SPARSE PAVING MATROIDS, BASIS-EXCHANGE PROPERTIES, AND CYCLIC FLATS

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ABSTRACT. We provide evidence for five long-standing, basis-exchange conjectures for matroids by proving them for the enormous class of sparse paving matroids. We also explore the role that these matroids may play in the following problem: as a function of the size of the ground set, what is the greatest number of cyclic flats that a matroid can have?

1. INTRODUCTION

A matroid is *paving* if the closure of each nonspanning circuit is a hyperplane; it is *sparse paving* if each nonspanning circuit is a hyperplane. Thus, a matroid M of rank r is sparse paving if and only if each r-subset of $E(M)$ is either a basis or a circuit-hyperplane. It follows that the class of sparse paving matroids is dual-closed. It is easy to show that this class is also minor-closed. Sparse paving matroids can also be characterized as the matroids M for which both M and its dual, M^*, are paving.

While paving and sparse paving matroids have received increasing attention recently (see, e.g., [12, 17, 21, 22, 23]), they have long played important roles in matroid theory. For instance, D. Knuth [20] constructed at least

$$\frac{2^{\binom{n}{\lfloor n/2\rfloor}/2n}}{n!}$$

nonisomorphic sparse paving matroids of rank $\lfloor n/2\rfloor$ on n elements; with the upper bound by M. Piff [26], it follows that the number g_n of nonisomorphic simple matroids on n elements satisfies

$$n-\frac{3}{2}\log_2 n+O(\log_2\log_2 n)\leq \log_2\log_2 g_n\leq n-\log_2 n+O(\log_2\log_2 n), \tag{1.1}$$

with sparse paving matroids accounting for the lower bound. Taking this further, in [21], D. Mayhew, M. Newman, D. Welsh, and G. Whittle have conjectured that, asymptotically, almost all matroids are sparse paving.

The five basis-exchange conjectures treated in this paper, all of which have been open for decades and have been proven for only a few classes of matroids, are part of the circle of ideas that revolve around the well-known symmetric basis-exchange property: for any bases B_1, B_2 of a matroid M, if $b_1\in B_1-B_2$, then, for some $b_2\in B_2-B_1$, both $(B_1-b_1)\cup b_2$ and $(B_2-b_2)\cup b_1$ are also bases of M.

The first conjecture concerns the *basis pair graph*, $G(M)$, of a matroid M, which is defined as follows. The vertices of $G(M)$ are the ordered triples (A_1,A_2,A_3) of subsets of $E(M)$ where A_1 and A_2 are disjoint bases of M and A_3 is $E(M)-(A_1\cup A_2)$. (Thus, the inequality $|E(M)|\geq 2\,r(M)$ must hold in order for $G(M)$ to have any vertices.) Two

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vertices, say $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$, of $G(M)$ are adjacent if $\mathbf{B}$ can be obtained from $\mathbf{A}$ by switching some pair of elements in two different sets in $\mathbf{A}$, that is, if

$$|A_1 - B_1| + |A_2 - B_2| + |A_3 - B_3| = 2.$$

If $E(M)$ is the disjoint union of two bases of M, then $G(M)$ is isomorphic to the basis-cobasis graph studied by R. Cordovil and M. Moreira [8]. The following conjecture was posed by M. Farber [9], who proved it for transversal matroids. (In [10], M. Farber, B. Richter and H. Shank proved it for graphic and cographic matroids.)

Conjecture 1.1. *The basis pair graph of any matroid is connected.*

The second conjecture involves a family of graphs that we can associate with a matroid. Fix an integer $k \geq 2$. Let M be a matroid of rank r and let S be a multiset of size kr with elements in $E(M)$. Define the graph $G_M(S)$ as follows: the vertices of $G_M(S)$ are all multisets of k bases of M whose multiset union is S; two vertices are adjacent if one can be obtained from the other by one symmetric exchange among one pair of bases in one of the vertices. Thus, vertices $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ and $\mathcal{B} = \{B_1, B_2, \dots, B_k\}$ are adjacent if, for some bases $B_i, B_j \in \mathcal{B}$ and elements $b_i \in B_i - B_j$ and $b_j \in B_j - B_i$, we obtain $\mathcal{A}$ from $\mathcal{B}$ by replacing B_i by $(B_i - b_i) \cup b_j$ and replacing B_j by $(B_j - b_j) \cup b_i$. (This graph may be empty.) The conjecture below is due to N. White [28, Conjecture 12].

Conjecture 1.2. *For any matroid M and multiset S of size $k\,r(M)$ with elements in $E(M)$ and with $k \geq 2$, the graph $G_M(S)$ is connected.*

Conjecture 1.2 is sometimes cast in terms of toric ideals. A routine argument shows that the conjecture holds for M if and only if it holds for M^*. It has been shown for graphic (and so for cographic) matroids by J. Blasiak [1] and for rank-3 (and so for nullity-3) matroids by K. Kashiwabara [19]. J. Herzog and T. Hibi [14] have shown that Conjecture 1.2 is equivalent to its counterpart for discrete polymatroids. J. Schweig [27] has proven the counterpart of the conjecture for certain discrete polymatroids.

While Conjecture 1.2 has received most attention, [28, Conjecture 12] has three parts, of which the next conjecture is the strongest. Consider the graph $G'_M(S)$ in which k-tuples of bases replace multisets of bases. Thus, its vertices are all k-tuples of bases of M whose multiset union is S; vertices $\mathbf{A} = (A_1, A_2, \dots, A_k)$ and $\mathbf{B} = (B_1, B_2, \dots, B_k)$ are adjacent if, for some integers i and j with $1 \leq i < j \leq k$ and some $b_i \in B_i - B_j$ and $b_j \in B_j - B_i$, we obtain $\mathbf{A}$ from $\mathbf{B}$ by replacing B_i by $(B_i - b_i) \cup b_j$ and replacing B_j by $(B_j - b_j) \cup b_i$.

