Counting packings of generic subsets in finite groups

Roland Bacher

November 4, 2010

Abstract[1]*: A packing of subsets* $\mathcal{S}_1,\dots,\mathcal{S}_n$ *in a group* G *is a sequence* $(g_1,\dots,g_n)$ *such that* $g_1\mathcal{S}_1,\dots,g_n\mathcal{S}_n$ *are disjoint subsets of* G*. We give a formula for the number of packings if the group* G *is finite and if the subsets* $\mathcal{S}_1,\dots,\mathcal{S}_n$ *satisfy a genericity condition.*

1 Introduction

A *(left-)packing* of n non-empty subsets $\mathcal{S}_1,\dots,\mathcal{S}_n$ in a group G is an element $(g_1,\dots,g_n)$ of G^n such that the left-translates $g_1\mathcal{S}_1, g_2\mathcal{G}_2,\dots,g_n\mathcal{G}_n$ of the sets $\mathcal{S}_i$ are disjoint. If G is a finite group with N elements, the number of packings of $\mathcal{S}_1,\dots,\mathcal{S}_n$ is bounded by N^n and thus finite. The sets $\mathcal{S}_1,\mathcal{S}_2,\dots$ are labelled by their indices. In particular, permuting the elements $g_1,\dots,g_n$ of a packing $(g_1,\dots,g_n)\in\mathcal{G}^n$ of $\mathcal{S}_1=\mathcal{S}_2=\cdots=\mathcal{S}_n$ yields a different packing. Moreover, in the case where $\mathcal{S}_1$ for example is of the form $\mathcal{S}_1=H\mathcal{S}_1$ for some subgroup H of G, a packing $(g_1,\dots,g_n)$ gives rise to $\sharp(H)$ distinct packings $(hg_1,g_2,\dots,g_n),\ h\in H$.

Given $(a_1,\dots,a_n)\in G^n$, remark that there is an obvious one-to-one map between packings of $\mathcal{S}_1,\dots,\mathcal{S}_n\subset G$ and packings of $a_1\mathcal{S}_1,\dots,a_n\mathcal{S}_n\subset G$.

This paper deals with enumerative properties of left-packings. Using the involutive antiautomorphism $g\longmapsto g^{-1}$, its content can easily be modified in order to deal with right-packings $\mathcal{S}_1g_1,\dots,\mathcal{S}_ng_n$.

Counting packings of arbitrary subsets $\mathcal{S}_1,\dots,\mathcal{S}_n$ in finite groups is probably difficult. There are however easy upper and lower bounds:

Proposition 1.1. *Let* $\alpha=\alpha(G;\mathcal{S}_1,\dots,\mathcal{S}_n)$ *denote the number of packings of subsets* $\mathcal{S}_1,\dots,\mathcal{S}_n$ *in a finite group* G *with* N *elements. Given an additional subset* $\mathcal{S}_{n+1}$ *of* G*, we denote by* $\tilde\alpha=\tilde\alpha(G;\mathcal{S}_1,\dots,\mathcal{S}_n,\mathcal{S}_{n+1})$ *the number of packings of* $\mathcal{S}_1,\dots,\mathcal{S}_n,\mathcal{S}_{n+1}$*. We have*

$$\left(N-\sharp(\mathcal{S}_{n+1})\sum_{i=1}^n\sharp(\mathcal{S}_i)\right)\alpha\le\tilde\alpha\le\left(N-\sum_{i=1}^n\sharp(\mathcal{S}_i)\right)\alpha\ .$$

[1]Keywords: Packings in groups, additive combinatorics, additive number theory, Stirling number, enumeration of hypertrees Math. class: 05C30, 11B73, 11P99

In particular, we have

$$\tilde{\alpha} = \left(N - \sum_{i=1}^{n} \sharp(\mathcal{S}_i)\right) \alpha \tag{1}$$

if $\mathcal{S}_{n+1}$ *is a singleton.*

A family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group G with identity element e is *generic* if for every sequence $i_1, \dots, i_k$ of $k \leq n$ distinct elements in $\{1, \dots, n\}$ and for every choice of elements $g_{i_j} \in \mathcal{S}_{i_j}^{-1}\mathcal{S}_{i_j} \setminus \{e\}$, we have

$$g_{i_1} g_{i_2} \cdots g_{i_k} \neq e\ .$$

Otherwise stated, a subset $\mathcal{S}_1, \dots, \mathcal{S}_n$ of a group G is generic if every non-trivial relation, written as a word with letters in the alphabets $\mathcal{G}_i = \mathcal{S}_i^{-1}\mathcal{S}_i \setminus \{e\}$, in the subgroup generated by the sets $\mathcal{G}_1, \dots, \mathcal{G}_n$ involves at least two elements in one of the sets $\mathcal{G}_1, \dots, \mathcal{G}_n$.

In the case of an additive abelian group G, the genericity condition boils down to the fact that the subset $(\mathcal{S}_1 - \mathcal{S}_1) \times \cdots \times (\mathcal{S}_n - \mathcal{S}_n)$ of the group G^n intersects the subgroup $\{(x_1, \dots, x_n) \in G^n \mid \sum_{i=1}^n x_i = 0\}$ of G^n only in the identity element $(0, \dots, 0)$.

Remark 1.2. *A generic family* $\mathcal{S}_1, \dots, \mathcal{S}_n$ *of subsets in the additive group* $\mathbb{Z}$ *with prescribed cardinalities* $s_i = \sharp(\mathcal{S}_i)$ *can be constructed by starting with* $\mathcal{S}_1 = \{0, \dots, s_1 - 1\}$ *and by defining* $\mathcal{S}_i$ *recursively as* $\mathcal{S}_i = \{0, k_i, 2k_i, \dots, (s_i - 1)k_i\}$ *where* k_i *is an arbitrary natural integer strictly larger than* $\sum_{j=1}^{i-1} \max(\mathcal{S}_j) - \min(\mathcal{S}_j)$. *A generic family is thus for example given by the sets* $\mathcal{S}_1 = \{0, 1\}, \mathcal{S}_2 = \{0, 2\}, \dots, \mathcal{S}_i = \{0, 2^{i-1}\}, \dots, \mathcal{S}_n = \{0, 2^{n-1}\}$.

Reducing such a generic family modulo a natural integer N *yields a generic family in the finite group* $\mathbb{Z}/N\mathbb{Z}$ *except if* N *is a divisor of a non-zero integer in the finite set* $\{\sum_{i=1}^n \mathcal{S}_i - \mathcal{S}_i\}$.

The aim of this paper is to describe a universal formula for the number of packings for a generic family of subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G. The number of associated packings depends then only on the cardinalities of G and $\mathcal{S}_1, \dots, \mathcal{S}_n$. A trivial example is given by n subsets reduced to singletons. The associated number of packings in a finite group with N elements is then easily seen to be given by $n!\binom{N}{n} = N(N-1)\cdots(N-n+1)$.

