Dynamical Casimir effect on surface waves

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Abstract

We consider the quantum radiation of scalar particles from a surface wave excited on a plane surface of a mirror. It is assumed that the field obeys Dirichlet condition on the boundary of the mirror. In both cases of running and standing surface waves the expression is given for the spectral-angular distribution of the number of the radiated quanta.

1 Introduction

The Casimir effect [1] is one of the most interesting manifestations of nontrivial properties of the vacuum state in a quantum field theory and can be viewed as a polarization of vacuum by boundary conditions. A new phenomenon, a quantum creation of particles (the dynamical Casimir effect) occurs when the geometry of the system varies in time (for a recent review see [4]). In the present talk, based on [2, 3], we discuss the quantum radiation from a surface wave excited on a flat boundary.

2 General formula for the number of the radiated quanta

Consider a quantum scalar field $\varphi(x)$ in a spacetime region with the boundary S. We assume that on the boundary the field obeys Dirichlet boundary condition. The field equation and the boundary condition have the form

$$\left(\nabla_l \nabla^l + m^2\right)\varphi(x) = 0, \ \varphi(x)|_{x \in S} = 0. \tag{1}$$

In the case of a moving boundary, the interaction with quantum fluctuations of the field can lead to the creation of real quanta out of vacuum. Let the vector $\xi^l(x) = n^l \xi(x)$ describes the displacement of the hypersurface S from the static hypersurface S_0: $x^l + \xi^l(x) \in S$ if $x^l \in S_0$. Here n^l is the unit normal to the boundary S_0. Assuming that the displacement is small, in

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the first approximation, for the number of quanta radiated in the interval of quantum numbers $(\nu, \nu + d\nu)$ one has the formula [2]

$$n(\nu)d\nu = \int d\nu' \left| \int_{S_0} d\Sigma \, n^i n^l \xi(x) [\partial_i \varphi_{0\nu'}(x)][\partial_l \varphi_{0\nu}(x)] \right|^2 d\nu, \tag{2}$$

where $\{\varphi_{0\nu}(x), \varphi_{0\nu}^*(x)\}$ is a complete set of eigenfunctions with the set of quantum numbers ν which are solutions of the boundary-value problem (1) with $S = S_0$. Below we consider applications of the general formula (2) to special cases.

3 Two-dimensional spacetime

First we consider the case of a two-dimensional spacetime with the coordinates (t, x) and with the line $x = 0$ as S_0. The corresponding eigenfunctions have the form

$$\varphi_{0k}(t, x) = \frac{\sin(kx)}{\sqrt{\pi\omega}} e^{-i\omega t}, \ \omega = \sqrt{k^2 + m^2}, \tag{3}$$

with $0 \leqslant k < \infty$. From (2), for the density of the number of quanta emitted to the region $x > 0$ one finds

$$n(k) = \frac{k^2}{\pi\omega} \int_0^\infty dk' \frac{k'^2}{\omega'} |\xi(\omega + \omega')|^2, \ \omega' = \sqrt{k'^2 + m^2}, \tag{4}$$

where

$$\xi(\omega) = \int_{-\infty}^{+\infty} dt \, \xi(t) e^{-i\omega t}. \tag{5}$$

For harmonic oscillations of the boundary, $\xi(t) = \xi_0 \cos(\omega_0 t)$, the expression for the density of the number of quanta radiated per unit time takes the form

$$n(k) = \frac{\xi_0^2 k^2}{2\pi\omega} \sqrt{(\omega_0 - \omega)^2 - m^2}, \ m \leqslant \omega \leqslant \omega_0 - m. \tag{6}$$

As a necessary condition for the radiation one has $\omega_0 > 2m$ and the maximal energy of the radiated quanta is equal to $\omega_0 - m$. The energy radiated per unit time, evaluated by using Eq. (6), is given by

$$E = \int_0^\infty dk \, n(k)\omega = \frac{\xi_0^2 \omega_0^4}{24\pi} f(m/\omega_0), \tag{7}$$

with the function

$$f(y) = 12 \int_y^{1/2} dz \sqrt{z^2 - y^2} \sqrt{(1 - z)^2 - y^2}, \ 0 \leqslant y \leqslant 1/2. \tag{8}$$

This function is plotted in figure 1. From the graph it is seen that the maximum energy is radiated in the case of a massless field. Formulae (6) and (7) are valid under the condition $|d\xi/dt| \sim \xi_0\omega_0 \ll 1$.

