PICARD GROUP OF HYPERSURFACES IN TORIC VARIETIES

UGO BRUZZO[¶§†] AND ANTONELLA GRASSI[¶]

[¶] Department of Mathematics, University of Pennsylvania,
David Rittenhouse Laboratory, 209 S 33rd Street,
Philadelphia, PA 19104, USA

[§] Istituto Nazionale di Fisica Nucleare, Sezione di Trieste

ABSTRACT. We show that the usual sufficient criterion for a generic hypersurface in a smooth projective manifold to have the same Picard number as the ambient variety can be generalized to hypersurfaces in complete simplicial toric varieties. This sufficient condition is always satisfied by generic K3 surfaces embedded in Fano toric 3-folds.

1. INTRODUCTION

We prove a Noether-Lefshetz type result for a generic $K3$ hypersurface in the (ample) anticanonical system of a simplicial toric Fano threefold, namely that the corresponding Picard numbers are equal.

The rank $\rho(\mathbb{P}_\Sigma)$ of the Picard group of a toric variety $\mathbb{P}_\Sigma$ can be easily computed from the combinatorial data of Σ. In the 80s and 90s it was shown [8, 2, 10, 1], that the Picard number of a generic hypersurface X in the anticanonical system of a Fano variety $\mathbb{P}_\Sigma$ can be explicitly computed from combinatorial data if $\dim X \geq$

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E-mail: `bruzzo@math.upenn.edu, bruzzo@sissa.it, grassi@sas.upenn.edu`.

[†] On leave of absence from Scuola Internazionale Superiore di Studi Avanzati, Via Bonomea 265, 34136 Trieste, Italy

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3. This result was a pivotal ingredient in describing the toric version of mirror symmetry (see for example [8]). The argument in the above papers is essentially topological and computes the dimension of the second cohomology group of X, which is the rank of $\mathrm{Pic}(X)$ if $\dim(X) \geq 3$; but this reasoning cannot be applied if $\dim(X) = 2$. On the other hand, it is known that $\rho(X) = \rho(\mathbb{P}_\Sigma)$ for particular cases of toric threefolds, namely certain weighted projective spaces [6, 11], as in the higher dimensional case.

Our argument is partly inspired by Cox's paper [6] and combines and generalizes classical infinitesimal techniques introduced in the 70s by Griffiths, Steenbrink and collaborators to solve the Noether-Lefschetz problem in the smooth case (see for example [4]) and revisited by Dolgachev for the weighted projective case [12], as well as more recent results about toric varieties, their Cox ring and their cohomology [3]. The crucial point is that X and $\mathbb{P}_\Sigma$ are projective orbifolds and a pure Hodge structure can also be defined in our case.

In Section 2 we mostly recall some relevant results from [3]. We start with basic properties of simplicial toric varieties and general hypersurfaces defined by ample divisors. It turns out that both objects are orbifolds, in particular, they have a pure Hodge structure. Moreover we note that the exact sequence defining the primitive cohomology in middle dimension of such a hypersurface splits orthogonally with respect to the intersection pairing. The middle cohomology is the sum of the primitive cohomology and the "fixed" cohomology, i.e., the cohomology inherited from the ambient toric variety; the split is consistent with the Hodge decomposition. We then state some results of [3] which express the primitive cohomology in top dimension in terms of the Jacobian ring of the hypersurface; here the crucial hypothesis is that the ambient space has odd dimension.

Section 3 contains the bulk of the argument: we proceed along the lines of the infinitesimal arguments of Griffiths for smooth varieties and adapt it to the toric case. We start from the moduli space of quasi-smooth hypersurfaces constructed in [3] and focus on the case of $K3$ hypersurfaces in the canonical system of a simplicial toric Fano threefold. We consider the natural Gauss-Manin connection and proceed to prove an infinitesimal Noether-Lefschetz theorem and then the needed global Noether-Lefshetz theorem.

Some evidence for the result we present here was given by Rohsiepe in [14], by using different arguments.

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2. Hypersurfaces in simplicial complete toric varieties

In this section we recall some basic facts about hypersurfaces in toric varieties and their cohomology. We recall the condition under which a generic hypersurface in a complete simplicial variety has sufficiently mild singularities. Basically, one wants that the hypersurface has a pure Hodge structure; this holds true if the hypersurface is an orbifold. Moreover we see how the middle cohomology of such a hypersurface splits, compatibly with the Hodge structure, and orthogonally with respect to the intersection pairing, as the sum of the primitive cohomology and the "fixed" cohomology, i.e., the cohomology inherited from the ambient toric variety.