The study of generic packings in groups is, as far as I am aware, a new addition to the already large set of classical notions of packings. Well-known and well-studied examples are Euclidean lattice-packings and sphere-packings in metric spaces. The corresponding theory has however a different flavour since one tries to pack a huge (often infinite) number of spheres as tightly as possible. Packings of generic families in finite groups are not dense at all: Typically the cardinalities of the sets $\mathcal{S}_i \subset G$ of a generic family are very small compared to the cardinality N of G and we are not interested in density but in enumerative properties. Another source of packing-related notions is additive and elementary number theory: The existence of an infinity of twin primes for example is obviously equivalent to the question whether the set $\mathcal{P}\cap(2+\mathcal{P})$ is infinite with $\mathcal{P}\subset\mathbb{Z}$ denoting the set of prime numbers.

Concerning the dual notion of packings, the following question is natural: Is there an interesting notion for generic packings in arbitrary groups?

The rest of the paper is organized as follows: Section 2 contains the main result, Theorem 2.1. It expresses the number of packings of a generic family $\mathcal{S}_1,\dots,\mathcal{S}_n$ in a finite group in terms of a formal power series $U=U(x,\sigma_1,\sigma_2,\dots)\in A[[x]]$ with coefficients in the ring $A=\mathbb{Z}[\sigma_1,\sigma_2,\dots]$ of polynomials in elementary symmetric functions $\sigma_1,\sigma_2,\dots$ defined by $\sum_{k=0}^\infty \sigma_k t^k=\prod_{j=1}^n(1+\sharp(\mathcal{S}_j)t)$. The series U is given explicitely by formula (4) and involves combinatorial integers $t_{i,j}(n)$ (defined recursively by formula (3)) which extend Stirling numbers of the first kind. The first few coefficients of U are given by

$$\begin{aligned}
&1-\sigma_2x-((1-\sigma_1)\sigma_3+\sigma_4)x^2\\
&-((2-3\sigma_1+\sigma_1^2)\sigma_4+(5-3\sigma_1)\sigma_5+3\sigma_6)x^3\\
&-((6-11\sigma_1+6\sigma_1^2-\sigma_1^3)\sigma_5+(26-26\sigma_1+6\sigma_2^2)\sigma_6\\
&\quad+(35-15\sigma_1)\sigma_7+15\sigma_8)x^4+\dots
\end{aligned}$$

with omitted terms divisible by x^5. Theorem 2.1 and Proposition 1.1 imply easily that U satisfies the functional equation

$$(1-\sigma_1x)U(x,\sigma_1,\sigma_2,\sigma_3,\dots)=U(x,1+\sigma_1,\sigma_1+\sigma_2,\sigma_2+\sigma_3,\dots)\ . \qquad (2)$$

Section 3 discusses the combinatorics of packings associated to arbitrary (not necessarily generic) families $\mathcal{S}_1,\dots,\mathcal{S}_n$ of subsets in a group.

In Section 4, we refine the results of section 3 by applying them to generic packings. The underlying combinatorics are then simpler and imply the existence of a formal power series $\tilde U(x,\sigma_1,\sigma_2,\dots)$ such that the formula

$$N^n\tilde U(N^{-1},\sigma_1,\sigma_2,\dots)$$

(with $\sigma_1,\sigma_2,\dots$ defined by $\sum_{k=0}^\infty\sigma_kt^k=\prod_{j=1}^n(1+\sharp(\mathcal{S}_j)t)$) gives the number of packings for a generic family $\mathcal{S}_1,\dots,\mathcal{S}_n$ of n non-empty subsets in a finite group with N elements. Although this approach does not yield an explicit formula for $\tilde U$ it gives some useful information on the coefficients of $\tilde U$. Moreover, such a series $\tilde U$ is unique and satisfies the functional equation (2) as an easy consequence (equivalent to identity (1) of Proposition 1.1) of its very definition.

Section 5 starts with establishing the uniqueness of a solution to the functional equation (2) under certain conditions satisfied by the series $\tilde U$ considered in Section 4. Since our formulae for the series U satisfy these conditions also, it is enough to show that U satisfies the functional equation (2) in order to prove that it coincides with $\tilde{U}$. This completes the proof of Theorem 2.1.

Section 6 uses Theorem 2.1 and its proof for computing the Moebius function of the poset of finite labelled hypertrees.

Section 7 illustrates the main theorem and its proof by giving a formula for the weighted number of labelled hypertrees with weights given by the Moebius function computed in Section 6.

Section 8 deals with computational aspects and examples.

The paper ends with section 9 describing a few experimental observation concerning arithmetic and analytic properties of the coefficients involved in U.

2 Main result

We consider the set $t_{i,j}(n)$ of strictly positive integers depending on $n \in \{1, 2, \dots\}$, indexed by $i \in \{n+1, \dots, 2n\}$, $j \in \{0, 1, \dots, 2n-i\}$ and defined recursively by $t_{2,0}(1) = 1$ and

$$t_{i,j}(n) = (i-2)t_{i-1,j}(n-1) + t_{i-1,j-1}(n-1) + (i-3)t_{i-2,j}(n-1) \qquad (3)$$

for $n \geq 2$. We set $t_{i,j}(n) = 0$ in all other cases, ie. if $i \leq n$ or $j < 0$ or $i + j > 2n$.

Given a natural integer $n \geq 1$, the set of all $\binom{n+1}{2}$ non-zero integers $t_{i,j}(n)$ can be organized into a triangular array $T(n)$ with $T(n)$ determining $T(n+1)$ recursively by formula (3) reminiscent of the recurrence relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for binomial coefficients. The first six triangular arrays $T(1), \dots, T(6)$ are given by

$$\begin{matrix} 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & \end{matrix} \qquad \begin{matrix} 2 & 3 & 1 \\ 5 & 3 & \\ 3 & & \end{matrix} \qquad \begin{matrix} 6 & 11 & 6 & 1 \\ 26 & 26 & 6 & \\ 35 & 15 & & \\ 15 & & & \end{matrix}$$

$$\begin{matrix} 24 & 50 & 35 & 10 & 1 \\ 154 & 200 & 80 & 10 & \\ 340 & 255 & 45 & & \\ 315 & 105 & & & \\ 105 & & & & \end{matrix} \qquad \begin{matrix} 120 & 274 & 225 & 85 & 15 & 1 \\ 1044 & 1604 & 855 & 190 & 15 & \\ 3304 & 3325 & 1050 & 105 & & \\ 4900 & 2940 & 420 & & & \\ 3465 & 945 & & & & \\ 945 & & & & & \end{matrix}$$

Observe that the first rows of $T(1), T(2), \dots$ coincide, up to signs, with Stirling numbers of the first kind. This is of course an easy consequence of the recurrence relation (3).

We consider the formal power series $U \in A[[x]]$ with coefficients in the ring $A = \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3, \dots]$ of integral polynomials in $\sigma_1, \sigma_2, \dots$ defined by

$$U(x, \sigma_1, \sigma_2, \dots) = 1 - \sum_{n=1}^{\infty} x^n \sum_{i=n+1}^{2n} \sigma_i \sum_{j=0}^{2n-i} t_{i,j}(n)(-\sigma_1)^j \ . \tag{4}$$

Theorem 2.1. *The number of packings of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in a finite group G with N elements is given by the formula*

$$N^n U(N^{-1}, \sigma_1, \sigma_2, \dots) \tag{5}$$

where

$$\sum_{i=0}^{n} \sigma_i t^i = \prod_{j=1}^{n} (1 + \sharp(\mathcal{S}_j)t)$$

and where the series $U(x, \sigma_1, \sigma_2, \dots)$ is given by formula (4).

