some exponent α_i. In the proof it will be convenient to

impose this behaviour by means of a hard constraint. For this purpose we introduce the stopping time τ^K, which is defined for $K > 0$ by

$$\tau^K \stackrel{\text{def}}{=} \tau_*^K \wedge \inf\{t \,:\, \exists i : \mathbf{\Psi}_i^\varepsilon(t) \geq \varepsilon^{\alpha_i - \kappa}\} \,. \tag{2.9}$$

From now on, we will write C_K to denote a constant which may depend on K (and T) and is allowed to change from line to line. Similarly, κ will be a positive universal constant which is sufficiently small and whose value is allowed to change from line to line. However, the final value of κ is independent of ε, K and T.

The remainder of this section is devoted to the proof of Theorem 2.2.

Proof of Theorem 2.2. Most of the work in the proof consists of bounding the difference between $\widetilde{v}^\gamma$ and $\bar{v}^\gamma$ in H^α. This bound will be obtained in several steps, using the intermediate processes $v_\varepsilon^{(i)}$, $i = 1, \dots, 4$, defined by

$$v_\varepsilon^{(1)}(t) = S(t)v_0 + \psi^\gamma(t) + S * \big(\overline{F}(v_\varepsilon^{(1)}) + \partial_x G(v_\varepsilon^{(1)})\big)(t) \,, \tag{2.10a}$$

$$v_\varepsilon^{(2)}(t) = S(t)v_0 + \psi^\gamma(t) + S * \big(\overline{F}(v_\varepsilon^{(2)}) + D_\varepsilon G(v_\varepsilon^{(2)})\big)(t) \,, \tag{2.10b}$$

$$v_\varepsilon^{(3)}(t) = S_\varepsilon(t)v_0 + \widetilde{\psi}^\gamma(t) + S_\varepsilon * \big(\overline{F}(v_\varepsilon^{(3)}) + D_\varepsilon G(v_\varepsilon^{(3)})\big)(t) \,, \tag{2.10c}$$

$$\begin{aligned} v_\varepsilon^{(4)}(t) = {} & S_\varepsilon(t)v_0 + \widetilde{\psi}^\gamma(t) \\ & + S_\varepsilon * \big(F(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) + \nabla G(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) D_\varepsilon(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi)\big)(t) \,. \end{aligned} \tag{2.10d}$$

Recall furthermore the definition of the stopping time τ^K given in (2.9). With this definition at hand, we set $\tau_0^K \stackrel{\text{def}}{=} \tau^K$ as well as $v_\varepsilon^{(0)} \stackrel{\text{def}}{=} \bar{v}^\gamma$ and $v_\varepsilon^{(5)} \stackrel{\text{def}}{=} \widetilde{v}^\gamma$, and we define recursively a sequence of stopping times τ_j^K with $j = 1, \dots, 5$ by

$$\tau_j^K = \tau_{j-1}^K \wedge \inf\{t \,:\, \|v_\varepsilon^{(j)}(t) - v_\varepsilon^{(j-1)}(t)\|_\alpha \geq K\} \,. \tag{2.11}$$

With these notations at hand, Propositions 5.1 – 5.7 state that, for every fixed values $K, \kappa > 0$ and every $j = 1, \dots, 5$, one has

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big(\sup_{t \leq \tau_j^K} \|v_\varepsilon^{(j)}(t) - v_\varepsilon^{(j-1)}(t)\|_\alpha > \varepsilon^{\frac{1}{8} - \kappa} \Big) = 0 \,, \tag{2.12}$$

Combining all of these bounds, we conclude immediately that, for every fixed time horizon $T > 0$ and every choice of values K and κ, we have

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big(\sup_{t \leq \tau_5^K} \|\widetilde{v}^\gamma(t) - \bar{v}^\gamma(t)\|_\alpha > \varepsilon^{\frac{1}{8} - \kappa} \Big) = 0 \,.$$

This is formally very close to (2.7), except that we still have the values $T, K > 0$ appearing in our statement and consider the solutions only up to the stopping time τ_5^K.

Since $\tau_* \wedge T \to \tau_*$ as $T \to \infty$ and since we already argued that $\tau_*^K \to \tau_* \wedge T$ as $K \to \infty$, the bound (2.7) follows if we are able to show that, for every fixed choice of K and T,

$$\lim_{\varepsilon \to 0} \mathbb{P}(\tau_5^K \neq \tau_*^K) = 0 \,. \tag{2.13}$$

Since the statement of our theorem is stronger, the smaller the value of κ, we can assume without loss of generality that $\kappa < \frac{1}{8}$. In this case, $\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{8} - \kappa} = 0$, so that (2.12) and (2.11) together imply that

$$\lim_{\varepsilon \to 0} \mathbb{P}(\tau_j^K \neq \tau_{j-1}^K) = 0 \,,$$

for $j = 1, \dots, 5$, from which we conclude that $\lim_{\varepsilon \to 0} \mathbb{P}(\tau_5^K \neq \tau^K) = 0$.

In order to finish the proof of (2.7), it now suffices to show that $\lim_{\varepsilon\to 0}\mathbb{P}(\tau^K \neq \tau_*^K) = 0$. Fix an arbitrary $T > 0$ and $\kappa > 0$. It then follows from Propositions 4.3, 4.4 and 4.5 that for each of the terms $\mathbf{\Psi}_j$ appearing in (2.9), there exists a constant $C_j > 0$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\mathbf{\Psi}_j^\varepsilon(t) \leq C_j\varepsilon^{\alpha_j-\frac{\kappa}{2}}\ ,$$

uniformly for all $\varepsilon \leq 1$. It then follows from Chebychev's inequality that

$$\mathbb{P}(\tau^K \neq \tau_*^K) \leq \sum_j \mathbb{P}\Big(\sup_{t\in[0,T]}\mathbf{\Psi}_j^\varepsilon(t) \geq \varepsilon^{\alpha_j-\kappa}\Big) \leq \sum_j C_j\varepsilon^{\kappa/2}\ ,$$

from which the claim follows.

Since (2.6) follows from (2.7) and (2.8), the proof of the theorem is complete if we show that (2.8) holds. Since it follows from Proposition 4.3 and Chebychev's inequality that

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big(\sup_{t\leq T}\|\widetilde{\psi}_\gamma(t) - \psi_\gamma(t)\|_{L^\infty} > \varepsilon^{\frac{1}{2}-\kappa}\Big) = 0\ ,$$

for every $T > 0$, this claim follows at once. □

3. Analytic tools

3.1. **Products and compositions of functions in Sobolev spaces.** In this subsection we collect bounds for products and compositions of functions in Sobolev spaces. As is usual in the analysis literature, we use the notation $\Phi \lesssim \Psi$ as a shorthand for 'there exists a constant C such that $\Phi \leq C\Psi$'. These estimates will be useful in order control the various terms that arise in the Taylor expansion of the nonlinearity that will be performed in Section 5 below.

Lemma 3.1. *Let $r, s, t \geq 0$ be such that $r \wedge s > t$ and $r + s > \frac{1}{2} + t$.*

(1) *For $f \in H^r$ and $g \in H^s$ we have $fg \in H^t$ and*

$$\|fg\|_t \lesssim \|f\|_r\|g\|_s\ . \tag{3.1}$$

(2) *For $f \in H^r$ and $g \in H^{-t}$ we have $fg \in H^{-s}$ and*

$$\|fg\|_{-s} \lesssim \|f\|_r\|g\|_{-t}\ . \tag{3.2}$$

Proof. The bound (3.1) is well-known and an elementary proof can be found for example in [Hai09, Theorem 6.18]. To prove (3.2), we take $\varphi \in H^s$ and use (3.1) to obtain

$$\langle\varphi, fg\rangle = \langle\varphi f, g\rangle \leq \|\varphi f\|_t\|g\|_{-t} \lesssim \|\varphi\|_s\|f\|_r\|g\|_{-t}\ ,$$

which implies the desired result. □

Lemma 3.2. *Let $s \in (\frac{1}{2}, 1)$. There exists $C > 0$ such that for any $u \in H^s$ and any $G \in C^1(\mathbf{R}^n;\mathbf{R}^n)$ satisfying*

$$\|G_u\|_{C^1} := \sup\{|G(x)| + |\nabla G(x)|\ :\ |x| \leq \|u\|_{L^\infty}\} < \infty\ ,$$

we have

$$\|G\circ u\|_s \leq C\|G_u\|_{C^1}(1 + \|u\|_s)\ .$$

Proof. Let τ_h be the shift operator defined by $\tau_h u(x) := u(x-h)$. It is well known (see, e.g., [DS93] or, for functions defined on $\mathbf{R}^n$, [AF03, Theorem 7.47]) that the expression

$$\|u\|_{L^2} + \Bigl(\int_0^1 \Bigl[t^{-s} \sup_{|h|<t} \|u - \tau_h u\|_{L^2}\Bigr]^2 \frac{dt}{t}\Bigr)^{\frac{1}{2}} \tag{3.3}$$

defines an equivalent norm on H^s. The result then follows by inserting the estimates

$$\|G \circ u\|_{L^2} \leq \|G \circ u\|_{L^\infty} \leq C\|G_u\|_{C^1}$$
$$\|G \circ u - \tau_h(G \circ u)\|_{L^2} \leq C\|G_u\|_{C^1}\|u - \tau_h u\|_{L^2}$$

into (3.3). □

3.2. **Semigroup bounds.** We will frequently use the fact that for $\alpha \geq \beta$ and $T > 0$, there exists a constant $C > 0$ such that

$$\|S(t)u\|_\alpha \leq Ct^{-\frac{1}{2}(\alpha-\beta)}\|u\|_\beta \;, \tag{3.4}$$

for any $\varepsilon \in (0,1]$, $t \in [0,T]$ and $u \in H^\beta$. This is a straightforward consequence of standard analytic semigroup theory [Lun95, Hai09]. Since the generator of S is selfadjoint in all of the H^s, it is also straightforward to prove (3.4) by hand. As a consequence, we have

Lemma 3.3. *Let $\alpha, \beta \in \mathbf{R}$ be such that $0 \leq \alpha - \beta < 2$ and let $T > 0$. There exists $C > 0$ such that for all $t \in [0,T]$ and $u \in C([0,t]; H^\beta)$ we have*

$$\Bigl\| \int_0^t S(t-s)u(s)\,ds \Bigr\|_\alpha \leq Ct^{1-\frac{1}{2}(\alpha-\beta)} \sup_{s\in[0,t]} \|u(s)\|_\beta \;. \tag{3.5}$$

Proof. It suffices to integrate the bound (3.4). □

The following bounds measure how well S_ε approximates S in these interpolation spaces. The general philosophy is that every power of ε has to be paid with one spatial derivative worth of regularity. This type of power-counting is a direct consequence of the fact that the function f that measures how much Δ_ε differs from ∂_x^2, is evaluated at $\varepsilon|k|$ in the definition of Δ_ε. The precise bounds are the following:

Lemma 3.4. *Let $\kappa \in [0,2]$. For $T > 0$ there exists $C > 0$ such that for any $t \in [0,T]$, $\varepsilon \in (0,1]$, and $u \in H^\beta$, we have*

$$\|S_\varepsilon(t)u - S(t)u\|_\alpha \leq C\varepsilon^\kappa t^{-\frac{1}{2}(\alpha-\beta+\kappa)}\|u\|_\beta \qquad (\beta \leq \alpha + 2\kappa)\;, \tag{3.6}$$

$$\|S_\varepsilon(t)u\|_\alpha \leq Ct^{-\frac{1}{2}(\alpha-\beta)}\|u\|_\beta \qquad (\beta \leq \alpha)\;. \tag{3.7}$$

Proof. We set $\bar f \stackrel{\text{def}}{=} f - 1$ and assume $\nu = 1$ for notational simplicity, since the case $\nu \neq 1$ is virtually identical. The assumptions on f imply that $|\bar f(\varepsilon n)| \leq c\varepsilon^2 n^2$ whenever $n < \delta/\varepsilon$ and δ is some sufficiently small constant. Using the mean value theorem and the fact that we can assume $\delta < 1$ without loss of generality, we obtain for $n < \delta/\varepsilon$ and $\kappa \in [0,2]$,

$$\begin{aligned}|\exp(-tn^2\bar f(\varepsilon n)) - 1| &\leq (2 \wedge ct\varepsilon^2 n^4)e^{ct\varepsilon^2 n^4} \leq Ct^{\frac{\kappa}{2}}\varepsilon^\kappa n^{2\kappa} e^{ct\delta^2 n^2} \\ &\leq C\varepsilon^\kappa t^{\frac{\kappa}{2}} n^{2\kappa} e^{c\delta^2 t(1+n^2)}\;.\end{aligned}$$

Inserting this bound into the identity

$$(S_\varepsilon(t)u - S(t))e_n = (e^{-tn^2\bar{f}(\varepsilon n)} - 1)e^{-t(1+n^2)}e_n \ ,$$

it then follows from (3.4) that

$$\begin{aligned} \|\Pi_{\delta/\varepsilon}(S_\varepsilon(t) - S(t))u\|_\alpha &\le C\varepsilon^\kappa t^{\frac{\kappa}{2}}\|S((1-\delta^2 c)t)u\|_{\alpha+2\kappa} \\ &\le C\varepsilon^\kappa t^{-\frac{1}{2}(\alpha-\beta+\kappa)}\|u\|_\beta \ , \end{aligned} \tag{3.8}$$

provided that we choose δ sufficiently small so that $\delta^2 c \le \frac{1}{2}$, say.