We mainly follow the notation in [3].

2.1. **Preliminaries and notation.** Let M be a free abelian group of rank d, let $N = \operatorname{Hom}(M, \mathbb{Z})$, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.1. [3, Def. 1.3]

(i) *A convex subset $\sigma \subset N_{\mathbb{R}}$ is a rational k-dimensional simplicial cone if there exist k linearly independent primitive elements $e_1, \ldots, e_k \in N$ such that $\sigma = \{\mu_1 e_1 + \cdots + \mu_k e_k\}$, with μ_i nonnegative real numbers.*

(ii) *Given two rational simplicial cones σ, σ', one says that σ' is a face of σ (we then write $\sigma' < \sigma$) if the set of integral generators of σ' is a subset of the set of integral generators of σ.*

(iii) *A finite set $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of rational simplicial cones is called a rational simplicial complete d-dimensional fan if*
   - (a) *all faces of cones in Σ are in Σ;*
   - (b) *if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' < \sigma$ and $\sigma \cap \sigma' < \sigma'$;*
   - (c) *$N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_s$.*

A rational simplicial complete 3-dimensional fan Σ defines a projective d-fold $\mathbb{P}_\Sigma$ having only Abelian quotient singularities. As a consequence, $\mathbb{P}_\Sigma$ is an orbifold. We shall use the term "orbifold" in the following sense (see, e.g., [8], Def. A.2.1): an n-dimensional variety Y is an orbifold if every point $y \in Y$ has a neighborhood which is isomorphic to U/G as an analytic space, where G is a subgroup of $Gl_n(\mathbb{C})$ with no nontrivial complex reflections, and U is a G-invariant neighborhood of the origin of $\mathbb{C}^n$. (A complex reflection is an element in $Gl_n(\mathbb{C})$ with $n-1$ eigenvalues equal to 1.) A sub-orbifold of an orbifold Y is a subvariety $Y' \subset Y$ with the property that for every $y \in Y'$ there is a local chart $(U/G, 0)$ of Y at y such that the inverse image of Y' in U is smooth at 0. Intuitively, a sub-orbifold is a subvariety whose only singularities come from the ambient variety. These notions of orbifold and sub-orbifold are synonymous to those of V-manifold and sub-V-manifold, which is indeed the terminology used in [3]. The notion of V-manifold is originally due to Satake [16].

Let $Cl(\Sigma)$ be the group of Weil divisors in $\mathbb{P}_\Sigma$ modulo rational equivalence. It is a free finitely generated Abelian group, whose rank ρ_Σ is called the Picard number of $\mathbb{P}_\Sigma$. The group $\mathbf{D}(\Sigma) = \operatorname{Spec}\mathbb{C}[Cl(\Sigma)]$ is a affine algebraic D-group whose character group is isomorphic to $Cl(\Sigma)$. Since there is a surjection $\mathbb{Z}^n \to Cl(\Sigma)$, we have an embedding $D(\Sigma) \to (\mathbb{C}^*)^n$, and a natural action of $D(\Sigma)$ on the affine space $\mathbb{A}^n$. The quotient $\mathbf{T}(\Sigma) = (\mathbb{C}^*)^n/\mathbf{D}(\Sigma)$ is an algebraic torus. Below we shall show that this group acts naturally on $\mathbb{P}_\Sigma$.

Definition 2.2. *(The Cox Ring* [7]*) Given a fan Σ, consider a variable z_i for each 1-dimensional cone ς_i in Σ, and let $S(\Sigma)$ be the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$. For every $\sigma \in \Sigma$, let $z_\sigma = \prod_{\varsigma_i \not\subset \sigma} z_i$, and let $B(\Sigma)$ the ideal in $S(\Sigma)$ generated by the z_σ's.*

$S(\Sigma)$ is a graded ring, with grading provided by the class group, $S(\Sigma) = \oplus_{\beta \in Cl(\Sigma)} S_\beta$. We identify the affine space $\mathbb{A}^n$ with $\operatorname{Spec} S(\Sigma)$, and denote by $Z(\Sigma)$ the affine variety in $\mathbb{A}^n$ given by the ideal $B(\Sigma)$. If we set $U(\Sigma) = \mathbb{A}^n - Z(\Sigma)$, the group $D(\Sigma)$ acts on $U(\Sigma)$, and the toric variety $\mathbb{P}_\Sigma$ may be represented as $U(\Sigma)/D(\Sigma)$. This yields an action of $T(\Sigma)$ on $\mathbb{P}_\Sigma$. For every face τ in Σ we shall denote by $\mathbf{T}_\tau \subset \mathbb{P}_\Sigma$ the orbit of τ in $\mathbb{P}_\Sigma$ under this action.