Remark that formula (5) of Theorem 2.1 is polynomial in N for fixed complex numbers $\sigma_1, \sigma_2, \dots$ such that $\sigma_{n+1} = \sigma_{n+2} = \dots = 0$. Indeed, the coefficient of x^n in $U(x, \sigma_1, \sigma_2, \dots)$ belongs to the ideal generated by $\sigma_{n+1}, \sigma_{n+2}, \dots, \sigma_{2n} \in \mathbb{Z}[\sigma_1, \sigma_2, \dots]$.

The proof of Theorem 2.1 is based on combinatorial properties of generic packings and on a functional equation for U described by the following result which is an almost obvious consequence of Theorem 2.1 and equality (1) in Proposition 1.1.

Proposition 2.2. *Suppose that $\tilde{U} \in \mathbb{C}[[x, \sigma_1, \sigma_2, \dots]]$ gives the number of packings $N^n \tilde{U}(N^{-1}, \sigma_1, \sigma_2, \dots)$ with $\sum_{i=0}^{n} \sigma_i t^i = \prod_{j=1}^{n}(1 + \sharp(\mathcal{S}_j)t)$ for every generic family of n non-empty subsets $\mathcal{S}_1, \dots, \mathcal{S}_n \subset G$ in a finite group G with $N = \sharp(G)$ elements.*

We have then

$$(1 - \sigma_1 x)\tilde{U}(x, \sigma_1, \sigma_2, \sigma_3, \dots) = \tilde{U}(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \dots) \tag{6}$$

where $\tilde{\sigma}_i = \sigma_{i-1} + \sigma_i$, using the convention $\sigma_0 = 1$.

Proof Equation (6) corresponds to equation (1) if $\sigma_1, \sigma_2, \dots$ are symmetric elementary functions of a finite set of natural integers. The general case follows by remarking that the algebra of symmetric polynomials is a free polynomial algebra on the set of elementary symmetric polynomials. $\Box$

Remark 2.3. *Iterating the identity (6) n times with $\tilde{U} = U$ given by formula (4) we have*

$$U(x, \sigma_1, \sigma_2, \dots) \prod_{j=0}^{n-1} (1 - (\sigma_1 + j)x) = U(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \dots)$$

where

$$\tilde{\sigma}_k = \sum_{j=0}^{\min(k,n)} \binom{n}{j} \sigma_{k-j} \ .$$

A particular case is the specialization

$$U\left(x, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots\right) = \prod_{j=1}^{n-1}(1 - jx)$$

associated to generic families $\mathcal{S}_1, \dots, \mathcal{S}_n$ *given by* n *singletons.*

3 Combinatorics of packings for arbitrary families $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group G

Proof of Proposition 1.1 Let $\mathcal{S}_1, \dots \mathcal{S}_n$ be n subsets in a finite group G with N elements and let $\mathcal{S}_{n+1}$ be an additional subset of G. A packing of $\mathcal{S}_1, \dots, \mathcal{S}_n$ given by $(g_1, \dots, g_n) \in G^n$ extends to a packing $(g_1, \dots, g_n, g_{n+1}) \in G^{n+1}$ of $\mathcal{S}_1, \dots, \mathcal{S}_{n+1}$ if and only if $g_{n+1} \in G \setminus \left(\cup_{i=1}^n g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1}\right)$ where $\mathcal{S}^{-1}$ denotes the set of inverses. Since $g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1}$ contains at most $\sharp(\mathcal{S}_{n+1})\sharp(\mathcal{S}_i)$ elements, we have the first inequality.

Consider now a fixed element $x \in \mathcal{S}_{n+1}$. We have

$$\sharp\left(\cup_{i=1}^n g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1}\right) \quad \geq \quad \sharp\left(\cup_{i=1}^n g_i \mathcal{S}_i x^{-1}\right) = \sharp\left(\cup_{i=1}^n g_i \mathcal{S}_i\right)$$

Since $(g_1, \dots, g_n)$ is a packing, we have

$$\sharp\left(\cup_{i=1}^n g_i \mathcal{S}_i\right) = \sum_{i=1}^n \sharp(\mathcal{S}_i)$$

showing the second inequality.

Equality (1) is obvious for $\sharp(\mathcal{S}_{n+1}) = 1$. □

We fix a group G and a family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in G. Given an element $\mathbf{g} = (g_1, \dots, g_n)$ of G^n, we consider the corresponding *intersection graph* $\mathcal{I}(\mathbf{g})$ with vertices $1, \dots, n$ and edges $\{i, j\}$ between distinct vertices i, j if $g_i \mathcal{S}_i \cap g_j \mathcal{S}_j \neq \emptyset$ in G. Observe that $\mathbf{g} = (g_1, \dots, g_n)$ in G^n defines a packing if and only if $\mathcal{I}(\mathbf{g})$ is the trivial graph with n isolated vertices.

Given a finite simple graph Γ with vertices $1, \dots, n$ and edges $E(\Gamma)$, we consider the sets

$$\mathcal{R}_\Gamma = \{(g_1, \dots, g_n) \in G^n \mid g_i \mathcal{S}_i \cap g_j \mathcal{S}_j \neq \emptyset \text{ for every } \{i, j\} \in E(\Gamma)\} \ .$$

An element $\mathbf{g}$ in G^n belongs thus to $\mathcal{R}_\Gamma$ if and only if Γ is a subgraph of the intersection graph $\mathcal{I}(\mathbf{g})$.

We denote by $\mathcal{E}_\Gamma$ the equivalence classes of $\mathcal{R}_\Gamma$ defined by $(g_1, \dots, g_n) \sim (h_1, \dots, h_n)$ if $g_i h_i^{-1} = g_j h_j^{-1}$ for every edge $\{i, j\}$ of Γ. Two elements $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathbf{h} = (h_1, \dots, h_n)$ of $\mathcal{R}_\Gamma$ represent thus the same equivalence class of $\mathcal{E}_\Gamma$ if and only if the function $i \longmapsto g_i h_i^{-1}$ is constant on (vertices of) connected components.

Proposition 3.1. *Suppose that G is a finite group having N elements. We have then*

$$\sharp(\mathcal{R}_\Gamma) = \sharp(\mathcal{E}_\Gamma) N^{c(\Gamma)}$$

where $c(\Gamma)$ denotes the number of connected components of Γ.

