Generalized quantum isotonic nonlinear oscillator in d dimensions

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Abstract. We present a supersymmetric analysis for the d-dimensional Schrödinger equation with the generalized isotonic nonlinear-oscillator potential $V(r) = B^2/r^2 + \omega^2 r^2 + 2g(r^2 - a^2)/(r^2 + a^2)^2$, $B \geq 0$. We show that the eigenequation for this potential is exactly solvable provided $g = 2$ and $(\omega a^2)^2 = B^2 + (\ell + (d - 2)/2)^2$. Under these conditions, we obtain explicit formulae for all the energies and normalized bound-state wavefunctions.

Keywords: Isotonic nonlinear oscillator, supersymmetry, Gol'dman and Krivchenkov potential.

1. Introduction

Recently, Cariñena *et al.* [1] studied Schrödinger's equation with a quantum nonlinear oscillator potential of the form

$$\left[-\frac{d^2}{dx^2} + x^2 + 8\frac{2x^2 - 1}{(2x^2 + 1)^2}\right]\psi_n(x) = E_n\psi_n(x). \tag{1}$$

Their interest in this study came from the fact that Eq.(1), is exactly solvable, in a sense that the eigenenergies and eigenfunctions can be obtained explicitly; they were able to show that [1]

$$\begin{cases} \psi_n(x) = \frac{P_n(x)}{(2x^2+1)}e^{-x^2/2}, \\ \\ E_n = -3 + 2n, \quad n = 0, 3, 4, 5, \ldots, \end{cases} \tag{2}$$

where the polynomial factors $P_n(x)$ are related to the Hermite polynomials by

$$P_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ \\ H_n(x) + 4nH_{n-2}(x) + 4n(n-3)H_{n-4}(x) & \text{if } n = 3, 4, 5, \ldots \end{cases} \tag{3}$$

Soon afterwards, Fellows and Smith [2] considered another interesting case and, in particular, they showed that for certain values of the parameters w and g, the potential in the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + w^2x^2 + 2g\frac{x^2 - a^2}{(x^2 + a^2)^2}\right]\psi_n(x) = 2E_n\psi_n(x) \tag{4}$$

is indeed a supersymmetric partner of the harmonic oscillator potential. By means of the supersymmetric approach, Fellows and Smith [2] were able to construct an infinite set of exactly soluble *partner* potentials, along with their eigenfunctions and eigenvalues. Very recently, Sesma [3], using a Möbius transformation, was able to transform Eq.(4) into a confluent Heun equation and thereby obtain a simple and efficient algorithm to solve the Schrödinger equation (4) numerically, no matter what values are chosen for the parameters w and g. Furthermore, using suitable mass distributions, the position-dependent effective mass Schrödinger equation has been solved for a new nonlinear oscillator [4]. We note that the term $2(x^2 - a^2)/(x^2 + a^2)^2$ in Eq.(4) can be written as the sum of two centripetal barriers in the complex plane

$$\frac{2(x^2 - a^2)}{(x^2 + a^2)^2} = \frac{1}{(x + \imath a)^2} + \frac{1}{(x - \imath a)^2}, \quad \imath = \sqrt{-1},$$

that is to say, a rational potential with two imaginary poles symmetrically placed with respect to the origin [1].

The purpose of the present work is to use the supersymmetric approach to analyze the exact *analytic* solutions for the more general potential class (4). We consider the $d-$dimensional Schrödinger eigenvalue problem

$$H\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r}), \quad H = -\Delta + V, \tag{5}$$

where Δ is the $d-$dimensional Laplacian operator, $d \geq 2$, and $V(r)$ is the central potential given by

$$V(r) = \frac{B^2}{r^2} + \omega^2r^2 + 2g\frac{(r^2 - a^2)}{(r^2 + a^2)^2}, \quad B^2 \geq 0. \tag{6}$$

We show, through the factorization method, that the eigenequation for this potential, Eq.(5), is indeed exactly solvable provided $g = 2$ and $(\omega a^2)^2 = B^2 + (\ell + (d - 2)/2)^2$. The article is organized as follows. In section 2, we consider the d-dimensional Schrödinger equation with the nonlinear oscillator potential (6). In section 3, we briefly review the factorization method and the basic fomulae from supersymmetric theory that we need for our investigation [5, 6]. In section 4, we show that the isotonic nonlinear-oscillator potential [1, 2] and the Gol'dman and Krivchenkov potential are isospectral supersymmetric partner potentials. In section 6, the explicit construction of all the exact solutions of Schrödinger's equation (5) with $V(r)$ given by (6) is presented.