On the other hand, note that

$$(I - \Pi_{\delta/\varepsilon})(S_\varepsilon(t)u - S(t))e_n = \mathbf{1}_{\{|n|>\delta/\varepsilon\}}(e^{-tn^2\bar{f}(\varepsilon n)} - 1)e^{-t(1+n^2)}e_n \ .$$

Recall that $\bar{f}(\varepsilon n) \ge q-1$ for all n, and that $q \in (0,1]$. Then we can find a constant C such that

$$|\exp(-tn^2\bar{f}(\varepsilon n)) - 1|e^{-t(1+n^2)} \le Ce^{-q(1+n^2)t} \ .$$

Moreover, for any $\kappa > 0$ we have $\mathbf{1}_{\{|n|>\delta/\varepsilon\}} \le |\varepsilon n/\delta|^\kappa$. It thus follows, using (3.4) again, that

$$\|(I - \Pi_{\delta/\varepsilon})(S_\varepsilon(t) - S(t))u\|_\alpha \le C\varepsilon^\kappa\|S(qt)u\|_{\alpha+\kappa} \le C\varepsilon^\kappa t^{-\frac{1}{2}(\alpha-\beta+\kappa)}\|u\|_\beta \ .$$

The bound (3.6) now follows by combining this inequality with (3.8). The inequality (3.7) follows by combining the special case $\kappa = 0$ with (3.4). □

3.3. **Estimates for the gradient term.** In this section, we similarly show how well the operator D_ε approximates ∂_x. Again, the guiding principle is that every power of ε "costs" the loss of one derivative. However, we are also going to use the fact that D_ε is a bounded operator. In this case, we can gain up to one spatial derivative with respect to the operator ∂_x, but we have to "pay" with the same number of inverse powers of ε. The rigorous statement for the latter fact is the following:

Lemma 3.5. *Let $\beta \in \mathbf{R}$ and $\alpha \in [0,1]$. There exists $C > 0$ such that for all $\varepsilon \in (0,1]$ and $u \in H^\beta$ the estimate*

$$\|D_\varepsilon u\|_{\beta-\alpha} \le C\varepsilon^{\alpha-1}\|u\|_\beta$$

holds.

Proof. Using the assumption that $M := |\mu|(\mathbf{R}) < \infty$, together with Jensen's inequality and Fubini's theorem we obtain

$$\begin{aligned} \|D_\varepsilon u\|_{L^2}^2 &\le \frac{1}{\varepsilon^2}\int\left(\int_{\mathbf{R}}|u(x+\varepsilon y)|\,|\mu|(dy)\right)^2 dx \\ &\le \frac{M}{\varepsilon^2}\int\int_{\mathbf{R}}|u(x+\varepsilon y)|^2\,|\mu|(dy)\,dx = \frac{M^2}{\varepsilon^2}\|u\|_{L^2}^2 \ . \end{aligned}$$

On the other hand, assuming for the moment that u is smooth, we use the assumption that $\mu(\mathbf{R}) = 0$, and apply Jensen's inequality and Minkowski's integral

inequality to obtain

$$\begin{aligned}
\|D_\varepsilon u\|_{L^2}^2 &= \frac{1}{\varepsilon^2}\int\Big(\int_{\mathbf{R}} u(x+\varepsilon y)\,\mu(dy)\Big)^2\,dx\\
&= \frac{1}{\varepsilon^2}\int\Big(\int_{\mathbf{R}}\int_0^{\varepsilon y} u'(x+z)\,dz\,\mu(dy)\Big)^2\,dx\\
&\le \frac{M}{\varepsilon^2}\int\int_{\mathbf{R}}\Big(\int_0^{\varepsilon y}|u'(x+z)|\,dz\Big)^2|\mu|(dy)\,dx\\
&\le \frac{M}{\varepsilon^2}\int_{\mathbf{R}}\Big(\int_0^{\varepsilon y}\Big(\int|u'(x+z)|^2\,dx\Big)^{\frac12}dz\Big)^2|\mu|(dy)\\
&= M\|u'\|_{L^2}^2\int_{\mathbf{R}} y^2|\mu|(dy) \le C\|u\|_1^2\;.
\end{aligned}$$

Using complex interpolation, it follows that $\|D_\varepsilon u\|_{L^2} \le C\varepsilon^{\alpha-1}\|u\|_\alpha$ for every $\alpha\in[0,1]$. The desired result then follows from the fact that D_ε commutes with every Fourier multiplier. □

The announced approximation result on the other hand is the following:

Lemma 3.6. *Let $\beta\in\mathbf{R}$ and $\alpha\in[0,1]$. There exists $C>0$ such that for all $\varepsilon\in(0,1]$ and $u\in H^\beta$ the estimate*

$$\|D_\varepsilon u-\partial_x u\|_{\beta-1-\alpha}\le C\varepsilon^\alpha\|u\|_\beta$$

holds.

Proof. In view of (1.3) we have, assuming for the moment that u is smooth,

$$(D_\varepsilon-\partial_x)u(x)=\frac{1}{\varepsilon}\int_{\mathbf{R}}\int_0^{\varepsilon y}\int_0^w u''(x+z)\,dz\,dw\,\mu(dy)\;.$$

Integrating against a test function φ and applying Fubini's theorem, we arrive at

$$\begin{aligned}
\Big|\int\varphi(x)(D_\varepsilon-\partial_x)u(x)\,dx\Big| &\le \frac{C}{\varepsilon}\int_{\mathbf{R}}\int_0^{\varepsilon y}\int_0^w\|\varphi\|_{2-\beta}\|u\|_\beta\,dz\,dw\,|\mu|(dy)\\
&\le C\varepsilon\|\varphi\|_{2-\beta}\|u\|_\beta\int_{\mathbf{R}}|y|^2\,|\mu|(dy)\;,
\end{aligned}$$

which implies that

$$\|(D_\varepsilon-\partial_x)u\|_{\beta-2}\le C\varepsilon\|u\|_\beta\;.$$

On the other hand, Lemma 3.5 implies that

$$\|(D_\varepsilon-\partial_x)u\|_{\beta-1}\le C\|u\|_\beta\;,$$

and the result then follows as before by interpolating between these estimates. □

As an immediate corollary of these bounds, we obtain the following useful fact:

Corollary 3.7. *Let $\beta\in[0,1)$. There exists $C>0$ such that for $\varepsilon\in(0,1]$, $u\in H^\beta$, and $G\in C^1(\mathbf{R}^n)$ we have*

$$\|D_\varepsilon G(u)-\partial_x G(u)\|_{-1}\le C\varepsilon^\beta\|G_u\|_{C^1}(1+\|u\|_\beta)\;,$$

where $\|G_u\|_{C^1}$ is defined as in Lemma 3.2.

Proof. Using Lemmas 3.6 and 3.2 we obtain

$$\|D_\varepsilon G(u) - \partial_x G(u)\|_{-1} \le C\varepsilon^\beta \|G(u)\|_\beta \le C\varepsilon^\beta \|G_u\|_{C^1}(1 + \|u\|_\beta) \ ,$$

which is the stated claim. □

4. Probabilistic tools

In this section we prove some sharp estimates for certain expressions involving stochastic convolutions. Our main tool is the following version of Kolmogorov's continuity criterion, which follows immediately from the one given for example in [RY94]. The reason why we state condition (4.1) in this form, is that it is automatically satisfied (by hypercontractivity) for random fields taking values in a Wiener chaos of fixed (finite) order.

Lemma 4.1. *Let $(\varphi(t))_{t\in[0,1]^n}$ be a Banach space-valued random field having the property that for any $q \in (2,\infty)$ there exists a constant $K_q > 0$ such that*

$$\begin{aligned} \big(\mathbb{E}\|\varphi(t)\|^q\big)^{\frac{1}{q}} &\le K_q\big(\mathbb{E}\|\varphi(t)\|^2\big)^{\frac{1}{2}} \ , \\ \big(\mathbb{E}\|\varphi(s)-\varphi(t)\|^q\big)^{\frac{1}{q}} &\le K_q\big(\mathbb{E}\|\varphi(s)-\varphi(t)\|^2\big)^{\frac{1}{2}} \ , \end{aligned} \tag{4.1}$$

for all $s,t \in [0,1]^n$. Furthermore, suppose that the estimate

$$\mathbb{E}\|\varphi(s)-\varphi(t)\|^2 \le K_0|s-t|^\delta$$

holds for some $K_0, \delta > 0$ and all $s,t \in [0,1]^n$. Then, for every $p > 0$ there exists $C > 0$ such that

$$\mathbb{E}\sup_{t\in[0,1]^n}\|\varphi(t)\|^p \le C\big(K_0 + \mathbb{E}\|\varphi(0)\|^2\big)^{\frac{p}{2}} \ .$$

Throughout this subsection we shall use θ_k and $\widetilde{\theta}_k$ for the Fourier coefficients of ψ and $\widetilde{\psi}$, so that

$$\psi(t) = \sum_{k\in\mathbf{Z}} \theta_k(t) e_k \ , \qquad \widetilde{\psi}(t) = \sum_{k\in\mathbf{Z}} \widetilde{\theta}_k(t) e_k \ .$$

With this notation at hand, we first state the following approximation bound, which shows that we can again trade powers of k for powers of ε, provided that we look at the difference squared:

Lemma 4.2. *For $t \ge 0$, $k \in \mathbf{Z}$ and $\varepsilon \in (0,1]$ we have*

$$\mathbb{E}|\widetilde{\theta}_k(t) - \theta_k(t)|^2 \le C(k^{-2} \wedge \varepsilon^4 k^2) \ . \tag{4.2}$$

Proof. We write again $\bar{f} = f - 1$ and assume $\nu = 1$ for simplicity. The Itô isometry then implies that

$$\begin{aligned} \mathbb{E}|\widetilde{\theta}_k(t) - \theta_k(t)|^2 &= C\int_0^\infty e^{-2t(1+k^2)}\Big(1 - h(\varepsilon|k|)e^{-tk^2\bar{f}(\varepsilon|k|)}\Big)^2 dt \\ &\le C\int_0^\infty e^{-2t(1+k^2)}\Big(1 - e^{-tk^2\bar{f}(\varepsilon|k|)}\Big)^2 dt \\ &\quad + C\int_0^\infty e^{-2t(1+k^2)}e^{-2tk^2\bar{f}(\varepsilon|k|)}\Big(1 - h(\varepsilon|k|)\Big)^2 dt \\ &\stackrel{\text{def}}{=} I_1 + I_2 \ . \end{aligned} \tag{4.3}$$

Let $\delta > 0$ be a (small) constant to be determined later and consider first the term I_1 with $|\varepsilon k| \leq \delta$. Since f is twice differentiable near the origin, we can find δ small enough so that $|\bar{f}(|\varepsilon k|)| \leq c\varepsilon^2 k^2$ for some $c > 0$. Therefore, for $t \geq 0$,

$$|1 - e^{-tk^2 \bar{f}(\varepsilon|k|)}| \leq ct\varepsilon^2 k^4 e^{ct\varepsilon^2 k^4} \leq ct\varepsilon^2 k^4 e^{c\delta^2 tk^2} \ , \tag{4.4}$$

so that

$$|I_1| \leq C\varepsilon^4 k^8 \int_0^\infty t^2 e^{-2t(1+k^2)+2c\delta^2 k^2 t}\, dt \ .$$

If we ensure that δ is small enough so that $2c\delta^2 \leq 1$, we obtain

$$|I_1| \leq C\varepsilon^4 k^8 \int_0^\infty t^2 e^{-k^2 t}\, dt \leq C\varepsilon^4 k^2 \leq C\big(k^{-2} \wedge \varepsilon^4 k^2\big) \ ,$$

where the last inequality follows from the fact that $|\varepsilon k| \leq \delta$ by assumption.

To treat the case $|\varepsilon k| > \delta$, we use the fact that by assumption there exists $q \in (0, 1]$ such that $f \geq q$, so that

$$\begin{aligned} |I_1| &\leq \int_0^\infty e^{-2tk^2} \Big(1 - e^{-tk^2(q-1)}\Big)^2 dt \leq C \int_0^\infty e^{-2tqk^2}\, dt \\ &\leq Ck^{-2} \leq C\big(k^{-2} \wedge \varepsilon^4 k^2\big) \ . \end{aligned} \tag{4.5}$$

The bound on I_2 works in pretty much the same way, using the fact that the assumptions on h imply that

$$|1 - h(\varepsilon|k|)| \leq C\big(1 \wedge \varepsilon^2 k^2\big) \ .$$

Using again the fact that $f \geq q$, we then obtain

$$I_2 \leq C \int_0^\infty e^{-2tqk^2} \big(1 \wedge \varepsilon^4 k^4\big)\, dt \leq C\big(k^{-2} \wedge \varepsilon^4 k^2\big) \ ,$$

as required. □

We continue with a sequence of propositions, in which the estimates obtained in the previous lemma are used to establish various bounds for stochastic convolutions.

Proposition 4.3. *Let $0 < \gamma < \chi$. For $\kappa > 0$ and $\varepsilon \in (0, 1]$ we have*

$$\mathbb{E} \sup_{t\in[0,T]} \|\psi_\gamma(t)\|_{L^\infty} \leq C\varepsilon^{\frac{\gamma}{2}-\kappa} \ , \qquad \mathbb{E} \sup_{t\in[0,T]} \|\widetilde{\psi}_\gamma(t)\|_{L^\infty} \leq C\varepsilon^{\frac{\gamma}{2}-\kappa} \ ,$$

$$\mathbb{E} \sup_{t\in[0,T]} \|\widetilde{\psi}^\chi_\gamma(t)\|_{L^\infty} \leq C\varepsilon^{\frac{\gamma}{2}-\kappa} \ , \qquad \mathbb{E} \sup_{t\in[0,T]} \|\widetilde{\psi}_\gamma(t) - \psi_\gamma(t)\|_{L^\infty} \leq C\varepsilon^{\frac{1}{2}-\kappa} \ .$$

Proof. We start with the proof of the second estimate. Observe that $\widetilde{\theta}_k$ is a complex one-dimensional stationary Ornstein-Uhlenbeck process with variance $h^2/(2(1 + \nu k^2 f))$ and characteristic time $1 + \nu k^2 f$. This implies that

$$\mathbb{E}|\widetilde{\theta}_k(t)|^2 = \frac{h^2(\varepsilon|k|)}{2(1 + \nu k^2 f(\varepsilon|k|))} \leq C(1 \wedge k^{-2}) \tag{4.6}$$

and

$$\mathbb{E}|\widetilde{\theta}_k(t) - \widetilde{\theta}_k(s)|^2 \leq Ch^2(\varepsilon|k|)|t - s| \leq C|t - s| \ . \tag{4.7}$$

These bounds imply that, on the one hand,

$$\mathbb{E}|\widetilde{\theta}_k(t)e_k(x) - \widetilde{\theta}_k(s)e_k(y)|^2 \leq C\mathbb{E}|\widetilde{\theta}_k(t)|^2 + C\mathbb{E}|\widetilde{\theta}_k(s)|^2 \leq C(1 \wedge k^{-2}) \ ,$$

while on the other hand, one has

$$\begin{aligned}&\mathbb{E}|\widetilde{\theta}_k(t)e_k(x)-\widetilde{\theta}_k(s)e_k(y)|^2\\&\le C\mathbb{E}|\widetilde{\theta}_k(t)-\widetilde{\theta}_k(s)|^2+Ck^2|x-y|^2\mathbb{E}|\widetilde{\theta}_k(s)|^2\\&\le C|t-s|+C|x-y|^2\;.\end{aligned}$$

Combining these inequalities we find that, for every $\kappa\in[0,2]$,

$$\mathbb{E}|\widetilde{\theta}_k(t)e_k(x)-\widetilde{\theta}_k(s)e_k(y)|^2\le C(1\wedge k^{-2})^{1-\frac12\kappa}(|t-s|+|x-y|^2)^{\frac{\kappa}{2}}\;.$$

Since the $\widetilde{\theta}_k$'s are independent except for the reality condition $\widetilde{\theta}_{-k}=\overline{\widetilde{\theta}_k}$, we infer that

$$\begin{aligned}\mathbb{E}|\widetilde{\psi}_\gamma(t,x)-\widetilde{\psi}_\gamma(s,y)|^2&\le C\sum_{|k|>\varepsilon^{-\gamma}}\mathbb{E}|\widetilde{\theta}_k(t)e_k(x)-\widetilde{\theta}_k(s)e_k(y)|^2\\&\le C(|t-s|+|x-y|^2)^{\frac{\kappa}{2}}\sum_{|k|>\varepsilon^{-\gamma}}(1\wedge k^{-2})^{1-\frac12\kappa}\\&\le C\varepsilon^{(1-\kappa)\gamma}(|t-s|+|x-y|^2)^{\frac{\kappa}{2}}\;.\end{aligned}$$

Arguing similarly, we obtain

$$\mathbb{E}|\widetilde{\psi}_\gamma(0,0)|^2\le C\sum_{|k|>\varepsilon^{-\gamma}}\mathbb{E}|\widetilde{\theta}_k(0)|^2\le C\sum_{|k|>\varepsilon^{-\gamma}}(1\wedge k^{-2})\le C\varepsilon^\gamma\;.$$

The result now follows by combining these two bounds with Lemma 4.1.

