Six-dimensional nilpotent Lie algebras

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Abstract

We give a full classification of 6-dimensional nilpotent Lie algebras over an arbitrary field, including fields that are not algebraically closed and fields of characteristic 2. To achieve the classification we use the action of the automorphism group on the second cohomology space, as isomorphism types of nilpotent Lie algebras correspond to orbits of subspaces under this action. In some cases, these orbits are determined using geometric invariants, such as the Gram determinant or the Arf invariant. As a byproduct, we completely determine, for a 4-dimensional vector space V, the orbits of $\mathrm{GL}(V)$ on the set of 2-dimensional subspaces of $V \wedge V$.

1   Introduction

The classification of small-dimensional Lie algebras is a classical problem. The history of the classification problem of 6-dimensional nilpotent Lie algebras goes back to Umlauf ([17]). In the 1950's Morozov ([10]) published a classification of 6-dimensional nilpotent Lie algebras valid over fields of characteristic 0. Recently several classifications have appeared, over various ground fields. We mention [8] (over algebraically closed fields, and over the real field), [14] (over various finite fields) [5] (over fields of characteristic not 2). However, no classification that treats all ground fields, in particular fields of characteristic 2, is known up to now. It is the purpose of this paper to complete the classification of 6-dimensional nilpotent Lie algebras over an arbitrary field.

Nilpotent Lie algebras up to dimension five are well-known. There is just one isomorphism type of nilpotent Lie algebras with dimension two, two isomorphism types in dimension 3, three isomorphism type in dimension 4, and 9 isomorphism types in dimension 5. The classification of nilpotent Lie algebras with dimension up to 5 is independent of the field in the sense that the isomorphism types can be described by uniform structure constant tables with integer entries. This is not, however, the case in dimension 6, as the number of isomorphism types of 6-dimensional nilpotent Lie algebras may depend on the characteristic of the underlying field. There are 36 isomorphism types over $\mathbb{F}_2$, but only 34 isomorphism types over $\mathbb{F}_3$; see [14]. Over fields $\mathbb{F}$ of characteristic not 2, the number of isomorphism types is $26 + 4s$ where s is the (possibly infinite) index of $(\mathbb{F}^*)^2$ in the multiplicative group $\mathbb{F}^*$ of $\mathbb{F}$ ([5]). In the present paper we give a classification that covers all ground fields, in particular also those of characteristic 2. For a field $\mathbb{F}$ of characteristic 2, define an equivalence relation $\overset{*+}{\sim}$ on $\mathbb{F}$ as follows: $\alpha \overset{*+}{\sim} \beta$ if and only if $\alpha = \gamma^2\beta + \delta^2$ with some $\gamma \in \mathbb{F}^*$ and $\delta \in \mathbb{F}$. Let t be the (possibly infinite) number of equivalence classes of $\overset{*+}{\sim}$ in $\mathbb{F}$.

Theorem 1.1. *The number of isomorphism types of nilpotent Lie algebras with dimension* 6 *over fields of characteristic different from* 2 *is* $26+4s$, *while this number is* $30+2s+4t$ *over fields of characteristic* 2.

In Section 3, a list of the isomorphism classes of the 6-dimensional nilpotent Lie algebras over an arbitrary field is given. The first part of this theorem concerning fields of characteristic different from 2 was already proved in the article [5] by the second author (see also [4] for more details).

As stressed above, our paper treats fields of characteristic 2, and, for the first time, gives a full classification of 6-dimensional nilpotent Lie algebras over an arbitrary field of characteristic 2. As far as we are aware, in characteristic 2, the only existing classification of such Lie algebras was given over algebraically closed fields by Gong's Ph.D. dissertation [8]. Comparing Gong's results to ours, we found that Gong's classification contains one mistake: namely, his Lie algebras $N_{6,2,10}$ and (E) are both isomorphic to our Lie algebra $L_{6,24}(0)$ defined in Section 3. Apart from this, our classification agrees with Gong's.

The original aim of the research presented in this paper was to extend the results of [5] to fields of characteristic 2. As in the course of this work some proofs of [5] were revised, we decided, in this paper, to present a full classification of 6-dimensional nilpotent Lie algebras that is valid over all fields. In addition, some results in [5] relied on computer calculations (specifically, computing a Gröbner bases for ideals in a polynomial rings), while the arguments of the present paper are all theoretical with no computer calculations involved. Nevertheless, we should mention here that several of the theoretical arguments in Section 5 were inspired by Gröbner basis computations in the computational algebra system Magma [1] and it would have been significantly more difficult, maybe even impossible, to obtain the classification in Theorem 1.1 without performing such computations.

Our methodology, which is the same as in [5], is explained in Section 2. We construct the 6-dimensional nilpotent Lie algebras as certain central extensions, descendants in our terminology, of lower-dimensional algebras. To separate the isomorphism classes of the descendants, we use the action of the automorphism group on the subspaces of the second cohomology space. It is interesting to note that, in some examples, the automorphism group preserves a non-degenerate quadratic form on the second cohomology space and the isomorphism classes of the descendants can be characterized by purely geometric means. An example of this situation is the abelian Lie algebra L with dimension 4, whose automorphism group $GL(4, \mathbb{F})$, by the Klein correspondence, preserves a quadratic form on the second cohomology space $H^2(L, \mathbb{F}) = (\mathbb{F}^4) \wedge (\mathbb{F}^4)$ with dimension 6. In characteristic different from 2, the 6-dimensional descendants of L can be completely determined using the Gram determinant of the restriction of the quadratic form to the 2-dimensional subspaces of $H^2(L, \mathbb{F})$. In characteristic 2, we determined these descendants using the Arf invariant of the restriction of the quadratic form to these 2-dimensional subspaces. See Sections 4 and 5 for the details. The use of the Klein correspondence in the classification of nilpotent Lie algebras of nilpotency class 2 was also explored in [16]. An interesting byproduct of our work is the determination of the $GL(4, \mathbb{F})$-orbits on the 2-dimensional subspaces of $(\mathbb{F}^4) \wedge (\mathbb{F}^4)$ (Theorem 4.4).

Here is an outline of the paper. In Section 2 we describe the cohomological method that we use to classify nilpotent Lie algebras, which also appeared in [5], [15], [8]. In Section 3 we present the main result of this paper, that is the classification of the 6-dimensional nilpotent Lie algebras. Section 4 has a number of results on bilinear and quadratic forms that we need. Then in Section 5 the main work is performed to prove the main result.

The main result of this paper can be accessed electronically using the LieAlgDB package [2] of the computational algebra system GAP [7].

2 A summary of the method

The main idea that we use here is to obtain the nilpotent Lie algebras of dimension n as central extensions of Lie algebras of smaller dimension. The central extensions are defined using the second cohomology

space, and the isomorphism classes of the central extensions correspond to the orbits of the automorphism group on the set of the subspaces of this cohomology space. This method has been described for Lie algebras by Skjelbred and Sund ([15]). Similar ideas appear in the recent work concerning the classification of p-groups; see, e.g., [6], [11], [12], [13]. We summarize the method in this section without giving proofs or explanations; the details can, for instance, be found in [5].

For a Lie algebra L, let L^i denote the terms of the lower central series. If L is nilpotent then $L^{i+1} = 0$ with some i and the smallest such i is called the *nilpotency class* of L. The second term L^2 of the lower central series will usually be written as L'. We denote the center of L by $C(L)$. Adapting terminology from [12] to our context, a Lie algebra K is said to be a *descendant* of the Lie algebra L if $K/C(K) \cong L$ and $C(K) \leqslant K'$. If $\dim C(K) = s$ then K is also referred to as a *step-s descendant*. A descendant of a nilpotent Lie algebra is nilpotent. Conversely, if K is a finite-dimensional nilpotent Lie algebra over a field $\mathbb{F}$, then K is either a descendant of a smaller-dimensional nilpotent Lie algebra, or $K = K_1 \oplus \mathbb{F}$ where K_1 is an ideal of K and $\mathbb{F}$ is viewed as a 1-dimensional Lie algebra. Hence determining the isomorphism types of the descendants of the nilpotent Lie algebras with dimension at most 5 suffices for the classification of the nilpotent Lie algebras with dimension 6.

The main idea of the method is that, for a nilpotent Lie algebra L over a field $\mathbb{F}$, the isomorphism types of the descendants of L are in 1-1 correspondence with the $\text{Aut}(L)$-orbits of some of the subspaces of the second cohomology space $H^2(L, \mathbb{F})$. The second cohomology spaces for nilpotent Lie algebras are defined as follows. For a vector space V, let $Z^2(L, V)$ denote the set of alternating bilinear maps $\vartheta : L \times L \to V$ with the property that

$$\vartheta([x_1, x_2], x_3) + \vartheta([x_3, x_1], x_2) + \vartheta([x_2, x_3], x_1) = 0 \text{ for all } x_1, \ x_2, \ x_3 \in L.$$

The set $Z^2(L, V)$ is viewed as a vector space over $\mathbb{F}$ and the elements of $Z^2(L, V)$ are said to be *cocycles*. We define, for a linear map $\nu \colon L \to V$, a map $\eta_\nu \colon L \times L \to V$ as $\eta_\nu(x, y) = \nu([x, y])$. The set $\{\eta_\nu \mid \nu : L \to V \text{ is linear}\}$ is denoted by $B^2(L, V)$. It is routine to check that $B^2(L, V)$ is a subspace of $Z^2(L, V)$, and the elements of $B^2(L, V)$ are called *coboundaries*. The *second cohomology space* $H^2(L, V)$ is defined as the quotient $Z^2(L, V)/B^2(L, V)$.

The vector spaces defined in the previous paragraph can be viewed as $\text{Aut}(L)$ modules. Indeed, for $\varphi \in \text{Aut}(L)$ and $\vartheta \in Z^2(L, V)$ define $\varphi\vartheta \in Z^2(L, V)$ by the equation $(\varphi\vartheta)(x, y) = \vartheta(\varphi(x), \varphi(y))$. The action $\vartheta \mapsto \varphi\vartheta$ makes $Z^2(L, V)$ an $\text{Aut}(L)$-module and it is easy to see that $B^2(L, V)$ is an $\text{Aut}(L)$-submodule. Hence the quotient $H^2(L, V)$ can also be viewed as an $\text{Aut}(L)$-module.

Let L be a Lie algebra and V a vector space over a field $\mathbb{F}$. For $\vartheta \in Z^2(L, V)$, define a Lie algebra L_ϑ as follows. The underlying space of L_ϑ is $L \oplus V$. The product of two elements $x + v, \ y + u \in L_\vartheta$, is defined as $[x + v, y + w] = [x, y]_L + \vartheta(x, y)$ where $[x, y]_L$ denotes the product in L. Then L_ϑ is a Lie algebra and V is an ideal of L_ϑ such that $V \leqslant C(L_\vartheta)$. In addition, $L \cong L_\vartheta/V$, and hence L_ϑ is a central extension of L. Further, if $\vartheta_1, \ \vartheta_2 \in Z^2(L, V)$ such that $\vartheta_1 - \vartheta_2 \in B^2(L, V)$ then $L_{\vartheta_1} \cong L_{\vartheta_2}$, and so the isomorphism type of L_ϑ only depends on the element $\vartheta + B^2(L, V)$ of $H^2(L, V)$. Conversely let K be a Lie algebra such that $C(K) \neq 0$, and set $V = C(K)$ and $L = K/C(K)$. Let $\pi \colon K \to L$ be the projection map. Choose an injective linear map $\sigma \colon L \to K$ such that $\pi(\sigma(x)) = x$ for all $x \in L$. Define $\vartheta \colon L \times L \to V$ by $\vartheta(x, y) = [\sigma(x), \sigma(y)] - \sigma([x, y])$. Then ϑ is a cocycle such that $K \cong L_\vartheta$. Though ϑ depends on the choice of σ, the coset $\vartheta + B^2(L, B)$ is independent of σ. Hence the central extension K of L determines a well-defined element of $H^2(L, V)$.

Let us now fix a basis $\{e_1, \ldots, e_s\}$ of V. A cocycle $\vartheta \in Z^2(L, V)$ can be written as

$$\vartheta(x, y) = \sum_{i=1}^{s} \vartheta_i(x, y) e_i,$$

where $\vartheta_i \in Z^2(L, \mathbb{F})$. Furthermore, ϑ is a coboundary if and only if all ϑ_i are. For $\vartheta \in Z^2(L, V)$, let $\vartheta^\perp$

denote the *radical* of ϑ; that is, the set of elements $x \in L$ such that $\vartheta(x, y) = 0$ for all $y \in L$. Then

$$\vartheta^\perp = \bigcap_{\eta \in \langle \vartheta_1, \ldots, \vartheta_s \rangle} \eta^\perp = \vartheta_1^\perp \cap \cdots \cap \vartheta_s^\perp.$$

Theorem 2.1 (Lemmas 2–4 in [5])**.** *Let L be a Lie algebra, let V be a vector space with fixed basis $\{e_1, \ldots, e_s\}$ over a field $\mathbb{F}$, and let ϑ, η be elements of $Z^2(L, V)$.*

(i) *The Lie algebra L_ϑ is a step-s descendant of L if and only if $\vartheta^\perp \cap C(L) = 0$ and the image of the subspace $\langle \vartheta_1, \ldots, \vartheta_s \rangle$ in $H^2(L, \mathbb{F})$ is s-dimensional .*

(ii) *Suppose that η is an other element of $Z^2(L, V)$ and that L_ϑ, L_η are descendants of L. Then $L_\vartheta \cong L_\eta$ if and only if images of the subspaces $\langle \vartheta_1, \ldots, \vartheta_s \rangle$ and $\langle \eta_1, \ldots, \eta_s \rangle$ in $H^2(L, \mathbb{F})$ are in the same orbit under the action of $\mathrm{Aut}(L)$.*

A subspace U of $H^2(L, \mathbb{F})$ is said to be allowable if $\bigcap_{\vartheta \in U} \vartheta^\perp \cap C(L) = 0$. By Theorem 2.1, there is a one-to-one correspondence between the set of isomorphism types of step-s descendants of L and the $\mathrm{Aut}(L)$-orbits on the s-dimensional allowable subspaces of $H^2(L, \mathbb{F})$. Hence the classification of 6-dimensional nilpotent Lie algebras requires that we determine these orbits for all nilpotent Lie algebras of dimension at most 5. The determination of these orbits is achieved in Section 5.