2. Schrödinger's equation in d dimensions

By considering the action of the Laplacian operator on a wavefunction of the form $\Psi(\boldsymbol{r}) = u(r)Y_l(\theta_1, \theta_2, \ldots, \theta_{d-1})$ with a spherically symmetric factor $u(r)$ and a generalized spherical harmonic factor Y_l, we obtain the radial Schrödinger equation in d dimensions for a spherically symmetric potential $V(r)$ as [7,8]:

$$-\frac{d^2 u}{dr^2} - \frac{d-1}{r}\frac{du}{dr} + \frac{\ell(\ell+d-2)}{r^2}u + V(r)u = Eu, \tag{7}$$

where, if $\boldsymbol{r} \in \Re^d$, then $r = \|\boldsymbol{r}\|$, and

$$u(r) \in L^2([0,\infty), r^{d-1}dr).$$

The first-order derivative term can be removed by using the new radial function $\psi(r)$ given by

$$\psi(r) = r^{\frac{d-1}{2}}u(r), \quad d \geq 2, \quad \psi(0) = 0, \quad \psi(r) \in L^2([0,\infty), dr).$$

Thus, we find

$$-\frac{d^2\psi}{dr^2} + U(r)\psi = E\psi \tag{8}$$

where

$$U(r) = V(r) + \frac{(2\ell+d-1)(2\ell+d-3)}{4r^2}. \tag{9}$$

We now recall the definition of $V(r)$ in (6) and combine the terms in $1/r^2$ to obtain,

$$U(r) = \frac{k(k+1)}{r^2} + \omega^2 r^2 + 2g\frac{(r^2-a^2)}{(r^2+a^2)^2}, \tag{10}$$

where k is defined so that

$$B^2 + \frac{1}{4}(2\ell+d-1)(2\ell+d-3) = k(k+1),$$

that is to say,

$$k = \left[B^2 + \left(\ell + \frac{d-2}{2}\right)^2\right]^{\frac{1}{2}} - \frac{1}{2}. \tag{11}$$

We shall find it convenient to label energy eigenstates $\psi_{nk} \equiv \psi_{n\gamma_d}$, where $n = 0, 1, 2, \ldots$ is the number of radial nodes, and $\gamma_d = k + \frac{1}{2}$.

3. The factorization method

In this section we give a brief review of some concepts of supersymmetric quantum mechanics (SUSY QM) that we shall need in the following sections, namely the factorization method, and supersymmetric partner potentials. We start with Schrödinger's time-independent equation for a one-dimensional radial problem (in the units $\hbar = 2m = 1$)

$$H\psi(r) = \left[-\frac{d^2}{dr^2} + U(r)\right]\psi(r) = E\psi(r) \tag{12}$$

where the potential $U(r)$ is real and possibly singular at $r = 0$. The wave function $\psi(r)$ must be square-integrable and normalized in a sense that $\int |\psi(r)|^2 dr = 1$. The main idea of the factorization method, as introduced in this context by Schrödinger and Dirac, is to write the second order differential operator H in (12) as the product of two first order differential operators $\mathcal{A}^\dagger$ and $\mathcal{A}$, such that $H = \mathcal{A}^\dagger \mathcal{A}$ and

$$\mathcal{A}^\dagger = -\frac{d}{dr} + W(r), \quad \mathcal{A} = \frac{d}{dr} + W(r) \tag{13}$$

where the function $W(r)$ is a real function of r and is known as the superpotential of the problem. The operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ are Hermitian conjugates of each other: $(\mathcal{A}^\dagger)^\dagger = \mathcal{A}$. Let us define

$$H_1 = \mathcal{A}^\dagger \mathcal{A} = -\frac{d^2}{dr^2} + U_1(r), \quad H_2 = \mathcal{A}\mathcal{A}^\dagger = -\frac{d^2}{dr^2} + U_2(r). \tag{14}$$