The proof of the first and third estimates being very similar, we do not reproduce them here. In order to prove the last estimate, we use Lemma 4.2 to obtain

$$\mathbb{E}|\widetilde{\theta}_k(t)-\theta_k(t)|^2\le C(k^{-2})^{\frac34+\frac14\kappa}(\varepsilon^4k^2)^{\frac14-\frac14\kappa}\le C\varepsilon^{1-\kappa}k^{-1-\kappa}\;.$$

This bound then replaces (4.6), and the rest of the proof is again analogous to the proof of the second estimate. □

Proposition 4.4. *Let $\zeta>0$. For $\kappa>0$ and $\varepsilon\in(0,1]$ we have*

$$\mathbb{E}\sup_{t\in[0,T]}\|\psi^\zeta(t)\|_\alpha\le C\varepsilon^{-\zeta(\alpha-\frac12)-\kappa}\;,\qquad(\alpha>\tfrac12)\;,\tag{4.8}$$

$$\mathbb{E}\sup_{t\in[0,T]}\|\widetilde{\psi}^\zeta(t)-\psi^\zeta(t)\|_\alpha\le C\varepsilon^{2-\zeta(\alpha+\frac32)-\kappa}\;,\qquad(\alpha>-\tfrac32)\;.\tag{4.9}$$

Proof. In view of the estimates

$$\mathbb{E}|\theta_k(t)|^2\le Ck^{-2}\;,\qquad\mathbb{E}|\theta_k(t)-\theta_k(s)|^2\le C|t-s|\;,\tag{4.10}$$

we obtain

$$\begin{aligned}\mathbb{E}\|\psi^\zeta(t)-\psi^\zeta(s)\|_\alpha^2&\le C|t-s|^\kappa\sum_{|k|\le\varepsilon^{-\zeta}}(1+k^2)^{\alpha-1+\kappa}\\&\le C|t-s|^\kappa\varepsilon^{-2\zeta(\alpha-\frac12+\kappa)}\end{aligned}$$

and

$$\mathbb{E}\|\psi^\zeta(0)\|_\alpha^2\le C\varepsilon^{-2\zeta(\alpha-\frac12)}\;.$$

The inequality (4.8) thus follows from Lemma 4.1.

In order to prove (4.9) we argue similarly, but the estimates are slightly more involved. Write $\delta_k := \widetilde{\theta}_k - \theta_k$ so that $\widetilde{\psi}^\zeta - \psi^\zeta = \sum_{|k|\le\varepsilon^{-\zeta}} \delta_k e_k$. Using (4.7) and (4.10) we have for $s,t \ge 0$,

$$\mathbb{E}|\delta_k(t) - \delta_k(s)|^2 \le C|t-s| \ .$$

Combining this bound with Lemma 4.2, we infer that for $\kappa \in [0, \frac{1}{2})$,

$$\mathbb{E}|\delta_k(t) - \delta_k(s)|^2 \le C(k^{-2})^\kappa (\varepsilon^4 k^2)^{1-2\kappa}|t-s|^\kappa = C\varepsilon^{4-8\kappa}k^{2-6\kappa}|t-s|^\kappa \ .$$

For $\kappa \in (0, \frac{1}{3}\alpha + \frac{1}{2})$, we thus obtain

$$\begin{aligned}\mathbb{E}\|\big(\widetilde{\psi}^\zeta - \psi^\zeta\big)(t) - \big(\widetilde{\psi}^\zeta - \psi^\zeta\big)(s)\|_\alpha^2 &\le C|t-s|^\kappa \varepsilon^{4-8\kappa} \sum_{|k|\le\varepsilon^{-\zeta}} (1+k^2)^{\alpha+1-3\kappa} \\ &\le C|t-s|^\kappa \varepsilon^{4-\zeta(2\alpha+3)-8\kappa}\end{aligned}$$

and similarly

$$\mathbb{E}\sup_{t\in[0,T]} \|\widetilde{\psi}^\zeta(t) - \psi^\zeta(t)\|_\alpha^2 \le C\varepsilon^{4-\zeta(2\alpha+3)-8\kappa} \ .$$

The desired estimate (4.9) now follows from Lemma 4.1. □

Proposition 4.5. *Let $\zeta > 0$. For every $\kappa > 0$ there exists $C > 0$ such that*

$$\mathbb{E}\sup_{t\in[0,T]} \Theta(\widetilde{\psi}_\zeta(t)) \le C\varepsilon^{-1+(\zeta-1)^+-\kappa} \ ,$$

for all $\varepsilon \in (0,1]$, where we wrote $(\zeta-1)^+ \stackrel{\text{def}}{=} 0 \vee (\zeta-1)$.

Proof. As in the proof of Propositions 4.3 and 4.4, we shall apply Kolmogorov's continuity criterion from Lemma 4.1, this time for L^2-valued random fields. It follows from (4.6) that

$$\begin{aligned}\mathbb{E}\|\widehat{D}_{\varepsilon y}\big(\widetilde{\psi}_\zeta(t) - \widetilde{\psi}_\zeta(s)\big)\|_{L^2}^2 &= \sum_{|k|>\varepsilon^{-\zeta}} \left|\frac{e^{ik\varepsilon y}-1}{\varepsilon y}\right|^2 \mathbb{E}|\widetilde{\theta}_k(t) - \widetilde{\theta}_k(s)|^2 \\ &\le C\sum_{k>\varepsilon^{-\zeta}} \frac{1-\cos(k\varepsilon y)}{|\varepsilon k y|^2} \ .\end{aligned}$$

Note that, up to a factor $\varepsilon|y|$, this sum can be interpreted as a Riemann sum for the function $H(t) \stackrel{\text{def}}{=} t^{-2}(1-\cos(t))$. In fact, since $H(t) \le 2(1\wedge t^{-2})$,

$$\varepsilon|y| \sum_{k>\varepsilon^{-\zeta}} \frac{1-\cos(k\varepsilon y)}{|k\varepsilon y|^2} = \sum_{k>\varepsilon^{-\zeta}} \varepsilon|y| H(k\varepsilon y) \le 2\int_{\varepsilon^{1-\zeta}}^\infty (1\wedge t^{-2})\,dt \le C\varepsilon^{(\zeta-1)^+}. \tag{4.11}$$

It thus follows that

$$\mathbb{E}\|\widehat{D}_{\varepsilon y}\big(\widetilde{\psi}_\zeta(t) - \widetilde{\psi}_\zeta(s)\big)\|_{L^2}^2 \le C|\varepsilon y|^{-1}\varepsilon^{(\zeta-1)^+}. \tag{4.12}$$

On the other hand, (4.6) and (4.7) imply that

$$\mathbb{E}|\widetilde{\theta}_k(t) - \widetilde{\theta}_k(s)|^2 \le C(1\wedge k^{-2})^{\frac{3}{4}}|t-s|^{\frac{1}{4}} \ ,$$

and therefore

$$\begin{aligned}\mathbb{E}\|\widehat{D}_{\varepsilon y}\big(\widetilde{\psi}_\zeta(t)-\widetilde{\psi}_\zeta(s)\big)\|_{L^2}^2 &= \sum_{|k|>\varepsilon^{-\zeta}}\left|\frac{e^{ik\varepsilon y}-1}{\varepsilon y}\right|^2\mathbb{E}|\widetilde{\theta}_k(t)-\widetilde{\theta}_k(s)|^2\\ &\le C|\varepsilon y|^{-2}|t-s|^{\frac14}\sum_{|k|>\varepsilon^{-\zeta}}(1\wedge k^{-2})^{\frac34}\\ &\le C|\varepsilon y|^{-2}|t-s|^{\frac14}\ .\end{aligned}\tag{4.13}$$

Combining (4.12) and (4.13), we find that

$$\mathbb{E}\|\widehat{D}_{\varepsilon y}\big(\widetilde{\psi}_\zeta(t)-\widetilde{\psi}_\zeta(s)\big)\|_{L^2}^2 \le |\varepsilon y|^{-1-\kappa}|t-s|^{\frac14\kappa}\varepsilon^{(1-\kappa)(\zeta-1)^+}.$$

Similarly, we obtain

$$\begin{aligned}\mathbb{E}\|\widehat{D}_{\varepsilon y}\widetilde{\psi}_\zeta(0)\|_{L^2}^2 &= \sum_{|k|>\varepsilon^{-\zeta}}\left|\frac{e^{ik\varepsilon y}-1}{\varepsilon y}\right|^2\mathbb{E}|\widetilde{\theta}_k(0)|^2\\ &\le C\sum_{|k|>\varepsilon^{-\zeta}}\frac{1-\cos(k\varepsilon y)}{|\varepsilon k y|^2}\le C|\varepsilon y|^{-1}\varepsilon^{(\zeta-1)^+}.\end{aligned}$$

In view of Lemma 4.1, the latter two estimates imply that

$$\mathbb{E}\sup_{t\in[0,T]}\|\widehat{D}_{\varepsilon y}\widetilde{\psi}_\zeta(t)\|_{L^2}^2 \le C|\varepsilon y|^{-1-\kappa}\varepsilon^{(1-\kappa)(\zeta-1)^+}.$$

Using this bound, the desired result for $\Theta(\widetilde{\psi}_\zeta(t))$ can be obtained easily, since

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\Theta(\widetilde{\psi}_\zeta(t)) &= \mathbb{E}\sup_{t\in[0,T]}\int_{\mathbf{R}}|y|^2\|\widehat{D}_{\varepsilon y}\widetilde{\psi}_\zeta(t)\|_{L^2}^2\,|\mu|(dy)\\ &\le \int_{\mathbf{R}}|y|^2\mathbb{E}\sup_{t\in[0,T]}\|\widehat{D}_{\varepsilon y}\widetilde{\psi}_\zeta(t)\|_{L^2}^2\,|\mu|(dy)\\ &\le C\varepsilon^{-1-\kappa+(1-\kappa)(\zeta-1)^+}\int_{\mathbf{R}}|y|^{1-\kappa}\,|\mu|(dy)\\ &\le C\varepsilon^{-1-\kappa+(1-\kappa)(\zeta-1)^+}.\end{aligned}$$

The result now follows by rescaling κ. □

The next and final result of this section involves the term which gives rise to the correction term in the limiting equation. Before stating the result, we introduce the notation

$$\Xi_\varepsilon^y(u)\stackrel{\text{def}}{=}\frac{\varepsilon y^2}{2}\widehat{D}_{\varepsilon y}u\otimes\widehat{D}_{\varepsilon y}u\ ,$$

$$\Lambda^y\stackrel{\text{def}}{=}\frac{1}{2\pi\nu}\int_{\mathbf{R}_+}\frac{h^2(t)}{t^2f(t)}\big(1-\cos(yt)\big)\,dt\ ,$$

and

$$\Lambda_\varepsilon^y\stackrel{\text{def}}{=}\sum_{\varepsilon^{-\gamma}<k<\varepsilon^{-\chi}}\Lambda_{\varepsilon,k}^y\stackrel{\text{def}}{=}\sum_{\varepsilon^{-\gamma}<k<\varepsilon^{-\chi}}\frac{(1-\cos(\varepsilon k y))h^2(\varepsilon k)}{2\pi\varepsilon(1+\nu k^2f(\varepsilon k))}\ .$$

Note that one has the identities

$$\Xi_\varepsilon(u)=\int_{\mathbf{R}}\Xi_\varepsilon^y(u)\,\mu(dy)\ ,\qquad \Lambda=\int_{\mathbf{R}}\Lambda^y\,\mu(dy)\ ,\qquad \mathbb{E}\Xi_\varepsilon^y(\widetilde{\psi}_\gamma^\chi)=\Lambda_\varepsilon^y I\ ,$$

where the constant Λ is given by (1.5).

Proposition 4.6. *Let $\alpha > \frac{1}{2}$, $\gamma \leq \frac{1}{2}$, and $\chi \geq \frac{3}{2}$. For $\varepsilon \in (0,1]$, we then have*

$$\mathbb{E} \sup_{t\in[0,T]} \|\Lambda - \Xi_\varepsilon(\widetilde{\psi}^\chi_\gamma(t))\|_{-\alpha} \leq C\varepsilon^{\frac{1}{2}} \ .$$

Proof. The proof is an application of Lemma 4.1 with $\xi = \Lambda - \Xi_\varepsilon(\widetilde{\psi}^\chi_\gamma)$. For brevity we shall write $A := \Xi_\varepsilon(\widetilde{\psi}^\chi_\gamma)$ and $A^y := \Xi^y_\varepsilon(\widetilde{\psi}^\chi_\gamma)$. We divide the proof into several steps.

Step 1. First we claim that $\xi(t) = \Lambda - A(t)$ satisfies the condition (4.1) concerning the equivalence of all q-moments.