For a massless field we can obtain general formula for the radiated power in the case of general motion of the mirror. Indeed, by taking into account (4), the energy radiated per unit time to the region $x > 0$ is presented in the form

$$E = \int_0^\infty d\omega \, n(\omega)\omega = \frac{2i}{\pi^2} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \frac{\xi(t)\xi(t')}{(t - t' + i\varepsilon)^5}, \ \varepsilon > 0. \tag{9}$$
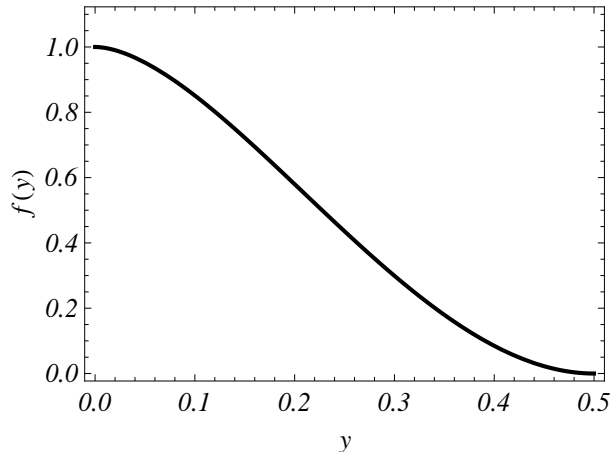
Figure 1: The graph of the function $f(y)$ defined by (8).

Introducing new integration variables $t_1 = t' - t$, $t_2 = t' + t$, the integral over t_1 is evaluated by using the relation

$$\int_{-\infty}^{+\infty} dt \frac{g(t_1)}{(t_1 - i\varepsilon)^5} = \frac{i\pi}{4} g^{(4)}(0),$$
(10)

with

$$g(t_1) = \xi\left(\frac{t_2 + t_1}{2}\right) \xi\left(\frac{t_2 - t_1}{2}\right).$$
(11)

Integrating by parts, one finds

$$E = \frac{1}{12\pi} \int_{-\infty}^{+\infty} dt\, |\xi''(t)|^2.$$
(12)

This result coincides with the formula derived in [5] by using the expectation value of the energy-momentum tensor: $E = \int dV \left\langle T^{00} \right\rangle$.

4 Examples for 4-dimensional spacetime

In a 4-dimensional spacetime, as the hypersurface S_0 we take the plane $x = 0$. For the corresponding eigenfunctions one has

$$\varphi_{0\mathbf{k}}(t, \mathbf{x}) = \frac{\sin(k_1 x)}{2\pi\sqrt{\pi\omega}} e^{i\mathbf{k}_\perp \mathbf{x}_\perp - i\omega t}, \ \omega = \sqrt{k^2 + m^2},$$
(13)

where $\mathbf{x} = (x, y, z)$, $\mathbf{x}_\perp = (y, z)$, $\mathbf{k}_\perp = (k_2, k_3)$. For the density of the number of quanta from (2) we find the expression

$$n(\mathbf{k}) = \frac{1}{16\pi^6} \int d\mathbf{k}' \frac{k_1^2 k_1'^2}{\omega\omega'} |\xi(\omega + \omega', \mathbf{k}_\perp + \mathbf{k}'_\perp)|^2,$$
(14)

where

$$\xi(\omega, \mathbf{k}_\perp) = \int_{-\infty}^{+\infty} dt dy dz\, \xi(t, y, z) e^{i\mathbf{k}_\perp \mathbf{x}_\perp - i\omega t}.$$
(15)

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First let us consider the case when the plane oscillates as a whole: $\xi = \xi(t)$. For a massless field we find

$$n(\mathbf{k}) = \frac{k_1^2}{4\pi^4\omega} \int_{k_\perp}^\infty d\omega' \sqrt{\omega'^2 - k_\perp^2} \, |\xi(\omega + \omega')|^2. \tag{16}$$

For the total energy radiated into the region $x > 0$ we obtain the expression

$$E = \int d\mathbf{k} \, n(\mathbf{k})\omega = \frac{1}{i\pi^3} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \frac{\xi(t)\xi(t')}{(t - t' + i\varepsilon)^7}. \tag{17}$$

After the evaluation of the integral, in a way similar to the two-dimensional case, we find

$$E = \frac{1}{720\pi^2} \int_{-\infty}^{+\infty} dt \, |\xi'''(t)|^2. \tag{18}$$

By taking into account the radiation into the region $x < 0$, this result coincides with the formula derived in [5] by using the expectation value of the energy-momentum tensor.