In general $H_1 \equiv H$ and H_2 are two different Hamiltonians. They are known as partner Hamiltonians in SUSY QM and are given explicitly by:

$$H_1 = \mathcal{A}^\dagger \mathcal{A} = \left(-\frac{d}{dr} + W(r)\right)\left(\frac{d}{dr} + W(r)\right) = -\frac{d^2}{dr^2} + W^2(r) - \frac{dW(r)}{dr} = -\frac{d^2}{dr^2} + U_1(r) \tag{15}$$

and

$$H_2 = \mathcal{A}\mathcal{A}^\dagger = \left(\frac{d}{dr} + W(r)\right)\left(-\frac{d}{dr} + W(r)\right) = -\frac{d^2}{dr^2} + W^2(r) + \frac{dW(r)}{dr} = -\frac{d^2}{dr^2} + U_2(r) \tag{16}$$

where the potentials U_1 and U_2 are known as supersymmetric partner potentials defined by:

$$U_1(r) = W^2(r) - \frac{dW(r)}{dr}, \quad \text{and} \quad U_2(r) = W^2(r) + \frac{dW(r)}{dr}. \tag{17}$$

Suppose that $\psi_n^{(1)}(r)$ and $\psi_n^{(2)}(r)$ are the eigenfunctions of H_1 and H_2, respectively. In the unbroken SUSY case, the ground state is not degenerate with a vanishing energy $E_0 = 0$ and it is usually expressed in terms of the superpotential $W(r)$ as:

$$\mathcal{A}\,\psi_0^{(1)}(r) = \left(\frac{d}{dr} + W(r)\right)\,\psi_0^{(1)}(r) = 0 \Rightarrow \psi_0^{(1)}(r) = C\exp\left(-\int^r W(\tau)d\tau\right), \tag{18}$$

where C is the normalization constant. The key result is the iso-spectrality between the two Hamiltonians H_1 and H_2 for all but the ground state $(n = 0)$, which can be shown as follows. Since the energy eigenvalues of H_1 and H_2 are positive semi-definite $E_n^{1,2} \geq 0$, we have for $H_2\psi_n^{(2)} = \mathcal{A}\mathcal{A}^\dagger\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)}$ and by multiplying through by $\mathcal{A}^\dagger$ we see that

$$H_1(\mathcal{A}^\dagger\psi_n^{(2)}) = \mathcal{A}^\dagger\mathcal{A}(\mathcal{A}^\dagger\psi_n^{(2)}) = \mathcal{A}^\dagger H_2\psi_n^{(2)} = E_n^{(2)}(\mathcal{A}^\dagger\psi_n^{(2)}). \tag{19}$$

Thus $A^\dagger\psi_n^{(2)}$ is an eigenstate of H_1 with same energy eigenvalue $E_n^{(2)}$, and there must be $\psi_n^{(1)}(r)$ such that

$$\psi_n^{(1)}(r) = c_n\mathcal{A}^\dagger\psi_n^{(2)}(r) \tag{20}$$

where c_n are constants for $n = 1, 2, \ldots$ Similarly, for $H_1\psi_n^{(1)} = \mathcal{A}^\dagger\mathcal{A}\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)}$ and, multiplying through by $\mathcal{A}$, we find

$$H_2(\mathcal{A}\psi_n^{(1)}) = \mathcal{A}\mathcal{A}^\dagger(\mathcal{A}\psi_n^{(1)}) = \mathcal{A}H_1\psi_n^{(1)} = E_n^{(1)}(\mathcal{A}\psi_n^{(1)}). \tag{21}$$