3   The 6-dimensional nilpotent Lie algebras

Let $\mathbb{F}$ be an arbitrary field. In order to classify the 6-dimensional nilpotent Lie algebras over $\mathbb{F}$, we determine the isomorphism classes of the 6-dimensional descendants of the nilpotent Lie algebras with dimension at most 5. In this section we summarize the result by listing these isomorphism classes for each of the Lie algebras with dimension at most 5, while in the Section 5 we provide with a detailed proof. The notation we use to describe nilpotent Lie algebras of dimension at most 5 is the same as in [5].

Unlike the 5-dimensional algebras, nilpotent Lie algebras of dimension 6 cannot be described uniformly over all fields. In some cases, the isomorphism types of descendants will depend on a parameter. In order to describe these cases, we need some notation. First, for a field $\mathbb{F}$, let $\mathbb{F}^*$ denote the multiplicative group of non-zero elements of $\mathbb{F}$. If $\mathbb{F}$ is a field, then let $\overset{*}{\sim}$ denote the equivalence relation on $\mathbb{F}$ defined as $\alpha \overset{*}{\sim} \beta$ if and only if $\alpha = \gamma^2 \beta$ with some $\gamma \in \mathbb{F}^*$. If $\mathrm{char}\,\mathbb{F} = 2$ then define the equivalence relation $\overset{*+}{\sim}$ as $\alpha \overset{*+}{\sim} \beta$ if and only if $\alpha = \gamma^2 \beta + \delta^2$ with some $\gamma \in \mathbb{F}^*$ and $\delta \in \mathbb{F}$. Using that $\mathrm{char}\,\mathbb{F} = 2$, it is easy to show that $\overset{*+}{\sim}$ is indeed an equivalence relation. For a set X and equivalence relation $\sim$, let $X/(\sim)$ denote a transversal of the equivalence relation $\sim$; that is, $X/(\sim)$ is a set that contains precisely one element from each of the equivalence classes of $\sim$. Now let $\mathbb{F}$ be a field of characteristic 2. View $\mathbb{F}$ as a vector space over $\mathbb{F}_2$, the map $\psi \colon \mathbb{F} \to \mathbb{F}$ defined by $\psi(x) = x^2 + x$ is $\mathbb{F}_2$-linear with kernel $\mathbb{F}_2$. Hence the image $\psi(\mathbb{F})$ is a subspace of codimension 1. Let ω denote a fixed element of $\mathbb{F} \setminus \psi(\mathbb{F})$. Then the set $\{0, \omega\}$ is a set of coset representatives for $\psi(\mathbb{F})$ in $\mathbb{F}$.

Following and extending the notation in [5], the nilpotent Lie algebras in this paper are denoted by $L_{d,k}$, $L_{d,k}(\varepsilon)$, $L_{d,k}^{(2)}$, or $L_{d,k}^{(2)}(\varepsilon)$ where d is the dimension of the algebra, k is its index among the nilpotent Lie algebras with dimension d, ε is a possible parameter, and the superscript "(2)" refers to the fact that the algebra is defined over a field of characteristic 2. The list of nilpotent Lie algebras with dimension at most 5, described in the same notation, can be found in [5]. In particular, there are 9 isomorphism types of nilpotent Lie algebras of dimension 5: $L_{5,1}, \ldots, L_{5,9}$. Hence there are 9 isomorphism types of nilpotent Lie algebras with dimension 6 that are not descendants of smaller-dimensional Lie algebras, namely $L_{6,1}, \ldots, L_{6,9}$, where $L_{6,i} = L_{5,i} \oplus \mathbb{F}$.

Next we describe the 6-dimensional descendants of nilpotent Lie algebras with dimension at most 5. The Lie algebras in this section are given with multiplication tables with respect to fixed bases with trivial products of the form $[x_i, x_j] = 0$ omitted. With respect to the list of [5] we have made a few small changes. The multiplication table of $L_{6,19}(\varepsilon)$, for $\varepsilon \neq 0$, is different from (but isomorphic to) the Lie algebra denoted with the same symbol in [5]. The Lie algebras $L_{6,19}(0)$ and $L_{6,21}(0)$ from [5] are denoted here by $L_{6,27}$ and $L_{6,28}$, respectively. We have made these changes because the structure of $L_{6,k}(\epsilon)$, $k = 19, 21$, is different for $\varepsilon = 0$ and $\varepsilon \neq 0$. Furthermore, in characteristic 2 a few new algebras appear that are not contained in [5]. These Lie algebras are $L_{6,k}^{(2)}$ or $L_{6,k}^{(2)}(\varepsilon)$ (here $k \in \{1, \ldots, 8\}$).

Step-1 descendants of 5-dimensional Lie algebras

(5/1) The abelian Lie algebra $L_{5,1}$ has no step-1 descendants.

(5/2) The Lie algebra $L_{5,2} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3 \rangle$ has only one isomorphism class of step-1 descendants namely

$$L_{6,10} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_4, x_5] = x_6 \rangle.$$

(5/3) The Lie algebra $L_{5,3} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$ has two isomorphism classes of step-1 descendants namely

$$\begin{aligned} L_{6,11} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_6, [x_2, x_3] = x_6, [x_2, x_5] = x_6 \rangle \text{ and} \\ L_{6,12} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_6, [x_2, x_5] = x_6 \rangle. \end{aligned}$$

(5/4) The Lie algebra $L_{5,4} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_5, [x_3, x_4] = x_5 \rangle$ has no step-1 descendants.

(5/5) Let $L_{5,5} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle$. If $\operatorname{char} \mathbb{F} \neq 2$, then $L_{5,5}$ has a unique isomorphism type of step-1 descendants, namely

$$L_{6,13} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_5, [x_3, x_4] = x_6 \rangle.$$

If $\operatorname{char} \mathbb{F} = 2$ then $L_{5,5}$ has two isomorphism classes of step-1 descendants, namely $L_{6,13}$ above and

$$L_{6,1}^{(2)} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_5 + x_6, [x_3, x_4] = x_6 \rangle.$$

(5/6) Set $L_{5,6} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle$. If $\operatorname{char} \mathbb{F} \neq 2$, then $L_{5,6}$ has two isomorphism classes of step-1 descendants, namely

$$\begin{aligned} L_{6,14} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, \\ & \qquad [x_2, x_3] = x_5, [x_2, x_5] = x_6, [x_3, x_4] = -x_6 \rangle; \\ L_{6,15} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, \\ & \qquad [x_1, x_5] = x_6, [x_2, x_3] = x_5, [x_2, x_4] = x_6 \rangle. \end{aligned}$$

If $\operatorname{char} \mathbb{F} = 2$ then $L_{6,15}$ and the Lie algebras

$$\begin{aligned} L_{6,2}^{(2)} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, \\ & \qquad [x_1, x_5] = x_6, [x_2, x_3] = x_5 + x_6, [x_2, x_4] = x_6 \rangle; \\ L_{6,3}^{(2)}(\varepsilon) &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, \\ & \qquad [x_2, x_3] = x_5 + \varepsilon x_6, [x_2, x_5] = x_6, [x_3, x_4] = x_6 \rangle \text{ with } \varepsilon \in \mathbb{F}/(\overset{*+}{\sim}), \end{aligned}$$

form a complete and irredundant list of representatives of the isomorphism classes of step-1 descendants of $L_{5,6}$.

(5/7) Set $L_{5,7} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 \rangle$. If char $\mathbb{F} \neq 2$, then the Lie algebra $L_{5,7}$ has three isomorphism classes of step-1 descendants namely

$$
\begin{aligned}
L_{6,16} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_5] = x_6, [x_3, x_4] = -x_6 \rangle; \\
L_{6,17} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, [x_2, x_3] = x_6 \rangle; \\
L_{6,18} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6 \rangle.
\end{aligned}
$$

If char $\mathbb{F} = 2$ then $L_{6,17}$, $L_{6,18}$ and the Lie algebras

$$
\begin{aligned}
L_{6,4}^{(2)}(\varepsilon) = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, \\
[x_2, x_3] = \varepsilon x_6, [x_2, x_5] = x_6, [x_3, x_4] = x_6 \rangle,
\end{aligned}
$$

where ε runs through the elements of $\mathbb{F}/(\overset{*+}{\sim})$, form a complete and irredundant set of representatives of the isomorphism types of step-1 descendants of $L_{5,7}$.

(5/8) Set $L_{5,8} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle$. If char $\mathbb{F} \neq 2$, then the Lie algebras

$$
\begin{aligned}
L_{6,20} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6 \rangle; \\
L_{6,19}(\varepsilon) &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6, [x_3, x_5] = \varepsilon x_6 \rangle, \\
&\quad \text{where } \varepsilon \in \mathbb{F}^*/(\overset{*}{\sim}),
\end{aligned}
$$

form a complete and irredundant set of representatives of the isomorphism classes of step-1 descendants of $L_{5,8}$. If char $\mathbb{F} = 2$ then such a set of representatives is formed by the Lie algebras $L_{6,19}(\varepsilon)$ with $\varepsilon \in \mathbb{F}^*/(\overset{*}{\sim})$, $L_{6,20}$ and the Lie algebra

$$
L_{6,5}^{(2)} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_5] = x_6, [x_3, x_4] = x_6 \rangle.
$$

(5/9) Set $L_{5,9} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 \rangle$. If char $\mathbb{F} \neq 2$ then the Lie algebras

$$
L_{6,21}(\varepsilon) = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_6, [x_2, x_3] = x_5, [x_2, x_5] = \varepsilon x_6 \rangle,
$$

where ε runs through the elements of $\mathbb{F}^*/(\overset{*}{\sim})$ form a complete and irredundant set of representatives of the isomorphism classes of step-1 descendants of $L_{5,9}$. If char $\mathbb{F} = 2$ then such a set of representatives is formed by the Lie algebras $L_{6,21}(\varepsilon)$ with $\varepsilon \in \mathbb{F}^*/(\overset{*}{\sim})$ and by the Lie algebra

$$
L_{6,6}^{(2)} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_5] = x_6, [x_2, x_3] = x_5, [x_2, x_4] = x_6 \rangle.
$$

Step-2 descendants of 4-dimensional Lie algebras

(4/1) Let $L_{4,1}$ be the abelian Lie algebra of dimension 4. If char $\mathbb{F} \neq 2$ the following is a complete and irredundant list of the representatives of the isomorphism classes of the step-2 descendants of $L_{4,1}$:

$$
L_{6,22}(\varepsilon) = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = \varepsilon x_6, [x_3, x_4] = x_5 \rangle
$$

where $\varepsilon \in \mathbb{F}/(\overset{*}{\sim})$. If char $\mathbb{F} = 2$, then such a list is formed by the Lie algebras $L_{6,22}(\nu)$ as above, where, in this case, $\nu \in \mathbb{F}/(\overset{*+}{\sim})$, and by the Lie algebras

$$
L_{6,7}^{(2)}(\eta) = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = \eta x_6, [x_3, x_4] = x_5 + x_6 \rangle
$$

where $\eta \in \{0, \omega\}$.

6

(4/2) Set $L_{4,2} = \langle x_1, \ldots, x_4 \mid [x_1, x_2] = x_3 \rangle$. If char $\mathbb{F} \neq 2$, then the following Lie algebras form a complete and irredundant set of representatives of the isomorphism classes of the step-2 descendants of $L_{4,2}$:

$$
\begin{aligned}
L_{6,27} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_6 \rangle; \\
L_{6,23} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_2, x_4] = x_5 \rangle; \\
L_{6,25} &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6 \rangle; \\
L_{6,24}(\varepsilon) &= \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = \varepsilon x_6, \\
& \quad [x_2, x_3] = x_6, [x_2, x_4] = x_5 \rangle \text{ where } \varepsilon \in \mathbb{F}/(\overset{*}{\sim}).
\end{aligned}
$$

If char $\mathbb{F} = 2$ then such a set of representatives is formed by the Lie algebras $L_{6,27}$, $L_{6,23}$, $L_{6,24}(\nu)$, where $\nu \in \mathbb{F}/(\overset{*+}{\sim})$, $L_{6,25}$ and by the Lie algebras

$$
\begin{aligned}
L_{6,8}^{(2)}(\eta) = \langle x_1, \ldots, x_6 \mid & [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = \eta x_6, \\
& [x_2, x_3] = x_6, [x_2, x_4] = x_5 + x_6 \rangle \text{ where } \eta \in \{0, \omega\}.
\end{aligned}
$$

(4/3) The Lie algebra $L_{4,3} = \langle x_1, \ldots, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$ has only one isomorphism class of step-2 descendants namely

$$
L_{6,28} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_6 \rangle.
$$

Step-3 descendants of 3-dimensional Lie algebras

(3/1) The abelian Lie algebra $L_{3,1}$ has a unique isomorphism type of step-3 descendants, namely

$$
L_{6,26} = \langle x_1, \ldots, x_6 \mid [x_1, x_2] = x_4, \ [x_1, x_3] = x_5, \ [x_2, x_3] = x_6 \rangle.
$$

(3/2) The Lie algebra $L_{3,2} = \langle x_1, x_2, x_3 | [x_1, x_2] = x_3 \rangle$ has no step-3 descendants.

By explicitly listing the isomorphism classes of 6-dimensional nilpotent Lie algebras, the following theorem gives a summary of the results stated in this section. Recall that in a field $\mathbb{F}$ of characteristic 2, ω denotes a fixed element from $\mathbb{F} \setminus \{x^2 + x \mid x \in \mathbb{F}\}$.