2.2. **Quasi-smooth hypersurfaces.** Let ρ_Σ denote the rank of the Picard group of $\mathbb{P}_\Sigma$. Let L be an ample line bundle on $\mathbb{P}_\Sigma$, and denote by $\beta \in Cl(\Sigma) \simeq \mathbb{Z}^{\oplus \rho_\Sigma}$ its degree. Note that a section of L is a polynomial in S_β.

Definition 2.3. [3, Def. 3.1] *Let f be a section of L, and let $\mathbf{V}(f)$ be the zero locus of f in* $\operatorname{Spec} S(\Sigma)$*. We say that the hypersurface X cut in $\mathbb{P}_\Sigma$ by the equation $f = 0$ is quasi-smooth if $\mathbf{V}(f)$ is smooth outside $Z(\Sigma)$.*

Definition 2.4. [3, Def. 4.13] *If L is an ample line bundle on $\mathbb{P}_\Sigma$, a hypersurface X cut by a section f of L is said to be* nondegenerate *if $X \cap \mathbf{T}_\tau$ is a smooth 1-codimensional subvariety of $\mathbf{T}_\tau$ for all τ in Σ.*

Proposition 2.5. [3, Prop. 3.5, 4.15] *If f is the generic section of an ample invertible sheaf, then X is nondegenerate. Moreover, every nondegenerate hypersurface $X \subset \mathbb{P}_\Sigma$ is quasi-smooth. Thus, if f is a generic section of L, its zero locus is a quasi-smooth hypersurface X in $\mathbb{P}_\Sigma$, hence it is an orbifold.*

The advantage from X being an orbifold lies in the fact that the complex cohomology of an orbifold has a pure Hodge structure in each dimension [17, 18].

2.3. **Primitive cohomology of a hypersurface.** Let L be an ample line bundle on $\mathbb{P}_\Sigma$, and let X be a hypersurface in $\mathbb{P}_\Sigma$ cut by a section f of L (note that by [3], Proposition 10.8, f lies in $B(\Sigma)$).

Denote by $i\colon X \to \mathbb{P}_\Sigma$ the inclusion, and by $i^*\colon H^\bullet(\mathbb{P}_\Sigma, \mathbb{C}) \to H^\bullet(X, \mathbb{C})$ the associated morphism in cohomology. It is shown in [3], Proposition 10.8, that $i^*\colon H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C}) \to H^{d-1}(X, \mathbb{C})$ is injective.

Definition 2.6. *The primitive cohomology group $PH^{d-1}(X)$ is the quotient $H^{d-1}(X, \mathbb{C})/\, i^*(H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C}))$.*

Lemma 2.7. *The exact sequence*

$$0 \to i^*(H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C})) \to H^{d-1}(X, \mathbb{C}) \to PH^{d-1}(X) \to 0$$

splits orthogonally with respect to the intersection pairing in $H^\bullet(X, \mathbb{C})$. The same is true with coefficients in $\mathbb{Q}$.

Proof. Cupping by the first Chern class of L is an isomorphism $\ell\colon H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C}) \to H^{d+1}(\mathbb{P}_\Sigma, \mathbb{C})$ by the hard Lefschetz theorem (which has been proved for projective orbifolds by Saito in [15]). Let $i_*\colon H^{d-1}(X, \mathbb{C}) \to H^{d+1}(\mathbb{P}_\Sigma, \mathbb{C})$ be the Gysin map.

We may draw the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C}) & \longrightarrow & H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \ell} & & \\
0 & \longrightarrow & \ker i_* & \longrightarrow & H^{d-1}(X, \mathbb{C}) & \xrightarrow[i_*]{} & H^{d+1}(\mathbb{P}_\Sigma, \mathbb{C}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \uparrow{\scriptstyle s} & & \downarrow & & \\
0 & \longrightarrow & \ker i_* & \longrightarrow & PH^{d-1}(X) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

which provides a splitting s of the above exact sequence. Let us denote by $\langle\, , \rangle$ the intersection pairing in cohomology both in $H^\bullet(X, \mathbb{C})$ and $H^\bullet(\mathbb{P}_\Sigma, \mathbb{C})$, and recall that i^* and i_* are adjoint with respect to the intersection pairing. The upper-right square commutes since by Poincaré duality

$$\langle i_* i^* \alpha, \beta \rangle = \langle i^* \alpha, i^* \beta \rangle = \langle [X] \cup \alpha, \beta \rangle = \langle \ell(\alpha), \beta \rangle\,.$$