Proof We set $c = c(\Gamma)$ and we denote the connected components of Γ by $\Gamma_1, \dots, \Gamma_c$. We get a free action of G^c on $\mathcal{R}_\Gamma$ by considering

$$(a_1, \dots, a_c) \cdot (g_1, \dots, g_n) \longmapsto (a_{\gamma(1)}^{-1} g_1, \dots, a_{\gamma(n)}^{-1} g_n)$$

where $\gamma(i) \in \{1, \dots, c\}$ is defined by the inclusion of the vertex i in the $\gamma(i)-$th connected component $\Gamma_{\gamma(i)}$ of Γ. This action is transitive and elements of $\mathcal{E}_\Gamma$ are thus in one-to-one correspondence with orbits of this action on the set G^n. $\square$

Remark 3.2. *Fixing an element $(g_1, \dots, g_n)$ representing an equivalence class of $\mathcal{E}_\Gamma$ and choosing elements $a_{i,j} \in \mathcal{S}_i, a_{j,i} \in \mathcal{S}_j$ such that $g_i a_{i,j} = g_j a_{j,i}$ for every edge $\{i, j\}$ in a spanning forest of Γ, one sees that $\mathcal{E}_\Gamma$ consists of at most $(\max_i \sharp(\mathcal{S}_i))^{2c-2}$ distinct equivalence classes.*

Proposition 3.3. *The number $\alpha = \alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n)$ of packings of a family $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G with N elements is given by*

$$\alpha = \sum_{\Gamma \in \mathcal{B}} (-1)^{e(\Gamma)} \sharp(\mathcal{E}_\Gamma) N^{c(\Gamma)}$$

where the sum is over the Boolean poset $\mathcal{B}$ of all $2^{\binom{n}{2}}$ simple graphs with vertices $1, \dots, n$ and where $e(\Gamma) = \sharp(E(\Gamma))$, respectively $c(\Gamma)$, denotes the number of edges, respectively connected components, of a graph $\Gamma \in \mathcal{B}$.

Proof Proposition 3.1 shows that it is enough to prove the equality

$$\alpha = \sum_{\Gamma \in \mathcal{B}} (-1)^{e(\Gamma)} \sharp(\mathcal{R}_\Gamma) \ .$$

An element $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ defines a packing if and only if its intersection graph $\mathcal{I}(\mathbf{g})$ is trivial. It provides thus a contribution of 1 to α in this case since it is only involved as an element of $\mathcal{R}_\Gamma$ if Γ is the trivial graph with isolated vertices $1, \dots, n$ and no edges.

An element $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ with non-trivial intersection graph $\mathcal{I}(\mathbf{g})$ containing $e \geq 1$ edges yields a contribution of 0 to α since contributions coming from the 2^{e-1} subgraphs of $\mathcal{I}(\mathbf{g})$ containing an even number of edges cancel out with contributions associated to the 2^{e-1} subgraphs having an odd number of edges. $\square$

Remark 3.4. *Introducing*

$$\alpha_\Gamma = \{\mathbf{g} \in G^n \mid \mathcal{I}(\mathbf{g}) = \Gamma\} \ ,$$

we have $\alpha = \alpha_T$ where T denotes the trivial graph with n isolated vertices $1, \dots, n$. The above proof of Proposition 3.3 computes α by applying Moebius inversion (more precisely, its dual form, see Proposition 3.7.2 of [4])

$$\alpha = \sum_{\Gamma \in \mathcal{B}} \mu(\Gamma) \sharp(\mathcal{R}_\Gamma)$$

(with $\mu(\Gamma) = (-1)^{e(\Gamma)}$ denoting the Moebius function of the Boolean lattice $\mathcal{B}$ of all simple graphs on $1, \dots, n$) to the numbers

$$\sharp(\mathcal{R}_\Gamma) = \sum_{\Gamma \subset \Gamma'} \alpha_{\Gamma'}$$

given by Proposition 3.1.

4 Proving the existence of U

We consider a fixed generic family of n non-empty finite subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G having N elements.

In this section we prove the existence of a series $\tilde{U}(x, \sigma_1, \sigma_2, \dots)$ such that the number of associated packings is given by

$$N^n \tilde{U}(N^{-1}, \sigma_1, \sigma_2, \dots)$$

(see equation 5) with $\sigma_1, \sigma_2, \dots$ defined by $\sum_{i=0}^n \sigma_i t^i = \prod_{j=1}^n (1 + \sharp(\mathcal{S}_j) t)$.

We recall that a simple graph Γ is a *block graph* (or a cordal and diamond-free graph) if all its cycles occur in maximal cliques (ie. in maximal complete subgraphs) of Γ. Block graphs can be identified with hyperforests, see Section 7. Examples are given by forests and disjoint unions of complete graphs.

Proposition 4.1. *All intersection graphs $\mathcal{I}(\mathbf{g})$, $\mathbf{g} \in G^n$ associated to a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n \subset G$ are block graphs.*

Proof Consider an oriented cycle formed by k cyclically consecutive vertices $i_1, i_2, \dots, i_{k-1}, i_k, i_{k+1} = i_1$ of $\mathcal{I}(\mathbf{g})$. For every $j \in \{1, \dots, k\}$ there exist thus two (not necessarily distinct) elements $a_{i_j}, b_{i_j} \in \mathcal{S}_{i_j}$ such that $g_{i_j} a_{i_j} = g_{i_{j+1}} b_{i_{j+1}}$. This implies the relation

$$g_{i_1} a_{i_1} (g_{i_2} b_{i_2})^{-1} \cdots g_{i_k} a_{i_k} (g_{i_1} b_{i_1})^{-1} = e \ . \tag{7}$$

Setting $c_{i_j} = b_{i_j}^{-1} a_{i_j}$, relation (7) is equivalent to the relation $c_{i_1} \cdots c_{i_k} = e$ with $c_{i_j} \in \mathcal{S}_{i_j}^{-1} \mathcal{S}_{i_j}$. Genericity of the family $\mathcal{S}_1, \dots, \mathcal{S}_n$ shows $a_{i_j} = b_{i_j}$ for all j and the sets $g_{i_j} \mathcal{S}_{i_j}$ intersect in the common element $g_{i_1} a_{i_1} = \cdots = g_{i_k} a_{i_k}$. All vertices $i_1, \dots, i_k$ of $\mathcal{I}(\mathbf{t})$ are thus adjacent and contained in a maximal clique of $\mathcal{I}(\mathbf{t})$. □

Lemma 4.2. *The intersection $g_i\mathcal{S}_i \cap g_j\mathcal{S}_j$ associated to an edge $\{i,j\}$ in an intersection graph $\mathcal{I}(\mathbf{g})$ is reduced to a unique element if $\mathcal{S}_1,\dots,\mathcal{S}_n$ is a generic family of G.*

Proof Otherwise there exist two distinct elements $a_i, b_i \in \mathcal{S}_j$ and two distinct elements $a_j, b_j \in \mathcal{S}_j$ such that $g_ia_i = g_ja_j$ and $g_ib_i = g_jb_j$. This implies the relation $a_i^{-1}b_ib_j^{-1}a_j = e$ in contradiction with the definition of genericity. □

Proposition 4.3. *Let $\mathcal{S}_1,\dots,\mathcal{S}_n$ be a generic family of subsets in a group G and let Γ be a block graph with vertices $1,\dots,n$.*

We have

$$\sharp(\mathcal{E}_\Gamma) = \prod_{j=1}^{n} (\sharp(\mathcal{S}_j))^{d_h(j)}$$

where $d_h(j)$ denotes the number of non-trivial maximal cliques containing j.