Conjecture 1.3. *For any matroid M and multiset S of size $k\,r(M)$ with elements in $E(M)$ and with $k \geq 2$, the graph $G'_M(S)$ is connected.*

We show that the conclusion of Conjecture 1.3 holds for a matroid M if Conjecture 1.2 holds for M and Conjecture 1.1 holds for all of its minors. It follows that Conjecture 1.3 holds for all sparse paving matroids.

The fourth conjecture was made by Y. Kajitani, S. Ueno, and H. Miyano [18]. A matroid M is *cyclically orderable* if there is a cyclic permutation $(a_1, a_2, \dots, a_n)$ of $E(M)$ in which each set of $r(M)$ cyclically-consecutive elements is a basis of M.

Conjecture 1.4. *A matroid M is cyclically orderable if and only if, for all nonempty subsets A of $E(M)$,*

$$r(M)\,|A| \leq r(A)\,|E(M)|. \tag{1.2}$$

A counting argument shows that inequality (1.2) holds if M is cyclically orderable. Recent progress on this conjecture was made by J. van den Heuvel and S. Thomassé [15]

The fifth conjecture was first raised as a problem by H. Gabow [11] and has been pursued in [8, 15, 18]. To match our work below, we state the conjecture in the case of disjoint bases; it is easy to show that this implies its counterpart for arbitrary bases.

Conjecture 1.5. *If B_1 and B_2 are disjoint bases of a rank-r matroid M, then some cycle $(b_1, b_2, \dots, b_r, b_{r+1}, \dots, b_{2r})$ has $B_1 = \{b_1, b_2, \dots, b_r\}$ and $B_2 = \{b_{r+1}, b_{r+2}, \dots, b_{2r}\}$, and has each set of r cyclically-consecutive elements being a basis of M.*

It is not hard to show that if this conjecture holds for M, then it holds for M^* and for all minors of M. H. Gabow [11] noted that the conjecture holds for transversal matroids. It has also been proven for graphic matroids [8, 18]. A. de Mier [24] observed that this conjecture holds for strongly base-orderable matroids. Recall that a matroid is *strongly base-orderable* if for each pair of bases B_1 and B_2 of M, there is a bijection $\phi : B_1 \to B_2$ such that for every subset $X \subseteq B_1$, both $(B_1 - X) \cup \phi(X)$ and $(B_2 - \phi(X)) \cup X$ are bases. If M is strongly base-orderable, then listing the elements of B_1 in any order followed by their images under ϕ, in the corresponding order, gives the required cycle. The class of strongly base-orderable matroids is both minor-closed and dual-closed, and it strictly contains the class of all gammoids (which include transversal matroids).

In Section 2, we prove Conjectures 1.1–1.5 for sparse paving matroids. Section 3 treats another aspect of these matroids as we study the greatest number of cyclic flats in any matroid on n elements. We give an upper bound on this number and note that a lower bound follows from work of R. Graham and N. Sloane [13] which, in a different setting, essentially constructs sparse paving matroids. The gap between these bounds is similar to that in inequality (1.1). We provide the relevant background on cyclic flats in that section.

Our notation follows J. Oxley [25]. The symmetric difference, $(X - Y) \cup (Y - X)$, of two sets X and Y is denoted by $X \triangle Y$. We let $[n]$ denote the set $\{1, 2, \dots, n\}$.

2. Proofs of Conjectures 1.1–1.5 in the Case of Sparse Paving Matroids

We will use the lemmas below. The first follows easily from the definition of sparse paving.

Lemma 2.1. *Let M be a sparse paving matroid of rank r. Let H and B be two r-subsets of $E(M)$ with $|H \triangle B| = 2$. If H is a circuit-hyperplane of M, then B is a basis.*

Although we will not use it, we note that the following strengthening of Lemma 2.1 is easy to prove: a matroid M of rank r is sparse paving if and only if whenever H and B are two r-subsets of $E(M)$ with $|H \triangle B| = 2$ and H is not a basis, then B is a basis. (We remark that the analogous condition on discrete polymatroids winds up being too restrictive to be of interest.)

Lemma 2.2. *Let B and B' be distinct bases of a sparse paving matroid M. For $a \in B - B'$ and $X \subseteq B' - B$, there are at least $|X| - 2$ elements $x \in X$ for which both $(B - a) \cup x$ and $(B' - x) \cup a$ are bases of M.*

Proof. The lemma follows since, by Lemma 2.1, at most one set $(B - a) \cup x$ with $x \in X$, and at most one set $(B' - x') \cup a$ with $x' \in X$, is a circuit-hyperplane. □

We now turn to Conjecture 1.1.

Theorem 2.3. *Conjecture 1.1 holds for sparse paving matroids.*

Proof. We first prove the result when $E(M)$ is the disjoint union of two bases; we will then reduce the general case to this one. In this case, vertices have the form $(B_1, B_2, \emptyset)$, which we simplify to (B_1, B_2) in the next two paragraphs. We must show that for each pair (A_1, A_2) and (B_1, B_2) of vertices in $G(M)$ with $|A_1 \triangle B_1| \geq 4$, there is a path between them. For this, it suffices to show that there is a path from (B_1, B_2) to a vertex (B'_1, B'_2) with $|A_1 \triangle B'_1| < |A_1 \triangle B_1|$.

If $|B_1 - A_1| \geq 3$, then fix $x \in B_1 - A_1$ and set $X = A_1 - B_1$. We have $|X| \geq 3$ and $X \subseteq B_2$, so, by Lemma 2.2, the pair $\big((B_1 - x) \cup y, (B_2 - y) \cup x\big)$ is a vertex of $G(M)$ for some $y \in X$. Also, $|A_1 \triangle \big((B_1 - x) \cup y\big)| < |A_1 \triangle B_1|$, as needed.

In the remaining case, $|B_1 - A_1| = 2$, let $B_1 - A_1 = \{b_1, b_2\}$ and $A_1 - B_1 = \{a_1, a_2\}$. Thus, $a_1, a_2 \in B_2$. If any of the following four symmetric exchanges yields only bases, it would provide the desired vertex (B'_1, B'_2) adjacent to (B_1, B_2):

(a) $(B_1 - b_1) \cup a_1$ and $(B_2 - a_1) \cup b_1$,
(b) $(B_1 - b_1) \cup a_2$ and $(B_2 - a_2) \cup b_1$,
(c) $(B_1 - b_2) \cup a_1$ and $(B_2 - a_1) \cup b_2$,
(d) $(B_1 - b_2) \cup a_2$ and $(B_2 - a_2) \cup b_2$.