If $\alpha \in H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C})$ and $\beta \in PH^{d-1}(X)$, we have

$$\langle i^*(\alpha), s(\beta) \rangle = \langle \alpha, i_*(s(\beta)) \rangle = 0$$

since $(i^* \circ \ell^{-1} \circ i_*)(s(\beta)) = 0$. If the statement is true with coefficients in $\mathbb{C}$ it also true with coefficients in $\mathbb{Q}$ since $H^\bullet(X, \mathbb{C}) \simeq H^\bullet(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. □

Remark 2.8. The kernel of i_* acting on $H^{d-1}(X, \mathbb{C})$ is sometimes called the "variable cohomology" $H^{d-1}_{\text{var}}(X, \mathbb{C})$. So we have shown that in degree $d-1$ the variable and primitive cohomologies of X are isomorphic. △

Both $H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C})$ and $H^{d-1}(X, \mathbb{C})$ have pure Hodge structures, and the morphism i^* is compatible with them, so that $PH^{d-1}(X)$ inherits a pure Hodge structure. We shall write

$$PH^{d-1}(X) = \bigoplus_{p=0}^{d-1} PH^{p,d-1-p}(X).$$

A key point in what follows is the possibility of representing the primitive cohomology classes in terms of meromophic forms on $\mathbb{P}_\Sigma$.

Proposition 2.9. *There is a natural isomorphism*

$$PH^{p,d-p-1}(X) \simeq \frac{H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)}{H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X) + dH^0(\mathbb{P}_\Sigma, \Omega^{d-1}_{\mathbb{P}_\Sigma}((d-p)X)}$$

Proof. This follows from comparing Corollaries 10.2 and 10.12 in [3]. □

The resulting projection map, multiplied by the factor $(-1)^{p-1}/(d-p+1)!$, will be denoted by

$$r_p \colon H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X) \to PH^{p,d-p-1}(X) \tag{1}$$

and is called the *p-th residue map* in analogy with the classical case.

Remark 2.10. The result in Proposition 2.9 implicitly uses a generalization of Bott's vanishing theorem, called the Bott-Steenbrink-Danilov theorem, which indeed holds true under our assumptions. The exact statement is that $H^i(\mathbb{P}_\Sigma, \Omega^p_{\mathbb{P}_\Sigma}(L)) = 0$ for all $i > 0$ and $p \geq 0$ if L is an ample line bundle on $\mathbb{P}_\Sigma$. This was stated without proof by Danilov [9] and proved in [3] (Theorem 7.1). △

To any hypersurface in $\mathbb{P}_\Sigma$, cut by a section f of L, one can associate a ring: this is the Jacobian ring $R(f)$ of f, defined as the quotient $R(f) = S(\Sigma)/J(f)$ of $S(\Sigma)$ by the ideal generated by the derivatives of f, with the induced grading. This ring encodes all the information about the primitive cohomology of X. Let $\beta_0 = -\deg K_{\mathbb{P}_\Sigma}$, $\beta = \deg L$.

Proposition 2.11. *If $p \neq d/2 - 1$, there is an isomorphism $PH^{p,d-p-1}(X) \simeq R(f)_{(d-p)\beta-\beta_0}$.*

Proof. [3] Theorem 10.13. □

3. The Picard group of the generic toric threefold

3.1. **The Gauss-Manin connection.** We need to consider a Gauss-Manin connection on the local system given by the middle cohomology groups of the quasi-smooth hypersurfaces of the linear system of L. Let $\mathcal{Z}$ be the open subscheme of $|L|$ parametrizing the quasi-smooth hypersurfaces in $|L|$, and let $\pi \colon \mathscr{F} \to \mathcal{Z}$ be the tautological family on $\mathcal{Z}$. Let $\mathscr{H}^{d-1}$ be the local system on $\mathcal{Z}$ whose fiber at z is the cohomology $H^{d-1}(X_z)$, i.e., $\mathscr{H}^{d-1} = R^{d-1}\pi_*\mathbb{C}$. It defines a flat connection ∇ in the vector bundle $\mathscr{E}^{d-1} = \mathscr{H}^{d-1} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{Z}}$, the *Gauss-Manin connection* of $\mathscr{E}^{d-1}$. Since the hypersurfaces X_z are quasi-smooth, the Hodge structure of the fibres

$H^{d-1}(X_z)$ of $\mathscr{E}^{d-1}$ varies analytically with z [17], hence there are holomorphic subbundles $\mathscr{E}^{p,d-p-1}$ of $\mathscr{E}^{d-1}$. We also have quotient bundles $\mathscr{P}\mathscr{E}^{p,d-p-1}$ of $\mathscr{E}^{p,d-p-1}$ corresponding to the primitive cohomologies of the hypersurfaces X_z.