To see this, note that $\widetilde{\psi}^\chi_\gamma$ admits the representation $\widetilde{\psi}(t) = \sum_k \alpha_k(t) e_k$ where each $\alpha_k(t)$ is a Gaussian random vector in $\mathbf{R}^n$. As a consequence, for every $y \in \mathbf{R}$, each component of $\Lambda^y_\varepsilon - A^y$ is a polynomial of Gaussian random variables of degree at most two. It thus belongs to the direct sum of Wiener chaoses of order ≤ 2 and the same is true for $\Lambda_\varepsilon - A$, since each Wiener chaos is a closed subspace of the space of square integrable random variables. The claim thus follows from the well-known equivalence of moments for Hilbert space-valued Wiener chaos (see, e.g., [KW92]).

Step 2. In this step, we estimate how well Λ^y_ε approximates Λ^y. Since $|1 - \cos x| \leq C(1 \wedge x^2)$, we have the bound $|\Lambda^y_{\varepsilon,k}| \leq C\big(\varepsilon y^2 \wedge (\varepsilon k^2)^{-1}\big)$ for some constant C. As an immediate consequence, we have the bound

$$\Big|\Lambda^y_\varepsilon - \sum_{k\geq 1} \Lambda^y_{\varepsilon,k}\Big| \leq C\big(\varepsilon^{1-\gamma} y^2 + \varepsilon^{\chi-1}\big) \ . \tag{4.14}$$

Define now the function

$$\Phi_y(t) = \frac{(1-\cos(yt))h^2(t)}{2\pi\nu t^2 f(t)} \ ,$$

so that, since h^2/f is bounded by assumption, we obtain the bound

$$|\Lambda^y_{\varepsilon,k} - \varepsilon\Phi_y(\varepsilon k)| \leq C\frac{\varepsilon y^2}{k^2} \ .$$

Combining this bound with (4.14), we have

$$\Big|\Lambda^y_\varepsilon - \sum_{k\geq 1} \varepsilon\Phi_y(\varepsilon k)\Big| \leq C\big(\varepsilon^{1-\gamma} y^2 + \varepsilon^{\chi-1}\big) \ .$$

At this stage, we recall that for any function Φ of bounded variation, one has the approximation

$$\Big|\sum_{k\geq 1} \varepsilon\Phi(\varepsilon k) - \int_0^\infty \Phi(t)\,dt\Big| \leq \varepsilon\|\Phi\|_{\mathrm{BV}} \ ,$$

where $\|\Phi\|_{\mathrm{BV}}$ denotes the variation of Φ over $\mathbf{R}_+$. Furthermore, for any pair Φ, Ψ, we have the bound

$$\|\Phi\Psi\|_{\mathrm{BV}} \leq \|\Phi\|_{L^\infty}\|\Psi\|_{\mathrm{BV}} + \|\Psi\|_{L^\infty}\|\Phi\|_{\mathrm{BV}} \ . \tag{4.15}$$

If we set $\Psi_y(t) = (1-\cos(yt))/t^2$, we have

$$\|\Psi_y\|_{\mathrm{BV}} = \int_0^\infty |\Psi_y'(t)|\,dt = \int_0^\infty \frac{|yt\sin yt + 2\cos yt - 2|}{t^3}\,dt$$

$$\leq C|y|^3 \int_0^\infty \Big(1 \wedge \frac{1}{y^2 t^2}\Big)\, dt \leq C y^2 \;.$$

Since $\Psi(0) = y^2/2$, a similar bound holds for its L^∞ norm, and we conclude from (4.15) that

$$\|\Phi_y\|_{\mathrm{BV}} \leq C y^2 \;.$$

It follows immediately that we have the bound

$$|\Lambda_\varepsilon^y - \Lambda^y| \leq C\big(\varepsilon^{1-\gamma} y^2 + \varepsilon y^2 + \varepsilon^{\chi-1}\big) \;. \tag{4.16}$$

Step 3. We now use these bounds in order to obtain control over $\|\Lambda - A\|_{-\alpha}^2$ for a fixed time $t \geq 0$ (which is often suppressed in the notation).

In order to shorten the notations, note that, we can write

$$\widetilde{\psi}(x,t) = \sum_{k\in\mathbf{Z}} \frac{h(\varepsilon|k|)}{\sqrt{2}\sqrt{1+\nu k^2 f(\varepsilon|k|)}} \eta_k(t)\, e_k(x) \;,$$

where the η_k are a sequence of i.i.d. $\mathbf{C}^n$-valued Ornstein-Uhlenbeck processes with

$$\mathbb{E}\big(\eta_k(t) \otimes \eta_\ell(s)\big) = \mathcal{E}_k^{t-s} \delta_{k,-\ell} I \;, \qquad \mathcal{E}_k^t = \exp(-(1+\nu k^2 f(\varepsilon|k|))|t|) \;,$$

and satisfying the reality condition $\eta_{-k} = \bar{\eta}_k$. Here, I denotes the identity matrix. We will also use the notational shortcut

$$\mathbf{A}_{k,\ell}^t \stackrel{\text{def}}{=} \eta_k(t) \otimes \eta_\ell(t) \;.$$

Set now

$$q_\varepsilon^k = \frac{e^{ik\varepsilon y} - 1}{\sqrt{2}} \frac{h(\varepsilon|k|)}{\sqrt{1+\nu k^2 f(\varepsilon|k|)}} \;,$$

as a shorthand. With all of these notations in place, it follows from the definition of Λ_ε^y that

$$A^y(t) - \Lambda_\varepsilon^y I = \sum_{\varepsilon^{-\gamma} < |k|, |\ell| \leq \varepsilon^{-\chi}} q_\varepsilon^k q_\varepsilon^\ell \big(\mathbf{A}_{k,\ell}^t - \delta_{k,-\ell} I\big)\, e_{k+\ell} \;,$$

As a consequence, we have the identity

$$\begin{aligned}\mathbb{E}\|\Lambda_\varepsilon^y I - A^y(t)\|_{-\alpha}^2 = \sum_{k\in\mathbf{Z}} (1+|k|^2)^{-\alpha} \sum_{\ell,m} q_\varepsilon^\ell q_\varepsilon^{k-\ell} \bar{q}_\varepsilon^m \bar{q}_\varepsilon^{k-m} \\ \times\, \mathbb{E}\,\mathrm{tr}\Big(\big(\mathbf{A}_{\ell,k-\ell}^t - \delta_{k,0} I\big)\big(\bar{\mathbf{A}}_{m,k-m}^t - \delta_{k,0} I\big)\Big) \;,\end{aligned}$$

where the second sum ranges over all $\ell, m \in \mathbf{Z}$ for which $\ell, k-\ell, m, k-m$ belong to $(\varepsilon^{-\gamma}, \varepsilon^{-\chi}]$. A straightforward case analysis allows to check that

$$\mathbb{E}\,\mathrm{tr}\Big(\big(\mathbf{A}_{\ell,k-\ell}^t - \delta_{k,0} I\big)\big(\bar{\mathbf{A}}_{m,k-m}^t - \delta_{k,0} I\big)\Big) = n\delta_{\ell,m} + n^2 \delta_{\ell,k-m} \;, \tag{4.17}$$

so that

$$\mathbb{E}\|\Lambda_\varepsilon^y I - A^y(t)\|_{-\alpha}^2 \leq C \sum_{k\in\mathbf{Z}} (1+|k|^2)^{-\alpha} \sum_{\ell\in\mathbf{Z}} |q_\varepsilon^\ell|^2 |q_\varepsilon^{k-\ell}|^2 \;.$$

Note now that there exists a constant C such that the bound

$$|q_\varepsilon^k| \leq C\sqrt{\varepsilon}\Big(|y| \wedge \frac{1}{\varepsilon|k|}\Big) \leq C\varepsilon^{\frac{1-\beta}{2}} |k|^{-\frac{\beta}{2}} |y|^{1-\frac{\beta}{2}} \;,$$

is valid for all $\varepsilon < 1$, $k \in \mathbf{Z}$, $y \in \mathbf{R}$, and $\beta \in [0,1]$. It follows that there exists a constant $C > 0$ such that we have the bound

$$\begin{aligned}\mathbb{E}\|\Lambda_\varepsilon^y I - A^y(t)\|_{-\alpha}^2 &\leq C \sum_{\ell,m\geq 1} \frac{|q_\varepsilon^\ell|^2 |q_\varepsilon^m|^2}{|\ell+m|^{2\alpha}} \leq C \sum_{\ell,m\geq 1} \frac{|q_\varepsilon^\ell|^2 |q_\varepsilon^m|^2}{|\ell|^\alpha |m|^\alpha} \\ &\leq C\varepsilon^{2-2\beta} \sum_{\ell,m\geq 1} \frac{|y|^{4-2\beta}}{|\ell|^{\alpha+\beta}|m|^{\alpha+\beta}} \leq C\varepsilon |y|^3 \,,\end{aligned}$$

where we made the choice $\beta = \frac{1}{2}$ to obtain the last bound, using the fact that $\alpha > \frac{1}{2}$ by assumption. Combining this bound with (4.16), the constraints $\gamma \leq \frac{1}{2}$ and $\chi \geq \frac{3}{2}$, and using the fact that μ has finite fourth moment, we have

$$\begin{aligned}\mathbb{E}\|\Lambda I - A(t)\|_{-\alpha}^2 &\leq C \int \mathbb{E}\|\Lambda^y - A^y(t)\|_{-\alpha}^2 \, |\mu|(dy) \\ &\leq C \int \mathbb{E}\|\Lambda_\varepsilon^y - A^y(t)\|_{-\alpha}^2 \, |\mu|(dy) + C\varepsilon \leq C\varepsilon \,.\end{aligned}$$

Step 4. Finally we shall estimate $\mathbb{E}\|A(t) - A(s)\|_{-\alpha}^2$. Similarly to (4.17), this involves the identity

$$\mathbb{E}\,\mathrm{tr}\left(\mathbf{A}_{\ell,k-\ell}^t \bar{\mathbf{A}}_{m,k-m}^s\right) = n\delta_{k,0} + (n\delta_{l,m} + n^2\delta_{l,k-m})\mathcal{E}_\ell^{t-s}\mathcal{E}_{k-m}^{t-s} \,.$$

As a consequence, we infer that

$$\begin{aligned}D_{k\ell m}(t,s) &\stackrel{\text{def}}{=} \mathbb{E}\,\mathrm{tr}\Big(\big(\mathbf{A}_{\ell,k-\ell}^t - \mathbf{A}_{\ell,k-\ell}^s\big)\big(\bar{\mathbf{A}}_{m,k-m}^t - \bar{\mathbf{A}}_{m,k-m}^s\big)\Big) \\ &= 2(n\delta_{\ell,m} + n^2\delta_{\ell,k-m})(1 - \mathcal{E}_\ell^{t-s}\mathcal{E}_{k-m}^{t-s}) \,.\end{aligned}$$

It thus follows that for any $\delta \in [0,1]$,

$$\begin{aligned}&D_{k\ell m}(t,s) \\ &\leq C(\delta_{\ell,m} + \delta_{\ell,k-m})\Big(1 \wedge \big(2 + \nu\ell^2 f(\varepsilon|\ell|) + \nu(k-m)^2 f(\varepsilon|k-m|)\big)|t-s|\Big) \\ &\leq C(\delta_{\ell,m} + \delta_{\ell,k-m})|t-s|^\delta \Big(1 + \ell^{2\delta} f(\varepsilon|\ell|)^\delta + (k-m)^{2\delta} f(\varepsilon|k-m|)^\delta\Big) \,.\end{aligned}$$

Using this bound we obtain

$$\begin{aligned}\mathbb{E}\big\|A^y(t) - A^y(s)\big\|_{-\alpha}^2 &= \sum_{k\in\mathbf{Z}} (1+|k|^2)^{-\alpha} \sum_{\ell,m} q_\varepsilon^\ell q_\varepsilon^{k-\ell} \bar{q}_\varepsilon^m \bar{q}_\varepsilon^{k-m} D_{k\ell m}(t,s) \\ &\leq \sum_{k\in\mathbf{Z}} (1+|k|^2)^{-\alpha} \sum_\ell |q_\varepsilon^\ell|^2 |q_\varepsilon^{k-\ell}|^2 \big(D_{k\ell\ell}(t,s) + D_{k,\ell,k-\ell}(t,s)\big) \\ &\leq C|t-s|^\delta \sum_{k\in\mathbf{Z}} (1+|k|^2)^{-\alpha} \sum_\ell |q_\varepsilon^\ell|^2 |q_\varepsilon^{k-\ell}|^2 \\ &\qquad \times \Big(1 + \ell^{2\delta} f(\varepsilon|\ell|)^\delta + |k-\ell|^{2\delta} f(\varepsilon|k-\ell|)^\delta\Big) \,.\end{aligned}$$

Note that this expression is almost the same as in Step 3. Using the calculations done there and taking into account that h and h/f^2 are bounded functions, we infer that

$$\mathbb{E}\|A^y(t) - A^y(s)\|_{-\alpha}^2 = C|t-s|^\delta \sum_{\ell,m\geq 1} \frac{|q_\varepsilon^\ell|^2 |q_\varepsilon^m|^2}{|\ell|^{\alpha-2\delta}|m|^{\alpha-2\delta}} \leq C|t-s|^\delta \varepsilon |y|^3 \,,$$

and therefore, using Jensen's inequality (which can be applied since $|\mu|$ has finite mass), and Fubini's theorem,

$$\begin{aligned}\mathbb{E}\|A(t)-A(s)\|_{-\alpha}^2 &= \mathbb{E}\Big\| \int_{\mathbf{R}} \big(A^y(t)-A^y(s)\big)\,\mu(dy)\Big\|_{-\alpha}^2 \\ &\le C\int_{\mathbf{R}} \mathbb{E}\|A^y(t)-A^y(s)\|_{-\alpha}^2\,|\mu|(dy) \\ &\le C\varepsilon|t-s|^\delta \int_{\mathbf{R}} |y^3|\,|\mu|(dy) \le C\varepsilon|t-s|^\delta\,,\end{aligned}$$

which is the desired bound.