Thus, we may assume that each pair contains a circuit-hyperplane. By symmetry, we may assume that $(B_1 - b_1) \cup a_1$ is a circuit-hyperplane; then $(B_1 - b_1) \cup a_2$ and $(B_1 - b_2) \cup a_1$ are bases by Lemma 2.1, so $(B_2 - a_2) \cup b_1$ and $(B_2 - a_1) \cup b_2$ are circuit-hyperplanes; thus, $(B_2 - a_2) \cup b_2$ is a basis by Lemma 2.1, so $(B_1 - b_2) \cup a_2$ is a circuit-hyperplane. For all four sets just identified to be circuit-hyperplanes, we must have $r(M) \geq 3$, so there is an element x in $B_1 \cap A_1$. By comparison with the four known circuit-hyperplanes, it follows that each set in the following symmetric exchanges is a basis:

(e) $(B_1 - x) \cup a_1$ and $(B_2 - a_1) \cup x$,
(f) $B'_1 = (B_1 - \{x, b_2\}) \cup \{a_1, a_2\}$ and $B'_2 = (B_2 - \{a_1, a_2\}) \cup \{x, b_2\}$.

Since (B'_1, B'_2) is adjacent to (A_1, A_2), the needed path from (B_1, B_2) to (A_1, A_2) exists.

In the general case, for two vertices $\mathbf{A} = (A_1, A_2, A_3)$ and (B_1, B_2, B_3) of $G(M)$, we will show that there is a path in $G(M)$ from $\mathbf{A}$ to a vertex of the form (C_1, C_2, B_3); the theorem then follows by applying the case just treated to the basis pair graph of $M \backslash B_3$. (Recall that the third set in these triples need not be a basis.)

Assume $|A_3 \triangle B_3| \geq 4$. By symmetry, we may assume $|A_1 \cap B_3| \geq 1$; fix some $a_1 \in A_1 \cap B_3$. Since M is sparse paving, the hyperplane $\text{cl}(A_1 - a_1)$ contains at most one element in $A_3 - B_3$, so $A'_1 = (A_1 - a_1) \cup a_3$ is a basis for some $a_3 \in A_3 - B_3$. The vertex (A'_1, A_2, A'_3), where $A'_3 = (A_3 - a_3) \cup a_1$, is adjacent to $\mathbf{A}$ and has $|A'_3 \triangle B_3| < |A_3 \triangle B_3|$.

By iterating the argument above, it now suffices to treat the case $|A_3 \triangle B_3| = 2$. Let $A_3 - B_3 = \{a_3\}$ and $B_3 - A_3 = \{b_3\}$. We may assume $b_3 \in A_1$. If $(A_1 - b_3) \cup a_3$ is a basis of M, then the claim holds, so assume instead that this set is a circuit-hyperplane. By symmetrically exchanging any element $a_1 \in A_1 - b_3$ with some element $a_2 \in A_2$, we get a vertex $((A_1 - a_1) \cup a_2, (A_2 - a_2) \cup a_1, A_3)$ that is adjacent to $\mathbf{A}$ and in which, by Lemma 2.1, we can exchange b_3 in $(A_1 - a_1) \cup a_2$ with a_3 in A_3, which completes the proof of the claim and so of the theorem. □

We now turn to Conjecture 1.2.

Theorem 2.4. *Conjecture 1.2 holds for sparse paving matroids.*

Proof. Let M be a sparse paving matroid. We prove that $G_M(S)$ is connected by induction on k, where $|S| = k\, r(M)$. The base case $k = 1$ is trivial: $G_M(S)$ is connected since it has at most one vertex. For $k \geq 2$, we claim that for any two vertices

$$\mathcal{A} = \{A_1, A_2, \ldots, A_k\} \quad \text{and} \quad \mathcal{B} = \{B_1, B_2, \ldots, B_k\}$$

of $G_M(S)$, there are (possibly trivial) paths from $\mathcal{A}$ to some vertex $\{A'_1, A'_2, \ldots, A'_k\}$ and from $\mathcal{B}$ to some vertex $\{B'_1, B'_2, \ldots, B'_k\}$ with $A'_1 = B'_1$. Proving this claim gives the result by induction since having a path from $\mathcal{A}$ to $\mathcal{B}$ in $G_M(S)$ follows from having a path from $\{A'_2, A'_3, \ldots, A'_k\}$ to $\{B'_2, B'_3, \ldots, B'_k\}$ in $G_M(S - A'_1)$, where $S - A'_1$ is the multiset difference. List the sets in $\mathcal{A}$ and $\mathcal{B}$ so that $|A_1 \triangle B_1| \leq |A_h \triangle B_j|$ for all $h, j \in [k]$. Set $|A_1 \triangle B_1| = 2i$. To prove the claim, it suffices to show that if $i > 0$, then

(*) there is a path from $\mathcal{B}$ to a vertex $\{B''_1, B''_2, \ldots, B''_k\}$ with $|A_1 \triangle B''_1| < 2i$.

Set $A_1 - B_1 = \{a_1, a_2, \ldots, a_i\}$ and $B_1 - A_1 = \{b_1, b_2, \ldots, b_i\}$. By symmetry, we may assume that the sum of the multiplicities of the elements in $A_1 - B_1$ in S is at least as large as the corresponding sum for $B_1 - A_1$. It follows that some basis in $\mathcal{B}$, say B_2, has more elements from $A_1 - B_1$ than from $B_1 - A_1$. We consider several options for B_2.