Theorem 3.1. *(I) Over a field $\mathbb{F}$ of characteristic different from 2, the list of the isomorphism types of 6-dimensional nilpotent Lie algebras is the following: $L_{5,k} \oplus \mathbb{F}$ with $k \in \{1, \ldots, 9\}$; $L_{6,k}$ with $k \in \{10, \ldots, 18, 20, 23, 25, \ldots, 28\}$; $L_{6,k}(\varepsilon_1)$ with $k = \{19, 21\}$ and $\varepsilon_1 \in \mathbb{F}^*/(\overset{*}{\sim})$; $L_{6,k}(\varepsilon_2)$ with $k \in \{22, 24\}$ and $\varepsilon_2 \in \mathbb{F}/(\overset{*}{\sim})$.*

(II) Over a field $\mathbb{F}$ of characteristic 2, the isomorphism types of 6-dimensional nilpotent Lie algebras are $L_{5,k} \oplus \mathbb{F}$ with $k \in \{1, \ldots, 9\}$; $L_{6,k}$ with $k \in \{10, \ldots, 13, 15, 17, 18, 20, 23, 25, \ldots, 28\}$; $L_{6,k}(\varepsilon_1)$ with $k = \{19, 21\}$ and $\varepsilon_1 \in \mathbb{F}^*/(\overset{*}{\sim})$; $L_{6,k}(\varepsilon_2)$ with $k = \{22, 24\}$ and $\varepsilon_2 \in \mathbb{F}/(\overset{*+}{\sim})$; $L_{6,k}^{(2)}$ with $k = \{1, 2, 5, 6\}$; $L_{6,k}^{(2)}(\varepsilon_3)$ with $k = \{3, 4\}$ and $\varepsilon_3 \in \mathbb{F}/(\overset{*+}{\sim})$; $L_{6,k}^{(2)}(\varepsilon_4)$ with $k \in \{7, 8\}$ and $\varepsilon_4 \in \{0, \omega\}$.

Theorem 3.1 follows from the statements concerning the descendants of this section. These statements are proved in Section 5. Noting that $\mathbb{F}/(\overset{*}{\sim}) = \mathbb{F}^*/(\overset{*}{\sim}) \cup \{0\}$, the main Theorem 1.1 is a consequence of Theorem 3.1. If $\mathbb{F}$ is an algebraically closed field or a perfect field of characteristic 2, then $s = 1$. If $\mathbb{F}$ is a finite field of size q then $s = \gcd(q - 1, 2)$. Further, if $\mathbb{F}$ is a perfect field of characteristic 2 then $t = 1$. This gives that the number of isomorphism types of 6-dimensional nilpotent Lie algebras is 34 over a finite field of characteristic different from 2; is 30 over an algebraically closed field of characteristic different from 2; and it is 36 over a perfect field (including finite fields) of characteristic 2.

4 Vector spaces with forms

As cocycles are alternating bilinear forms, and vector spaces with quadratic forms will play an important role in determining isomorphisms within certain descendants of $L_{4,1}$ and $L_{4,2}$, we summarize in this section some basic facts concerning quadratic and bilinear forms. Suppose that V is a vector space over a field $\mathbb{F}$ and f is a function from $V \times V$ to $\mathbb{F}$. The function f is said to be a *bilinear form* if f is linear in both of its variables. Further, f is said to be *symmetric* if $f(u,v) = f(v,u)$, f is said to be *alternating* if $f(v,v) = 0$, while f is said to be *skew-symmetric* if $f(u,v) = -f(v,u)$ for all v, $u \in V$. An alternating form is always skew-symmetric, while the converse of this statement is only valid if char $\mathbb{F} \neq 2$. In characteristic 2, there are skew-symmetric forms that are not alternating. For a map $Q : V \to \mathbb{F}$ define $f_Q : V \times V \to \mathbb{F}$ as

$$f_Q(u,v) = Q(u+v) - Q(u) - Q(v).$$

Then Q is said to be a *quadratic form* if $Q(\alpha v) = \alpha^2 Q(v)$ for all $\alpha \in \mathbb{F}$ and $v \in V$ and f_Q is a bilinear form. In this case the bilinear form f is called the *associated bilinear form* of Q. If f is a symmetric or skew-symmetric bilinear form on a vector space V and $U \subseteq V$ then let $U^\perp$ denote the *orthogonal complement* of U in V:

$$U^\perp = \{v \in V \mid f(u,v) = 0 \text{ for all } u \in U\}.$$

The *radical* of the form f is defined as $V^\perp$ and f is said to be *non-singular* if $V^\perp = 0$; otherwise f is said to be *singular*. If U is a subspace of V then an orthogonal form Q or a bilinear form f can be restricted to U and the restriction is a form with the same symmetrical properties as f. Such a space U is called *singular* or *non-singular*, if the restriction of the form to U is singular or non-singular, respectively.

If V is a vector space with basis $\{b_1, \ldots, b_n\}$ then, for i, $j \in \{1, \ldots, n\}$ with $i \neq j$, let $\Delta_{i,j}$ denote the alternating bilinear form defined as $\Delta_{i,j}(b_i, b_j) = -\Delta_{i,j}(b_j, b_i) = 1$ and $\Delta_{i,j}(b_k, b_l) = 0$ otherwise. Then the set of forms $\Delta_{i,j}$ with $i < j$ is a basis for the linear space of alternating bilinear forms on V. Suppose that V is a vector space with a bilinear form f. If $g \in \mathrm{GL}(V)$ then the form gf defined by $(gf)(u,v) = f(gu, gv)$ is also a bilinear form on V. Further, gf is alternating, skew-symmetric, or symmetric if and only if f is alternating, skew-symmetric, or symmetric, respectively. This defines a $\mathrm{GL}(V)$-action on the set of bilinear forms on V. The following lemma is well-known, see for example [9, Theorem 8.10.1].

Lemma 4.1. *Let V be a vector space with a fixed basis $\{b_1, \ldots, b_n\}$ and set $n_1 = n$ if n is even, $n_1 = n-1$ if n is odd.*

(i) *If Δ is a non-singular alternating bilinear form on V then n is even.*

(ii) *The group $\mathrm{GL}(V)$ has $\lfloor n/2 \rfloor$ orbits on the set of alternating bilinear forms on V with orbit representatives*

$$\Delta_{1,2}, \ \Delta_{1,2} + \Delta_{3,4}, \ldots, \Delta_{1,2} + \Delta_{3,4} + \cdots + \Delta_{n_1-1, n_1}.$$

Let f be a form of one of the types above on a vector space V over a field $\mathbb{F}$. If $\{b_1, \ldots, b_n\}$ is a fixed basis of V then the Gram matrix $\mathbb{G}$ is defined with respect to this basis as the matrix whose (i,j) entry is $f(b_i, b_j)$. The Gram determinant will be crucial for separating isomorphism types within parametric families of Lie algebras in characteristic different from 2. However, in characteristic 2, another invariant, namely the Arf invariant, will be needed. Let Q be a quadratic form on a vector space V over a field of characteristic 2 with associated bilinear form f. Note, in this case, that f is alternating. Assume that f is non-singular, which implies by Lemma 4.1(i), that $\dim V$ is even. Let $e_1, \ldots, e_k, f_1, \ldots, f_k$ be a symplectic basis of V; that is $f(e_i, e_j) = f(f_i, f_j) = 0$ and $f(e_i, f_j) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker-delta. Define the *Arf invariant* δ_Q of Q with respect to the given basis as

$$q = \sum_{i=1}^{k} Q(e_i)Q(f_i).$$

8

Of course, the Arf invariant of Q depends on the chosen symplectic basis of V. However, the following is valid.

Lemma 4.2. *Let V be a vector space over a field and let Q be a quadratic form with non-singular associated bilinear form.*

(i) *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ denote the Gram matrices of V with respect to two bases of V. Then $\det \mathbb{G}_1/\det \mathbb{G}_2$ is an element of the multiplicative subgroup $\{x^2 \mid x \in \mathbb{F}^*\}$ of $\mathbb{F}^*$.*

(ii) *Assume that $\operatorname{char} \mathbb{F} = 2$ and suppose that q_1 and q_2 are the values of the Arf invariant of Q with respect to two symplectic bases. Then $q_1 + q_2$ is an element of the additive subgroup $\{x^2 + x \mid x \in \mathbb{F}\}$ of $\mathbb{F}$.*

Proof. Statement (i) is well-known, see for instance equation (8.1.8) in [3]. Statement (ii) is proved in [3, Theorem 8.11.12]. □

Suppose that Q is a quadratic form on a vector space V with associated bilinear form f_Q. A vector $v \in V$ is said to be *singular* if $Q(v) = 0$, and it is *isotropic* if $f(v, v) = 0$. A subspace U of V is *totally singular* if $Q(u) = 0$ for all $u \in U$, while U is said to be totally isotropic if $f(u, u) = 0$ for all $u \in U$. If $\operatorname{char} \mathbb{F} \neq 2$, then the notions of singular and isotropic, and those of totally singular and totally isotropic can be freely interchanged. This, however, is no longer true if $\operatorname{char} \mathbb{F} = 2$. Let V be a vector space with a quadratic form Q and let G be a group acting on V. Then we say that G *preserves Q modulo scalars* if for each $g \in G$ there is some $\alpha_g \in \mathbb{F}$ such that

$$Q(gv) = \alpha_g Q(v).$$

Lemma 4.3. *Suppose that V is a vector space over a field $\mathbb{F}$ with a quadratic form Q whose associated bilinear form is f, and let G be a subgroup of $\operatorname{GL}(V)$ preserving Q modulo scalars. Suppose that S_1 and S_2 are 2-dimensional subspaces of V in the same G-orbit, and let $\{b_1, b_2\}$ and $\{c_1, c_2\}$ be bases of S_1 and S_2, respectively.*

(i) *Suppose that S_1 and S_2 are non-singular subspaces, and let $\mathbb{G}_1$ and $\mathbb{G}_2$ be the Gram matrices of S_1 and S_2 with respect to the given bases. Then $(\det \mathbb{G}_1)/(\det \mathbb{G}_2) \in \{\alpha^2 \mid \alpha \in \mathbb{F}^*\}$.*

(ii) *Suppose that $\operatorname{char} \mathbb{F} = 2$, that f is non-singular, that the given bases of S_i are symplectic, and that q_1 and q_2 are the Arf invariants of S_1 and S_2 with respect to the given bases. Then $q_1 + q_2 \in \{\alpha + \alpha^2 \mid \alpha \in \mathbb{F}\}$.*

(iii) *Suppose that $\operatorname{char} \mathbb{F} = 2$ and that f is identically zero on S_i, for $i = 1, 2$, and set $q_1 = Q(b_1)Q(b_2)$ and $q_2 = Q(c_1)Q(c_2)$. Then there exist $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$ such that $q_2 = \alpha^2 q_1 + \beta^2$.*

Proof. Suppose that $g \in G$ such that $gS_1 = S_2$. Since G preserves the form Q modulo scalars, there is some $\alpha \in \mathbb{F}$ such that $Q(gv) = \alpha Q(v)$, and $f(gu, gv) = \alpha f(u, v)$ for all $u, v \in V$. Let us prove statement (i) first. The elements gb_1 and gb_2 form a basis for S_2 and the Gram matrix of S_2 with respect to this basis is $\alpha \mathbb{G}_1$ with determinant $\alpha^2 \det \mathbb{G}_1$. By a remark above, there is some $\beta \in \mathbb{F}^*$ such that $\alpha^2 \det \mathbb{G}_1 = \beta^2 \det \mathbb{G}_2$, and hence $(\det \mathbb{G}_1)/(\det \mathbb{G}_2) = (\beta/\alpha)^2$ as claimed.

Let us now prove the second assertion. Since b_1, b_2 is a symplectic basis of S_1, we obtain that $f(gb_1, gb_1) = f(gb_2, gb_2) = 0$ and $f(gb_1, gb_2) = f(gb_2, gb_1) = \alpha$. Hence the basis $(1/\alpha)gb_1, gb_2$ is a symplectic basis of S_2 and the Arf invariant with respect to this basis is

$$Q\left(\frac{1}{\alpha}gb_1\right) Q(gb_2) = \frac{1}{\alpha^2} \alpha^2 Q(b_1) Q(b_2) = q_1.$$

9

Thus Lemma 4.3 gives that $q_1 + q_2 \in \{\alpha + \alpha^2 \mid \alpha \in \mathbb{F}\}$.

(iii) First we note that if f is identically zero, then $Q(\gamma b_1 + \delta b_2) = \gamma^2 Q(b_1) + \delta^2 Q(b_2)$. Since $gS_1 = S_2$, there are $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2 \in \mathbb{F}$ such that $c_1 = g(\alpha_1 b_1 + \alpha_2 b_2)$ and $c_2 = g(\beta_1 b_1 + \beta_2 b_2)$. As c_1 and c_2 are linearly independent, we obtain that $\alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$. Then

$$
\begin{aligned}
q_2 &= Q(c_1)Q(c_2) \\
&= Q(g(\alpha_1 b_1 + \alpha_2 b_2))Q(g(\beta_1 b_1 + \beta_2 b_2)) \\
&= \alpha^2(\alpha_1^2 Q(b_1) + \alpha_2^2 Q(b_2))(\beta_1^2 Q(b_1) + \beta_2^2 Q(b_2)) \\
&= \alpha^2(\alpha_1^2\beta_1^2 Q(b_1)^2 + (\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2)Q(b_1)Q(b_2) + \alpha_2^2\beta_2^2 Q(b_2)^2) \\
&= (\alpha\alpha_1\beta_1 Q(b_1) + \alpha\alpha_2\beta_2 Q(b_2))^2 + (\alpha\alpha_1\beta_2 + \alpha\alpha_2\beta_1)^2 q_2.
\end{aligned}
$$

Since $\alpha \neq 0$ and $\alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$, the assertion follows. $\qquad\square$

We close this section with an interesting byproduct of our work. In the course of determining the isomorphism classes of the descendants of the 4-dimensional abelian Lie algebra, we determined, for an arbitrary 4-dimensional vector space V, the $\mathrm{GL}(V)$-orbits on the 2-dimensional subspaces of $V \wedge V$. As this result may have applications elsewhere, we state it separately.