Note that the cup product

$$H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) \to H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X))$$

induces maps

$$\tilde{\gamma}_p\colon H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))/\mathbb{C}(f) \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) \\ \to H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X))\,. \quad (2)$$

If z_0 is the point in $\mathcal{Z}$ corresponding to X, the space $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))/\mathbb{C}(f)$ can be identified with $T_{z_0}\mathcal{Z}$. The following proposition states that the morphism $\tilde{\gamma}_p$ induces in cohomology the Gauss-Manin connection. Let π_p be the projection $\mathscr{E}^{d-1} \to \mathscr{P}\mathscr{E}^{p,d-p+1}$.

Lemma 3.1. *Let σ_0 be a primitive class in $PH^{p,d-p-1}(X)$, let $v \in T_{z_0}\mathcal{Z}$, and let σ be a section of $\mathscr{E}^{p,d-p-1}$ along a curve in $\mathcal{Z}$ whose tangent vector at z_0 is v. Then*

$$\pi_{p-1}(\nabla_v(\sigma)) = r_{p-1}(\tilde{\gamma}_p(v \otimes \tilde{\sigma})) \quad (3)$$

where $\tilde{\sigma}$ is an element in $H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X))$ such that $r_p(\tilde{\sigma}) = \sigma_0$.

Here r_p, r_{p-1} are the residue morphisms defined in equation (1).

Proof. This is a standard computation, see [4], Proposition 5.4.3. Let f_i be local representatives, with respect to a suitable cover $\{U_i\}$ of $\mathbb{P}_\Sigma$, of the section f. Via the isomorphism of Proposition 2.9, we locally represent σ_0 by the meromorphic differential forms ω_i/f_i^{d-p+1}. A tangent vector $v \in T_{z_0}\mathcal{Z}$ represents a deformation $f_i \mapsto f_i + tg_i$ where t is a complex parameter, and g_i are holomorphic functions. Then $\nabla_v(\sigma)$ is represented by

$$\left[\frac{d}{dt}\frac{\omega_i}{(f_i+tg_i)^{d-p+1}}\right]_{t=0} = -(d-p+1)\frac{g_i\,\omega_i}{f_i^{d-p+2}}\,.$$

But the right-hand side of this equation is, up to a suitable factor, the argument of the map r_{p-1} in the right-hand side of equation (3). □

The tangent space $T_{z_0}\mathcal{Z}$ is isomorphic to $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))/\mathbb{C}(f)$. Let ϕ be the projection $\phi\colon H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \to T_{z_0}\mathcal{Z}$. Equation (3) states that the following diagram commutes:

$$\begin{array}{ccc} \dfrac{H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))}{\mathbb{C}(f)} \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) & \xrightarrow{\tilde{\gamma}_p} & H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)) \\ \downarrow{\scriptstyle \phi\otimes r_p} & & \downarrow{\scriptstyle r_{p-1}} \\ T_{z_0}\mathcal{Z} \otimes PH^{p,d-1-p}(X) & \xrightarrow{\gamma_p} & PH^{p-1,d-p}(X) \end{array} \tag{4}$$

where γ_p is the morphism that maps $v \otimes \alpha$ to $\nabla_v \alpha$. (Also, we have used the same notation for the morphism $\tilde{\gamma}_p$ in this diagram and the one in equation (2).)

Another property of the Gauss-Manin connection that we shall need is the following.

Lemma 3.2. *If α and η are sections of $\mathscr{E}^{p,d-p-1}$ and $\mathscr{E}^{d-p,p-1}$ respectively, then for every tangent vector $v \in T_{z_0}\mathcal{Z}$,*

$$\nabla_v \alpha \cup \eta = -\alpha \cup \nabla_v \eta\,. \tag{5}$$

Proof. The Gauss-Manin connection is compatible with the cup product by definition, i.e.,

$$\nabla_v(\alpha \cup \eta) = \nabla_v \alpha \cup \eta + \alpha \cup \nabla_v \eta\,.$$