For the case $i \geq 3$, first assume $B_2 \cap (B_1 - A_1) = \emptyset$. We may assume $a_1 \in B_2$. Apply Lemma 2.2 with $x = a_1$ and $X = B_1 - A_1$ (so $|X| \geq 3$): for some $b_h \in B_1 - A_1$, both $(B_1 - b_h) \cup a_1$ and $(B_2 - a_1) \cup b_h$ are bases, so statement (*) follows.

Now, along with $i \geq 3$, assume $|B_2 \cap (A_1 - B_1)| \geq 3$. Let $X = B_2 \cap (A_1 - B_1)$. Since B_2 has more elements from $A_1 - B_1$ than from $B_1 - A_1$, some element in $B_1 - A_1$, say b_1, is not in B_2. Apply Lemma 2.2 to B_1 and B_2 with $x = b_1$ and X: for some $a_h \in X$, both $(B_1 - b_1) \cup a_h$ and $(B_2 - a_h) \cup b_1$ are bases. Statement (*) now follows.

We now address the case with $B_2 \cap (A_1 \triangle B_1) = \{a_1, a_2, b_3\}$, thereby completing the argument for $i \geq 3$. If we can symmetrically exchange one of a_1, a_2 in B_2 for one of b_1, b_2 in B_1 to get bases, then statement (*) holds. Assume that none of these four symmetric exchanges yields only bases. An argument like that in the third paragraph of the proof of Theorem 2.3 shows that we may assume that

$$(B_1 - b_1) \cup a_1, \quad (B_2 - a_2) \cup b_1, \quad (B_2 - a_1) \cup b_2, \quad \text{and} \quad (B_1 - b_2) \cup a_2$$

are circuit-hyperplanes. In order to have $|A_1 \triangle B_1| \leq |A_1 \triangle B_2|$ given that $B_2 \cap (A_1 \triangle B_1)$ is $\{a_1, a_2, b_3\}$, there must be an element, say y, in $B_2 - (A_1 \cup B_1)$. From Lemma 2.1 and the circuit-hyperplanes above, we have that $(B_1 - b_1) \cup y$ and $(B_2 - y) \cup b_1$ are bases, as are $(B_1 - \{b_1, b_2\}) \cup \{y, a_1\}$ and $(B_2 - \{y, a_1\}) \cup \{b_1, b_2\}$. Statement (*) now follows, which completes the argument for $i \geq 3$.

Now assume $i = 2$. By symmetry, there are two cases: $B_2 \cap \{b_1, b_2\}$ is either $\emptyset$ or $\{b_1\}$. First assume $B_2 \cap \{b_1, b_2\} = \emptyset$. We may assume $a_1 \in B_2$. If a_1 in B_2 can be symmetrically exchanged with either b_1 or b_2 in B_1 to yield two bases, then statement (*) holds, so assume this fails. By symmetry, $H_1 = (B_1 - b_1) \cup a_1$ and $H_2 = (B_2 - a_1) \cup b_2$ can be assumed to be circuit-hyperplanes. Since $|A_1 \triangle B_1| \leq |A_1 \triangle B_2|$, there are least two elements, say z_2 and z_3, in $B_2 - A_1$. By Lemma 2.1, either $(B_2 - z_2) \cup b_1$ or $(B_2 - z_3) \cup b_1$ is a basis; assume the former is. Comparison with H_1 shows that $(B_1 - b_1) \cup z_2$ and $(B_1 - \{b_1, b_2\}) \cup \{z_2, a_1\}$ are bases; similarly, $(B_2 - \{z_2, a_1\}) \cup \{b_1, b_2\}$ is a basis by comparison with H_2. Statement (*) now follows.

We now address the case with $B_2 \cap \{b_1, b_2\} = \{b_1\}$, thus completing the argument for $i = 2$. Note that B_2 must also contain a_1 and a_2. Statement (*) holds if b_2 in B_1 can be symmetrically exchanged with either a_1 or a_2 in B_2 to yield two bases. If neither exchange yields only bases, then, by symmetry, we may assume that $H_1 = (B_1 - b_2) \cup a_1$ and $H_2 = (B_2 - a_2) \cup b_2$ are circuit-hyperplanes. At least two elements in $A_1 \cap B_1$, say x_3 and x_4, are not in B_2 since $|A_1 \triangle B_1| \leq |A_1 \triangle B_2|$. At least one of $(B_2 - a_1) \cup x_3$ and $(B_2 - a_1) \cup x_4$ is a basis by Lemma 2.1; assume the first is. Now $(B_1 - x_3) \cup a_1$ is a basis by comparison with H_1. The sets $(B_1-\{x_3,b_2\})\cup\{a_1,a_2\}$ and $(B_2-\{a_1,a_2\})\cup\{x_3,b_2\}$ are also bases by comparison with H_1 and H_2, respectively. It follows that statement (*) holds. This completes the argument for $i=2$.

Finally, assume $i=1$, so $A_1-B_1=\{a_1\}$ and $B_1-A_1=\{b_1\}$. Thus, B_2 contains a_1 and not b_1. Let $X=B_2-a_1$. If $X\cup b_1$ is a basis (as it must be if k is 2), then exchanging a_1 and b_1 in B_2 and B_1 shows that statement (*) holds. Thus, assume $k\geq 3$ and

(A) $X\cup b_1$ is a circuit-hyperplane.

If $3\leq h\leq k$ and $b_1\notin B_h$, and if there is an element $y\in X-B_h$, then there is a $z\in B_h-B_2$ for which both $(B_h-z)\cup y$ and $(B_2-y)\cup z$ are bases; from Lemma 2.1 and statement (A), it follows that we can symmetrically exchange a_1 in $(B_2-y)\cup z$ with b_1 in B_1 to get two bases, which yields statement (*). Thus, we may assume

(B) each basis B_h contains either b_1 or all of X.

If $B_h\cap\{a_1,b_1\}=\{b_1\}$ for some h with $3\leq h\leq k$, then the assumption about the multiplicities of a_1 and b_1 implies that $B_{h'}\cap\{a_1,b_1\}=\{a_1\}$ for some h' with $3\leq h'\leq k$. Symmetrically exchange a_1 in $B_{h'}-B_h$ for some $z\in B_h-B_{h'}$ to get bases; since $B_{h'}-a_1$ is X by statement (B), statement (A) gives $z\neq b_1$. Thus, we may assume

(C) for $3\leq h\leq k$, if $b_1\in B_h$, then $a_1\in B_h$.