Theorem 4.4. *Let V be a vector space of dimension 4 over a field $\mathbb{F}$ with basis $\{v_1, v_2, v_3, v_4\}$. If $\mathrm{char}\,\mathbb{F} \neq 2$ then set*

$$
\mathcal{P} = \{\langle v_1 \wedge v_2, v_1 \wedge v_3 \rangle\} \cup \{\langle v_1 \wedge v_2 + v_3 \wedge v_4, v_1 \wedge v_3 + \varepsilon v_2 \wedge v_4 \rangle \mid \varepsilon \in \mathbb{F}/(\overset{*}{\sim})\}.
$$

If $\mathrm{char}\,\mathbb{F} = 2$ then let $\omega \in \mathbb{F}$ be a fixed element such that $\omega \notin \{x^2 + x \mid x \in \mathbb{F}\}$ and set

$$
\begin{aligned}
\mathcal{P} &= \{\langle v_1 \wedge v_2, v_1 \wedge v_3 \rangle, \langle v_1 \wedge v_2 + v_3 \wedge v_4, v_1 \wedge v_3 + v_3 \wedge v_4 \rangle, \\
&\qquad \langle v_1 \wedge v_2 + v_3 \wedge v_4, v_1 \wedge v_3 + \omega v_2 \wedge v_4 + v_3 \wedge v_4 \rangle\} \\
&\cup \{\langle v_1 \wedge v_2 + v_3 \wedge v_4, v_1 \wedge v_3 + \varepsilon v_2 \wedge v_4 \rangle \mid \varepsilon \in \mathbb{F}/(\overset{*+}{\sim})\}.
\end{aligned}
$$

Then $\mathcal{P}$ is a complete and irredundant set of representatives of the $\mathrm{GL}(V)$-orbits on the two-dimensional subspaces of $V \wedge V$.

The proof of Theorem 4.4 will be given in Section 5.

5   The calculation of 6-dimensional descendants

In this section we prove the results stated in Section 3. In order to make the calculations more compact, we introduce uniform notation. In each of the sections, L will denote a fixed nilpotent Lie algebra with dimension d, where $d < 6$, for which the descendants will be computed. The algebra L will be given by a multiplication table as in Section 3. Recall that the forms $\Delta_{i,j}$ with $i \leqslant j$ form a basis for the space of alternating bilinear forms on L, and the cohomology spaces $Z^2(L, \mathbb{F})$, $B^2(L, \mathbb{F})$, and $H^2(L, \mathbb{F})$ will be determined in terms of this basis. These spaces will be described for each of the Lie algebras without proof, as it is an easy exercise to compute them in these cases. If Δ is an element of $Z^2(L, \mathbb{F})$, then $\overline{\Delta}$ denotes its image in $H^2(L, \mathbb{F})$.

The automorphism group of L will be described as a group of $(d \times d)$-matrices with respect to the given basis of L. The automorphism groups will also be presented without proof as it is in general easy to verify that the given matrices do indeed form the full automorphism group. We use column notation to describe the automorphisms; that is, the i-th column of the given matrix will contain the image of the i-th basis vector of L. By Theorem 2.1, we will need to compute the $\mathrm{Aut}(L)$-orbits on the $(6-d)$-dimensional

allowable subspaces of $H^2(L, \mathbb{F})$. The set of these subspaces will be denoted by $\mathcal{S}$. The quotient $H^2(L, \mathbb{F})$ will be given with a fixed basis $\Gamma_1, \ldots, \Gamma_k$. The element $\alpha_1 \Gamma_1 + \cdots + \alpha_k \Gamma_k$ of $H^2(L, F)$ will be written simply as $(\alpha_1, \ldots, \alpha_k)$. The action of $\mathrm{Aut}(L)$ on $H^2(L, \mathbb{F})$ will be computed explicitly.

Now we can start determining the 6-dimensional descendants of the nilpotent Lie algebras with dimension at most 5.

$\mathbf{L}_{5,1}$, $\mathbf{L}_{5,4}$, $\mathbf{L}_{4,3}$, $\mathbf{L}_{3,1}$, $\mathbf{L}_{3,2}$

First we compute the descendants in the easy cases. As there are no non-singular alternating bilinear forms on an odd-dimensional space (Lemma 4.1(i)), the Lie algebra $L_{5,1}$ has no step-1 descendants. If $L = L_{5,4}$ then $Z^2(L, F) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{3,4} \rangle$, $B^2(L, F) = \langle \Delta_{1,2} + \Delta_{3,4} \rangle$, and $H^2(L, \mathbb{F}) = \langle \overline{\Delta_{1,3}}, \overline{\Delta_{1,4}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,4}}, \overline{\Delta_{3,4}} \rangle$. Note that each element of $H^2(L, \mathbb{F})$ has x_5 in its radical. Hence $\mathcal{S} = \emptyset$, and so there are no step-1 descendants of L.

If $L = L_{4,3}$ then $H^2(L, \mathbb{F})$ is 2-dimensional, spanned by $\overline{\Delta_{1,4}}$, and $\overline{\Delta_{2,3}}$. Since $H^2(L, \mathbb{F})$ is allowable, $\mathcal{S} = H^2(L, \mathbb{F})$ which gives that L has a single isomorphism type of step-2 descendants, as claimed in Section 3. Similarly, if $L = L_{3,1}$, we obtain that $H^2(L, \mathbb{F})$ is 3-dimensional spanned by $\overline{\Delta_{1,2}}$, $\overline{\Delta_{1,3}}$, and $\overline{\Delta_{2,3}}$. Since $H^2(L, \mathbb{F})$ is allowable, we have that $\mathcal{S} = H^2(L, \mathbb{F})$, which shows that L has only one isomorphism class of step-3 descendants. Finally, if $L = L_{3,2}$, then H^2 is 2-dimensional spanned by $\overline{\Delta_{1,3}}$ and $\overline{\Delta_{2,3}}$ which shows, in this case, that $\mathcal{S} = \emptyset$.

$\mathbf{L}_{5,2}$

Set $L = L_{5,2}$. It is routine to check that $\mathrm{Aut}(L)$ is the group of invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & u & a_{34} & a_{35} \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{pmatrix} \tag{1}$$

with $u = a_{11}a_{22} - a_{12}a_{21}$. As $Z^2(L, \mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{2,5}, \Delta_{4,5} \rangle$ and $B^2(L, \mathbb{F}) = \langle \Delta_{1,2} \rangle$, we obtain that $H^2(L, \mathbb{F}) = \langle \overline{\Delta_{1,3}}, \overline{\Delta_{1,4}}, \overline{\Delta_{1,5}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,4}}, \overline{\Delta_{2,5}}, \overline{\Delta_{4,5}} \rangle$ and

$$\mathcal{S} = \{ \langle (a, b, c, d, e, f, g) \rangle \mid g \neq 0 \text{ and } (a, d) \neq (0, 0) \}.$$

As the matrix A is invertible, $u \neq 0$. For $\vartheta = (a, b, c, d, e, f, g) \in H^2(L, \mathbb{F})$, we compute that $A\vartheta = (\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g})$ where

$$
\begin{aligned}
\bar{a} &= u(a_{11}a + a_{21}d); \\
\bar{b} &= a_{11}a_{34}a + a_{11}a_{44}b + a_{11}a_{54}c + a_{21}a_{34}d + a_{21}a_{44}e + a_{21}a_{54}f + (a_{41}a_{54} - a_{44}a_{51})g; \\
\bar{c} &= a_{11}a_{35}a + a_{11}a_{45}b + a_{11}a_{55}c + a_{21}a_{35}d + a_{21}a_{45}e + a_{21}a_{55}f + (a_{41}a_{55} - a_{45}a_{51})g; \\
\bar{d} &= u(a_{12}a + a_{22}d); \\
\bar{e} &= a_{12}a_{34}a + a_{12}a_{44}b + a_{12}a_{54}c + a_{22}a_{34}d + a_{22}a_{44}e + a_{22}a_{54}f + (a_{42}a_{54} - a_{44}a_{52})g; \\
\bar{f} &= a_{12}a_{35}a + a_{12}a_{45}b + a_{12}a_{55}c + a_{22}a_{35}d + a_{22}a_{45}e + a_{22}a_{55}f + (a_{42}a_{55} - a_{45}a_{52})g; \\
\bar{g} &= (a_{44}a_{55} - a_{45}a_{54})g.
\end{aligned}
$$

Choose $S = \langle(a,b,c,d,e,f,g)\rangle \in \mathcal{S}$ and let B be the first of the following matrices if $a \neq 0$, or the second if $a = 0$:

$$
\begin{pmatrix}
1 & -dg & 0 & 0 & 0 \\
0 & ag & 0 & 0 & 0 \\
0 & 0 & ag & -b & -c \\
0 & -(af - cd) & 0 & a & 0 \\
0 & ae - bd & 0 & 0 & a
\end{pmatrix},
\qquad
\begin{pmatrix}
0 & -dg & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & dg & -e & -f \\
0 & cd & 0 & d & 0 \\
0 & -bd & 0 & 0 & d
\end{pmatrix}.
$$

Then $BS = \langle(1,0,0,0,0,0,1)\rangle$ and hence the group of matrices of the form (1) has only one orbit on $\mathcal{S}$. Thus L has only one step-1 descendant, namely $L_{6,10}$.

$\mathbf{L}_{5,3}$

Set $L = L_{5,3}$. The group $\mathrm{Aut}(L)$ consists of the invertible matrices of the form

$$
A = \begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{11}a_{22} & 0 & 0 \\
a_{41} & a_{42} & a_{11}a_{32} & a_{11}^2 a_{22} & a_{45} \\
a_{51} & a_{52} & 0 & 0 & a_{55}
\end{pmatrix}.
\tag{2}
$$

We have that $Z^2(L,\mathbb{F}) = \langle\Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{2,3}, \Delta_{2,5}\rangle$, $B^2(L,\mathbb{F}) = \langle\Delta_{1,2}, \Delta_{1,3}\rangle$, and so $H^2(L,\mathbb{F}) = \langle\overline{\Delta_{1,4}}, \overline{\Delta_{1,5}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,5}}\rangle$. Further,

$$
\mathcal{S} = \{\langle(a,b,c,d)\rangle \mid a, d \neq 0\}.
$$

If $\vartheta = (a,b,c,d) \in H^2(L,\mathbb{F})$, then $A\vartheta = (\bar{a}, \bar{b}, \bar{c}, \bar{d})$ where

$$
\bar{a} = a_{11}^3 a_{22} a, \quad \bar{b} = a_{11} a_{45} a + a_{11} a_{55} b + a_{21} a_{55} d, \quad \bar{c} = a_{11} a_{22}^2 c, \quad \bar{d} = a_{22} a_{55} d.
$$

Choose $S = \langle(a,b,c,d)\rangle \in \mathcal{S}$. Set $c_1 = 1$ if $c = 0$ and set $c_1 = c/(ad^2)$ otherwise. Let B denote the matrix

$$
\begin{pmatrix}
c_1 & 0 & 0 & 0 & 0 \\
0 & c_1 & 0 & 0 & 0 \\
0 & 0 & c_1^2 & 0 & 0 \\
0 & 0 & 0 & c_1^3 & 0 \\
0 & 0 & 0 & 0 & c_1^3
\end{pmatrix}
\begin{pmatrix}
d & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & d & 0 & 0 \\
0 & 0 & 0 & d^2 & -bd^2 \\
0 & 0 & 0 & 0 & ad^2
\end{pmatrix}.
$$

Easy computation shows that $BS = \langle(1,0,0,1)\rangle$ if $c = 0$ while $BS = \langle(1,0,1,1)\rangle$ otherwise. Hence the group of matrices of the form (2) has two orbits on $\mathcal{S}$, namely $\langle(1,0,0,1)\rangle$ and $\langle(1,0,1,1)\rangle$. The corresponding Lie algebras are $L_{6,12}$ and $L_{6,11}$.

The derived subalgebra $\langle x_3, x_4, x_6\rangle$ of both $L_{6,11}$ and $L_{6,12}$ is 3-dimensional. However, in $L_{6,11}$ the centralizer $\langle x_3, x_4, x_5, x_6\rangle$ of the derived subalgebra is 4-dimensional, while in $L_{6,12}$ the centralizer $\langle x_2, x_3, x_4, x_5, x_6\rangle$ of $L'_{6,12}$ is 5-dimensional. Hence $L_{6,11}$ and $L_{6,12}$ are not isomorphic.

$\mathbf{L}_{5,5}$

Let $L = L_{5,5}$. Invertible matrices of the form

$$
A = \begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{11}a_{22} & -a_{11}a_{21} & 0 \\
a_{41} & a_{42} & 0 & a_{11}^2 & 0 \\
a_{51} & a_{52} & u & a_{54} & a_{11}^2 a_{22,}
\end{pmatrix}
\tag{3}
$$

where $u = a_{11}a_{32} + a_{21}a_{42} - a_{22}a_{41}$, form the group $\mathrm{Aut}(L)$. We have, in addition, that $Z^2(L, \mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5} + \Delta_{3,4}, \Delta_{2,3}, \Delta_{2,4} \rangle$, $B^2(L, \mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3} + \Delta_{2,4} \rangle$, and so $H^2(L, \mathbb{F}) = \langle \overline{\Delta_{1,4}}, \overline{\Delta_{1,5} + \Delta_{3,4}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,4}} \rangle$. The set of allowable subspaces of $H^2(L, \mathbb{F})$ is

$$\mathcal{S} = \{ \langle (a, b, c, d) \rangle \mid b \neq 0 \}.$$

If $\vartheta = (a, b, c, d) \in H^2(L, \mathbb{F})$, then $A\vartheta = (\bar{a}, \bar{b}, \bar{c}, \bar{d})$ where

$$\bar{a} = a_{11}^3 a + (a_{11}a_{54} + a_{11}^2 a_{31} + a_{11}a_{21}a_{41})b - a_{11}a_{21}^2 c + a_{11}^2 a_{21} d;$$

$$\bar{b} = a_{11}^3 a_{22} b, \quad \bar{c} = -a_{11}a_{22}a_{42} b + a_{11}a_{22}^2 c, \quad \bar{d} = 2a_{11}a_{22}a_{41} b - 2a_{11}a_{21}a_{22} c + a_{11}^2 a_{22} d.$$

Choose $S = \langle (a, b, c, d) \rangle \in \mathcal{S}$. If $\mathrm{char}\,\mathbb{F} \neq 2$ then

$$\begin{pmatrix} 2b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ -2a & 0 & 2b^2 & 0 & 0 \\ -d & c & 0 & 4b^2 & 0 \\ 0 & 0 & bd & 0 & 4b^3 \end{pmatrix} S = \langle (0, 1, 0, 0) \rangle,$$

which shows that the group of matrices of the form (3) is transitive on $\mathcal{S}$, and that L has only one isomorphism type of step-1 descendants, namely $L_{6,13}$, as claimed.