But $\alpha \cup \eta = 0$ because it is an element in $\mathscr{E}^{d,d-2}$. □

3.2. **The moduli space of hypersurfaces in $\mathbb{P}_\Sigma$.** There is natural way to construct a coarse moduli space $\mathcal{M}_\beta$ for the generic quasi-smooth hypersurfaces in $\mathbb{P}_\Sigma$ having divisor class β, see [3], Section 13. The moduli space $\mathcal{M}_\beta$ may be constructed as a quotient

$$U/\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)\,, \tag{6}$$

where U is an open subset of $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))$, and $\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ is the unique nontrivial extension

$$1 \to D(\Sigma) \to \widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma) \to \mathrm{Aut}_\beta(\mathbb{P}_\Sigma) \to 1\,.$$

Here $\mathrm{Aut}_\beta(\mathbb{P}_\Sigma)$ is the subgroup of $\mathrm{Aut}_\beta(\mathbb{P}_\Sigma)$ which preserves the grading β. By differentiating, we have a surjective map

$$\kappa_\beta \colon H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \to T_X\mathcal{M}_\beta\,,$$

which is in a way a Kodaira-Spencer map. According to [13], Proposition 2.1, $T_X\mathcal{M}_\beta$ embeds into $H^1(X, \Theta_X)$, where Θ_X is the tangent sheaf to X, and its direct sum complement is a space which classifies "nonpolynomial" infinitesimal deformations of X, i.e., deformations that are not obtained by moving X in the linear system $|L|$.

Proposition 3.3. *There is a morphism*

$$\gamma_p\colon T_X\mathcal{M}_\beta \otimes PH^{p,d-1-p}(X) \to PH^{p-1,d-p}(X) \tag{7}$$

such that the diagram

$$\begin{array}{ccc} H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) & \xrightarrow{\cup} & H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)) \\ \Big\downarrow{\scriptstyle \kappa_\beta\otimes r_p} & & \Big\downarrow{\scriptstyle r_{p-1}} \\ T_X\mathcal{M}_\beta \otimes PH^{p,d-1-p}(X) & \xrightarrow{\gamma_p} & PH^{p-1,d-p}(X) \end{array}$$

commutes.

Proof. Ideally one would like that the local system $\mathscr{H}^{d-1}$ and its various subsystems would descend to the moduli space $\mathcal{M}_\beta$. This is not possible however because the group $\operatorname{Aut}_\beta(\mathbb{P}_\Sigma)$ is not connected. Nevertheless, this group has a connected subgroup $\operatorname{Aut}^0_\beta(\mathbb{P}_\Sigma)$ of finite order, and, perhaps after suitably shrinking U, the quotient $U/\operatorname{Aut}^0_\beta(\mathbb{P}_\Sigma)$ is a finite étale covering of $\mathcal{M}^0_\beta$ of $\mathcal{M}_\beta$ [5]. Since the tangent spaces at points $\mathcal{M}^0_\beta$ are canonically isomorphic to the tangent spaces at the image points in $\mathcal{M}_\beta$, it suffices to prove the Proposition with $\mathcal{M}_\beta$ replaced by $\mathcal{M}^0_\beta$.

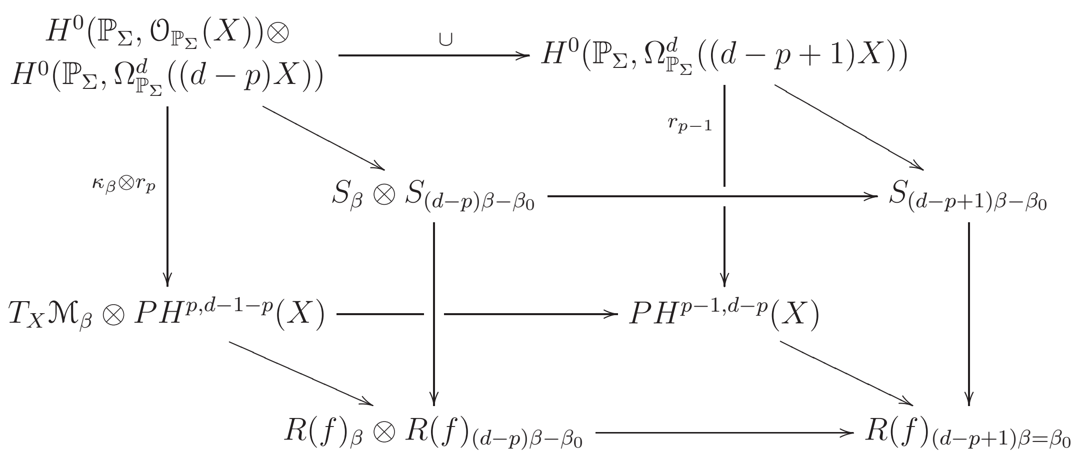
Now, if $\rho\colon \mathcal{Z} \to \mathcal{M}^0_\beta$ is the induced map (where $\mathcal{Z}$ has been suitably restricted), the local system $\mathscr{H}^{d-1}$ descend to a local system $\rho_*\mathscr{H}^{d-1}$ on $\mathcal{M}^0_\beta$, and $\rho^*\rho_*\mathscr{H}^{d-1} \simeq \mathscr{H}^{d-1}$ (one has indeed a natural morphism $\mathscr{H}^{d-1} \to \rho^*\rho_*\mathscr{H}^{d-1}$ which on the stalks is an isomorphism due topological base change; note that ρ is proper). Thus we obtain on $\mathcal{M}^0_\beta$ holomorphic bundles that are equipped with a Gauss-Manin connection, which is trivial in the direction of the fibers of ρ. If we define again γ_p by $\gamma_p(v \otimes \alpha) = \nabla_v(\alpha)$ (where ∇ is now the Gauss-Manin connection on $\mathcal{M}^0_\beta$), the commutavity of the diagram in the statement follows from the commutativity of the diagram (4). □