Assume $3\leq h\leq k$ and $a_1,b_1\in B_h$. If $|B_2\triangle B_h|\geq 4$, then for $x\in(B_h-b_1)-B_2$, we can symmetrically exchange $x\in B_h$ with some $y\in B_2$ (which cannot be a_1) to yield two bases; with statement (A), this allows us to exchange b_1 in B_1 with a_1 in $(B_2-y)\cup x$ to yield statement (*). Thus, we may assume

(D) if $a_1,b_1\in B_h$, then $|B_2\triangle B_h|=2$.

The proof is completed by showing that statements (A)–(D) yield a contradiction. Consider the multisets $\mathscr{A}=\{\{a_1\},A_2,A_3,\ldots,A_k\}$ and $\mathscr{B}=\{\{b_1\},B_2,B_3,\ldots,B_k\}$ of sets. Their multiset unions, $\bigcup_{A\in\mathscr{A}}A$ and $\bigcup_{B\in\mathscr{B}}B$, are equal. Let b_1 have multiplicity $t+1$ in these unions. Statements (B)–(D) imply that the sum of the multiplicities of the elements in X in the sets in $\mathscr{B}$ is $|X|(k-t-1)+(|X|-1)t$, that is, $|X|(k-1)-t$. By statement (A), $X\cup b_1$ is not in $\mathscr{A}$, so the sum of the multiplicities of the elements in X in the sets in $\mathscr{A}$ is at most $|X|(k-t-2)+(|X|-1)(t+1)$, that is, $|X|(k-1)-t-1$, which, as desired, contradicts the equality $\bigcup_{A\in\mathscr{A}}A=\bigcup_{B\in\mathscr{B}}B$. $\square$

We now prove a general connection between Conjectures 1.1, 1.2, and 1.3.

Theorem 2.5. *Let M be a matroid for which the basis pair graph of each of its minors is connected. For $k\geq 2$, let S be a multiset of size $k\,r(M)$ with elements in $E(M)$. If $G_M(S)$ is connected, then so is $G'_M(S)$.*

Proof. Since $G_M(S)$ is connected, to show that $G'_M(S)$ is connected it suffices to show that for each vertex $\mathbf{A}=(A_1,A_2,\ldots,A_k)$ of $G'_M(S)$ and each permutation σ of $[k]$, there is a path in $G'_M(S)$ from $\mathbf{A}$ to $\mathbf{A}_\sigma=(A_{\sigma(1)},A_{\sigma(2)},\ldots,A_{\sigma(k)})$. Since every permutation is a composition of transpositions, we focus on a transposition σ, say permuting i and j with $i<j$. The desired result follows if we show that there is a path from $\mathbf{A}$ to $\mathbf{A}_\sigma$ in which all bases but the i-th and j-th are fixed. This follow by noting that the sequence of symmetric exchanges that gives a path from $(A_i-A_j,A_j-A_i,\emptyset)$ to $(A_j-A_i,A_i-A_j,\emptyset)$ in the basis pair graph of the minor $M|(A_i\cup A_j)/(A_i\cap A_j)$ also gives the desired path from $\mathbf{A}$ to $\mathbf{A}_\sigma$ in $G'_M(S)$. $\square$

Corollary 2.6. *For any minor-closed class of matroids for which Conjecture 1.1 holds, Conjectures 1.2 and 1.3 are equivalent. In particular, Conjecture 1.3 holds for all sparse paving matroids.*

For Conjecture 1.4, we start with a lemma. A k*-interval* in a cycle σ is a set of k cyclically-consecutive elements, that is, $\{x, \sigma(x), \sigma^2(x), \dots, \sigma^{k-1}(x)\}$ for some x.

Lemma 2.7. *Let M be a rank-r sparse paving matroid on n elements. If $2r \leq n$, then, over all cycles on $E(M)$, the average number of r-intervals that are circuit-hyperplanes of M is less than two.*

Proof. Let $\mathrm{b}(M)$ and $\mathrm{ch}(M)$ be, respectively, the numbers of bases and circuit-hyperplanes of M. By focusing on circuit-hyperplanes, it follows that the average of interest is

$$\frac{\mathrm{ch}(M)\, r!\, (n-r)!}{(n-1)!}.$$

The desired result follows easily from this expression and the assumed inequality, $2r \leq n$, once we show

$$\mathrm{ch}(M) \leq \frac{1}{n-r+1}\binom{n}{r}. \tag{2.1}$$

Consider the pairs (H, B) consisting of a circuit-hyperplane H of M and a basis B of M with $|H \triangle B| = 2$. The definition of sparse paving gives three properties that yield the inequality above: each circuit-hyperplane is in $r(n-r)$ such pairs, each basis is in at most r such pairs, and $\mathrm{b}(M) + \mathrm{ch}(M) = \binom{n}{r}$. □

Theorem 2.8. *Conjecture 1.4 holds for sparse paving matroids.*

Proof. As noted after Conjecture 1.4, inequality (1.2) holds in every cyclically orderable matroid. The conjecture is easy to verify for all sparse paving matroids that have rank or nullity at most two (this includes all disconnected sparse paving matroids, i.e., $U_{0,n}$, $U_{n,n}$, $U_{n-1,n} \oplus U_{1,1}$, $U_{1,n} \oplus U_{0,1}$, and $U_{1,2} \oplus U_{1,2}$; this also includes all cases in which inequality (1.2) fails), so below we assume that M has rank and nullity at least three.

We may assume $E(M) = [n]$. For a cycle σ on $E(M)$, all $r(M)$-intervals in σ are bases of M if and only if their complements, all $r(M^*)$-intervals in σ, are bases of M^*, so, by replacing M by M^* if needed, we may assume that $2r \leq n$ where $r = r(M)$. By Lemma 2.7, for some cycle, say $\sigma_1 = (1, 2, \dots, n)$, on $E(M)$, at most one of its r-intervals is a circuit-hyperplane. We may assume there is such an interval, say

$$H_1 = \{4, 5, \dots, r+3\},$$

otherwise the desired conclusion holds.