Suppose now that $\mathrm{char}\,\mathbb{F} = 2$. Set $b_1 = d/b^2$ if $b \neq 0$ and set $b_1 = 1$ otherwise. Let B denote the matrix

$$\begin{pmatrix} b_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 \\ 0 & 0 & 0 & b_1^2 & 0 \\ 0 & 0 & 0 & 0 & b_1^2 \end{pmatrix} \begin{pmatrix} b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ a & 0 & b^2 & 0 & 0 \\ 0 & c & 0 & b^2 & 0 \\ 0 & 0 & 0 & 0 & b^3 \end{pmatrix}.$$

Then $BS = \langle (0, 1, 0, 1) \rangle$ if $d \neq 0$ and $BS = \langle (0, 1, 0, 0) \rangle$ otherwise. Hence, in this case, the group of matrices of the form (3) has two orbits on $\mathcal{S}$, namely $\langle (0, 1, 0, 0) \rangle$ and $\langle (0, 1, 0, 1) \rangle$. The corresponding Lie algebras are $L_{6,13}$, and $L_{6,1}^{(2)}$, respectively.

We claim, for $\mathrm{char}\,\mathbb{F} = 2$ that $L_{6,1}^{(2)}$ and $L_{6,13}$ are not isomorphic. It suffices to show that the allowable subspaces $\langle (0, 1, 0, 0) \rangle$ and $\langle (0, 1, 0, 1) \rangle$ are in different orbits under $\mathrm{Aut}(L)$. Suppose by contradiction that A is of the form (3) such that $A\langle (0, 1, 0, 1) \rangle = \langle (0, 1, 0, 0) \rangle$. Then, as $\mathrm{char}\,\mathbb{F} = 2$, the expression for $\bar{d}$ gives that $0 = a_{11}^2 a_{22}$, which is impossible as A is invertible. Hence the two descendants are non-isomorphic.

$\mathbf{L}_{5,6}$

Set $L = L_{5,6}$. The group $\mathrm{Aut}(L)$ consists of the invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{11}^2 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}^3 & 0 & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^4 & 0 \\ a_{51} & a_{52} & u & v & a_{11}^5 \end{pmatrix},$$

where $u = -a_{11}^2 a_{31} + a_{11}a_{42} + a_{21}a_{32}$ and $v = a_{11}^3 a_{21} + a_{11}^2 a_{32}$. As $Z^2(L, \mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5} + \Delta_{2,4}, \Delta_{2,3}, \Delta_{2,5} - \Delta_{3,4} \rangle$ and $B^2(L, \mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4} + \Delta_{2,3} \rangle$, we obtain that $H^2(L, \mathbb{F}) = \langle \overline{\Delta_{1,5} + \Delta_{2,4}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,5} - \Delta_{3,4}} \rangle$. Moreover,

$$\mathcal{S} = \{ \langle (a, b, c) \rangle \mid (a, c) \neq (0, 0) \}.$$

13

If $\vartheta = (a, b, c) \in Z^2(L, \mathbb{F})$, then $A\vartheta = (\bar{a}, \bar{b}, \bar{c})$ where

$$\bar{a} = a_{11}^5(a_{11}a + a_{21}c), \quad \bar{b} = -2a_{11}^4 a_{21}a + a_{11}^5 b + (2a_{11}^3 a_{42} - a_{11}^3 a_{21}^2 - a_{11}a_{32}^2)c, \quad \bar{c} = a_{11}^7 c.$$

Choose $S = \langle(a, b, c)\rangle \in \mathcal{S}$. Suppose first that $\operatorname{char}\mathbb{F} \neq 2$. Let B denote the first of the following matrix if $c \neq 0$, or the second if $c = 0$:

$$\begin{pmatrix} 2c & 0 & 0 & 0 & 0 \\ -2a & 4c^2 & 0 & 0 & 0 \\ 0 & 0 & 8c^3 & 0 & 0 \\ 0 & -2(a^2 + bc) & 0 & 16c^4 & 0 \\ 0 & 0 & -4c(a^2 + bc) & -16ac^3 & 32c^5 \end{pmatrix}, \quad \begin{pmatrix} 2a & 0 & 0 & 0 & 0 \\ b & 4a^2 & 0 & 0 & 0 \\ 0 & 0 & 8a^3 & 0 & 0 \\ 0 & 0 & 0 & 16a^4 & 0 \\ 0 & 0 & 0 & 8a^3b & 32a^5 \end{pmatrix}.$$

Then, if $c \neq 0$, then $BS = \langle(0, 0, 1)\rangle$, while $BS = \langle(1, 0, 0)\rangle$ otherwise. This shows that $\operatorname{Aut}(L)$ has at most two orbits on $\mathcal{S}$, namely $\langle(0, 0, 1)\rangle$ and $\langle(1, 0, 0)\rangle$ and the corresponding Lie algebras are $L_{6,14}$ and $L_{6,15}$, respectively. The Lie algebras $L_{6,14}$ and $L_{6,15}$ are clearly non-isomorphic, as $(L_{6,15})'$ is abelian, while $(L_{6,14})'$ is not.

Assume now that $\operatorname{char}\mathbb{F} = 2$. If $b = c = 0$ then $S = \langle(1, 0, 0)\rangle$, and so we may suppose that $(b, c) \neq (0, 0)$. Let B be the first of the following matrices if $c = 0$ and $b \neq 0$; while if $c \neq 0$ then let B be the second matrix:

$$\begin{pmatrix} b & 0 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 & 0 \\ 0 & 0 & b^3 & 0 & 0 \\ 0 & 0 & 0 & b^4 & 0 \\ 0 & 0 & 0 & 0 & b^5 \end{pmatrix}, \quad \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ a & c^2 & 0 & 0 & 0 \\ 0 & ac & c^3 & 0 & 0 \\ 0 & 0 & ac^2 & c^4 & 0 \\ 0 & 0 & a^2c & 0 & c^5 \end{pmatrix}.$$

Now, if $c = 0$ and $b \neq 0$ then $BS = \langle(1, 1, 0)\rangle$, while $BS = \langle(0, b/c^3, 1)\rangle$ otherwise. Thus the set of the subspaces $\langle(1, 0, 0)\rangle$, $\langle(1, 1, 0)\rangle$ and $\langle(0, \varepsilon, 1)\rangle$ with $\varepsilon \in \mathbb{F}$ contains a representative of each of the $\operatorname{Aut}(L)$-orbits on $\mathcal{S}$. The Lie algebras corresponding to these subspaces are $L_{6,15}$, $L_{6,2}^{(2)}$ and $L_{6,3}^{(2)}(\varepsilon)$.

Let us now determine the possible isomorphisms among the algebras $L_{6,15}$, $L_{6,2}^{(2)}$ and $L_{6,3}^{(2)}(\varepsilon)$. First note that the derived subalgebras of $L_{6,15}$ and $L_{6,2}^{(2)}$ are abelian, while that of $L_{6,3}^{(2)}(\varepsilon)$ is not. Thus $L_{6,3}^{(2)}(\varepsilon) \not\cong L_{6,15}$ and $L_{6,3}^{(2)}(\varepsilon) \not\cong L_{6,2}^{(2)}$. If $L_{6,2}^{(2)} \cong L_{6,15}$, then there is some $A \in \operatorname{Aut}(L)$ such that $A\langle(1, 1, 0)\rangle = \langle(1, 0, 0)\rangle$. However, the expression for $\bar{b}$ implies in this case that $a_{11} = 0$, which makes A non-invertible. Thus $L_{6,2}^{(2)} \not\cong L_{6,15}$.

Therefore it remains to determine the isomorphisms among the algebras $L_{6,3}^{(2)}(\varepsilon)$ with different values of ε. We claim that $L_{6,3}^{(2)}(\varepsilon) \cong L_{6,3}^{(2)}(\nu)$ if and only of $\varepsilon \overset{*+}{\sim} \nu$ where $\overset{*+}{\sim}$ is the equivalence relation defined at the beginning of Section 3. To prove one direction of this claim, assume that $L_{6,3}^{(2)}(\varepsilon) \cong L_{6,3}^{(2)}(\nu)$. Then there is $A \in \operatorname{Aut}(L)$ such that $A\langle(0, \varepsilon, 1)\rangle = \langle(0, \nu, 1)\rangle$. The equations for $\bar{b}$ and $\bar{c}$ give that $a_{11}^5\varepsilon + a_{11}^3 a_{21}^2 + a_{11}a_{32}^2 = a_{11}^7\nu$. Since $a_{11} \neq 0$, we may divide both sides of this equation by a_{11}^5 and obtain that $\varepsilon \overset{*+}{\sim} \nu$, as required. Now suppose, conversely, that $\varepsilon \overset{*+}{\sim} \nu$; that is, there are $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$ such that $\nu = \alpha^2\varepsilon + \beta^2$. Then

$$\begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 & 0 \\ 0 & \alpha^{-2} & 0 & 0 & 0 \\ 0 & \beta\alpha^{-3} & \alpha^{-3} & 0 & 0 \\ 0 & 0 & \beta\alpha^{-4} & \alpha^{-4} & 0 \\ 0 & 0 & 0 & \beta\alpha^{-5} & \alpha^{-5} \end{pmatrix} \langle(0, \varepsilon, 1)\rangle = \langle(0, \nu, 1)\rangle.$$

This proves the claim about the isomorphisms among the algebras $L_{6,3}^{(2)}(\nu)$.

L$_{5,7}$

The group $\mathrm{Aut}(L)$ consists of the invertible matrices of the form

$$
A = \begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{11}a_{22} & 0 & 0 \\
a_{41} & a_{42} & a_{11}a_{32} & a_{11}^2 a_{22} & 0 \\
a_{51} & a_{52} & a_{11}a_{42} & a_{11}^2 a_{32} & a_{11}^3 a_{22}
\end{pmatrix}.
$$

We have $Z^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{2,3}, \Delta_{2,5} - \Delta_{3,4} \rangle$ and $B^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4} \rangle$, and so $H^2(L,\mathbb{F}) = \langle \overline{\Delta}_{1,5}, \overline{\Delta}_{2,3}, \overline{\Delta}_{2,5} - \overline{\Delta}_{3,4} \rangle$. Moreover,

$$
\mathcal{S} = \{ \langle (a,b,c) \rangle \mid (a,c) \neq (0,0) \}.
$$

If $\vartheta = (a,b,c) \in H^2(L,\mathbb{F})$, then $A\vartheta = (\bar{a}, \bar{b}, \bar{c})$ where

$$
\bar{a} = a_{11}^3 a_{22}(a_{11}a + a_{21}c), \quad \bar{b} = a_{11}(a_{22}^2 b + (2a_{22}a_{42} - a_{32}^2)c), \quad \bar{c} = a_{11}^3 a_{22}^2 c.
$$

Choose $S = \langle (a,b,c) \rangle \in \mathcal{S}$. If $b = c = 0$ then $S = \langle (1,0,0) \rangle$. Let B be the first, the second, or the third of the following matrices, in the cases when $c = 0$ and $b \neq 0$; $c \neq 0$ and $\mathrm{char}\,\mathbb{F} \neq 2$; or $c \neq 0$ and $\mathrm{char}\,\mathbb{F} = 2$; respectively:

$$
\begin{pmatrix}
b & 0 & 0 & 0 & 0 \\
0 & b^2 & 0 & 0 & 0 \\
0 & 0 & b^3 & 0 & 0 \\
0 & 0 & 0 & b^4 & 0 \\
0 & 0 & 0 & 0 & b^5
\end{pmatrix},
\quad
\begin{pmatrix}
c & 0 & 0 & 0 & 0 \\
-a & 2c & 0 & 0 & 0 \\
0 & 0 & 2c^2 & 0 & 0 \\
0 & -b & 0 & 2c^3 & 0 \\
0 & 0 & -bc & 0 & 2c^4
\end{pmatrix},
\quad
\begin{pmatrix}
c & 0 & 0 & 0 & 0 \\
a & 1 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & c^2 & 0 \\
0 & 0 & 0 & 0 & c^3
\end{pmatrix}.
$$

Then we obtain that $BS = \langle (1,1,0) \rangle$ if $c = 0$ and $b \neq 0$; $BS = \langle (0,0,1) \rangle$ if $c \neq 0$ and $\mathrm{char}\,\mathbb{F} \neq 2$; while $BS = \langle (0,\varepsilon,1) \rangle$ if $c \neq 0$ and $\mathrm{char}\,\mathbb{F} = 2$. Hence if $\mathrm{char}\,\mathbb{F} \neq 2$ then $\mathrm{Aut}(L)$ has at most three orbits on $\mathcal{S}$, namely $\langle (1,0,0) \rangle$, $\langle (1,1,0) \rangle$, $\langle (0,0,1) \rangle$ and the corresponding Lie algebras are $L_{6,18}$, $L_{6,17}$, and $L_{6,16}$, respectively. If $\mathrm{char}\,\mathbb{F} = 2$ then the 1-spaces $\langle (1,0,0) \rangle$, $\langle (1,1,0) \rangle$, $\langle (0,\varepsilon,1) \rangle$ contain a set of representatives for the $\mathrm{Aut}(L)$-orbits on $\mathcal{S}$ with corresponding Lie algebras $L_{6,18}$, $L_{6,17}$ and $L_{6,4}^{(2)}(\varepsilon)$, respectively.