The result follows by combining these steps with Lemma 4.1. □

5. Convergence of the approximations

This last section is devoted to the convergence result itself. Recall that we are considering a number of intermediate processes $v_\varepsilon^{(j)}$ with $j=1,\dots,4$ defined in (2.10). This section is correspondingly broken into five subsections, with the jth subsection yielding a bound on $\|v_\varepsilon^{(j)}-v_\varepsilon^{(j-1)}\|_\alpha$. To prove these bounds, we shall introduce in each step a stopping time that forces the difference between the processes considered in that step to remain bounded. We then show that this difference actually vanishes as $\varepsilon\to 0$ with an explicit rate. As a consequence, the process actually doesn't "see" the stopping time with high probability.

5.1. From $\bar v^\gamma$ to $v_\varepsilon^{(1)}$.

Define

$$\tau_1^K := \tau^K \wedge \inf\{t\le T\ :\ \|v_\varepsilon^{(1)}(t)-\bar v^\gamma(t)\|_\alpha \ge K\}\,.$$

We shall show that for $t\le\tau_K$, the H^α-norm of $v_\varepsilon^{(1)}(t)-\bar v^\gamma(t)$ is controlled by the L^∞-norm of ψ_γ, which is of order $\varepsilon^{\frac{\gamma}{2}-\kappa}$ for any $\kappa>0$, as shown in Section 4. The proof uses the mild formulations of the equations for $v_\varepsilon^{(1)}$ and $\bar v^\gamma(t)$ as well as the regularising properties of the semigroup S. Note that the next proposition would still be true if we had replaced the H^α-norm in the definition of τ_1^K by the L^∞-norm. However, in the proof of Proposition 5.2 below it will be important to have a bound on $v_\varepsilon^{(1)}$ in H^α.

Proposition 5.1. *For $\kappa>0$ we have*

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big(\sup_{t\le\tau_1^K}\|v_\varepsilon^{(1)}(t)-\bar v^\gamma(t)\|_\alpha > \varepsilon^{\frac{\gamma}{2}-\kappa}\Big)=0\,.$$

Proof. Let $0\le s\le t\le\tau^*$. It follows from (2.3) and (2.10a) that $\varrho_\varepsilon := v_\varepsilon^{(1)}-\bar v^\gamma$ satisfies the equation

$$\varrho_\varepsilon(t) = S(t-s)\varrho_\varepsilon(s) + \int_s^t S(t-r)(\sigma^1+\partial_x\sigma_\varepsilon^2)(r)\,dr\,,$$

where

$$\begin{aligned}\sigma_\varepsilon^1 &:= \overline{F}(\bar v^\gamma+\varrho_\varepsilon)-\overline{F}(\bar v^\gamma+\psi_\gamma)\,,\\ \sigma_\varepsilon^2 &:= G(\bar v^\gamma+\varrho_\varepsilon)-G(\bar v^\gamma+\psi_\gamma)\,.\end{aligned}$$

Lemma 3.3 yields the estimate

$$\begin{aligned}\|\varrho_\varepsilon(t)\|_\alpha &\le \|\varrho_\varepsilon(s)\|_\alpha + C(t-s)^{\frac12(1-\alpha)} \sup_{r\in(s,t)} \|(\sigma_\varepsilon^1 + \partial_x\sigma_\varepsilon^2)(r)\|_{-1} \\ &\le \|\varrho_\varepsilon(s)\|_\alpha + C(t-s)^{\frac12(1-\alpha)} \sup_{r\in(s,t)} \|\sigma_\varepsilon^1(r)\|_{L^\infty} + \|\sigma_\varepsilon^2(r)\|_{L^\infty} \ .\end{aligned}$$

Since $\bar v^\gamma$, ϱ_ε, and ψ_γ are bounded in L^∞-norm for $r \le \tau_1^K$, and F, G are $\mathcal{C}^3$, it follows that

$$\|\sigma_\varepsilon^1(r)\|_{L^\infty} + \|\sigma_\varepsilon^2(r)\|_{L^\infty} \le C_K\|\varrho_\varepsilon(r)\|_{L^\infty} + C_K\|\psi_\gamma(r)\|_{L^\infty},$$

from which we infer that

$$\|\varrho_\varepsilon(t)\|_\alpha \le \|\varrho_\varepsilon(s)\|_\alpha + C_K'(t-s)^{\frac12(1-\alpha)} \sup_{r\in(s,t)} \Big(\|\varrho_\varepsilon(r)\|_{L^\infty} + \|\psi_\gamma(r)\|_{L^\infty}\Big) \ .$$

Choose $\delta_K > 0$ so small that $C_K'\delta_K^{\frac12(1-\alpha)} \le \frac12$, and set for $k \ge 0$,

$$r_k := \sup\big\{\|\varrho_\varepsilon(t)\|_\alpha \ : \ t \in [k\delta_K \wedge \tau_1^K, (k+1)\delta_K \wedge \tau_1^K]\big\} \ .$$

Taking into account that $H^\alpha \subseteq L^\infty$, we obtain the inequality

$$r_{k+1} \le r_k + \frac12 r_{k+1} + \frac12 \sup_{t\in[0,T]} \|\psi_\gamma(t)\|_{L^\infty} \ ,$$

which reduces to

$$r_{k+1} \le 2r_k + \sup_{t\in[0,T]} \|\psi_\gamma(t)\|_{L^\infty} \ .$$

Combined with the estimate

$$r_0 \le 2 \sup_{r\le\delta_K\wedge\tau_1^K} \|\psi_\gamma(r)\|_{L^\infty}$$

which can be derived similarly, it then follows that

$$\sup_{t\in[0,\tau_1^K]} \|\varrho_\varepsilon(t)\|_\alpha \le \sup_{0\le k\le T/\delta_K} r_k \le C_K \sup_{t\in[0,T]} \|\psi_\gamma(t)\|_{L^\infty} \ ,$$

which together with Proposition 4.3 implies the desired result. □

5.2. **From $v_\varepsilon^{(1)}$ to $v_\varepsilon^{(2)}$.** For the purpose of this section, we define the stopping time

$$\tau_2^K := \tau_1^K \wedge \inf\{t \le T \ : \ \|v_\varepsilon^{(2)}(t) - v_\varepsilon^{(1)}(t)\|_\alpha \ge K\} \ ,$$

as well as the exponent

$$\widetilde\alpha \stackrel{\text{def}}{=} (1-\gamma)\alpha + \frac\gamma2 = \frac23 \ .$$

Proposition 5.2. *For $\kappa > 0$ we have*

$$\lim_{\varepsilon\to0} \mathbb{P}\Big(\sup_{t\le\tau_2^K} \|v_\varepsilon^{(2)}(t) - v_\varepsilon^{(1)}(t)\|_\alpha > \varepsilon^{\widetilde\alpha-\kappa}\Big) = 0 \ .$$

Proof. Let $0 \le s \le t \le \tau^*$ and note that $\varrho_\varepsilon := v_\varepsilon^{(2)} - v_\varepsilon^{(1)}$ satisfies

$$\varrho_\varepsilon(t) = S(t-s)\varrho_\varepsilon(s) + \int_s^t S(t-r)\sigma_\varepsilon(r)\,dr \ ,$$

where

$$\sigma_\varepsilon := \overline F(v_\varepsilon^{(2)}) - \overline F(v_\varepsilon^{(1)}) + D_\varepsilon(G(v_\varepsilon^{(1)} + \varrho_\varepsilon)) - \partial_x G(v_\varepsilon^{(1)}) \ .$$

From the definition of τ_2^K we know that $v_\varepsilon^{(1)}$ and ϱ_ε are bounded in L^∞ by a constant depending on K. Moreover, we have the bound $\|v_\varepsilon^{(1)}\|_\alpha \le C_K \varepsilon^{-\gamma(\alpha-\frac12)-\kappa}$. Using these facts together with Corollary 3.7 we obtain, for $r \le \tau_2^K$,

$$\begin{aligned}\|\sigma_\varepsilon\|_{-1} &\le \|\overline{F}(v_\varepsilon^{(2)}) - \overline{F}(v_\varepsilon^{(1)})\|_{L^\infty} \\ &\quad + \|(D_\varepsilon - \partial_x)G(v_\varepsilon^{(1)})\|_{-1} + \|D^\varepsilon(G(v_\varepsilon^{(1)} + \varrho_\varepsilon) - G(v_\varepsilon^{(1)}))\|_{-1} \\ &\le C_K\|\varrho_\varepsilon\|_{L^\infty} + C_K\varepsilon^\alpha(1 + \|v_\varepsilon^{(1)}\|_\alpha) + \|G(v_\varepsilon^{(1)} + \varrho_\varepsilon) - G(v_\varepsilon^{(1)})\|_{L^\infty} \\ &\le C_K(\varepsilon^{\widetilde{\alpha}-\kappa} + \|\varrho_\varepsilon\|_{L^\infty})\ ,\end{aligned}$$

hence

$$\begin{aligned}\|\varrho_\varepsilon(t)\|_\alpha &\le \|\varrho_\varepsilon(s)\|_\alpha + C(t-s)^{\frac12(1-\alpha)} \sup_{r\in(s,t)} \|\sigma_\varepsilon(r)\|_{-1} \\ &\le \|\varrho_\varepsilon(s)\|_\alpha + C_K(t-s)^{\frac12(1-\alpha)} \sup_{r\in(s,t)} \Big(\varepsilon^{\widetilde{\alpha}-\kappa} + \|\varrho_\varepsilon(r)\|_{L^\infty}\Big)\ .\end{aligned}$$

Arguing as in the proof of Proposition 5.1, it follows that

$$\sup_{t\in[0,\tau_2^K]} \|\varrho_\varepsilon(t)\|_\alpha \le C_K\varepsilon^{\widetilde{\alpha}-\kappa}\ ,$$

which immediately yields the desired result. □

5.3. **From $v_\varepsilon^{(2)}$ to $v_\varepsilon^{(3)}$.** Define

$$\tau_3^K := \tau_2^K \wedge \inf\{t \le T\ :\ \|v_\varepsilon^{(3)}(t) - v_\varepsilon^{(2)}(t)\|_\alpha \ge K\}\ .$$

In this case, the singularity $(t-s)^{-\frac12\alpha}$ which arises in the proof below, prevents us from arguing as in Proposition 5.1. We nevertheless have:

Proposition 5.3. *For $\kappa > 0$ we have*

$$\lim_{\varepsilon\to0} \mathbb{P}\Big(\sup_{t\le\tau_3^K} \|v_\varepsilon^{(3)}(t) - v_\varepsilon^{(2)}(t)\|_\alpha > \varepsilon^{\zeta-\kappa}\Big) = 0\ ,$$

where the exponent ζ is given by

$$\zeta \stackrel{\text{def}}{=} \widetilde{\alpha} \wedge (\tfrac32 - \alpha) \wedge \big(2 - \gamma(\alpha + \tfrac32)\big) = \frac23\ .$$

Remark 5.4. The exponent ζ arises by collecting the bounds (5.2), (5.3), and (5.5).

Proof. Let $0 \le s \le t \le \tau^*$. It follows from (2.10b) and (2.10c) that $\varrho_\varepsilon := v_\varepsilon^{(3)} - v_\varepsilon^{(2)}$ satisfies

$$\varrho_\varepsilon(t) = S_\varepsilon(t-s)\varrho_\varepsilon(s) + (S_\varepsilon(t-s) - S(t-s))v_\varepsilon^{(2)}(s) + \mathcal{R}_1(s,t) + \mathcal{R}_2(s,t)\ , \quad (5.1)$$

where

$$\mathcal{R}_1(s,t) \stackrel{\text{def}}{=} (\widetilde{\psi}^\gamma(t) - \psi^\gamma(t)) - \big(S_\varepsilon(t-s)\widetilde{\psi}^\gamma(s) - S(t-s)\psi^\gamma(s)\big)$$

and

$$\begin{aligned}\mathcal{R}_2(s,t) := &\int_s^t (S_\varepsilon(t-r) - S(t-r))\Big(\overline{F}(v_\varepsilon^{(3)}(r)) + D_\varepsilon G(v_\varepsilon^{(3)}(r)\Big)\,dr \\ &+ \int_s^t S(t-r)\big(\overline{F}(v_\varepsilon^{(3)}(r)) - \overline{F}(v_\varepsilon^{(2)}(r)) \\ &\qquad + D_\varepsilon G(v_\varepsilon^{(3)}(r)) - D_\varepsilon G(v_\varepsilon^{(2)}(r))\big)\,dr\ .\end{aligned}$$

We shall first prove a bound on $\mathcal{R}_1(s,t)$. Using both inequalities from Lemma 3.4, we obtain

$$\begin{aligned}
&\|\big(S_\varepsilon(t-s)\widetilde{\psi}^\gamma(s) - S(t-s)\psi^\gamma(s)\big)\|_\alpha \\
&\le \|S_\varepsilon(t-s)\big(\widetilde{\psi}^\gamma(s) - \psi^\gamma(s)\big)\|_\alpha + \|\big(S_\varepsilon(t-s) - S(t-s)\big)\psi^\gamma(s)\|_\alpha \\
&\le C\|\big(\widetilde{\psi}^\gamma(s) - \psi^\gamma(s)\big)\|_\alpha + C\varepsilon^2\|\psi^\gamma(s)\|_{\alpha+2}\,,
\end{aligned}$$

and therefore

$$\begin{aligned}
\|\mathcal{R}_1(s,t)\|_\alpha \le \|\big(\widetilde{\psi}^\gamma(t) - \psi^\gamma(t)\big)\|_\alpha + C\|\big(\widetilde{\psi}^\gamma(s) - \psi^\gamma(s)\big)\|_\alpha \\
+ C\varepsilon^2\|\psi^\gamma(s)\|_{\alpha+2}\,.
\end{aligned}$$

It thus follows from Proposition 4.4 that

$$\mathbb{E}\sup_{s,t\in[0,T]}\|\mathcal{R}_1(s,t)\|_\alpha \le C\varepsilon^{2-\gamma(\alpha+\frac{3}{2})-\kappa}\,. \tag{5.2}$$