Now we turn to the case of a running surface wave excited on the surface of a plane mirror (see figure 2). The corresponding displacement function has the form

$$\xi(t, y, z) = f(z - v_0 t). \tag{19}$$

In this case for the Fourier-transform we have

$$\xi(\omega, \mathbf{k}_\perp) = (2\pi)^2 \delta(k_2)\delta(\omega - k_3 v_0) \int_{-\infty}^{+\infty} dz \, f(z)e^{ik_3 z}. \tag{20}$$

One has $k_3 \leqslant \omega$ and for $v_0 < 1$ this expression vanishes. In this case the radiation is absent.
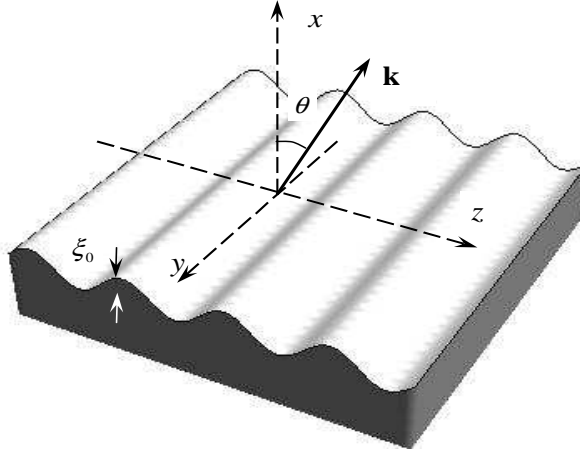

Figure 2: Quantum radiation from a surface wave excited on a plane boundary.

This result becomes obvious if we pass to the reference frame moving with the surface wave. In the case of harmonic oscillations,

$$\xi(t, y, z) = \xi_0 \cos(k_0 z - \omega_0 t), \tag{21}$$

with $\omega_0 > k_0$, the quanta are radiated satisfying the condition

$$\omega_0 - \omega \geqslant \sqrt{k_2^2 + (k_0 - k_3)^2}. \tag{22}$$

The number density of the quanta radiated from unit surface of the mirror per unit time is given by the expression

$$n(\mathbf{k}) = \frac{k_1^2 \xi_0^2}{8\pi^3 \omega} [(\omega_0 - \omega)^2 - k_2^2 - (k_0 - k_3)^2]^{1/2}. \tag{23}$$

Introducing spherical coordinates in the momentum space, $k_1 = \omega \cos\theta$, $k_2 = \omega \sin\theta \sin\phi$, $k_3 = \omega \sin\theta \cos\phi$, $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \phi \leqslant 2\pi$, for the spectral-angular density of the number of the radiated quanta we have

$$\frac{dN}{d\omega d\Omega} = \frac{\xi_0^2 \omega^3}{8\pi^3} \cos^2\theta (\omega_0^2 - 2\omega_0\omega - k_0^2 + 2k_0\omega \sin\theta \cos\phi + \omega^2 \cos^2\theta)^{1/2}, \tag{24}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle element and θ is the angle between the normal to the mirror and the direction of the radiation (see figure 2). For a given ω, the allowed angular region for the radiation is determined by the non-negativity condition for the expression under the square root in (24).

The limit $k_0 \to 0$ of this formula corresponds to the mirror oscillating as a whole. In this case, for the number of quanta radiated in the energy range $(\omega, \omega + d\omega)$, in the solid angle $d\Omega$ from the unit surface of the mirror per unit time one has

$$\begin{aligned} \frac{dN}{d\omega d\Omega} &= \frac{\xi_0^2 \omega^3}{8\pi^3} \cos^2\theta (\omega_0^2 - 2\omega_0\omega + \omega^2 \cos^2\theta)^{1/2}, \\ \sin\theta &\leqslant \omega_0/\omega - 1, \ 0 \leqslant \theta \leqslant \pi/2. \end{aligned} \tag{25}$$