Proof Let $\mathbf{g} \in G^n$ represent a class of $\mathcal{E}_\Gamma$. The proof of Proposition 4.1 and Lemma 4.2 show that each maximal clique of Γ with vertices $i_1,\dots,i_k$ corresponds to a unique element $a = \bigcap_{j=1}^{k} g_{i_j}\mathcal{S}_{i_j}$ of G. This defines maps $\mathbf{g} \longrightarrow g_{i_j}^{-1}a$ which depend only on the class of $\mathbf{g}$ in $\mathcal{E}_\Gamma$ and extend to a map from $\mathcal{E}_\Gamma$ to $\prod_{j=1}^n \mathcal{S}_j^{d_h(j)}$ which is one-to-one. □

Proposition 4.4. *There exists a series $\tilde{U} \in \mathbb{Z}[x,\sigma_1,\sigma_2,\dots]$ such that the number of packings of a generic family $\mathcal{S}_1,\dots\mathcal{S}_n$ of n non-empty subsets in a finite group G with N elements is given by*

$$N^n\tilde{U}(N^{-1},\sigma_1,\sigma_2,\dots)$$

where $\sum_{i=0}^n \sigma_i t^i = \prod_{j=1}^n (1+\sharp(\mathcal{S}_j)t)$.

Moreover, every monomial in $\sigma_1,\sigma_2,\dots$ contributing to the coefficient x^k of $\tilde{U}$ is divisible by an element of the set $\{\sigma_{k+1},\cdots,\sigma_{2k}\}$ and is of degree at most $2k$ with respect to the grading given by $\deg(\sigma_i)=i$ for $i=1,2,\dots$.

Proof Using Moebius inversion in the poset of block graphs (ordered by inclusion) with n labelled vertices $1,\dots,n$, one gets an expression for the number of packings which involves only symmetric functions of $\sharp(\mathcal{S}_1),\dots,\sharp(\mathcal{S}_n)$ by Proposition 4.3. The contribution to N^{n-k} for $k \geq 1$ comes only from block graphs involving $k+c$ non-isolated vertices in exactly c non-trivial connected components. Such a block graph contributes a monomial divisible by σ_{k+c}. Since $c \leq \frac{k+c}{2}$, the degree of such a monomial contribution is maximal and equals $2k$ if and only if Γ is a forest consisting of k disjoint edges involving $2k$ vertices. □

Proposition 4.5. *Let $\tilde{U} \in \mathbb{Z}[[x,\sigma_1,\sigma_2,\dots]]$ be a series such that $N^n\tilde{U}(N^{-1},\sigma_1,\sigma_2,\dots)$ is the number of packings for every generic family $\mathcal{S}_1,\dots,\mathcal{S}_n$ of n non-empty subsets in a finite group G with N elements. Then $\tilde{U}$ is unique and satisfies the functional equation (6) of Proposition 2.2.*

Proof If $\tilde{U}_1 \neq \tilde{U}_2$ are two such series, the difference

$$D = \sum_{n=0}^{\infty} D_n x^n = \tilde{U}_1(x, \sigma_1, \sigma_2, \sigma_3, \dots) - \tilde{U}_2(x, \sigma_1, \sigma_2, \sigma_3, \dots)$$

is not identically zero. Thus there exists an integer n such that the polynomial $D_n \in \mathbb{C}[\sigma_1, \sigma_2, \dots, \sigma_{2n}]$ is non-zero. The definition of $\tilde{U}_i$ shows that the symmetric polynomial $P_n(s_1, \dots, s_{2n})$ obtained by the polynomial substitution given by $\sum_{k=0}^{2n} \sigma_k t^k = \prod_{j=1}^{2n}(1 + s_j t)$ in D_n is identically zero for all $s_1, \dots, s_{2n} \in \mathbb{N}$. This implies $P_n(s_1, \dots, s_{2n}) = 0$ and $D_n = 0$ since P determines $D_n \in \mathbb{C}[\sigma_1, \dots, \sigma_{2n}]$ uniquely.

Proposition 2.2 states that $\tilde{U}$ satisfies the functional equation (6). □

5 Proof of Theorem 2.1

Proposition 5.1. *There exists at most a unique series $\tilde{U} \in \mathbb{Z}[[x, \sigma_1, \sigma_2, \dots]]$ with the following properties:*

(i) $\tilde{U}$ satisfies the identity (6) of Proposition 2.2,

(ii) $\tilde{U}$ is of the form $\tilde{U} = 1 + \sum_{n=1}^{\infty} a_n x^n$ with $a_n \in \mathbb{Z}[\sigma_1, \sigma_2, \dots]$ a polynomial of degree $\leq 2n$ with respect to the grading $\deg(\sigma_i) = i$ and a_n is an element of the ideal generated by $\sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}, \dots$.

Proof Consider $D = \tilde{U}_1 - \tilde{U}_2 = \sum_{n=1}^{\infty} D_n x^n$ for two different series $\tilde{U}_1, \tilde{U}_2$ satisfying the conditions of Proposition 5.1. Let $n \geq 1$ be the smallest integer such that $D_n \neq 0$. Let $m \geq n + 1$ be the smallest integer such that $D_n = \sum_{k=m}^{2n} \sigma_k C_{n,k}$ with $C_{n,k} \in \mathbb{C}[\sigma_1, \sigma_2, \dots]$ and $C_{n,m} \neq 0$. Since D_n is of degree $\leq 2n$ with respect to the grading given by $\deg(\sigma_i) = i$, we have $C_{n,m} \in \mathbb{C}[\sigma_1, \dots, \sigma_{2n-m}] \subset \mathbb{C}[\sigma_1, \dots, \sigma_{n-1}]$.

Equation (6) and minimality of n imply

$$D_n(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = D_n(\sigma_1, \sigma_2, \sigma_3, \dots)$$

or equivalently

$$\sum_{k=m}^{2n} (\sigma_{k-1} + \sigma_k) C_{n,k}(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = \sum_{k=m}^{2n} \sigma_k C_{n,k}(\sigma_1, \sigma_2, \sigma_3, \dots) \, .$$

Comparison of both sides modulo the ideal I generated by $\sigma_m, \sigma_{m+1}, \sigma_{m+2}, \dots$ gives

$$C_{n,m}(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = 0$$

contradicting $C_{n,m} \neq 0$. □.