We shall now prove a bound on $\mathcal{R}_2(s,t)$. For this purpose, we note that the definitions of the various stopping times imply that $v_\varepsilon^{(2)}(t)$ is bounded in H^α-norm by $C_K\varepsilon^{-\gamma(\alpha-\frac{1}{2})-\kappa}$. Using this fact, together with Lemmas 3.4, 3.5 and 3.2, we obtain

$$\begin{aligned}
&\Big\|\int_s^t (S_\varepsilon(t-r) - S(t-r))\Big(F(v_\varepsilon^{(3)}(r)) + D_\varepsilon G(v_\varepsilon^{(3)}(r))\Big)\,dr\Big\|_\alpha \\
&\quad\le \varepsilon^\alpha\int_s^t (t-r)^{-\frac{1}{2}(1+\alpha)}\|F(v_\varepsilon^{(3)}(r)) + D_\varepsilon G(v_\varepsilon^{(3)}(r))\|_{\alpha-1}\,dr \\
&\quad\le C\varepsilon^\alpha(t-s)^{\frac{1}{2}(1-\alpha)}\sup_{r\in[s,t]}\big(\|F(v_\varepsilon^{(3)}(r))\|_\alpha + \|G(v_\varepsilon^{(3)}(r))\|_\alpha\big) \\
&\quad\le C_K\varepsilon^\alpha(t-s)^{\frac{1}{2}(1-\alpha)}(1 + \sup_{r\in[s,t]}\|v_\varepsilon^{(3)}(r)\|_\alpha) \\
&\quad\le C_K\varepsilon^\alpha(t-s)^{\frac{1}{2}(1-\alpha)}\big(1 + \sup_{r\in[s,t]}(\|v_\varepsilon^{(2)}(r)\|_\alpha + \|\varrho_\varepsilon(r)\|_\alpha)\big) \\
&\quad\le C_K\varepsilon^{\widetilde{\alpha}-\kappa}(t-s)^{\frac{1}{2}(1-\alpha)}\,.
\end{aligned}$$

Furthermore, taking into account the L^∞-bounds on $v_\varepsilon^{(2)}$ and ϱ_ε enforced by the stopping times, Lemma 3.5 implies that

$$\begin{aligned}
&\Big\|\int_s^t S(t-r)\Big(F(v_\varepsilon^{(3)}(r)) - F(v_\varepsilon^{(2)}(r)) + D_\varepsilon G(v_\varepsilon^{(3)}(r)) - D_\varepsilon G(v_\varepsilon^{(2)}(r))\Big)\,dr\Big\|_\alpha \\
&\quad\le C(t-s)^{\frac{1}{2}(1-\alpha)}\sup_{r\in(s,t)}\|F(v_\varepsilon^{(3)}(r)) - F(v_\varepsilon^{(2)}(r)) \\
&\qquad\qquad + D_\varepsilon G(v_\varepsilon^{(3)}(r)) - D_\varepsilon G(v_\varepsilon^{(2)}(r))\|_{-1} \\
&\quad\le C(t-s)^{\frac{1}{2}(1-\alpha)}\sup_{r\in(s,t)}\Big(\|F(v_\varepsilon^{(3)}(r)) - F(v_\varepsilon^{(2)}(r))\|_{L^\infty} \\
&\qquad\qquad + \|G(v_\varepsilon^{(3)}(r)) - G(v_\varepsilon^{(2)}(r))\|_{L^\infty}\Big) \\
&\quad\le C_K(t-s)^{\frac{1}{2}(1-\alpha)}\sup_{r\in(s,t)}\|\varrho_\varepsilon(r)\|_{L^\infty}\,.
\end{aligned}$$

It thus follows that

$$\|\mathcal{R}_2(s,t)\|_\alpha \le C'_K(t-s)^{\frac{1}{2}(1-\alpha)}\Big(\varepsilon^{\widetilde{\alpha}-\kappa} + \sup_{r\in(s,t)}\|\varrho_\varepsilon(r)\|_{L^\infty}\Big)\ , \tag{5.3}$$

where we gave the constant a name, since it will be reused below.

Choose $\delta_K \in (0,1)$ sufficiently small so that $C'_K\big(\delta_K^{\frac{1}{2}(1-\alpha)} + \delta_K^{\frac{1}{2}\alpha}\big) \le \frac{1}{4}$. For $k \ge 0$ put $\ell_k := k\delta_K \wedge \tau_3^K$, and for $k \ge 1$ set

$$r_k := \sup\big\{\|\varrho_\varepsilon(t)\|_\alpha \ :\ t \in [\ell_{k-1}, \ell_{k+1}]\big\}\ .$$

Our next aim is to find a bound for r_1. Observe that, when $s = 0$, (5.1) simplifies to

$$\varrho_\varepsilon(t) = (S_\varepsilon(t) - S(t))v_0 + (\widetilde{\psi}^\gamma(t) - \psi^\gamma(t)) + \mathcal{R}_2(0,t)\ , \tag{5.4}$$

with $\mathcal{R}_2$ defined as previously. Using Lemma 3.4 and the definition of τ^K, we obtain

$$\|(S_\varepsilon(t) - S(t))v_0\|_\alpha \le C\varepsilon^{\frac{3}{2}-\alpha-\kappa}\|v_0\|_{\frac{3}{2}-\kappa} \le C_K\varepsilon^{\frac{3}{2}-\alpha-\kappa}\ . \tag{5.5}$$

Since $t \le 2\delta_K$ and $C'_K\delta_K^{\frac{1}{2}(1-\alpha)} \le \frac{1}{4}$, it follows from (5.3) and (5.4) that

$$r_1 \le C_K\varepsilon^{\frac{3}{2}-\alpha-\kappa} + \sup_{t\in[0,T]}\|\widetilde{\psi}^\gamma(t) - \psi^\gamma(t)\|_\alpha + \frac{1}{2}(\varepsilon^{\widetilde{\alpha}-\kappa} + r_1)\ ,$$

hence, by definition of τ^K,

$$r_1 \le C_K\varepsilon^{(\frac{3}{2}-\alpha)\wedge(2-\gamma(\alpha+\frac{3}{2}))\wedge\widetilde{\alpha}}\varepsilon^{-\kappa} = C_K\varepsilon^{\zeta-\kappa}\ , \tag{5.6}$$

where ζ is defined as in the statement of the result.

Next we shall prove a recursive bound for r_k. Note that the nonnegativity of the function f in the definition of S_ε implies that

$$\|S_\varepsilon(t-s)\varrho_\varepsilon(s)\|_\alpha \le \|\varrho_\varepsilon(s)\|_\alpha\ .$$

Furthermore, by Lemma 3.4 and the fact that $\|v_\varepsilon^{(2)}\|_\alpha \le C_K\varepsilon^{-\gamma(\alpha-\frac{1}{2})-\kappa}$ before time τ_3^K, we have

$$\begin{aligned}\|(S_\varepsilon(t-s) - S(t-s))v_\varepsilon^{(2)}(s)\|_\alpha &\le C\varepsilon^\alpha(t-s)^{-\frac{1}{2}\alpha}\|v_\varepsilon^{(2)}(s)\|_\alpha\\ &\le C_K(t-s)^{-\frac{1}{2}\alpha}\varepsilon^{\widetilde{\alpha}-\kappa}\ .\end{aligned}$$

Combining these bounds with (5.1) and (5.3), we find that

$$\begin{aligned}\|\varrho_\varepsilon(t)\|_\alpha \le \|\varrho_\varepsilon(s)\|_\alpha &+ C_K(t-s)^{-\frac{1}{2}\alpha}\varepsilon^{\widetilde{\alpha}-\kappa} + \|\mathcal{R}_1(s,t)\|_\alpha\\ &+ C'_K(t-s)^{\frac{1}{2}(1-\alpha)}\big(\varepsilon^{\widetilde{\alpha}-\kappa} + \sup_{r\in(s,t)}\|\varrho_\varepsilon(r)\|_\alpha\big)\ .\end{aligned}$$

Taking $k \ge 1$, $s = \ell_{k-1}$, and $t \in [\ell_k, \ell_{k+2}]$, it then follows, since $|t-s| \in [\delta_K, 3\delta_K]$ and $C'_K\delta^{\frac{1}{2}(1-\alpha)} \le \frac{1}{2}$, that

$$\|\varrho_\varepsilon(t)\|_\alpha \le \|\varrho_\varepsilon(\ell_{k-1})\|_\alpha + C_K\varepsilon^{\widetilde{\alpha}-\kappa} + \|\mathcal{R}_1(\ell_{k-1},t)\|_\alpha + \tfrac{1}{2}\varepsilon^{\widetilde{\alpha}-\kappa} + \tfrac{1}{2}r_{k+1}\ .$$

Taking the supremum over $t \in [\ell_k, \ell_{k+2}]$, we obtain

$$r_{k+1} \le r_k + C_K\varepsilon^{\widetilde{\alpha}-\kappa} + \sup_{s,t\in[0,T]}\|\mathcal{R}_1(s,t)\|_\alpha + \tfrac{1}{2}r_{k+1}\ ,$$

hence

$$r_{k+1} \le 2r_k + C_K\varepsilon^{\widetilde{\alpha}-\kappa} + 2\sup_{s,t\in[0,T]}\|\mathcal{R}_1(s,t)\|_\alpha\ . \tag{5.7}$$

It readily follows from (5.6) and (5.7) that

$$\sup_{t\in[0,\tau_3^K]} \|\varrho_\varepsilon(t)\|_\alpha = \sup_{1\le k\le\lceil T/\delta_K\rceil} r_k \le C_K\big(\varepsilon^{\zeta-\kappa} + \varepsilon^{\widetilde{\alpha}-\kappa} + \sup_{s,t\in[0,T]} \|\mathcal{R}_1(s,t)\|_\alpha\big)\ ,$$

hence the result follows in view of the bound on $\mathcal{R}_1(s,t)$. □

5.4. **From $v_\varepsilon^{(3)}$ to $v_\varepsilon^{(4)}$.** Define

$$\tau_4^K := \tau_3^K \wedge \inf\{t\le T\ :\ \|v_\varepsilon^{(4)}(t) - v_\varepsilon^{(3)}(t)\|_\alpha \ge K\}\ .$$

Proposition 5.5. *For $\kappa > 0$ we have*

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{t\le\tau_4^K} \|v_\varepsilon^{(4)}(t) - v_\varepsilon^{(3)}(t)\|_\alpha > \varepsilon^{\xi-\kappa}\Big) = 0\ ,$$

where

$$\xi \stackrel{def}{=} \tfrac{\gamma}{2} \wedge \big(\widetilde{\alpha} - \tfrac12\big) \wedge \big(\tfrac12 - \chi(\alpha - \tfrac12)\big) = \frac{1}{8}\ .$$

Remark 5.6. Similarly to above, the exponent ξ arises from the bounds (5.9)–(5.14).

Proof. Let $0 \le s \le t \le \tau^*$. It follows from (2.10d) and (2.10c) that $\varrho_\varepsilon := v_\varepsilon^{(4)} - v_\varepsilon^{(3)}$ satisfies

$$\varrho_\varepsilon(t) = S_\varepsilon(t-s)\varrho_\varepsilon(s) + \int_s^t S_\varepsilon(t-r)\sigma_\varepsilon(r)\,dr\ ,$$

where

$$\begin{aligned}\sigma_\varepsilon &:= F(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - F(v_\varepsilon^{(3)})\\ &\quad + \nabla G(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) D_\varepsilon(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - D_\varepsilon G(v_\varepsilon^{(3)}) + \Lambda\Delta G(v_\varepsilon^{(3)})\ .\end{aligned}$$

The definition of D_ε together with (1.3) imply that for any function u the following identity holds:

$$\begin{aligned}D_\varepsilon G(u)(x) &= \nabla G(u(x)) D_\varepsilon u(x)\\ &\quad + \int_{\mathbf{R}} \frac{\varepsilon y^2}{2} D^2 G(u(x))[\widehat{D}_{\varepsilon y}u(x), \widehat{D}_{\varepsilon y}u(x)]\,\mu(dy)\\ &\quad + \int_{\mathbf{R}} \varepsilon^2 y^3 \int_0^1\int_0^t\int_0^s D^3 G\big((1-r)u(x) + ru(x+\varepsilon y)\big)\\ &\qquad [\widehat{D}_{\varepsilon y}u(x), \widehat{D}_{\varepsilon y}u(x), \widehat{D}_{\varepsilon y}u(x)]\,dr\,ds\,dt\,\mu(dy)\ ,\end{aligned} \tag{5.8}$$

where the operator $\widehat{D}_\varepsilon$ is defined by taking $\mu := \delta_1 - \delta_0$ in the definition of D_ε, i.e., $\widehat{D}_\varepsilon u(x) = \varepsilon^{-1}(u(x+\varepsilon) - u(x))$. As a consequence, we may write

$$\begin{aligned}\sigma_\varepsilon &= F(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - F(v_\varepsilon^{(3)}) + D_\varepsilon(G(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - G(v_\varepsilon^{(3)}))\\ &\quad + (\Lambda\Delta G(v_\varepsilon^{(3)}) - A(u_\varepsilon^{(4)}, u_\varepsilon^{(4)})) - B\\ &= F(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - F(v_\varepsilon^{(3)}) + D_\varepsilon(G(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - G(v_\varepsilon^{(3)}))\\ &\quad - A(v_\varepsilon^{(4)}, v_\varepsilon^{(4)}) - 2A(v_\varepsilon^{(4)}, \widetilde{\psi}_\gamma^\chi) + (\Lambda\Delta G(v_\varepsilon^{(3)}) - A(\widetilde{\psi}_\gamma^\chi, \widetilde{\psi}_\gamma^\chi)) - B\ ,\end{aligned}$$

where we have used

$$A(v,w)(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}} \frac{\varepsilon y^2}{2} D^2 G(u_\varepsilon^{(4)}(x))[\widehat{D}_{\varepsilon y} v(x), \widehat{D}_{\varepsilon y} w(x)]\, \mu(dy) \ ,$$

$$B(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}} \varepsilon^2 y^3 \int_0^1 \int_0^t \int_0^s D^3 G\big((1-r)u_\varepsilon^{(4)}(x) + r u_\varepsilon^{(4)}(x+\varepsilon y)\big) \\ [\widehat{D}_{\varepsilon y} u_\varepsilon^{(4)}(x), \widehat{D}_{\varepsilon y} u_\varepsilon^{(4)}(x), \widehat{D}_{\varepsilon y} u_\varepsilon^{(4)}(x)]\, dr\, ds\, dt\, \mu(dy) \ ,$$

and $u_\varepsilon^{(4)} := v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi$.

Our next aim is to prove the estimates (5.9)–(5.14) below in order to bound $\|\sigma_\varepsilon\|_{-1}$.