Consider $\sigma_2 = (1, 2, \underline{4}, \underline{3}, 5, \dots, n)$. (To aid the reader, we underline the entries that differ from σ_1.) Only two of its r-intervals differ from their counterparts in σ_1, namely, $\{3, 5, 6, \dots, r+3\}$, which is a basis (use Lemma 2.1 with H_1), and

$$H_2 = \{n-r+4, \dots, n, 1, 2, 4\}.$$

If H_2 is a basis, then σ_2 is the cycle we want. Thus, assume that H_2 is a circuit-hyperplane.

We repeatedly apply this type of argument below. For brevity, for each cycle we list its r-intervals that differ from their counterparts in σ_1 and, when possible, the circuit-hyperplanes that, with Lemma 2.1, show that these intervals are bases. For brevity, we omit the r-interval $\{i, 5, 6, \dots, r+3\}$, with $i \neq 4$, which is a basis (compare it to H_1). Since the permutations σ_i below differ from σ_1 in at most four consecutive places, the assumption that the nullity of M is at least three implies that an r-interval in σ_i cannot differ from its counterpart in σ_1 at both ends.

Consider $\sigma_3 = (1, \underline{3}, \underline{4}, \underline{2}, 5, \dots, n)$. The relevant intervals are

- ⋄ $\{4, 2, 5, 6, \dots, r+2\}$ (compare to H_1),
- ⋄ $\{n-r+4, \dots, n, 1, 3, 4\}$ (compare to H_2), and

$$H_3 = \{n-r+3, \dots, n, 1, 3\}.$$

Thus, σ_3 has the desired properties unless H_3 is a circuit-hyperplane, so we assume it is.

Consider $\sigma_4 = (1, \underline{4}, 3, \underline{2}, 5, \dots, n)$. The relevant intervals are

- ⋄ $\{n-r+4, \dots, n, 1, 4, 3\}$ and $\{n-r+3, \dots, n, 1, 4\}$ (compare to H_3), and

$$H_4 = \{3, 2, 5, 6, \dots, r+2\}.$$

Thus, σ_4 has the desired properties unless H_4 is a circuit-hyperplane, so we assume it is.

Consider $\sigma_5 = (\underline{3}, \underline{4}, \underline{1}, \underline{2}, 5, \dots, n)$. The relevant intervals are

- ⋄ $\{1, 2, 5, 6, \dots, r+2\}$ (compare to H_4),
- ⋄ $\{n-r+4, \dots, n, 3, 4, 1\}$ (compare to H_2),
- ⋄ $\{n-r+3, \dots, n, 3, 4\}$ and $\{n-r+2, \dots, n, 3\}$ (compare to H_3), and

$$H_5 = \{4, 1, 2, 5, 6, \dots, r+1\}.$$

Thus, σ_5 has the desired properties unless H_5 is a circuit-hyperplane, so we assume it is.

Consider $\sigma_6 = (\underline{4}, \underline{3}, \underline{1}, \underline{2}, 5, \dots, n)$. The relevant intervals are

- ⋄ $\{1, 2, 5, 6, \dots, r+2\}$ (compare to H_4),
- ⋄ $\{3, 1, 2, 5, 6, \dots, r+1\}$ (compare to H_5),
- ⋄ $\{n-r+4, \dots, n, 4, 3, 1\}$ (compare to H_2),
- ⋄ $\{n-r+3, \dots, n, 4, 3\}$ (compare to H_3), and

$$H_6 = \{n-r+2, \dots, n, 4\}.$$

Thus, σ_6 has the desired properties unless H_6 is a circuit-hyperplane, so we assume it is.

Finally, consider $\sigma = (\underline{2}, \underline{3}, \underline{4}, \underline{1}, 5, \dots, n)$. The relevant intervals are

- ⋄ $\{4, 1, 5, 6, \dots, r+2\}$ (compare to H_1),
- ⋄ $\{3, 4, 1, 5, 6, \dots, r+1\}$ (compare to H_5),
- ⋄ $\{n-r+4, \dots, n, 2, 3, 4\}$ (compare to H_2),
- ⋄ $\{n-r+3, \dots, n, 2, 3\}$ (compare to H_3), and
- ⋄ $\{n-r+2, \dots, n, 2\}$ (compare to H_6).

Thus, σ has the desired properties, which completes the proof. □

We now turn to Conjecture 1.5.

Theorem 2.9. *Conjecture 1.5 holds for sparse paving matroids.*

Proof. Consider disjoint bases $B = \{b_1, b_2, \dots, b_r\}$ and $C = \{c_1, c_2, \dots, c_r\}$ of a sparse paving matroid M. By the basis-exchange property, we may assume that in the cycle

$$\sigma = (b_1, b_2, \dots, b_r, c_1, c_2, \dots, c_r),$$

every r-interval of the form $\{b_i, b_{i+1}, \dots, b_r, c_1, \dots, c_{i-1}\}$ is a basis; such cycles are said to *start properly*. We say that a *problem occurs at* c_i if $\{c_i, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-1}\}$ is not a basis; clearly, $i > 1$. We will show how, if a problem occurs at c_i, then we can switch a few elements so that the number of problems decreases and the cycle starts properly; iterating this procedure produces the desired cycle.