Proposition 5.2. *The series U defined by formula (4) satisfies equation (6) of Proposition 2.2.*

Proof Remark first that both series $(1-\sigma_1 x)U(x,\sigma_1,\sigma_2,\dots)-1$ and $U(x,1+\sigma_1,\sigma_1+\sigma_2,\dots)-1$ are linear in $\sigma_2,\sigma_3,\dots$. Considering the coefficient of $\sigma_i\sigma_1^j x^n$ of both series, equation (6) amounts to the identity

$$t_{i,j}(n)+t_{i,j-1}(n-1)=\sum_{k=j}^{n}(-1)^{k+j}\binom{k}{j}\left(t_{i,k}(n)+t_{i+1,k}(n)\right)$$

or equivalently to

$$t_{i,j}(n)+t_{i,j-1}(n-1)=\sum_{k}(-1)^{k+j}\binom{k}{j}\left(t_{i,k}(n)+t_{i+1,k}(n)\right) \tag{8}$$

where $\sum_k f(k)=\sum_{k\in\mathbb{Z}} f(k)$ since $\binom{k}{j}\left(t_{i,k}(n)+t_{i+1,k}(n)\right)=0$ for $k<j$ or $k\geq n$. We prove (8) by induction on n. Setting $t_{i,j}(0)=0$ it holds for $n=1$ and $n=2$. Applying the recursion relation (3) which holds for all $i,j\in\mathbb{Z}$ if $n\geq 2$ to the right-hand side

$$R=\sum_{k}(-1)^{k+j}\binom{k}{j}(t_{i,k}(n)+t_{i+1,k}(n))$$

of (8) we get

$$\begin{aligned}R &= \sum_{k=j}^{n}(-1)^{k+j}\binom{k}{j}\Big(\\ &\quad (i-2)t_{i-1,k}(n-1)+t_{i-1,k-1}(n-1)+(i-3)t_{i-2,k}(n-1)\\ &\qquad +(i-1)t_{i,k}(n-1)+t_{i,k-1}(n-1)+(i-2)t_{i-1,k}(n-1)\Big)\\ &= L+C\end{aligned}$$

where

$$\begin{aligned}L &= (i-2)\sum_{k}(-1)^{k+j}\binom{k}{j}(t_{i-1,k}(n-1)+t_{i,k}(n-1))\\ &\quad +\sum_{k}(-1)^{k+j-1}\binom{k}{j-1}(t_{i-1,k}(n-1)+t_{i,k}(n-1))\\ &\quad +(i-3)\sum_{k}(-1)^{k+j}\binom{k}{j}(t_{i-2,k}(n-1)+t_{i-1,k}(n-1))\end{aligned}$$

and

$$
\begin{aligned}
C &= -\sum_k (-1)^{k+j-1}\binom{k}{j-1}(t_{i-1,k}(n-1)+t_{i,k}(n-1)) \\
&\quad +\sum_k (-1)^{k+j}\binom{k}{j}(t_{i-1,k-1}(n-1)+t_{i,k-1}(n-1)) \\
&\quad +\sum_k (-1)^{k+j}\binom{k}{j}(t_{i,k}(n-1)+t_{i-1,k}(n-1)) \\
&= \sum_k (-1)^{k+j}\binom{k}{j-1}(t_{i-1,k}(n-1)+t_{i,k}(n-1)) \\
&\quad +\sum_k (-1)^{k+j}\binom{k}{j}(t_{i-1,k}(n-1)+t_{i,k}(n-1)) \\
&\quad -\sum_k (-1)^{k+j}\binom{k+1}{j}(t_{i-1,k}(n-1)+t_{i,k}(n-1)) \\
&= \sum_k (-1)^{k+j}\left(\binom{k}{j-1}+\binom{k}{j}-\binom{k+1}{j}\right)(t_{i-1,k}(n-1)+t_{i,k}(n-1)) \\
&= 0
\end{aligned}
$$

Using induction on n and applying (8) we get

$$
\begin{aligned}
L &= (i-2)(t_{i-1,j}(n-1)+t_{i-1,j-1}(n-2)) \\
&\quad +(t_{i-1,j-1}(n-1)+t_{i-1,j-2}(n-2)) \\
&\quad +(i-3)(t_{i-2,j}(n-1)+t_{i-2,j-1}(n-2))
\end{aligned}
$$

We have thus

$$
\begin{aligned}
L &= (i-2)t_{i-1,j}(n-1)+t_{i-1,j-1}(n-1)+(i-3)t_{i-2,j}(n-1) \\
&\quad +(i-2)t_{i-1,j-1}(n-2)+t_{i-1,j-2}(n-2)+(i-3)t_{i-2,j-1}(n-2)
\end{aligned}
$$

and applying (3) we get

$$
L = t_{i,j}(n)+t_{i,j-1}(n-1)
$$

which is the left-hand-side involved in (8). □

Proof of Theorem 2.1 By Proposition 5.2, the series U defined by formula (4) satisfies condition (i) of Proposition 5.1. It satisfies condition (ii) by construction. By Proposition 4.5 it coincides with the series $\tilde{U}$ given by Proposition 4.4. □

6 The Moebius function for the poset of block graphs

Let $\mathcal{P}$ be a poset (partially ordered set) with a unique minimal element m such that $\{y \in \mathcal{P} \mid y < x\}$ is finite for all $x \in \mathcal{P}$. This allows the definition of a Moebius function μ given recursively by $\mu(m) = 1$ and $\mu(x) = -\sum_{y<x} \mu(y)$ for all $x > m$. Given a function $f : \mathcal{P} \longrightarrow \mathbb{C}$ with finite support, the value $f(m)$ can then be recovered from the function $g(x) = \sum_{y \geq x} f(y)$ using Moebius inversion

$$f(m) = \sum_{x \in \mathcal{P}} \mu(x) g(x) \ ,$$

see Proposition 3.7.2 of [4] (we use only the values $\mu(m, x)$ of the Moebius function and write $\mu(x) = \mu(m, x)$ in analogy with the usual, well-known number-theoretic Moebius function of natural integers). Moebius inversion was the main ingredient in the proof of Proposition 4.4. We did however not compute the Moebius function for the poset $\mathcal{HF}$ (with $\mathcal{HF}$ standing for HyperForest, see Remark 6.2 and Section 7) of all block graphs (ordered by inclusion) with vertices $1, 2, 3, 4, \ldots$ and finitely many edges. The poset $\mathcal{HF}$ has a minimal element given by the trivial graph having only isolated vertices. The subset

$$\{\Gamma' \in \mathcal{HF} \mid \Gamma' \subset \Gamma\}$$

of all block subgraphs of a fixed block graph $\Gamma \in \mathcal{HF}$ with n edges contains at most 2^n elements given by removing suitable subsets of edges from Γ. The poset $\mathcal{HF}$ has thus a Moebius function which is described by the following result.

Proposition 6.1. *The Moebius function $\mu(\Gamma)$ on a vertex-labelled block graph Γ with respect to the poset $\mathcal{HF}$ of all vertex-labelled block graphs having finitely many edges is given by*

$$\mu(\Gamma) = \prod_{j \geq 2} (-(j-2)!)^{\kappa_j}$$

where κ_j denotes the number of maximal cliques with j vertices in Γ.

Remark 6.2. *The poset $\mathcal{HF}$ coincides with the poset of all hyperforests (ordered by inclusion) having vertices $\mathbb{N}$ and a finite number of hyperedges. It is in fact a lattice with wedge $\Gamma_1 \wedge \Gamma_2$ given by the intersection and join $\Gamma_1 \vee \Gamma_2$ given by the smallest block graph containing Γ_1 and Γ_2 as subgraphs.*

Proof Remark first that the order relation on the poset of all (vertex-labelled) block subgraphs of Γ is the product over all order-relations on block graphs induced by maximal cliques of Γ. We have thus $\mu(\Gamma) = \prod_{j \geq 2} (\mu(K_j))^{\kappa_j}$ where K_j is a complete subgraph on j labelled vertices and where κ_j is the number of maximal cliques with j vertices in Γ.