First term. Since $v_\varepsilon^{(4)}$, $\widetilde{\psi}_\gamma^\chi$ and ϱ_ε are bounded in L^∞ by definition of τ_4^K, it follows that

$$\begin{aligned} \|F(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - F(v_\varepsilon^{(3)})\|_{-1} &\le C\|F(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - F(v_\varepsilon^{(4)} - \varrho_\varepsilon)\|_{L^\infty} \\ &\le C_K(\|\widetilde{\psi}_\gamma^\chi\|_{L^\infty} + \|\varrho_\varepsilon\|_{L^\infty}) \\ &\le C_K(\varepsilon^{\frac{\gamma}{2}-\kappa} + \|\varrho_\varepsilon\|_\alpha) \ . \end{aligned} \tag{5.9}$$

Second term. We use Lemma 3.5 and the fact that $v_\varepsilon^{(4)}$, $\widetilde{\psi}_\gamma^\chi$ and ϱ_ε are bounded in L^∞ by definition of τ_4^K, to estimate

$$\begin{aligned} \|D_\varepsilon(G(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - G(v_\varepsilon^{(3)}))\|_{-1} &\le C\|G(v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi) - G(v_\varepsilon^{(4)} - \varrho_\varepsilon)\|_{L^\infty} \\ &\le C_K(\|\widetilde{\psi}_\gamma^\chi\|_{L^\infty} + \|\varrho_\varepsilon\|_{L^\infty}) \\ &\le C_K(\varepsilon^{\frac{\gamma}{2}-\kappa} + \|\varrho_\varepsilon\|_\alpha) \ . \end{aligned} \tag{5.10}$$

Third and fourth term. First we note that for arbitrary functions v, w, one has

$$\|A(v,w)\|_{-1} \le C\|A(v,w)\|_{L^1} \le C\varepsilon \|D^2 G(u_\varepsilon^{(4)})\|_{L^\infty} \sqrt{\Theta_\varepsilon(v)\Theta_\varepsilon(w)} \ .$$

Since $\|u_\varepsilon^{(4)}(t)\|_{L^\infty} \le C_K$ for $t \le \tau_4^K$, we have

$$\|D^2 G(u_\varepsilon^{(4)})\|_{L^\infty} \le C_K \ .$$

Furthermore, we observe that $\|v_\varepsilon^{(4)}\|_\alpha \le C_K \varepsilon^{-\gamma(\alpha-\frac{1}{2})-\kappa}$ before time τ_4^K. Using this bound together with Lemma 3.5 and (1.3), we estimate

$$\begin{aligned} \Theta_\varepsilon(v_\varepsilon^{(4)}) &= \int_{\mathbf{R}} y^2 \|\widehat{D}_{\varepsilon y} v_\varepsilon^{(4)}\|_{L^2}^2\, |\mu|(dy) \\ &\le C_K \int_{\mathbf{R}} y^2 |\varepsilon y|^{2(\alpha-1)} \|v_\varepsilon^{(4)}\|_\alpha^2\, |\mu|(dy) \\ &\le C_K \int_{\mathbf{R}} y^2 |\varepsilon y|^{2(\alpha-1)} \varepsilon^{-2\gamma(\alpha-\frac{1}{2})-2\kappa}\, |\mu|(dy) \\ &\le C_K \varepsilon^{2\widetilde{\alpha}-2-\kappa} \ . \end{aligned}$$

Moreover, by definition of the stopping time τ^K we have

$$\Theta_\varepsilon(\widetilde{\psi}_\gamma^\chi) \le \Theta_\varepsilon(\widetilde{\psi}_\gamma) \le C_K \varepsilon^{-1-\kappa}.$$

Putting everything together, we obtain

$$\|A(v_\varepsilon^{(4)}, v_\varepsilon^{(4)})\|_{-1} \le C_K \varepsilon^{2\widetilde{\alpha}-1-2\kappa} \tag{5.11}$$

and

$$\|A(v_\varepsilon^{(4)}, \widetilde{\psi}_\gamma^\chi)\|_{-1} \leq C_K \varepsilon^{\widetilde{\alpha}-\frac{1}{2}-2\kappa} \,. \tag{5.12}$$

Fifth term. Finally we estimate $\Lambda\Delta G(v_\varepsilon^{(3)}) - A(\widetilde{\psi}_\gamma^\chi, \widetilde{\psi}_\gamma^\chi)$. By definition of τ_ε^K, we have $\|\widetilde{\psi}_\gamma^\chi\|_\alpha \leq C_K \varepsilon^{-\chi(\alpha-\frac{1}{2})-\kappa}$ before time τ_4^K. Since $\|v_\varepsilon^{(4)}\|_\alpha \leq C_K \varepsilon^{-\gamma(\alpha-\frac{1}{2})-\kappa}$ as observed before, we thus have

$$\|u_\varepsilon^{(4)}\|_\alpha \leq C_K \varepsilon^{-\chi(\alpha-\frac{1}{2})-\kappa} \,.$$

Furthermore, since $\alpha > \frac{1}{2}$, there exists a constant $C > 0$ such that we have the bound

$$\begin{aligned}\|\Lambda\Delta G(u_\varepsilon^{(4)}) - A(\widetilde{\psi}_\gamma^\chi, \widetilde{\psi}_\gamma^\chi)\|_{-\alpha} &= \Big\| \operatorname{tr}\Big(D^2 G(u_\varepsilon^{(4)})\big(\Lambda I - \Xi_\varepsilon(\widetilde{\psi}_\gamma^\chi)\big)\Big)\Big\|_{-\alpha} \\ &\leq C\|D^2 G(u_\varepsilon^{(4)})\|_\alpha \|\Lambda I - \Xi_\varepsilon(\widetilde{\psi}_\gamma^\chi)\|_{-\alpha} \,.\end{aligned}$$

Since the stopping time τ^K enforces that $\|\Lambda I - \Xi_\varepsilon(\widetilde{\psi}_\gamma^\chi)\|_{-\alpha} \leq C_K \varepsilon^{\frac{1}{2}}$, we infer that

$$\|\Lambda\Delta G(u_\varepsilon^{(4)}) - A(\widetilde{\psi}_\gamma^\chi, \widetilde{\psi}_\gamma^\chi)\|_{-\alpha} \leq C_K \varepsilon^{\frac{1}{2}-\chi(\alpha-\frac{1}{2})-\kappa} \,.$$

Since $u_\varepsilon^{(4)} - v_\varepsilon^{(3)} = \varrho_\varepsilon + \widetilde{\psi}_\gamma^\chi$, we have by definition of τ^K,

$$\begin{aligned}\|\Delta G(u_\varepsilon^4) - \Delta G(v_\varepsilon^{(3)})\|_{-\alpha} &\leq \|\Delta G(u_\varepsilon^4) - \Delta G(v_\varepsilon^{(3)})\|_{L^\infty} \\ &\leq C_K \|\varrho_\varepsilon\|_{L^\infty} + C_K \|\widetilde{\psi}_\gamma^\chi\|_{L^\infty} \\ &\leq C_K \|\varrho_\varepsilon\|_\alpha + C_K \varepsilon^{\frac{\gamma}{2}-\kappa} \,.\end{aligned}$$

Putting these bounds together, we obtain

$$\|\Lambda\Delta G(v_\varepsilon^{(3)}) - A(\widetilde{\psi}_\gamma^\chi, \widetilde{\psi}_\gamma^\chi)\|_{-\alpha} \leq C_K \big(\varepsilon^{\frac{1}{2}-\chi(\alpha-\frac{1}{2})-\kappa} + \varepsilon^{\frac{\gamma}{2}-\kappa} + \|\varrho_\varepsilon\|_\alpha\big) \,. \tag{5.13}$$

Sixth term. To estimate B, we use the fact that $\|u_\varepsilon^{(4)}(t)\|_{L^\infty} \leq C_K$ for $t \leq \tau_4^K$, so that one has the bound

$$\|B\|_{L^1} \leq C_K \int\int_{\mathbf{R}} \varepsilon^2 y^3 |\widehat{D}_{\varepsilon y} u_\varepsilon^{(4)}(x)|^3 \, |\mu|(dy)\, dx.$$

We will split this expression into two parts, using the fact that $u_\varepsilon^{(4)} = \widetilde{\psi}_\gamma^\chi + v_\varepsilon^{(4)}$. First, using the fact that $\Theta(\widetilde{\psi}_\gamma^\chi) \leq C_K \varepsilon^{1-\kappa}$ before time τ_4^K by definition of the stopping time τ^K, we find that

$$\begin{aligned}&\int\int_{\mathbf{R}} \varepsilon^2 y^3 |\widehat{D}_{\varepsilon y} \widetilde{\psi}_\gamma^\chi(x)|^3 \, |\mu|(dy)\, dx \\ &\leq 2\|\widetilde{\psi}_\gamma^\chi\|_{L^\infty} \int\int_{\mathbf{R}} \varepsilon y^2 |\widehat{D}_{\varepsilon y} \widetilde{\psi}_\gamma^\chi(x)|^2 \, |\mu|(dy)\, dx \\ &= 2\varepsilon \|\widetilde{\psi}_\gamma^\chi\|_{L^\infty} \Theta(\widetilde{\psi}_\gamma^\chi) \leq C_K \varepsilon^{-2\kappa} \|\widetilde{\psi}_\gamma^\chi\|_{L^\infty} \leq C_K \varepsilon^{\frac{\gamma}{2}-3\kappa} \,.\end{aligned}$$

Second, using the fact that $H^{\frac{1}{6}} \subseteq L^3$, Lemma 3.5 and the fact that $\|v_\varepsilon^{(4)}\|_\alpha \leq C_K \varepsilon^{-\gamma(\alpha-\frac{1}{2})-\kappa}$, we obtain

$$\begin{aligned}\int\int_{\mathbf{R}} \varepsilon^2 y^3 |\widehat{D}_{\varepsilon y} v_\varepsilon^{(4)}(x)|^3 \, |\mu|(dy)\, dx &\leq C\varepsilon^2 \int_{\mathbf{R}} |y|^3 \|\widehat{D}_{\varepsilon y} v_\varepsilon^{(4)}\|_{\frac{1}{6}}^3 \, |\mu|(dy) \\ &\leq C\varepsilon^{3\alpha-\frac{3}{2}} \int_{\mathbf{R}} |y|^{3\alpha-\frac{1}{2}} \, |\mu|(dy) \, \|v_\varepsilon^{(4)}\|_\alpha^3\end{aligned}$$

$$\leq C_K \varepsilon^{3\widetilde{\alpha}-\frac{3}{2}-3\kappa} \ .$$

It thus follows that

$$\|B\|_{L^1} \leq C_K \varepsilon^{\frac{\gamma}{2}-3\kappa} + C_K \varepsilon^{3(\widetilde{\alpha}-\frac{1}{2})-3\kappa} \ . \tag{5.14}$$

Combining the inequalities (5.9)–(5.13), we find that

$$\sup_{r\in(s,t)} \|\sigma_\varepsilon(r)\|_{-1} \leq C_K \Big(\varepsilon^{\frac{\gamma}{2}-3\kappa} + \varepsilon^{\widetilde{\alpha}-\frac{1}{2}-2\kappa} + \varepsilon^{\frac{1}{2}-\chi(\alpha-\frac{1}{2})-\kappa} + \sup_{r\in(s,t)} \|\varrho_\varepsilon(r)\|_\alpha \Big) \ ,$$

and the result now follows as in Proposition 5.2. □

5.5. **From $v_\varepsilon^{(4)}$ to $\widetilde{v}^\gamma$.** Define

$$\tau_5^K := \tau_4^K \wedge \inf\{t \leq T \ : \ \|\widetilde{v}^\gamma(t) - v_\varepsilon^{(4)}(t)\|_\alpha \geq K\} \ .$$

Proposition 5.7. *For $\kappa > 0$ we have*

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(\sup_{t\leq \tau_5^K} \|\widetilde{v}^\gamma(t) - v_\varepsilon^{(4)}(t)\|_\alpha > \varepsilon^{\frac{1}{2}\chi-\frac{1}{2}-\kappa} \Big) = 0 \ .$$

Proof. Let $0 \leq s \leq t \leq \tau^*$. It follows from (2.4) and (2.10d) that $\varrho_\varepsilon := \widetilde{v}^\gamma - v_\varepsilon^{(4)}$ satisfies

$$\varrho_\varepsilon(t) = \int_s^t S_\varepsilon(t-r)\sigma_\varepsilon(r)\, dr \ ,$$

where

$$\begin{aligned}\sigma_\varepsilon := {} & \nabla G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) D_\varepsilon(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) - \nabla G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon) D_\varepsilon(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon) \\ & + F(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) - F(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon) \ .\end{aligned}$$

In order to estimate σ_ε we use (5.8) to write

$$\begin{aligned}\sigma_\varepsilon = {} & D_\varepsilon G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) - D_\varepsilon G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon) \\ & - \int_{\mathbf{R}} \frac{\varepsilon y^2}{2} \big(D^2 G(u_\varepsilon) - D^2 G(u_\varepsilon^{(4)}) \big) [\widehat{D}_{\varepsilon y} u_\varepsilon, \widehat{D}_{\varepsilon y} u_\varepsilon] \, \mu(dy) \\ & - \int_{\mathbf{R}} \frac{\varepsilon y^2}{2} D^2 G(u_\varepsilon^{(4)}) [\widehat{D}_{\varepsilon y}(u_\varepsilon + u_\varepsilon^{(4)}), \widehat{D}_{\varepsilon y}(u_\varepsilon - u_\varepsilon^{(4)})] \, \mu(dy) \\ & - \varepsilon^2 \big(R_\varepsilon(u_\varepsilon, u_\varepsilon) - R_\varepsilon(u_\varepsilon^{(4)}, u_\varepsilon^{(4)}) \big) \\ & + F(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) - F(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon) \\ =: {} & \sigma_{\varepsilon,1} + \ldots + \sigma_{\varepsilon,5} \ .\end{aligned}$$

where $u_\varepsilon := \widetilde{v}^\gamma + \widetilde{\psi}_\gamma$, $u_\varepsilon^{(4)} := v_\varepsilon^{(4)} + \widetilde{\psi}_\gamma^\chi$, and

$$\begin{aligned}R_\varepsilon(u^1, u^2)(x) := \int_{\mathbf{R}} \varepsilon^2 y^3 \int_0^1 \int_0^t \int_0^s & D^3 G((1-r)u^1(x) + r u^1(x+\varepsilon y)) \\ & [\widehat{D}_{\varepsilon y} u^2, \widehat{D}_{\varepsilon y} u^2, \widehat{D}_{\varepsilon y} u^2] \, dr\, ds\, dt\, d\mu(y) \ .\end{aligned}$$

We shall now estimate $\sigma_{\varepsilon,1}, \ldots, \sigma_{\varepsilon,5}$ individually.