First assume $1 < i < r$. We will show that one of the following cycles starts properly and has fewer problems (we underline the few elements that are permuted):

$$\sigma_1 = (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_i}, \underline{c_{i-1}}, \dots, c_r),$$

$$\sigma_2 = (b_1, b_2, \dots, \underline{b_i}, \underline{b_{i-1}}, \dots, b_r, c_1, c_2, \dots, c_r),$$
$$\sigma_3 = (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_{i+1}}, \underline{c_{i-1}}, \underline{c_i}, \dots, c_r).$$

Since $S_0 = \{c_i, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-1}\}$ is a circuit-hyperplane, Lemma 2.1 implies that $\{c_{i-1}, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-1}\}$ is a basis. Only one other r-interval in σ_1 differs from its counterpart in σ, namely, $S_1 = \{b_i, \dots, b_r, c_1, \dots, c_{i-2}, c_i\}$, so it follows that σ_1 starts properly and has fewer problems than σ unless S_1 is a circuit-hyperplane. Assume S_1 is a circuit-hyperplane. Only two r-intervals in σ_2 differ from their counterparts in σ; of these, the set $\{c_i, c_{i+1}, \dots, c_r, b_1, \dots, b_{i-2}, b_i\}$ is a basis by Lemma 2.1 (compare it to S_0); if its complement, $S_2 = \{b_{i-1}, b_{i+1}, \dots, b_r, c_1, \dots, c_{i-1}\}$, is a basis, then σ_2 starts properly and has fewer problems than σ, so we may assume that S_2 is also a circuit-hyperplane. Four r-intervals in σ_3 differ from their counterparts in σ, namely,

$$T_1 = \{c_{i-1}, c_i, c_{i+2}, \dots, c_r, b_1, \dots, b_{i-1}\}, \quad T_2 = \{c_i, c_{i+2}, \dots, c_r, b_1, \dots, b_i\},$$

and their complements. Each of these sets is a basis by Lemma 2.1 since each symmetric difference $T_1 \triangle S_0$, $T_2 \triangle S_0$, $(E(M) - T_1) \triangle S_1$, and $(E(M) - T_2) \triangle S_2$ has two elements, so σ_3 starts properly and has fewer problems than σ.

Now assume $i = r$, so $S_0 = \{c_r, b_1, \dots, b_{r-1}\}$ is a circuit-hyperplane. Consider

$$\sigma_1 = (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_r}, \underline{c_{r-1}}),$$
$$\sigma_2 = (b_1, b_2, \dots, \underline{b_r}, \underline{b_{r-1}}, c_1, c_2, \dots, c_r),$$
$$\sigma_3 = (b_1, b_2, \dots, b_r, c_1, c_2, \dots, \underline{c_{r-1}}, \underline{c_r}, \underline{c_{r-2}}).$$

An argument similar to that above shows that σ_1 starts properly and has fewer problems than σ unless $S_1 = \{b_r, c_1, c_2, \dots, c_{r-2}, c_r\}$ is a circuit-hyperplane; likewise, σ_2 starts properly and has fewer problems than σ unless $S_2 = \{b_{r-1}, c_1, c_2, \dots, c_{r-1}\}$ is a circuit-hyperplane. Assume both S_1 and S_2 are circuit-hyperplanes. Only four r-intervals in σ_3 differ from their counterparts in σ, namely:

$$T_1 = \{c_r, c_{r-2}, b_1, \dots, b_{r-2}\}, \quad T_2 = \{c_{r-2}, b_1, \dots, b_{r-1}\},$$

and their complements. These sets are bases since $T_1 \triangle S_0$, $T_2 \triangle S_0$, $(E(M) - T_1) \triangle S_2$, and $(E(M) - T_2) \triangle S_1$ each have two elements, so σ_3 is the desired cycle on $B \cup C$. $\square$

3. Sparse Paving Matroids and the Number of Cyclic Flats

A set X in a matroid M is *cyclic* if $M|X$ has no coloops. Such sets are precisely the (possibly empty) unions of circuits of M. Let $\mathcal{Z}(M)$ be the set of cyclic flats of M. As noted in [7], the cyclic flats, along with their ranks, determine the matroid; indeed, this data can be seen as distilling the essential geometric information about a matroid (see [2, 3] for constructions that exploit this perspective). Cyclic flats play many roles in matroid theory, especially in the theory of transversal matroids (see, e.g., [4, 5, 7, 16]).

Let z_n be $\max\{|\mathcal{Z}(M)| : |E(M)| = n\}$, that is, z_n is the greatest number of cyclic flats that any matroid on n elements can have. In [6], the problem of finding z_n was raised. The importance of this problem stems from the fact that cyclic flats and their ranks generally provide a relatively compact description of a matroid.

To deduce a simple upper bound on z_n, let a_i be the number of i-element cyclic flats in a matroid M with $|E(M)| = n$. Note that for $F \in \mathcal{Z}(M)$ and $e \in F$, the closure $\mathrm{cl}(F - e)$ is F; also, for $x \in E(M) - F$, the restriction $M|(F \cup x)$ has a unique coloop, namely x. It follows that the sets F and $F - e$, with $F \in \mathcal{Z}(M)$ and $e \in F$, all differ, as do the sets

F and $F \cup x$ with $F \in \mathcal{Z}(M)$ and $x \in E(M) - F$. Thus,

$$\sum_{i=0}^{n} a_i(i+1) \leq 2^n \qquad \text{and} \qquad \sum_{i=0}^{n} a_i(n-i+1) \leq 2^n.$$

Adding these inequalities gives $(n+2)\,|\mathcal{Z}(M)| \leq 2^{n+1}$, so $z_n \leq 2^{n+1}/(n+2)$. We next review a construction that yields sparse paving matroids in which the number of cyclic flats is not so far from this bound.

As mentioned in Section 1, Knuth [20] constructed a family of at least $\left(2^{\binom{n}{\lfloor n/2 \rfloor}/2n}\right)/n!$ nonisomorphic sparse paving matroids of rank $\lfloor n/2 \rfloor$ on n elements. To do this, he showed that there is a sparse paving matroid of rank $\lfloor n/2 \rfloor$ on n elements with at least $\binom{n}{\lfloor n/2 \rfloor}/2n$ circuit-hyperplanes; the circuit-hyperplane relaxations of this matroid, taking into account potential isomorphisms, give the family.