Note that the derived subalgebras $L_{6,17}$ and $L_{6,18}$ are abelian, while those of $L_{6,16}$ and $L_{6,4}^{(2)}(\varepsilon)$ are not. Hence $L_{6,16} \not\cong L_{6,17}$, $L_{6,16} \not\cong L_{6,18}$, $L_{6,4}^{(2)}(\varepsilon) \not\cong L_{6,17}$ and $L_{6,4}^{(2)}(\varepsilon) \not\cong L_{6,18}$. Further, the centralizer $\langle x_3, x_4, x_5, x_6 \rangle$ of $(L_{6,17})'$ is 4-dimensional while the centralizer $\langle x_2, x_3, x_4, x_5, x_6 \rangle$ of $(L_{6,18})'$ is 5-dimensional, and hence $L_{6,17} \not\cong L_{6,18}$. Thus we are left with having to determine the possible isomorphisms between the algebras $L_{6,4}^{(2)}(\varepsilon)$ with different values of ε. We claim that $L_{6,4}^{(2)}(\varepsilon) \cong L_{6,4}^{(2)}(\nu)$ if and only if $\varepsilon \overset{*+}{\sim} \nu$. To show this claim, suppose first that $L_{6,4}^{(2)}(\varepsilon) \cong L_{6,4}^{(2)}(\nu)$. Then there is $A \in \mathrm{Aut}(L)$ such that $A\langle (0,\varepsilon,1) \rangle = \langle (0,\nu,1) \rangle$. The equations for $\bar{b}$ and $\bar{c}$ and the fact that $a_{11} \neq 0$ imply that $a_{22}^2\varepsilon + a_{32}^2 = a_{11}^2 a_{22}^2 \nu$, which, in turn, gives that $\varepsilon \overset{*+}{\sim} \nu$. Conversely, let us assume that $\varepsilon \overset{*+}{\sim} \nu$. Then there is some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$ such that $\nu = \alpha^2\varepsilon + \beta^2$. Then

$$
\begin{pmatrix}
\alpha^{-1} & 0 & 0 & 0 & 0 \\
0 & \alpha^{-1}\beta^{-1} & 0 & 0 & 0 \\
0 & \alpha^{-2} & \alpha^{-2}\beta^{-1} & 0 & 0 \\
0 & 0 & \alpha^{-3} & \alpha^{-3}\beta^{-1} & 0 \\
0 & 0 & 0 & \alpha^{-4} & \alpha^{-4}\beta^{-1}
\end{pmatrix}
\langle (0,\varepsilon,1) \rangle = \langle (0,\nu,1) \rangle.
$$

This completes the proof of the claim concerning the isomorphism among the $L_{6,4}^{(2)}(\varepsilon)$.

L$_{5,8}$

The group $\mathrm{Aut}(L)$ consists of the invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}a_{22} & a_{11}a_{23} \\ a_{51} & a_{52} & a_{53} & a_{11}a_{32} & a_{11}a_{33} \end{pmatrix}.$$

We have $Z^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{2,5} + \Delta_{3,4}, \Delta_{3,5} \rangle$ and $B^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3} \rangle$, and so $H^2(L,\mathbb{F}) = \langle \overline{\Delta}_{1,4}, \overline{\Delta}_{1,5}, \overline{\Delta}_{2,3}, \overline{\Delta}_{2,4}, \overline{\Delta}_{2,5} + \overline{\Delta}_{3,4}, \overline{\Delta}_{3,5} \rangle$. Further,

$$\mathcal{S} = \left\{ \langle (a,b,c,d,e,f) \rangle \mid \mathrm{rank} \begin{pmatrix} a & b \\ d & e \\ e & f \end{pmatrix} = 2 \right\}.$$

If $\vartheta = (a,b,c,d,e,f) \in H$, then $A\vartheta = (\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f})$ where

$$\begin{aligned}
\bar{a} &= a_{11}^2 a_{22} a + a_{11}^2 a_{32} b + a_{11}a_{21}a_{22}d + a_{11}(a_{21}a_{32} + a_{22}a_{31})e + a_{11}a_{31}a_{32}f; \\
\bar{b} &= a_{11}^2 a_{23} a + a_{11}^2 a_{33} b + a_{11}a_{21}a_{23}d + a_{11}(a_{21}a_{33} + a_{23}a_{31})e + a_{11}a_{31}a_{33}f; \\
\bar{c} &= (a_{22}a_{33} - a_{23}a_{32})c + (a_{22}a_{43} - a_{23}a_{42})d + (a_{22}a_{53} - a_{23}a_{52} + a_{32}a_{43} - a_{33}a_{42})e; \\
&+ (a_{32}a_{53} - a_{33}a_{52})f; \\
\bar{d} &= a_{11}a_{22}^2 d + 2a_{11}a_{22}a_{32}e + a_{11}a_{32}^2 f; \\
\bar{e} &= a_{11}a_{22}a_{23}d + a_{11}(a_{22}a_{33} + a_{23}a_{32})e + a_{11}a_{32}a_{33}f; \\
\bar{f} &= a_{11}a_{23}^2 d + 2a_{11}a_{23}a_{33}e + a_{11}a_{33}^2 f.
\end{aligned}$$

Choose $S = \langle (a,b,c,d,e,f) \rangle \in \mathcal{S}$ and set $\delta_1 = ae - bd$, $\delta_2 = af - be$, $\delta_3 = df - e^2$. Suppose first that $d \neq 0$. If $\delta_1 \neq 0$, then let B be the first of the following two automorphisms; if $\delta_1 = 0$ (which implies that $\delta_3 \neq 0$), then let B be the second:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 \end{pmatrix} \begin{pmatrix} d & 0 & 0 & 0 & 0 \\ -a & 1 & e & 0 & 0 \\ 0 & 0 & -d & 0 & 0 \\ 0 & 0 & c & d & de \\ 0 & 0 & 0 & 0 & -d^2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \delta_3 & 0 & 0 & 0 \\ 1 & 0 & \delta_3 & 0 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 \\ 0 & 0 & 0 & 0 & \delta_3 \end{pmatrix} \begin{pmatrix} d & 0 & 0 & 0 & 0 \\ -a & 1 & e & 0 & 0 \\ 0 & 0 & -d & 0 & 0 \\ 0 & 0 & c & d & de \\ 0 & 0 & 0 & 0 & -d^2 \end{pmatrix}.$$

Then $BS = \langle (0,1,0,1,0,\delta_3) \rangle$. Suppose now that $d = 0$ and consider the following two cases: $e = 0$ which implies that a, $f \neq 0$; and $e \neq 0$. In these cases, let B denote the first or the second of the following transformations, respectively:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -b & af & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & af \\ 0 & 0 & ac & a & 0 \end{pmatrix}, \quad \begin{pmatrix} e^2 & 0 & 0 & 0 & 0 \\ af - be & 1 & 0 & 0 & 0 \\ -ae & 0 & e & 0 & 0 \\ 0 & c & 0 & e^2 & 0 \\ 0 & 0 & 0 & 0 & e^3 \end{pmatrix}.$$

Then, in the first case, $BS = \langle (0,1,0,1,0,0) \rangle$, while in the second case, $BS = \langle (0,0,0,0,1,f) \rangle$. Thus if $d = e = 0$ then we obtain that S is in the orbit of $\langle (0,1,0,1,0,0) \rangle$. Therefore we may assume that $d = 0$

and $e \neq 0$, and that S is in the orbit of $\langle (0,0,0,0,1,f) \rangle$. If $f \neq 0$, then let B_1 denote the first of the following transformations; if $f = 0$ and char $\mathbb{F} \neq 2$ then let B_1 be the second:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & f & 0 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & f & 0 \\
0 & 0 & 0 & -1 & 1
\end{pmatrix}, \qquad
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.
$$

We obtain, in both cases, that $B_1 \langle (0,0,0,0,1,f) \rangle = \langle (0,1,0,1,0,-1) \rangle$.

To summarize, in characteristic different from 2, the set of subspaces $\langle (0,1,0,1,0,\varepsilon) \rangle$ contain a representative from each of the $\mathrm{Aut}(L)$-orbits on $\mathcal{S}$. In characteristic 2, these orbits are covered by the subspaces $\langle (0,1,0,1,0,\varepsilon) \rangle$ and $\langle (0,0,0,0,1,0) \rangle$. The Lie algebras corresponding to these subspaces are $L_{6,19}(\varepsilon)$ and $L_{6,5}^{(2)}$. The Lie algebra $L_{6,19}(0)$ is written as $L_{6,20}$ in Section 3, to minimize the difference between our list and that in [5]. The expression for $\bar{d}$ shows, in characteristic 2, that the vectors in the $\mathrm{Aut}(L)$-orbit of $(0,0,0,0,1,0)$ all have 4-th coordinate 0, and so $\langle (0,1,0,1,0,\varepsilon) \rangle$ and $\langle (0,0,0,0,1,0) \rangle$ are indeed in different orbits. Thus we only need to verify the isomorphisms among the algebras $L_{6,19}(\varepsilon)$ with different values of ε. We claim that $L_{6,19}(\varepsilon) \cong L_{6,19}(\nu)$ if and only if $\varepsilon \overset{*}{\sim} \nu$. To prove one direction of this claim assume that $L_{6,19}(\varepsilon) \cong L_{6,19}(\nu)$. Then there is some $A \in \mathrm{Aut}(L)$ such that $\langle (0,1,0,1,0,\varepsilon) \rangle = \langle (0,1,0,1,0,\nu) \rangle$. Considering the equations for $\bar{a}$, $\bar{b}$ and $\bar{f}$ we obtain that

$$a_{22}^2 + \varepsilon a_{32}^2 \neq 0 \tag{4}$$

and that

$$
\begin{aligned}
a_{22}a_{23} + \varepsilon a_{32}a_{33} &= 0; \tag{5} \\
a_{23}^2 + \varepsilon a_{33}^2 - \nu a_{22}^2 - \varepsilon \nu a_{32}^2 &= 0. \tag{6}
\end{aligned}
$$

If $\varepsilon = 0$ then (4) implies that $a_{22} \neq 0$, and equations (5)–(6) give that $a_{23} = 0$ and that $\nu = 0$. Thus $L_{6,19}(0)$ is not isomorphic to $L_{6,19}(\nu)$ with $\nu \neq 0$. Therefore we may assume without loss of generality that ε, $\nu \neq 0$. Set $\delta = a_{22}^2 + \varepsilon a_{32}^2$. Then routine computation shows that

$$(a_{23}a_{22} - \varepsilon a_{23}a_{22})(a_{22}a_{23} + \varepsilon a_{32}a_{33}) + (\varepsilon a_{32}^2 - \delta)(a_{23}^2 + \varepsilon a_{33}^2 - \nu a_{22}^2 - \varepsilon \nu a_{32}^2) = \delta(\nu a_{22}^2 - \varepsilon a_{33}^2).$$

Since $\delta \neq 0$, equations (5) and (6) imply that $\nu a_{22}^2 - \varepsilon a_{33}^2$. Thus either $a_{22} = a_{33} = 0$ or $\varepsilon \overset{*}{\sim} \nu$, as required. Suppose that $a_{22} = a_{33} = 0$. Then $a_{23}^2 = \varepsilon \nu a_{32}^2$. Since the matrix A is invertible, we obtain that $a_{23} \neq 0$ and $a_{32} \neq 0$ and hence $1/\varepsilon \overset{*}{\sim} \nu$. Since $1/\varepsilon \overset{*}{\sim} \varepsilon$, this gives that $\varepsilon \overset{*}{\sim} \nu$.

Now we assume the converse; that is, let ε, $\nu \in \mathbb{F}$ such that $\varepsilon \overset{*}{\sim} \nu$. Let A be the automorphism of $L_{5,8}$ represented by the diagonal matrix with the entries $(1,1,\alpha,1,\alpha)$ in the diagonal. Then $A \langle (0,1,0,1,0,\varepsilon) \rangle = \langle (0,1,0,1,0,\alpha^2\varepsilon) \rangle = \langle (0,1,0,1,0,\nu) \rangle$. Hence $L_{6,19}(\varepsilon) \cong L_{6,19}(\nu)$.

$\mathbf{L}_{5,9}$

The group $\mathrm{Aut}(L)$ consists of the invertible matrices of the form

$$
A = \begin{pmatrix}
a_{11} & a_{12} & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 \\
a_{31} & a_{32} & u & 0 & 0 \\
a_{41} & a_{42} & a_{11}a_{32} - a_{12}a_{31} & a_{11}u & a_{12}u \\
a_{51} & a_{52} & a_{21}a_{32} - a_{22}a_{31} & a_{21}u & a_{22}u
\end{pmatrix},
$$

17

where $u = a_{11}a_{22} - a_{12}a_{21}$. Then $Z^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{1,5} + \Delta_{2,4}, \Delta_{2,3}, \Delta_{2,5} \rangle$ and $B^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{2,3} \rangle$, and hence $H^2(L,\mathbb{F}) = \langle \Delta_{1,4}, \Delta_{1,5} + \Delta_{2,4}, \Delta_{2,5} \rangle$. Further,

$$\mathcal{S} = \{ \langle (a,b,c) \rangle \mid ac - b^2 \neq 0 \}.$$

Let $\vartheta = (a,b,c) \in H^2(L,\mathbb{F})$. Then $A\vartheta = (\bar{a}, \bar{b}, \bar{c})$ where

$$\begin{aligned}
\bar{a} &= (a_{11}^2 a + 2a_{11}a_{21}b + a_{21}^2 c)u; \\
\bar{b} &= (a_{11}a_{12}a + (a_{11}a_{22} + a_{12}a_{21})b + a_{21}a_{22}c)u; \\
\bar{c} &= (a_{12}^2 a + 2a_{12}a_{22}b + a_{22}^2 c)u.
\end{aligned}$$

Choose $S = \langle (a,b,c) \rangle \in \mathcal{S}$. Let us consider three cases: $a \neq 0$; $a = 0$ and $c \neq 0$; $a = 0$, $c = 0$, and char $\mathbb{F} \neq 2$. Let, in these cases, B denote the first, the second, or the third of the following transformations, respectively:

$$
\begin{pmatrix}
1 & -b & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & a & -ab \\
0 & 0 & 0 & 0 & a^2
\end{pmatrix},
\begin{pmatrix}
0 & -c & 0 & 0 & 0 \\
1 & b & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & 0 & -c^2 \\
0 & 0 & 0 & c & bc
\end{pmatrix},
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & -2 \\
0 & 0 & 0 & 2 & 2
\end{pmatrix}.
$$

We obtain, in the first and the second case, that $BS = \langle (1, 0, ac - b^2) \rangle$, while, in the third case, we find that $BS = \langle (1, 0, -1) \rangle$.