First term. First we observe that $\widetilde{v}^\gamma$, $\widetilde{\psi}_\gamma$, $\widetilde{\psi}_\gamma^\chi$, and ϱ_ε are bounded in L^∞ before time τ_5^K. Using Lemma 3.5, the embedding $H^\alpha \subseteq L^\infty$, and the definition of the

stopping time to bound $\|\widetilde{\psi}_\chi\|_{L^\infty}$, we obtain

$$\begin{aligned}\|\sigma_{\varepsilon,1}\|_{-1} &= \|D_\varepsilon(G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) - G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon))\|_{-1} \\ &\le C\|G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) - G(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon)\|_{L^\infty} \\ &\le C_K(\|\widetilde{\psi}_\chi\|_{L^\infty} + \|\varrho_\varepsilon\|_{L^\infty}) \\ &\le C_K(\varepsilon^{\frac{1}{2}\chi-\kappa} + \|\varrho_\varepsilon\|_\alpha)\ .\end{aligned} \tag{5.15}$$

Second term. Using Lemma 3.5 and the fact that $\varepsilon^{\gamma(\alpha-\frac{1}{2})+\kappa}\|\widetilde{v}^\gamma\|_\alpha$ is bounded before time τ_5^K, we estimate

$$\begin{aligned}\Theta_\varepsilon(\widetilde{v}^\gamma) &= \int_{\mathbf{R}} y^2\|\widehat{D}_{\varepsilon y}\widetilde{v}^\gamma\|_{L^2}^2\,|\mu|(dy) \\ &\le C_K\int_{\mathbf{R}} y^2|\varepsilon y|^{2(\alpha-1)}\|\widetilde{v}^\gamma\|_\alpha^2\,|\mu|(dy) \\ &\le C_K\int_{\mathbf{R}} y^2|\varepsilon y|^{2(\alpha-1)}\varepsilon^{-2\gamma(\alpha-\frac{1}{2})-2\kappa}\,|\mu|(dy) \\ &\le C_K\varepsilon^{2\widetilde{\alpha}-2-\kappa}\ ,\end{aligned}$$

and by the definition of the stopping time τ^K,

$$\Theta_\varepsilon(\widetilde{\psi}_\gamma) \le C_K\varepsilon^{-1-\kappa}\ .$$

As a consequence,

$$\Theta_\varepsilon(u_\varepsilon) \le 2\big(\Theta_\varepsilon(\widetilde{v}^\gamma) + \Theta_\varepsilon(\widetilde{\psi}_\gamma)\big) \le C_K\big(\varepsilon^{2\widetilde{\alpha}-2-\kappa} + \varepsilon^{-1-\kappa}\big) \le \varepsilon^{-1-\kappa}\ . \tag{5.16}$$

Note that $\|u_\varepsilon\|_{L^\infty}$ and $\|u_\varepsilon^{(4)}\|_{L^\infty}$ are bounded before time τ_5^K. Using that $L^1 \subseteq H^{-1}$ we obtain

$$\begin{aligned}\|\sigma_{\varepsilon,2}\|_{-1} \le \|\sigma_{\varepsilon,2}\|_{L^1} &\le \varepsilon\|D^2G(u_\varepsilon) - D^2G(u_\varepsilon^{(4)})\|_{L^\infty}\Theta_\varepsilon(u_\varepsilon) \\ &\le C_K\varepsilon^{-2\kappa}\|u_\varepsilon - u_\varepsilon^{(4)}\|_{L^\infty} \\ &\le C_K\varepsilon^{-2\kappa}\big(\|\varrho_\varepsilon\|_{L^\infty} + \|\widetilde{\psi}_\chi\|_{L^\infty}\big) \\ &\le C_K\varepsilon^{-2\kappa}\big(\|\varrho_\varepsilon\|_\alpha + \varepsilon^{\frac{1}{2}\chi-\kappa}\big)\ .\end{aligned} \tag{5.17}$$

Third term. By Lemma 3.5 we have

$$\begin{aligned}\Theta_\varepsilon(\varrho_\varepsilon) &= \int_{\mathbf{R}} y^2\|\widehat{D}_{\varepsilon y}\varrho_\varepsilon\|_{L^2}^2\,|\mu|(dy) \\ &\le C\int_{\mathbf{R}} y^2|\varepsilon y|^{2(\alpha-1)}\|\varrho_\varepsilon\|_\alpha^2\,|\mu|(dy) \le C\varepsilon^{2\alpha-2}\|\varrho_\varepsilon\|_\alpha^2\ .\end{aligned} \tag{5.18}$$

Observe that $u_\varepsilon + u_\varepsilon^{(4)} = 2\widetilde{v}^\gamma - \varrho_\varepsilon + \widetilde{\psi}_\gamma + \widetilde{\psi}_\gamma^\chi$ and $u_\varepsilon - u_\varepsilon^{(4)} = \widetilde{\psi}_\chi + \varrho_\varepsilon$. Taking into account that

$$\varepsilon^{1+\kappa}\Theta_\varepsilon(\widetilde{\psi}_\gamma),\quad \varepsilon^{1+\kappa}\Theta_\varepsilon(\widetilde{\psi}_\gamma),\quad \varepsilon^{2-\chi+\kappa}\Theta_\varepsilon(\widetilde{\psi}_\chi),\quad \|\varrho_\varepsilon\|_\alpha$$

are all bounded before time τ_5^K, we obtain

$$\begin{aligned}\Theta_\varepsilon(u_\varepsilon + u_\varepsilon^{(4)}) &\le C\big(\Theta_\varepsilon(\widetilde{v}^\gamma) + \Theta_\varepsilon(\varrho_\varepsilon) + \Theta_\varepsilon(\widetilde{\psi}_\gamma) + \Theta_\varepsilon(\widetilde{\psi}_\gamma^\chi)\big) \\ &\le C_K\big(\varepsilon^{2\widetilde{\alpha}-2-\kappa} + \varepsilon^{2\alpha-1} + \varepsilon^{-1-\kappa}\big) \le C_K\varepsilon^{-1-\kappa}\ ,\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}\Theta_\varepsilon(u_\varepsilon - u_\varepsilon^{(4)}) &\le C\big(\Theta_\varepsilon(\widetilde{\psi}_\chi) + \Theta_\varepsilon(\varrho_\varepsilon)\big)\\ &\le C_K\varepsilon^{\chi-2-\kappa} + C\varepsilon^{2\alpha-2}\|\varrho_\varepsilon\|_\alpha^2 \ .\end{aligned} \tag{5.20}$$

Using that $\|u_\varepsilon^{(4)}\|_{L^\infty} \le C_K$ before time τ_5^K, we obtain

$$\begin{aligned}\|\sigma_{\varepsilon,3}\|_{-1} \le \|\sigma_{\varepsilon,3}\|_{L^1} &\le \varepsilon\|D^2G(u_\varepsilon^{(4)})\|_{L^\infty}\sqrt{\Theta_\varepsilon(u_\varepsilon + u_\varepsilon^{(4)})\Theta_\varepsilon(u_\varepsilon - u_\varepsilon^{(4)})}\\ &\le C_K\big(\varepsilon^{\frac12\chi-\frac12-2\kappa} + \varepsilon^{\alpha-\frac12-\kappa}\|\varrho_\varepsilon\|_\alpha\big) \ .\end{aligned}$$

Fourth term. We shall show that

$$\|\sigma_{\varepsilon,4}\|_{-1} \le C_K\big(\varepsilon^{-2\kappa}\|\varrho_\varepsilon\|_\alpha + \varepsilon^{\frac12\chi-\frac12-2\kappa}\big) \ . \tag{5.21}$$

First we use the L^∞-bound on $u_\varepsilon^{(4)}$ enforced by the stoppping time, to obtain the pointwise bound

$$\begin{aligned}&|R_\varepsilon(u_\varepsilon, u_\varepsilon) - R_\varepsilon(u_\varepsilon, u_\varepsilon^{(4)})|\\ &\le C_K\int_{\mathbf{R}} \varepsilon^2 y^3\Big(|\widehat{D}_{\varepsilon y}u_\varepsilon|^2 + |\widehat{D}_{\varepsilon y}u_\varepsilon||\widehat{D}_{\varepsilon y}u_\varepsilon^{(4)}| + |\widehat{D}_{\varepsilon y}u_\varepsilon^{(4)}|^2\Big)\\ &\qquad\qquad \times |\widehat{D}_{\varepsilon y}(u_\varepsilon - u_\varepsilon^{(4)})|\,|\mu|(dy)\\ &\le C_K\int_{\mathbf{R}} \varepsilon y^2\Big(|\widehat{D}_{\varepsilon y}u_\varepsilon| + |\widehat{D}_{\varepsilon y}u_\varepsilon^{(4)}|\Big)|\widehat{D}_{\varepsilon y}(u_\varepsilon - u_\varepsilon^{(4)})|\,|\mu|(dy) \ .\end{aligned}$$

In view of (5.20) it thus follows that

$$\begin{aligned}&\|R_\varepsilon(u_\varepsilon, u_\varepsilon) - R_\varepsilon(u_\varepsilon, u_\varepsilon^{(4)})\|_{L^1}\\ &\le C_K\varepsilon\int_{\mathbf{R}} y^2\|\big(|\widehat{D}_{\varepsilon y}u_\varepsilon| + |\widehat{D}_{\varepsilon y}u_\varepsilon^{(4)}|\big)|\widehat{D}_{\varepsilon y}(u_\varepsilon - u_\varepsilon^{(4)})|\ \|_{L^1}\,|\mu|(dy)\\ &\le C_K\varepsilon\int_{\mathbf{R}} y^2\big(\|\widehat{D}_{\varepsilon y}u_\varepsilon\|_{L^2} + \|\widehat{D}_{\varepsilon y}u_\varepsilon^{(4)}\|_{L^2}\big)\|\widehat{D}_{\varepsilon y}(u_\varepsilon - u_\varepsilon^{(4)})\|_{L^2}\,|\mu|(dy)\\ &\le C_K\varepsilon\sqrt{\big(\Theta_\varepsilon(u_\varepsilon) + \Theta_\varepsilon(u_\varepsilon^{(4)})\big)\Theta_\varepsilon(u_\varepsilon - u_\varepsilon^{(4)})} \ .\end{aligned}$$

Using (5.16), (5.18), and the definition of the stopping time to bound $\Theta_\varepsilon(\widetilde{\psi}_\chi)$, we find that

$$\begin{aligned}\Theta_\varepsilon(u_\varepsilon) + \Theta_\varepsilon(u_\varepsilon^{(4)}) &\le C\big(\Theta_\varepsilon(u_\varepsilon) + \Theta_\varepsilon(\varrho_\varepsilon) + \Theta_\varepsilon(\widetilde{\psi}_\chi)\big)\\ &\le C_K\big(\varepsilon^{-1-\kappa} + \varepsilon^{2\alpha-2} + \varepsilon^{\chi-1-\kappa}\big) \le \varepsilon^{-1-\kappa} \ .\end{aligned}$$

Using (5.20), we thus obtain

$$\|R_\varepsilon(u_\varepsilon, u_\varepsilon) - R_\varepsilon(u_\varepsilon, u_\varepsilon^{(4)})\|_{L^1} \le C_K\Big(\varepsilon^{\frac12\chi-\frac12-2\kappa} + \varepsilon^{\alpha-\frac12-\kappa}\|\varrho_\varepsilon\|_\alpha\Big) \ . \tag{5.22}$$

Furthermore, taking into account that

$$\|u_\varepsilon - u_\varepsilon^{(4)}\|_{L^\infty} \le C_K(\|\widetilde{\psi}_\chi\|_{L^\infty} + \|\varrho_\varepsilon\|_{L^\infty}) \le C_K(\varepsilon^{\frac12\chi-\kappa} + \|\varrho_\varepsilon\|_\alpha) \ ,$$

we have by (5.16),

$$\begin{aligned}&\|R_\varepsilon(u_\varepsilon, u_\varepsilon^{(4)}) - R_\varepsilon(u_\varepsilon^{(4)}, u_\varepsilon^{(4)})\|_{L^1}\\ &\qquad\le C_K\varepsilon^2\|u_\varepsilon - u_\varepsilon^{(4)}\|_{L^\infty}\int_{\mathbf{R}} y^3\||\widehat{D}_{\varepsilon y}u_\varepsilon^{(4)}|^3\|_{L^1}\,|\mu|(dy)\\ &\qquad\le C_K\varepsilon\|u_\varepsilon - u_\varepsilon^{(4)}\|_{L^\infty}\int_{\mathbf{R}} y^2\|\widehat{D}_{\varepsilon y}u_\varepsilon^{(4)}\|_{L^2}^2\,|\mu|(dy)\end{aligned} \tag{5.23}$$

$$= C_K \varepsilon \|u_\varepsilon - u_\varepsilon^{(4)}\|_{L^\infty} \Theta_\varepsilon^2(u_\varepsilon^{(4)})$$
$$\leq C_K(\varepsilon^{\frac{1}{2}\chi - 3\kappa} + \varepsilon^{-2\kappa}\|\varrho_\varepsilon\|_\alpha) \ .$$

The claim follows by adding (5.22) and (5.23) and using the embedding $L^1 \subseteq H^{-1}$.

Fifth term. As in the first step, we have

$$\begin{aligned} \|\sigma_{\varepsilon,4}\|_{-1} &= \|F(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma) - F(\widetilde{v}^\gamma + \widetilde{\psi}_\gamma^\chi - \varrho_\varepsilon)\|_{-1} \\ &\leq C_K\big(\|\widetilde{\psi}_\chi\|_{L^\infty} + \|\varrho_\varepsilon\|_{L^\infty}\big) \\ &\leq C_K\big(\varepsilon^{\frac{1}{2}\chi - \kappa} + \|\varrho_\varepsilon\|_\alpha\big) \ . \end{aligned} \tag{5.24}$$

Combining the five estimates, we obtain

$$\begin{aligned} \|\varrho_\varepsilon(t)\|_\alpha &\leq C(t-s)^{\frac{1}{2}(1-\alpha)} \sup_{r\in(s,t)} \|\sigma_\varepsilon(r)\|_{-1} \\ &\leq C(t-s)^{\frac{1}{2}(1-\alpha)} \sup_{r\in(s,t)} \Big(\varepsilon^{-2\kappa}\|\varrho_\varepsilon(r)\|_\alpha + \varepsilon^{\frac{1}{2}\chi - \frac{1}{2} - 2\kappa}\Big) \ . \end{aligned}$$

The result now follows as in the proof of Proposition 5.1. □

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