While exploring an equivalent problem in the context of coding theory, Graham and Sloane [13] generalized and strengthened Knuth's result by showing that for each rank r with $r \leq n$, there is a sparse paving matroid of rank r on n elements with at least $\binom{n}{r}/n$ circuit-hyperplanes. Their construction, which we sketch, has the same general flavor as Knuth's. Partition the set of all 0-1 vectors $(a_0, a_1, \ldots, a_{n-1})$ of length n with r ones into n classes according to the remainder, modulo n, of the sum of the positions that contain ones, i.e., $\sum_i a_i\, i$. They noted that any two vectors in the same class differ in at least four places. At least one of the classes has at least $\binom{n}{r}/n$ vectors; by interpreting these vectors as the characteristic functions of the circuit-hyperplanes, this class defines a sparse paving matroid with at least $\binom{n}{r}/n$ circuit-hyperplanes.

The cyclic flats of a sparse paving matroid M having rank and nullity at least two are $\emptyset$, $E(M)$, and its circuit-hyperplanes. A routine induction (treating even n and odd n separately) shows $\binom{n}{\lfloor n/2 \rfloor} \geq 2^{n-1}/\sqrt{n}$ (consistent with Stirling's approximation). Thus, it follows from Graham and Sloane's work that some sparse paving matroid on n elements has at least $2^{n-1}/n^{3/2} + 2$ cyclic flats. (For large n, the numbers of cyclic flats in these examples far surpass those mentioned in [6].) We summarize these remarks in the result below, which, if we apply $\log_2$ to each term in the inequality, bears a strong resemblance to inequality (1.1).

Theorem 3.1. *The maximum number of cyclic flats among matroids on n elements, z_n, satisfies*

$$\frac{2^{n-1}}{n^{3/2}} + 2 \leq z_n \leq \frac{2^{n+1}}{n+2}.$$

To close, we note that Graham and Sloane's examples cannot be substantially improved upon within the class of sparse paving matroids. The sparse paving matroids that they construct have $\binom{n}{\lfloor n/2 \rfloor}/n$ circuit-hyperplanes. It is routine to check that the right side of inequality (2.1) above is maximized when $r = \lfloor n/2 \rfloor$. The ratio of this upper bound to the number of circuit-hyperplanes in Graham and Sloane's examples tends to 2 as n goes to infinity. (Also see [13, Remark 2].) This supports the natural suspicion that the lower bound in Theorem 3.1 is close to optimal.

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REFERENCES

[1] J. Blasiak, The toric ideal of a graphic matroid is generated by quadrics, *Combinatorica* **28** (2008) 283–297.
[2] J. Bonin, A note on the sticky matroid conjecture, *Ann. Comb.* (to appear).
[3] J. Bonin, A construction of infinite sets of intertwines for pairs of matroids, *SIAM J. Discrete Math.* (to appear).
[4] J. Bonin, Transversal lattices, *Electron. J. Combin.* **15** (2008) Research Paper 15, 11 pp.
[5] J. Bonin, J. P S Kung, and A. de Mier, Characterizations of transversal and fundamental transversal matroids (submitted).
[6] J. Bonin and A. de Mier, The lattice of cyclic flats of a matroid, *Ann. Comb.* **12** (2008) 155–170.
[7] T. H. Brylawski, An affine representation for transversal geometries, *Studies in Appl. Math.* **54** (1975) 143–160.
[8] R. Cordovil and M. L. Moreira, Bases-cobases graphs and polytopes of matroids, *Combinatorica* **13** (1993) 157–165.
[9] M. Farber, Basis pair graphs of transversal matroids are connected, *Disc. Math.* **73** (1989) 245–248.
[10] M. Farber, B. Richter and H. Shank, Edge-disjoint spanning trees: a connectedness theorem, *J. Graph Theory* **9** (1985) 319–324.
[11] H. Gabow, Decomposing symmetric exchanges in matroid bases, *Math. Programming* **10** (1976) 271–276.
[12] J. Geelen and P. Humphries, Rota's basis conjecture for paving matroids, *SIAM J. Discrete Math.* **20** (2006) 1042–1045.
[13] R. L. Graham and N. J. A. Sloane, Lower bounds for constant weight codes, *IEEE Trans. Inform. Theory* **26** (1980) 37–43.
[14] J. Herzog and T. Hibi, Discrete polymatroids, *J. Algebraic Combin.* **16** (2002) 239–268.
[15] J. van den Heuvel and S. Thomassé, Cyclic orderings and cyclic arboricity of matroids, arXiv:0912.2929.
[16] A. W. Ingleton, Transversal matroids and related structures, in: *Higher Combinatorics*, M. Aigner, ed. (Proc. NATO Advanced Study Inst., Berlin, 1976; Reidel, Dordrecht-Boston, MA, 1977) 117–131.
[17] M. Jerrum, Two remarks concerning balanced matroids, *Combinatorica* **26** (2006) 733–742.
[18] Y. Kajitani, S. Ueno and H. Miyano, Ordering of the elements of a matroid such that its consecutive w elements are independent, *Discrete Math.* **72** (1988) 187–194.
[19] K. Kashiwabara, The toric ideal of a matroid of rank 3 is generated by quadrics, *Electron. J. Combin.* **17** (2010) Research Paper 28, 12 pp.
[20] D. Knuth, The asymptotic number of geometries, *J. Combin. Theory Ser. A* **16** (1974) 398–400.
[21] D. Mayhew, M. Newman, D. Welsh, and G. Whittle, On the asymptotic proportion of connected matroids, *European J. Combin.* (to appear).
[22] D. Mayhew and G. Royle, Matroids with nine elements, *J. Combin. Theory Ser. B* **98** (2008) 415–431.
[23] C. Merino, S. Noble, M. Ramírez-Ibañez, and R. Villarroel, On the structure of the h-vector of a paving matroid, arXiv:1008.2031v2, 14 August 2010.
[24] A. de Mier, personal communications (2010).
[25] J. G. Oxley, *Matroid Theory*, (Oxford University Press, Oxford, 1992).
[26] M. Piff, An upper bound for the number of matroids, *J. Combin. Theory Ser. B* **14** (1973) 241–245.
[27] J. Schweig, Toric ideals of lattice path matroids and polymatroids (preprint).
[28] N. White, A unique exchange property for bases, *Linear Algebra Appl.* **31** (1980) 81–91.

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