Thus $A\psi_n^{(1)}$ is an eigenstate of H_2 with the same eigenvalue $E_n^{(1)}$, and there must be $\psi_n^{(2)}(r)$ such that

$$\psi_n^{(2)}(r) = c_n'\mathcal{A}\psi_n^{(1)}(r). \tag{22}$$

Furthermore, since

$$1 = \langle\psi_n^{(2)}(r)|\psi_n^{(2)}(r)\rangle = |c_n'|^2\langle\mathcal{A}\psi_n^{(1)}(r)|\mathcal{A}\psi_n^{(1)}(r)\rangle = |c_n'|^2\langle\psi_n^{(1)}(r)|\mathcal{A}^\dagger\mathcal{A}\psi_n^{(1)}(r)\rangle = E_n^1|c_n'|^2\langle\psi_n^{(1)}(r)|\psi_n^{(1)}(r)\rangle,$$

we have

$$\psi_n^{(2)}(r) = (E_n^{(1)})^{-1/2}\mathcal{A}\psi_n^{(1)}(r) = (E_n^{(1)})^{-1/2}\left(\frac{d}{dx} + W(r)\right)\psi_n^{(1)}(r). \tag{23}$$

Similarly, we find

$$\psi_n^{(1)}(r) = (E_n^{(2)})^{-1/2}\mathcal{A}^\dagger\psi_n^{(2)}(r) = (E_n^{(2)})^{-1/2}\left(-\frac{d}{dx} + W(r)\right)\psi_n^{(2)}(r). \tag{24}$$

Thus, if we knew the eigenvalues and eigenfunctions of either of the two partner potentials, we could immediately derive the spectrum of the other. However, the above relations only give the relationship between the eigenvalues and eigenfunctions of the two partner Hamiltonians, but do not allow us to determine their spectra. In the next, we are guided by the idea of finding pairs of (essentially) isospectral Hamiltonians, one of which has a known soluble Hamiltonian.

4. Supersymmetric partner of the Isotonic potential

A key step in this work was the idea to consider a candidate for a superpotential $W(r)$ by means of the following Ansatz:

$$W(r) = \frac{k'}{r} + \omega' \, r + \frac{sr}{r^2 + a^2},$$

(25)

where k', ω' and s are real parameters to be determine shortly, and a is a fixed potential parameter. The potential U_1, then reads

$$U_1(r) = \omega'^2 r^2 + \frac{k'(k'+1)}{r^2} + \frac{s(-2a^2\omega' + s + 2k' + 1)r^2 - a^2 s(2\omega'a^2 - 2k' + 1)}{(r^2 + a^2)^2} + (2s + 2k' - 1)\omega'$$

(26)

and the potential U_2 is given by

$$U_2(r) = \omega'^2 r^2 + \frac{k'(k'-1)}{r^2} + \frac{s(s - 1 + 2k' - 2a^2\omega')r^2 + a^2 s(1 + 2k' - 2a^2\omega')}{(r^2 + a^2)^2} + (2s + 2k' + 1)\omega'$$

(27)

If we now compare the potential $U_1(r)$ with

$$V_1(r) = \frac{k(k+1)}{r^2} + \omega^2 r^2 + 2g\frac{(r^2 - a^2)}{(r^2 + a^2)^2}$$

(28)

we have for $k' = k$, $\omega' = \omega$, $s(-2a^2\omega' + s + 2k + 1) = 2g$, and $s(2\omega'a^2 - 2k' + 1) = 2g$, that

$$\begin{cases} s = 4(\omega a^2 - k), \\ \\ g = 2(wa^2 - k)(2wa^2 + 1 - 2k). \end{cases}$$

(29)

With these values of s and g we may reduce U_2 by the assumption that $s(s - 1 + 2k - 2a^2\omega) = 0$ and $a^2 s(1 + 2k - 2a^2\omega) = 0$ to

$$V_2(r) = \omega^2 r^2 + \frac{k(k-1)}{r^2} + (8(\omega a^2 - k) + 2k + 1)\omega.$$

(30)

These assumptions are valid under the following conditions:

$$\omega a^2 = k + \frac{1}{2}, \quad g = 2 \quad \text{and} \quad s = 2.$$

In summary, we have two partner potentials

$$V_{(1)}(r) = \frac{k(k+1)}{r^2} + \omega^2 r^2 + \frac{4(r^2 - a^2)}{(r^2 + a^2)^2} + \omega(2k + 3)$$

(31)

$$V_2(r) = \frac{k(k-1)}{r^2} + \omega^2 r^2 + \omega(2k + 5).$$

(32)

provided the parameters ω and a satisfy

$$\omega a^2 = k + \frac{1}{2} = \sqrt{B^2 + \left(\ell + \frac{(d-2)}{2}\right)^2}.$$