The tangent space $T_X\mathcal{M}_\beta$ at a point representing a hypersurface X is naturally isomorphic to the degree β summand of the Jacobian ring of f, that is, $T_X\mathcal{M}_\beta \simeq R(f)_\beta$ [3]. Moreover, there are isomorphisms $PH^{p,d-p-1}(X) \simeq R(f)_{(d-p)\beta-\beta_0}$ given by Proposition 2.11.

Proposition 3.4. *Under these isomorphisms, the morphism* γ_p *in equation* (7) *coincides with the multiplication in the ring* $R(f)$,

$$R(f)_\beta \otimes R(f)_{(d-p)\beta-\beta_0} \to R(f)_{(d-p+1)\beta-\beta_0}\,.$$

Proof. One has $H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X) \simeq S_{(d-p)\beta-\beta_0}$ [3, Thm. 9.7] and $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \simeq S_\beta$; under these isomorphisms, the cup product corresponds to the product in the ring S. This means that the "top square" of the 3-dimensional diagram



commutes. We need to show that the "bottom square" commutes as well. This will follow from the commutativity of the "side squares". The commutativity of the diagram on the right is contained in the proof of Theorem 10.6 in [3]. The commutativity of the diagram on the left follows from the commutativiy of the previous diagram, with $d-p+1$ replaced by $d-p$, and the commutativy of

$$\begin{array}{ccc} H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) & \xrightarrow{\sim} & S_\beta \\ \downarrow & & \downarrow \\ T_X\mathcal{M}_\beta & \xrightarrow{\sim} & R(f)_\beta \end{array}$$

which is shown in the proof of Proposition 13.7 in [3]. □

3.3. **Picard group.** Our aim is now to prove the following result.

Theorem 3.5. *Let* $\mathbb{P}_\Sigma$ *be a 3-dimensional complete simplicial toric variety,* L *an ample line bundle on* $\mathbb{P}_\Sigma$*, and* X *a generic quasi-smooth hypersurface in the linear system* $|L|$*. If the morphism* $\gamma_2\colon T_X\mathcal{M}_\beta \otimes PH^{2,0}(X) \to PH^{1,1}(X)$ *is surjective, then* X *and* $\mathbb{P}_\Sigma$ *have the same Picard number.* □

Theorem 3.5 will follow from two Lemmas, that one can actually prove in any dimension. We only assume that $p \neq d/2-1$ and $p \neq d/2$. The first Lemma is an "infinitesimal Noether-Lefschetz theorem", such as Theorem 7.5.1 in [4]. Denote by $H^{d-1}_T(X)$ the subspace of $H^{d-1}(X)$ made by the cohomology classes that are annihilated by the action of the Gauss-Manin connection. Coefficients may be taken indifferently in $\mathbb{C}$ or $\mathbb{Q}$.

Lemma 3.6. *Assume that the morphism*

$$\gamma_p \colon T_X\mathcal{M}_\beta \otimes PH^{p,d-1-p}(X) \to PH^{p-1,d-p}(X)$$

is surjective. Then $H^{d-1}_T(X) = i^*(H^{d-1}(\mathbb{P}_\Sigma))$.