In order to compute $\mu(K_j)$, the combinatorial description for the existence of a suitable series $\tilde{U}$ given by the proof of Proposition 4.4 shows that $\mu(K_{j+1})$ is given by the coefficient of $\sigma_{j+1} x^j$ in U. By Theorem 2.1 whose proof does not depend on Proposition 6.1 this coefficient equals $-t_{j+1,0}(j) = -(j-2)!$ where the last identity follows easily from formula (3) defining $t_{i,j}(n)$ recursively. $\square$

Remark 6.3. *It would be interesting to have a simple direct proof that* $\mu(K_n) = -(n-2)!$ *for a complete graph* K_n *with* $n \geq 2$ *vertices in the poset* $\mathcal{HF}$.

7 Enumeration of weighted hypertrees

A *hypergraph* is a generalized graph with edges replaced by *hyperedges* defined as finite subsets of at least two vertices. Calling $n-$simplex a hyperedge with $n+1$ vertices, allowing simplices reduced to one vertex and adding a closure property we get the definition of a simplicial complex.

A *path* in a hypergraph is a finite sequence $v_0, v_1, \ldots, v_n$ of vertices such that v_{i-1} and v_i belong to a common hyperedge for $i = 1, \ldots, n$. A hypergraph is *connected* if every pair v, w of vertices can be joined by a path $v_0 = v, v_1, \ldots, v_n = w$. A *cycle* of a hypergraph is a cyclic path $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ of n distinct vertices such that the n ordinary edges $\{v_i, v_{i+1}\}, i = 0, \ldots, n-1$ belong to n distinct hyperedges. A *hyperforest* is a hypergraph without cycles. A connected hyperforest is a *hypertree*. Associating to every maximal clique of a block graph the hyperedge with the same vertices we get a one-to-one map between block graphs and hyperforests.

We recall that Stirling numbers of the second kind $S_2(n,k)$, defined by the equality

$$e^{t(e^x-1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\sum_{k=0}^{n} S_2(n,k) t^k \right) ,$$

count the number of partitions of $\{1, \ldots, n\}$ into k non-empty subsets. The following result, first proven by Husimi, see [2] or [1] generalizes Cayley's theorem (corresponding to the case $k = n-1$ of ordinary trees) to hypertrees.

Theorem 7.1. *(Husimi) The number of hypertrees with* k *hyperedges and* n *labelled vertices is given by*

$$n^{k-1} S_2(n-1,k) .$$

Identifying hypertrees with the corresponding block graphs and introducing the resulting Moebius function $\mu(T) = \prod_{j\geq 2}(-(j-2)!)^{\kappa_j}$ of a hypertree having κ_j hyperedges involving j vertices for $j \geq 2$ we have the following result:

Theorem 7.2. *We have*

$$(-1)^n \sum_{T \in \mathcal{T}(n,k)} \mu(T) = n^{k-1} S_1(n-1,k)$$

where $\mathcal{T}(n,k)$ *denotes the set of all labelled hypertrees with* n *vertices and* k *hyperedges and where* $S_1(n,k)$ *denotes the Stirling number of the first kind* *defined by*

$$\sum_{k=0}^{n} S_1(n,k)x^k = x(x-1)(x-2)\cdots(x-n+1) = \prod_{j=0}^{n-1}(x-j)\ .$$

Proof Each hypertree T with vertices $1,\dots,n$ consisting of k hyperedges yields a contribution of $\mu(T)\prod_{i=1}^n s_i^{\deg_h(i)}$ (with $\deg_h(i)$ denoting the number of hyperedges containing the vertex i) to the coefficient $(-1)^k t_{n,k-1}(n-1) = -(-1)^n S_1(n-1,k)$ of the monomial $\sigma_n\sigma_1^{k-1}x^{n-1}$ in U and all contributions are of this form. Setting $s_1 = s_2 = \cdots = s_n = 1$ yields the result. $\Box$

8 Computational aspects and examples

The computation of $U(x,\sigma_1,\sigma_2,\dots)$ up to $o(x^n)$ is straightforward using the recurrence relation (3). For a given fixed numerical value of σ_1, the following trick reduces memory requirement and speeds the computation up: Setting

$$c_n(\sigma_1) = (\gamma_{n+1}(\sigma_1,n), \gamma_{n+2}(\sigma_1,n), \dots, \gamma_{2n}(\sigma_1,n))$$

with $\gamma_i(\sigma_1,n) = \sum_{j=0}^{n-1} t_{i,j}(n)(-\sigma_1)^j$ we have

$$U(x,\sigma_1,\sigma_2,\dots) = 1 - \sum_{n=1}^{\infty}\langle c_n(\sigma_1), (\sigma_{n+1},\dots,\sigma_{2n})\rangle x^n$$

where $\langle a,b\rangle = \sum_{i=1}^n a_i b_i$ for $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$. The coefficients $\gamma_i(\sigma_1,n)$ of $c_n(\sigma_1)$ can be computed from the coefficients of $c_{n-1}(\sigma_1)$ by the formula

$$\gamma_i(\sigma_1,n) = (i-2-\sigma_1)\gamma_{i-1}(\sigma_1,n-1) + (i-3)\gamma_{i-2}(\sigma_1,n-1)$$

with missing coefficients omitted in the case of $i = n+1$ or $i = 2n$.

The coefficients of the first vectors $c_1(0), c_2(0), c_3(0), \dots$ are given by the rows of

1				
1	1			
2	5	3		
6	26	35	15	
24	154	340	315	105 ,

see A112486 of [3].

8.1 The examples $U(x,-1,-1,-1,\dots)$ and $U(x,0,-1,-1,-1,\dots)$

The series

$$U(x,-1,-1,-1,-1,\dots) - 1$$

is the generating series of the sequence

$$S(n)=\sum_{i,j} t_{i,j}(n)$$

enumerating the sums of the triangles $T(n)$ defined by the integers $t_{i,j}(n)$. We have

$$\begin{aligned} & (1+x)U(x,-1,-1,-1,-1,\dots) \\ = & \ U(x,0,-2,-2,-2,-2,\dots) \\ = & \ 2U(x,0,-1,-1,-1,-1,\dots)-1 \end{aligned}$$

where $U(x,0,-1,-1,-1,\dots)-1$ corresponds to the generating series of the sequence

$$s(n)=\sum_{i=n+1}^{2n} t_{i,0}(n)$$

starting as

$$1,2,10,82,938,13778,247210,5240338,128149802,3551246162,\dots,$$

cf. A112487 of [3], and obtained by summing the integers of the first column of the triangles $T(1),T(2),\dots$. In particular, we have $2s(n)=S(n-1)+S(n)$ or equivalently

$$2\sum_{i=n+1}^{2n} t_{i,0}(n)=\sum_{i=n+1}^{2n}\sum_{j=0}^{2n-i} t_{i,j}(n)+\sum_{i=n}^{2n-2}\sum_{j=0}^{2n-2-i} t_{i,j}(n-1)$$

for all $n\geq 2$.