(33)

5. Gol'dman and Krivchenkov potential in d-dimensional

Schrödinger's equation with the Gol'dman and Krivchenkov potential [9] in d-dimensions is given by

$$\left(-\frac{d^2}{dr^2} + \frac{\Lambda(\Lambda+1)}{r^2} + \beta r^2 + \frac{\alpha}{r^2}\right)\psi_{n\gamma_d}(r) = E_{n\gamma_d}\psi_{n\gamma_d}(r), \quad \Lambda = l + \frac{1}{2}(d-3), \quad d \geq 2, \qquad (34)$$

and has exact eigenvalues given by [9]

$$E_{n\gamma_d} = 2\sqrt{\beta}\,(2n+\gamma_d), \quad n, l = 0, 1, 2, \ldots, \qquad (35)$$

where

$$\gamma_d = 1 + \sqrt{\alpha + (\Lambda + \frac{1}{2})^2}.$$

Meanwhile, the exact eigenfunctions are given by [9]

$$\psi_{n\gamma_d}(r) = (-1)^n \sqrt{\frac{2\beta^{\gamma_d/2}(\gamma_d)_n}{n!\Gamma(\gamma_d)}} r^{\gamma_d-1/2} e^{-\frac{1}{2}\sqrt{\beta}r^2} {}_1F_1(-n; \gamma_d; \sqrt{\beta}r^2). \qquad (36)$$

Here, we use the Pochhammer symbol $(\gamma)_n$, where

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},$$

and ${}_1F_1$ is the confluent hypergeometric function defined by ([10], chapter 7)

$${}_1F_1(-n; b; x) = \sum_{k=0}^{n} \frac{(-n)_k}{(b)_k\,k!} x^k. \qquad (37)$$

Using these exact solutions for the Gol'dman and Krivchenkov potential, we can show that the Schrödinger equation

$$\left(-\frac{d^2}{dr^2} + \frac{k(k-1)}{r^2} + w^2 r^2 + \omega(2k+5)\right)\psi_{nk}(r) = \epsilon_{nk}\psi_{nk}(r) \qquad (38)$$

has exact solutions given by

$$\epsilon_{nk} = 2\omega(2n+2k+3), \qquad (39)$$

and

$$\psi_{nk}(r) = (-1)^n \sqrt{\frac{2\omega^{k+1/2}(k+1/2)_n}{n!\Gamma(k+1/2)}} r^k e^{-\frac{1}{2}\omega r^2} {}_1F_1(-n; k+\frac{1}{2}; \omega r^2), \qquad (40)$$

where

$$k + \frac{1}{2} = \gamma_d = \omega a^2 = \left[B^2 + \left(\ell + \frac{d-2}{2}\right)^2\right]^{\frac{1}{2}}. \qquad (41)$$

It is clear from these relations that states having the same value for the combination $2\ell + d$ are degenerate.

6. Exact solutions for isotonic nonlinear-oscillator potentials

The supersymmetric approach, briefly discussed in section 3, along with exact solution of the Gol'dman and Krivchenkov potential allow us to obtain the exact solutions of Schrödinger's equation

$$\left(-\frac{d^2}{dr^2} + \frac{k(k+1)}{r^2} + \omega^2 r^2 + \frac{4(r^2 - a^2)}{(r^2 + a^2)^2} + \omega(2k+3)\right)\phi_{nk}(r) = \epsilon_{nk}\phi_k(r), \tag{42}$$

namely,

$$\epsilon_{nk} = 2\omega(2n + 2k + 3), \tag{43}$$

and the corresponding wavefunctions are given by

$$\phi_{nk}(r) = C'_n\left(-\frac{d}{dr} + \frac{k}{r} + \omega r + \frac{2r}{r^2 + a^2}\right)r^k e^{-\frac{1}{2}\omega r^2}L_n^{k-\frac{1}{2}}(\omega r^2) \tag{44}$$

where

$$C' = (-1)^n\sqrt{\frac{2\omega^{k+1/2}(k+1/2)_n}{2\omega(2n+2k+3)\ n!\Gamma(k+1/2)}}. \tag{45}$$