The spectral density of the radiation is obtained after the integration of (25) over the solid angle: $dN/d\omega = \omega^2 \int n(\mathbf{k}) d\Omega$. This leads to the result

$$\frac{dN}{d\omega} = \frac{\xi_0^2 \omega_0^4}{32\pi^2} \left[u(1-u)^3 + u^3(1-u) + (1/2)(1-2u)^2 \ln|1-2u| \right], \tag{26}$$

with the notation $u = \omega/\omega_0$. The expression on the right-hand side of (26) takes its maximum value at $u = 1/2$ with $(dN/d\omega)_{\max} = \xi_0^2 \omega_0^4/(2^8\pi^2)$. In figure 3 we have plotted the spectral density of the number of the radiated quanta as a function of ω/ω_0.
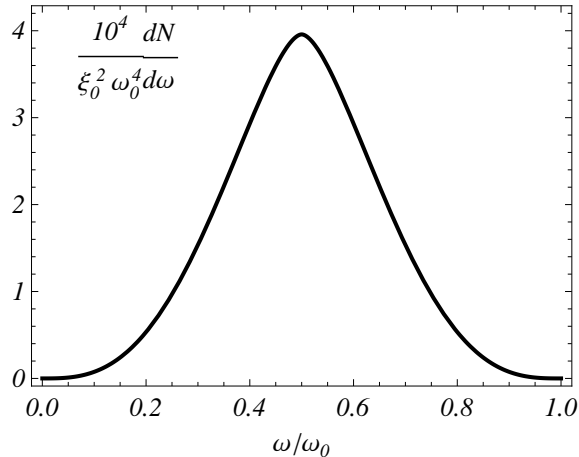

Figure 3: The spectral density of the radiated quanta given by (26).

For the total number of the radiated quanta and the total energy (per unit surface and per unit time) one finds (in standard units, for the radiated energy see also [5])

$$N = \frac{\xi_0^2 \omega_0^5}{720\pi^2 c^4}, \ E = \frac{\hbar \xi_0^2 \omega_0^6}{1440\pi^2 c^4}. \tag{27}$$

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The Planck constant in the expression for the radiated energy shows the quantum nature of the radiation.

Now we turn to the case of a standing surface wave excited on the surface of a plane mirror in the strip $0 \leqslant z \leqslant l$, $-\infty < y < +\infty$:

$$\xi(t, y, z) = \xi_0 \cos(\omega_0 t) \sin(k_0 z), \; k_0 = \pi n/l, \; n = 1, 2, 3, \ldots, \tag{28}$$

for $0 \leqslant z \leqslant l$ and $\xi(t, y, z) = 0$ otherwise. For the number of the radiated quanta per unit time and per unit length along y we find

$$n(\mathbf{k}) = \frac{k_1^2 \xi_0^2}{8\pi^4 \omega} \int_{w_1}^{w_2} du \, \frac{\sqrt{(w - w_1)(w_2 - w)}}{(1 - w^2)^2} \left[1 - (-1)^n \cos(\pi n w)\right], \tag{29}$$

where

$$w_{2,1} = [k_3 \pm \sqrt{(\omega_0 - \omega)^2 - k_2^2}]/k_0. \tag{30}$$

The following conditions should be satisfied

$$\omega = \sqrt{k_1^2 + k_2^2 + k_3^2} \leqslant \omega_0, \; |k_2| \leqslant \omega_0 - \omega, \tag{31}$$

for the presence of the radiation. In particular, the second inequality imposes a constraint on the angular region for the radiation. Note that the integrand in (29) is finite at $u = \pm 1$.

5   Conclusion

In the present talk we have discussed the quantum radiation from surface waves excited on a plane boundary. For simplicity we have considered the case of a scalar field with Dirichlet boundary condition. Other boundary conditions can be treated in a similar way. Assuming that the displacement of the moving boundary from the static one is small, in the first approximation, the expression for the density of the number of radiated quanta has the form (2). We give applications of this general formula to various special cases. First, we have considered the simplest case of two-dimensional spacetime when the general formula takes the form (4). For harmonic oscillations of the boundary this formula simplifies to (6). As a necessary condition for the presence of the quantum radiation one has $\omega_0 > 2m$ and the maximal energy of the radiated quanta is equal to $\omega_0 - m$. On the base of the number of the radiated quanta, we have evaluated the radiated energy. The latter coincides with the corresponding result evaluated by using the renormalized expectation value of the energy-momentum tensor.

More realistic case of 4-dimensional spacetime is discussed in section 4 with the expression (14) for the number of the radiated quanta. For a massless field we have considered various special cases. When the plane boundary oscillates as a whole the number of the radiated quanta and the total radiated energy are given by expressions (16) and (18), respectively. As in the two-dimensional case, the latter coincides with the result obtained on the base of the expectation value of the energy-momentum tensor. Next, we have considered the cases of running and standing surface waves. For the running wave the radiation is present in the case when the phase velocity of the wave is larger than the speed of light. The spectral-angular distribution of the radiated quanta is given by expression (25) for a running wave and by (29) for a standing surface wave.

References

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