To summarize, if char $\mathbb{F} \neq 2$ then, as $ac - b^2 \neq 0$, the set formed by the subspaces $\langle (1, 0, \varepsilon) \rangle$ with $\varepsilon \neq 0$ contain a representative in each of the $\mathrm{Aut}(L)$-orbits on $\mathcal{S}$. The Lie algebra corresponding to $\langle (1, 0, \varepsilon) \rangle$ is $L_{6,21}(\varepsilon)$. If char $\mathbb{F} = 2$ then the set consisting of the subspaces $\langle (1, 0, \varepsilon) \rangle$ and $\langle (0, 1, 0) \rangle$ contain such a system of representatives. The Lie algebra corresponding to $\langle (0, 1, 0) \rangle$ is $L_{6,6}^{(2)}$.

The expressions for $\bar{a}$, $\bar{b}$, $\bar{c}$ give in characteristic 2 that $\langle (0, 1, 0) \rangle$ is fixed by $\mathrm{Aut}(L)$, and hence $L_{6,6}^{(2)} \ncong L_{6,21}(\varepsilon)$. We claim, for $\varepsilon, \nu \in \mathbb{F}^*$ that $L_{6,21}(\varepsilon) \cong L_{6,21}(\nu)$ if and only if $\varepsilon \overset{*}{\sim} \nu$. Suppose first that $L_{6,21}(\varepsilon) \cong L_{6,21}(\nu)$. Then there is some automorphism $A \in \mathrm{Aut}(L)$ such that $A(1, 0, \varepsilon) = (1, 0, \nu)$. Using the equations for $\bar{a}$, $\bar{b}$, and $\bar{c}$ we obtain that

$$\begin{aligned}
a_{11}a_{12} + a_{21}a_{22}\varepsilon &= 0; & (7) \\
a_{12}^2 + a_{22}^2 \varepsilon - a_{11}^2 \nu - a_{21}^2 \varepsilon\nu &= 0. & (8)
\end{aligned}$$

Simple computation shows that

$$(a_{12}a_{22} - \nu a_{11}a_{21})(a_{11}a_{12} + a_{21}a_{22}\varepsilon) - a_{11}a_{22}(a_{12}^2 + a_{22}^2 \varepsilon - a_{11}^2 \nu - a_{21}^2 \varepsilon\nu) = (a_{11}a_{22} - a_{12}a_{21})(a_{11}^2 \nu - a_{22}^2 \varepsilon).$$

Since $a_{11}a_{22} - a_{12}a_{21} \neq 0$, we obtain that equations (7) and (8) imply that $a_{11}^2 \nu - a_{22}^2 \varepsilon$. Then either $\varepsilon \overset{*}{\sim} \nu$ or $a_{11} = a_{22} = 0$. If $a_{11} = a_{22} = 0$, then equation (8) becomes $a_{12}^2 = a_{21}^2 \varepsilon\nu$. Since A is invertible, we obtain, in this case, that $1/\varepsilon \overset{*}{\sim} \nu$. Since $\varepsilon \overset{*}{\sim} 1/\varepsilon$, this gives that $\varepsilon \overset{*}{\sim} \nu$.

Conversely, let us suppose that $\varepsilon, \nu \in \mathbb{F}^*$ such that $\varepsilon \overset{*}{\sim} \mu$. That is, there is some $\alpha \in \mathbb{F}^*$ such that $\nu = \alpha^2 \varepsilon$. Then let A be the automorphism of L represented by the diagonal matrix with $(1, \alpha, \alpha, \alpha, \alpha^2)$ as its diagonal. Then $A\langle (1, 0, \varepsilon) \rangle = \langle (1, 0, \alpha^2 \nu) \rangle = \langle (1, 0, \nu) \rangle$.

$\mathbf{L}_{4,1}$

The automorphism group of L is $\mathrm{GL}_4(\mathbb{F})$ and $H^2(L,\mathbb{F}) = Z^2(L,\mathbb{F})$ consists of all skew-symmetric bilinear forms on L, and hence $H^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{2,3}, \Delta_{2,4}, \Delta_{3,4} \rangle$. Note that $H^2(L,\mathbb{F})$ is naturally

isomorphic to the wedge product $\Lambda^2(\mathbb{F}^4)$ as $\mathrm{GL}_4(\mathbb{F})$-modules; therefore we do not write the explicit formulas for the action. The set of allowable subspaces $\mathcal{S}$ consists of the 2-dimensional subspaces $S = \langle \vartheta_1, \vartheta_2 \rangle$ such that $\vartheta_1^\perp \cap \vartheta_2^\perp = 0$. Let us compute the representatives of the $\mathrm{Aut}(L)$-orbits on $H^2(L, \mathbb{F})$. Let $S \in \mathcal{S}$ and write

$$S = \langle (a_1, b_1, c_1, d_1, e_1, f_1), (a_2, b_2, c_2, d_2, e_2, f_2) \rangle.$$

By Lemma 4.1(i), we may assume without loss of generality that $a_1 = 1$. Then

$$\begin{pmatrix} 1 & 0 & d_1 & e_1 \\ 0 & 1 & -b_1 & -c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} S = \langle (1,0,0,0,0,f_1'), (0, b_2', c_2', d_2', e_2', f_2') \rangle,$$

where f_1', b_2', c_2', d_2', e_2', $f_2' \in \mathbb{F}$. Thus no generality is lost by assuming that

$$S = \langle (1,0,0,0,0,f_1), (0, b_2, c_2, d_2, e_2, f_2) \rangle.$$

We claim that there is $B \in \mathrm{Aut}(L)$ such that

$$BS = \langle (1,0,0,0,0,f_1), (0,1,0,0,\bar{e}_2, f_2) \rangle.$$

Consider the following list of matrices:

$$\begin{pmatrix} 1 & -d_2 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & 1 & -c_2 \\ 0 & 0 & 0 & b_2 \end{pmatrix}, \begin{pmatrix} 1 & -e_2 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & 0 & -c_2 \\ 0 & 0 & 1 & b_2 \end{pmatrix}, \begin{pmatrix} 0 & -d_2 & 0 & 0 \\ 1 & b_2 & 0 & 0 \\ 0 & 0 & 1 & -e_2 \\ 0 & 0 & 0 & d_2 \end{pmatrix}, \begin{pmatrix} 0 & -e_2 & 0 & 0 \\ 1 & c_2 & 0 & 0 \\ 0 & 0 & 0 & -e_2 \\ 0 & 0 & 1 & d_2 \end{pmatrix}.$$

If $(b_2, c_2, d_2, e_2) \neq (0,0,0,0)$, then the list above contains at least one invertible matrix, and let B denote this matrix. On the other hand, if $(b_2, c_2, d_2, e_2) = (0,0,0,0)$, then $f_2 \neq 0$ and set, in this case,

$$B = \begin{pmatrix} 1 & -f_2 & f_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & f_2 \end{pmatrix}.$$

Having defined B as above, we obtain $BS = \langle (1,0,0,0,0,f_1), (0,1,0,0,\bar{e}_2, f_2) \rangle$ with some $\bar{e}_2$, as claimed. Let us hence suppose without loss of generality that $S = \langle (1,0,0,0,0,f_1), (0,1,0,0,e_2, f_2) \rangle$. Next consider the following three cases: $f_1 \neq 0$; $f_1 = 0$ and $e_2 \neq 0$; $f_1 = 0$ and $e_2 = 0$. Note that in the last case $f_2 \neq 0$, as otherwise $\vartheta_1^\perp \cap \vartheta_2^\perp \neq 0$. Define $C \in \mathrm{Aut}(L)$ in these cases, respectively, as follows:

$$\begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & f_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f_2 & -1 & 0 \\ 0 & -e_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f_2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain, in each of these cases, that $CS = \langle (1,0,0,0,0,1), (0,1,0,0,e_2', f_2) \rangle$ where $e_2' \in \mathbb{F}$. If $f_2 = 0$ then $CS = \langle (1,0,0,0,0,1), (0,1,0,0,\varepsilon,0) \rangle$ with some $\varepsilon \in \mathbb{F}$. On the other hand, if $f_2 \neq 0$ then

$$\begin{pmatrix} f_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & f_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \langle (1,0,0,0,0,1), (0,1,0,0,e_2', f_2) \rangle = \langle (1,0,0,0,0,1), (0,1,0,0,e_2'', 1) \rangle,$$

where $e_2'' \in \mathbb{F}$. Suppose now that char $\mathbb{F} \neq 2$, and let D be the first of the following two automorphisms if $e_2'' \in \{0, -1/4\}$, while we let D be the second otherwise:

$$\begin{pmatrix} 2e_2'' & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 4e_2''+1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Then $D\langle(1,0,0,0,0,1),(0,1,0,0,e_2'',1)\rangle = \langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon,0)\rangle$.

To summarize, in characteristic different from two, the set of 2-spaces $\langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon,0)\rangle$ where $\varepsilon \in \mathbb{F}$ contains at least one representative of each of the $\mathrm{Aut}(L)$-orbits on $\mathcal{S}$. In characteristic 2, such a set of 2-spaces is formed by the spaces $\langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon,0)\rangle$ and the spaces $\langle(1,0,0,0,0,1),(0,1,0,0,\nu,1)\rangle$ where ε, $\nu \in \mathbb{F}$. The corresponding Lie algebras are $L_{6,22}(\varepsilon)$ and $L_{6,7}^{(2)}(\varepsilon)$, respectively.

It remains to find the possible isomorphisms of the step-2 descendants of L. The group $\mathrm{Aut}(L)$ preserves, modulo scalars, a quadratic form Q on $H^2(L,\mathbb{F})$, defined, for $\vartheta = (a,b,c,d,e,f) \in H^2(L,\mathbb{F})$, as

$$Q(\vartheta) = af - be + cd. \tag{9}$$

Let f denote the symmetric bilinear form associated with Q. It is easy to see that Q is indeed preserved by the action of $\mathrm{GL}(V)$ modulo scalars; namely, for $A \in \mathrm{GL}(V)$ and $v \in V$ we have that $Q(Av) = (\det A)Q(v)$.

Assume first that char $\mathbb{F} \neq 2$. and consider two subspaces $S_1 = \langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon,0)\rangle$ and $S_2 = \langle(1,0,0,0,0,1),(0,1,0,0,\nu,0)\rangle$ such that S_1 and S_2 are in the same $\mathrm{Aut}(L)$-orbit. Since the determinants of the Gram matrices of the form f restricted to S_1 and S_2 are 4ε and 4ν, respectively, Lemma 4.2(2) implies that $\nu = \alpha^2 \varepsilon$ with some $\alpha \in \mathbb{F}^*$. Conversely if $\nu = \gamma^2 \varepsilon$ with some $\gamma \in \mathbb{F}^*$ and A is the automorphism of $L_{4,1}$ represented by the diagonal matrix with the entries $(1,\gamma,1,\gamma)$ in the diagonal, then $AS_1 = S_2$.

Suppose now that the characteristic of $\mathbb{F}$ is 2. Set $S_1 = \langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon,0)\rangle$ and $S_2 = \langle(1,0,0,0,0,1),(0,1,0,0,\nu,1)\rangle$ where ε, $\nu \in \mathbb{F}$. Since the restriction of f is identically zero on S_1 while it is non-singular on S_2, we obtain that S_1 and S_2 cannot be in the same $\mathrm{Aut}(L)$-orbit. Suppose now that $S_1 = \langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon_1,0)\rangle$ and $S_2 = \langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon_2,0)\rangle$ such that S_1 and S_2 are in the same $\mathrm{Aut}(L)$-orbit. Since f is identically zero on S_1 and S_2 and, for $i = 1$, 2, $Q(1,0,0,0,0,1)Q(0,1,0,0,\varepsilon_i,0) = \varepsilon_i$, we obtain from Lemma 4.2(iii) that $\varepsilon_2 = \alpha^2 \varepsilon_1 + \beta^2$ with some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Assume, conversely that $\varepsilon_2 = \alpha^2 \varepsilon_1 + \beta^2$ with some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Since there is nothing to prove if $\varepsilon_1 = \varepsilon_2 = 1$, we may assume without loss of generality that $\varepsilon_1 \neq 1$. Let A be the automorphism of $L_{4,1}$ represented by the matrix

$$\begin{pmatrix} \varepsilon_1 & \beta & 1 & \varepsilon_1\beta \\ 0 & \alpha & 0 & \alpha \\ 1 & \beta & 1 & \beta \\ 0 & \alpha & 0 & \varepsilon_1\alpha \end{pmatrix}.$$

Then $\det A = \alpha^2(\varepsilon_1^2 + 1)$ which, by assumption, is non-zero, and so A does define an isomorphism. Further, $A\langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon_1,0)\rangle = \langle(1,0,0,0,0,1),(0,1,0,0,\varepsilon_2,0)\rangle$.

Finally if $S_1 = \langle(1,0,0,0,0,1),(0,1,0,0,\nu_1,1)\rangle$ and $S_2 = \langle(1,0,0,0,0,1),(0,1,0,0,\nu_2,1)\rangle$, then the restriction of f on S_1 and on S_2 is non-singular, and the given bases of S_1 and S_2 are symplectic. Further, the Arf invariants of S_1 and S_2 with respect to these bases are ν_1 and ν_2, respectively. Thus Lemma 4.2(iii) gives that $\nu_1 + \nu_2 \in \{\alpha^2 + \alpha \mid \alpha \in \mathbb{F}^*\}$. Assume, conversely, that $\alpha^2 + \alpha + \nu_1 + \nu_2 = 0$ with some α. Let

20

A denote the isomorphism of $L_{4,1}$ represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $A\langle(1,0,0,0,0,1),(0,1,0,0,\nu_1,1)\rangle = \langle(1,0,0,0,0,1),(0,1,0,0,\nu_2,1)\rangle$.