Here we have used the well-known relation between the confluent hypergeometric function (37) and Laguerre polynomials ([10], page 203)

$$_1F_1(-n; \alpha+1; z) = \frac{n!}{(\alpha+1)_n}L_n^\alpha(z).$$

A straightforward computation, aided by the differential identity of Laguerre polynomials ([10], page 203, formula (11)), namely

$$\frac{d}{dz}L_n^\alpha(z) = -L_{n-1}^{\alpha+1}(z), \tag{46}$$

yields

$$\phi_{nk}(r) = \frac{C'_n r^{k+1}e^{-\omega r^2/2}}{r^2 + a^2}\left[(2k+2n+3)L_n^{k-1/2}(\omega r^2) - 2(n+1)L_{n+1}^{k-1/2}(\omega r^2) + 2\omega a^2 L_n^{k+1/2}(\omega r^2)\right], \tag{47}$$

where we have used the identity ([10], page 203, formula (8))

$$L_n^\alpha(z) = L_{n-1}^\alpha(z) + L_n^{\alpha-1}(z). \tag{48}$$

Since $wa^2 = k + 1/2$, we may write Eq.(47) as

$$\phi_{nk}(r) = \frac{C'_n r^{k+1}e^{-\omega r^2/2}}{r^2 + a^2}\left[(2k+2n+3)L_n^{k-1/2}(\omega r^2) - 2(n+1)L_{n+1}^{k-1/2}(\omega r^2) + (2k+1)L_n^{k+1/2}(\omega r^2)\right]. \tag{49}$$

This may be compared with the results of Fellow and Smith [2] by use of the well-known relations between Hermite and Laguerre polynomials ([10], page 216, problem (1))

$$L_n^{-1/2}(\omega r^2) = \frac{(-1)^n}{2^{2n}n!}H_{2n}(\sqrt{\omega}r) \tag{50}$$

and

$$L_n^{1/2}(\omega r^2) = \frac{(-1)^n}{2^{2n+1}n!\ \sqrt{\omega}\ r}H_{2n+1}(\sqrt{\omega}r). \tag{51}$$

For $k = 0$, $\omega = 1$ using the identity ([10], page 188, formulas (4) and (6))

$$2zH_k(z) = H_{k+1}(z) + 2kH_{k-1}(z). \tag{52}$$

we obtain

$$\phi_{nk}(r) = \frac{(-1)^n C_n' e^{-r^2/2}}{2^{2n+2} \; n! \; (r^2 + a^2)} \left[4(2n+3)H_{2n+1}(r) + 8n(2n+3)H_{2n-1}(r) + H_{2n+3}(r) \right], \qquad (53)$$

where

$$C' = (-1)^n \sqrt{\frac{\Gamma(n + \frac{1}{2})}{(2n+3) \; n! \; \pi}}. \qquad (54)$$

These general results agree with the results discussed by Cariñena et al [1], and Fellows et al. [2] for the comparable odd solutions $n = 3, 5, 7, \ldots$ in Eq.(2).

7. Conclusion

In this paper, we have studied a family of generalized quantum nonlinear oscillators in d-dimensions. These isotonic oscillator problems have potentials $V(r)$ of the form

$$V(r) = \frac{B^2}{r^2} + \omega^2 r^2 + 2g \frac{(r^2 - a^2)}{(r^2 + a^2)^2}. \qquad (55)$$

We show that if $g = 2$ and $(\omega a^2)^2 = B^2 + (\ell + (d-2)/2)^2$, then this potential can be regarded as a supersymmetric partner of the Gol'dman and Krivchenkov potential, for which exact solutions can be constructed. Thus, under these conditions, we are able to solve the eigenproblem $H\psi = (-\Delta + V)\psi = E\psi$ exactly in d dimensions, and provide formulae for all the discrete eigenvalues and corresponding normalized wave functions. We have shown that our solutions to the general problem agree with the results reported earlier in Refs. [1] and [2] for the comparable odd states $n = 3, 5, 7, \ldots$ in $d = 1$ dimension with $B = 0$.

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