8.2 A family of rational examples

Proposition 8.1. *Let $\sigma_1,\sigma_2,\dots$ be a sequence of complex numbers of the form $\sigma_n=(-1)^nP(n)$ for all $n\geq A$ where A is some natural integer and where $P(s)\in\mathbb{C}[s]$ is a polynomial. Then $U(x,\sigma_1,\sigma_2,\dots)$ is a rational series.*

Proof Let d denote the degree of P. Applying identity (6) of Proposition 2.2 iteratively $d+1$ times we get a series of the form $U(x,\tilde{\sigma}_1,\tilde{\sigma}_2,\dots,\tilde{\sigma}_{A+d+2},0,0,0,\dots)$ which is a polynomial. □

As an illustration we consider the series $U(x,y,1,-1,1,\dots)$. Proposition 2.2 shows

$$(1-xy)U(x,y,1,-1,1,-1,\dots)=(U(1+y,1+y,0,0,\dots)=1-(1+y)x\ .$$

We have thus $U(x,y,1,-1,1,\dots)=1-\frac{x}{1-xy}$.

8.3 Coefficients of $U(x, \sigma_1, P(2), P(3), P(4), \dots)$

Proposition 8.2. *Let $P(s) \in \mathbb{C}[s]$ be a polynomial of degree d. There exist constants $\alpha_0, \dots, \alpha_d \in \mathbb{C}$ such that*

$$[x^n]U(x, \sigma_1, P(2), P(3), P(4), \dots) = \sum_{h=0}^{d} \alpha_h [x^{n+h}]U(x, \sigma_1, 1, 1, 1, 1, 1, \dots)$$

for all $n \geq 1$ with $[x^n]U$ denoting the coefficient of x^n in the series U.

Proof The proof is by induction on d and holds certainly for $d = 0$. Setting $\gamma_i(n) = \sum_{j=n+1}^{2n} t_{i,j}(n)(-\sigma_1)^j$ we have

$$\begin{aligned} 0 &= -i^d \gamma_i(n+1) + i^d(i-2-\sigma_1)\gamma_{i-1}(n) + i^d(i-3)\gamma_{i-2}(n) \\ &= -i^d \gamma_i(n+1) + (i-1)^{d+1}\gamma_{i-1}(n) + (i-2)^{d-2}\gamma_{i-2}(n) + \\ &\quad +Q_1(i-1)\gamma_{i-1}(n) + Q_2(i-2)\gamma_{i-2}(n) \end{aligned}$$

where Q_1 and Q_2 are polynomials of degree $\leq d$.

Summing over i (for a fixed integer n) and using induction on d implies the result. □

9 Modular properties

Proposition 9.1. *The series $U(x, \sigma_1, \sigma_2, \dots) \in \mathbb{F}_p[[x]]$ is rational if $\sigma_1, \sigma_2, \dots$ is an ultimately periodic sequence of elements in $\mathbb{F}_p$.*

Proof Since we work over $\mathbb{F}_p$, one can restrict the indices i, j of the coefficients $t_{i,j}(n)$ to finite subsets. This implies that the coefficients of $U(x, \sigma_1, \dots)$ are ultimately periodic. □

The easiest non-trivial case of Proposition 9.1 is perhaps given by the generating series $U(x, 0, -1, -1, -1, \dots) - 1$ associated to the sequence

$$s(n) = \sum_{i=n+1}^{2n} t_{i,0}(n) = 1 - U(x, 0, -1, -1, -1, \dots)$$

obtained by summing all coefficients in the first column of the triangular arrays $T(1), T(2), \dots$.

Experimentally, there exists seemingly a sequence

$$\begin{aligned} &\alpha_0 = -1, \alpha_1 = 2, \alpha_2 = 0, \alpha_3 = \frac{1}{3}, \alpha_4 = \frac{5}{18}, \alpha_5 = \frac{149}{540}, \alpha_6 = \frac{553}{2025}, \\ &\alpha_7 = \frac{1849741}{6804000}, \alpha_8 = \frac{775167119}{2857680000}, \alpha_9 = \frac{325214957371}{1200225600000}, \dots \end{aligned}$$

of rational numbers such that

$$\left(1 + x^{p-1}\right) \sum_{n=1}^{\infty} s(n) x^n \equiv \sum_{n=0}^{p-1} \alpha_n x^{p-n} \pmod{p}$$

for every prime number p.

The rational sequence $\alpha_0, \alpha_1, \ldots$ has experimentally an asymptotic expansion given by

$$\alpha_n \sim \sum_{k=1}^{\infty} \frac{k^{k-n}}{k!} \left(\frac{2}{e^2}\right)^k .$$

In particular, it seems to converge to

$$\frac{2}{e^2} = .27067056647322538378799\ldots \ .$$

The generating series $\sum_{n=0}^{\infty} \alpha_n x^n$ seems to have a positive real root approximatively given by

$$.469988171695013162992878063240573355384683977952459810170161 \ .$$

The error term

$$\epsilon_n = \alpha_n - \sum_{k=1}^{\infty} \frac{k^{k-n}}{k!} \left(\frac{2}{e^2}\right)^k$$

seems to be of the form

$$\epsilon_n \sim \frac{(-1)^{n+1}}{s(n+1)} C(n)$$

with $C(n) = \gamma_0 - \frac{1}{12n^2} + \frac{\gamma_4}{n^4} + \frac{\gamma_6}{n^6} + \cdots \in \mathbb{R}[\frac{1}{n^2}]$ where

$$\begin{aligned} \gamma_0 &\sim 3.25889135327093 \\ \gamma_4 &\sim -.0120865999417078 \\ \gamma_6 &\sim .0312295100177430 \end{aligned}$$

More (probably correct) digits for γ_0 are given by

$$3.25889135327092945459791735692 \ .$$

Remark 9.2. *The constant γ_0 appearing in the error term $C(n)$ seems also to be related to the maximal index m_n such that $t_{m_n,0}(n) = \max_i(t_{i,0}(n))$ with m_n given asymptotically by $\frac{\gamma_0}{2}$ and it seems also to be involved in the asymptotics of $s(n)$, given experimentally by*

$$s(n) \sim \left(\frac{n\gamma_0}{e}\right)^n \sum_{k\geq 1} \tilde{\gamma}_k n^{-k}$$

where $\gamma_0 \sim 3.258891\ldots$ is as above and where

$$\begin{aligned} \tilde{\gamma}_1 &\sim .553942974899091 \\ \tilde{\gamma}_2 &\sim .239725616524017 \\ \tilde{\gamma}_3 &\sim .099867601803607 \ . \end{aligned}$$

References

[1] I.M. Gessel, L.H. Kalikow, *Hypergraphs and a functional equation of Bouwkamp and de Bruijn.* J. Combin. Theory Ser. A 110 (2005), no. 2, 275–289.

[2] K. Husimi, *Note on Mayer's theory of cluster integrals*, Journal of Chemical Physics **18** (1950), 682–684.

[3] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org.

[4] R. P. Stanley, Enumerative Combinatorics, Volume I, Cambridge University Press (1997).

Roland BACHER
INSTITUT FOURIER
Laboratoire de Mathématiques
UMR 5582 (UJF-CNRS)
BP 74
38402 St Martin d'Hères Cedex (France)

e-mail: Roland.Bacher@ujf-grenoble.fr