Proof. We note that we may replace $\mathcal{M}_\beta$ by $\mathcal{M}^0_\beta$, and consider the local systems $\mathscr{E}^{d-1}$ and $\mathscr{P}\mathscr{E}^{p,d-p-1}$ on $\mathcal{M}^0_\beta$. We also note that $H^{d-1}_T(X)$ has a Hodge decomposition. Let

$$\alpha \in H^{p,d-1-p}_T(X) \cap PH^{p,d-1-p}(X).$$

We regard classes in $PH^{p,d-1-p}(X)$ as elements in the fiber of $\mathscr{P}\mathscr{E}^{p,d-p-1}$ at the point $[X] \in M^0_\beta$. Any $\beta \in PH^{p-1,d-p}(X)$ may be written as $\beta = \sum_i \gamma_p(t_i \otimes \eta_i)$ with $\eta_i \in PH^{p,d-1-p}(X)$. Then by equation (3) and (5)

$$\langle \alpha, \beta \rangle = \sum_i \langle \alpha, \gamma_p(t_i \otimes \eta_i) \rangle = \sum_i \langle \alpha, \nabla_{t_i} \eta_i \rangle = -\sum_i \langle \nabla_{t_i} \alpha, \eta_i \rangle = 0.$$

So α is orthogonal to $PH^{d-1-p,p}(X)$. By Lemma 2.7, this means that α is orthogonal to the whole group $H^{d-1-p,p}(X)$, hence it is zero. Therefore $H^{d-1}_T(X) = i^*(H^{d-1}(\mathbb{P}_\Sigma))$. □

Lemma 3.7. *Under the same hypotheses of the previous Lemma, assume also also that d is odd, $d = 2m+1$. Then for z away from a countable union of subschemes of $\mathcal{Z}$ of positive codimension one has*

$$H^{m,m}(X_z, \mathbb{Q}) = \operatorname{im}[i^* \colon H^{m,m}(\mathbb{P}_\Sigma, \mathbb{Q}) \to H^{2m}(X_z.\mathbb{Q})].$$

Proof. Let $\bar{\mathcal{Z}}$ be the universal cover of $\mathcal{Z}$. On it the (pullback of the) local system $\mathscr{H}^{d-1}$ is trivial. Given a class $\alpha \in H^{m,m}(X)$ we can extend it to a global section $\mathscr{H}^{d-1}$ by parallel transport using the Gauss-Manin connection. Define the subset of $\bar{\mathcal{Z}}$

$$\bar{\mathcal{Z}}_\alpha = \{ z \in \bar{\mathcal{Z}} \,|\, \alpha(z) \in H^{m,m}(X_z, \mathbb{Q}) \}.$$

If $\bar{\mathcal{Z}}_\alpha = \bar{\mathcal{Z}}$ we are done because α is in $H^{d-1}_T(X)$ hence is in the image of i^* by the previous Lemma.

If $\bar{\mathcal{Z}}_\alpha \neq \bar{\mathcal{Z}}$, we note that $\bar{\mathcal{Z}}_\alpha$ is a subscheme of $\bar{\mathcal{Z}}$. Indeed it is the common zero locus of the sections $\pi_p(\alpha)$ of $\mathscr{E}^{p,d-1-p}$ for $p \neq m$. We subtract from $\mathcal{Z}$ the union of the projections of the subschemes $\bar{\mathcal{Z}}_\alpha$ where this does not hold. Note that these varieties are countable in number because we are considering rational classes. □

Theorem 3.5 now follows by taking rational coefficients, and $d = 3$.

We assume now that $\mathbb{P}_\Sigma$ is Fano, and that $L = -K_{\mathbb{P}_\Sigma}$, so that the hypersurfaces in the linear system $|L|$ are K3 surfaces. We have now $PH^{2,0} \simeq R(f)_0 \simeq \mathbb{C}$, $PH^{1,1}(X) \simeq R(f)_\beta$, and $T_X\mathcal{M}_\beta \simeq R(f)_\beta$. By Proposition 3.4, the morphism γ_2 corresponds to the multiplication $R(f)_\beta \otimes R(f)_0 \to R(f)_\beta$, and since $R(f)_0 \simeq \mathbb{C}$, this is an isomorphism. From Theorem 3.5 we have:

Theorem 3.8. *Let $\mathbb{P}_\Sigma$ be a 3-dimensional Fano complete simplicial toric variety, and X a generic hypersurface in the linear system $|-K_{\mathbb{P}_\Sigma}|$. Then X has the same Picard number as $\mathbb{P}_\Sigma$.*

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