The argument presented in this section give rise to the proof of Theorem 4.4.

Proof of Theorem 4.4. Let V be the vector space as in the statement of the theorem and let L be the 4-dimensional abelian Lie algebra $L_{4,1}$. As noted before, there is an isomorphism between the $\mathrm{GL}(4,\mathbb{F})$-modules $V \wedge V$ and $H^2(L,\mathbb{F})$ realized by the mapping $b_i \wedge b_j \mapsto \Delta_{i,j}$, and so we will identify $V \wedge V$ with $H^2(L,\mathbb{F})$. If S is a 2-dimensional subspace of $H^2(L,\mathbb{F})$, then the corresponding central extension L_S of L is a 6-dimensional nilpotent Lie algebra with 4 generators and central derived subalgebra of dimension 2. In addition if S is not allowable then $L_S = K \oplus \mathbb{F}$, which, using the classification of 5-dimensional nilpotent Lie algebras, gives that $L_S \cong L_{5,8} \oplus \mathbb{F}$. Hence $\mathrm{GL}(4,\mathbb{F})$ has a single orbit on the set of not allowable 2-dimensional subspaces. Since the orbits on the allowable 2-dimensional subspaces were determined in this section, the theorem follows. $\square$

$\mathbf{L}_{4,2}$

The group $\mathrm{Aut}(L)$ consists of the invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & u & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

where $u = a_{11}a_{22} - a_{12}a_{21}$. We have $Z^2(L,\mathbb{F}) = \langle \Delta_{1,2}, \Delta_{1,3}, \Delta_{1,4}, \Delta_{2,3}, \Delta_{2,4}\rangle$, $B^2(L,\mathbb{F}) = \langle \Delta_{1,2}\rangle$, and so $H^2(L,\mathbb{F}) = \langle \overline{\Delta_{1,3}}, \overline{\Delta_{1,4}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,4}}\rangle$. The set of allowable subspaces $\mathcal{S}$ consists of the 2-dimensional subspaces $S = \langle \vartheta_1, \vartheta_2\rangle$ such that $\vartheta_1^\perp \cap \vartheta_2^\perp \cap \langle x_3, x_4\rangle = 0$. If $\vartheta = (a,b,c,d) \in H^2(L,\mathbb{F})$, then $A\vartheta = (\bar{a}, \bar{b}, \bar{c}, \bar{d})$ where

$$\begin{aligned} \bar{a} &= (a_{11}a + a_{21}c)u; \\ \bar{b} &= a_{11}a_{34}a + a_{11}a_{44}b + a_{21}a_{34}c + a_{21}a_{44}d; \\ \bar{c} &= (a_{12}a + a_{22}c)u; \\ \bar{d} &= a_{12}a_{34}a + a_{12}a_{44}b + a_{22}a_{34}c + a_{22}a_{44}d. \end{aligned}$$

Choose a 2-dimensional subspace $S = \langle \vartheta_1, \vartheta_2\rangle$ of $\mathcal{S}$ where $\vartheta_1 = (a_1, b_1, c_1, d_1)$ and $\vartheta_2 = (a_2, b_2, c_2, d_2)$. If $a_1 = c_1 = 0$ and $a_2 = c_2 = 0$ then $x_3 \in \vartheta_1^\perp \cap \vartheta_2^\perp$, and hence S is not allowable. Thus, by possibly swapping ϑ_1 and ϑ_2, we may assume without loss of generality that $(a_1, c_1) \neq (0,0)$. Let B be the first of the following automorphisms if $a_1 \neq 0$ and let B be the second if $a_1 = 0$ (which implies that $c_1 \neq 0$):

$$\begin{pmatrix} 1 & -c_1 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & -a_1 b_1 \\ 0 & 0 & 0 & a_1^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & c_1 \end{pmatrix}.$$

Then the image BS is of the form $\langle (1,0,0,d_1'), (0, b_2', c_2', d_2')\rangle$ which implies that we may assume without loss of generality that $S = \langle (1,0,0,d_1), (0, b_2, c_2, d_2)\rangle$. We note that such an S is allowable if and only if $(d_1, b_2, d_2) \neq (0,0,0)$.

Suppose first that $c_2 = 0$ and $d_2 \neq 0$. Then

$$\begin{pmatrix} d_2 & 0 & 0 & 0 \\ -b_2 & 1 & 0 & 0 \\ 0 & 0 & d_2 & d_1 b_2 \\ 0 & 0 & 0 & d_2 \end{pmatrix} S = \langle (1,0,0,0), (0,0,0,1) \rangle.$$

Next, we assume that $c_2 = 0$ and $d_2 = 0$. If $d_1 = 0$ then $S = \langle (1,0,0,0), (0,1,0,0) \rangle$, while if $d_1 \neq 0$ then

$$\begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_1 \end{pmatrix} S = \langle (1,0,0,1), (0,1,0,0) \rangle.$$

Now suppose that $c_2 \neq 0$ and $d_2 = 0$ and let C be the first or the second of the following matrices, depending on whether $b_2 = 0$ or not:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & -b_2 & 0 \\ 0 & 0 & 0 & -b_2 c_2 \end{pmatrix}, \quad \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_1 \end{pmatrix}.$$

Then $CS = \langle (1,0,0,1), (0,\varepsilon,1,0) \rangle$. Finally, assume that $c_2 \neq 0$ and $d_2 \neq 0$ and let D be the first or the second of the following matrices depending on whether $b_2 \neq 0$:

$$\begin{pmatrix} d_2 & -d_2 & 0 & 0 \\ -b_2 & 0 & 0 & 0 \\ 0 & 0 & -b_2 d_2 & 0 \\ 0 & 0 & 0 & -b_2 c_2 \end{pmatrix}, \quad \begin{pmatrix} c_2 & (d_1 - 1)c_2 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & c_2 d_2 & 0 \\ 0 & 0 & 0 & c_2^2 \end{pmatrix}.$$

Then $DS = \langle (1,0,0,1), (0,\bar{b}_2,1,1) \rangle$, with $\bar{b}_2 \in \mathbb{F}$. If $\operatorname{char} \mathbb{F} \neq 2$ then this gives no new orbit as

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 \end{pmatrix} \langle (1,0,0,1), (0,\bar{b}_2,1,1) \rangle = \langle (1,0,0,1), (0,4\bar{b}_2 + 1,1,0) \rangle,$$

If $\operatorname{char} \mathbb{F} = 2$ then we obtain that $S = \langle (1,0,0,1), (0,\nu,1,1) \rangle$.

To summarize, if $\operatorname{char} \mathbb{F} \neq 2$ then the list of 2-spaces $\langle (1,0,0,0), (0,0,0,1) \rangle$, $\langle (1,0,0,0), (0,1,0,0) \rangle$, $\langle (1,0,0,1), (0,1,0,0) \rangle$, and $\langle (1,0,0,1), (0,\varepsilon,1,0) \rangle$ with $\varepsilon \in \mathbb{F}$ contains at least one representative for each of the $\operatorname{Aut}(L)$-orbits on $\mathcal{S}$. The corresponding Lie algebras are $L_{6,27}$, $L_{6,25}$, $L_{6,23}$, and $L_{6,24}(\varepsilon)$. In characteristic 2, such a set is formed by the subspaces above in addition to the subspaces $\langle (1,0,0,1), (0,\nu,1,1) \rangle$ with $\nu \in \mathbb{F}$. The Lie algebra that corresponds to the subspace $\langle (1,0,0,1), (0,\nu,1,1) \rangle$ is $L_{6,8}^{(2)}(\nu)$.

Finally, in this section we have to determine the possible isomorphisms of the Lie algebras in the previous paragraph. First we note, for $L = L_{6,24}(\varepsilon)$ and $L = L_{6,8}^{(2)}(\nu)$, that $C(L) = L^3$, while this equation is not valid for $L_{6,27}$, $L_{6,23}$, or $L_{6,25}$. In order to separate the Lie algebras $L_{6,27}$, $L_{6,25}$, $L_{6,23}$ we use the geometry of the $\operatorname{Aut}(L)$-action on $H^2(L, \mathbb{F})$. The expressions for $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ above give that the action of the automorphism A on $H^2(L, \mathbb{F})$ is represented, with respect to the basis $\{\overline{\Delta_{1,3}}, \overline{\Delta_{1,4}}, \overline{\Delta_{2,3}}, \overline{\Delta_{2,4}}\}$, by the matrix

$$\begin{pmatrix} a_{11}u & a_{11}a_{34} & a_{12}u & a_{12}a_{34} \\ 0 & a_{11}a_{44} & 0 & a_{12}a_{44} \\ a_{21}u & a_{21}a_{34} & a_{22}u & a_{22}a_{34} \\ 0 & a_{21}a_{44} & 0 & a_{22}a_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} u & a_{34} \\ 0 & a_{44} \end{pmatrix}. \tag{10}$$

It is well-known that $\mathrm{GL}(2,\mathbb{F}) \otimes \mathrm{GL}(2,\mathbb{F})$ preserves a quadratic form modulo scalars in its natural action on $\mathbb{F}^4$. To exploit this fact in our situation, define a quadratic form Q on $H^2(L,\mathbb{F})$ as

$$Q(\vartheta) = \alpha_1\alpha_4 - \alpha_2\alpha_3 \quad \text{where} \quad \vartheta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Let $A \in \mathrm{Aut}(L)$ and decompose A as $g_1 \otimes g_2$ as in (10). Then we have, for all $\vartheta \in H^2(L,\mathbb{F})$, that

$$Q(A\vartheta) = Q((g_1 \otimes g_2)\vartheta) = (\det g_1)(\det g_2)Q(\vartheta).$$

Now to show that the subspaces obtained above are in different $\mathrm{Aut}(L)$-orbits, notice that the subspace $\langle(1,0,0,0),(0,1,0,0)\rangle$ is totally singular. On the other hand, the singular vectors of the subspace $\langle(1,0,0,1),(0,1,0,0)\rangle$ are the elements of the 1-space $\langle(0,1,0,0)\rangle$, and hence two singular vectors are linearly dependent. Moreover, the singular vectors of the subspace $\langle(1,0,0,0),(0,0,0,1)\rangle$ are the elements of the 1-spaces $\langle(1,0,0,0)\rangle$ and $\langle(0,0,0,1)\rangle$, which shows that there is a pair of linearly independent singular vectors. Thus these three subspaces are in different $\mathrm{Aut}(L)$-orbits. Let us consider now two subspaces of the form $S_1 = \langle(1,0,0,1),(0,\varepsilon,1,0)\rangle$ and $S_2 = \langle(1,0,0,1),(0,\nu,1,0)\rangle$ and assume that they are in the same $\mathrm{Aut}(L)$-orbit; that is $S_2 = AS_1$ with some $A \in \mathrm{Aut}(L)$. If $\mathrm{char}\,\mathbb{F} \neq 2$, then the Gram determinants of S_1 and S_2 with respect to the given bases are ε and ν, respectively. Lemma 4.2(i) gives that $\varepsilon \overset{*}{\sim} \nu$. Conversely, assume that $\mathrm{char}\,\mathbb{F} \neq 2$ and that $\varepsilon \overset{*}{\sim} \nu$; that is $\nu = \varepsilon\alpha^2$ with some $\alpha \in \mathbb{F}^*$. Let A be the diagonal automorphism of L with $(\alpha, 1, \alpha, \alpha^2)$ in the diagonal. Then $A\langle(1,0,0,1),(0,\varepsilon,1,0)\rangle = \langle(1,0,0,1),(0,\nu,1,0)\rangle$. Hence the subspaces $\langle(1,0,0,1),(0,\varepsilon,1,0)\rangle$ and $\langle(1,0,0,1),(0,\nu,1,0)\rangle$ are in the same $\mathrm{Aut}(L)$-orbit if and only if $\varepsilon \overset{*}{\sim} \nu$. This settles the isomorphisms among the possible step-2 descendants of $L_{4,2}$ in the cases when $\mathrm{char}\,\mathbb{F} \neq 2$.

Suppose next that $\mathrm{char}\,\mathbb{F} = 2$ and let

$$S_1 = \langle(1,0,0,1),(0,\varepsilon_1,1,0)\rangle \quad \text{and} \quad S_2 = \langle(1,0,0,1),(0,\varepsilon_2,1,0)\rangle,$$

as above. Since f is identically zero on S_1 and S_2 and, for $i = 1, 2$, $Q(1,0,0,0,0,1)Q(0,1,0,0,\varepsilon_i,0) = \varepsilon_i$, we obtain from Lemma 4.2(iii) that $\varepsilon_2 = \alpha^2\varepsilon_1 + \beta^2$ with some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Suppose conversely that $\mathrm{char}\,\mathbb{F} = 2$ and that $\varepsilon_1 \overset{*+}{\sim} \varepsilon_2$; that is $\varepsilon_2 = \alpha^2\varepsilon_1 + \beta^2$ with some $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let A be the automorphism of L represented by the matrix

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & \alpha & \alpha\beta \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix}.$$

Then $AS_1 = S_2$, which gives, when $\mathrm{char}\,\mathbb{F} = 2$, that $\langle(1,0,0,1),(0,\varepsilon_1,1,0)\rangle$ and $\langle(1,0,0,1),(0,\varepsilon_2,1,0)\rangle$ are in the same $\mathrm{Aut}(L)$-orbit if and only if $\varepsilon_1 \overset{*+}{\sim} \varepsilon_2$.

Suppose now that $S_1 = \langle(1,0,0,1),(0,\nu_1,1,1)\rangle$ and $S_2 = \langle(1,0,0,1),(0,\nu_2,1,1)\rangle$ and assume that S_1 and S_2 are in the same $\mathrm{Aut}(L)$-orbit; that is, there is some $A \in \mathrm{Aut}(L)$ such that $S_1A = S_2$. The restriction of f on S_1 and on S_2 is non-singular, and the given bases of S_1 and S_2 are symplectic. Further, the Arf invariants of S_1 and S_2 with respect to these bases are ν_1 and ν_2, respectively. Thus Lemma 4.2(iii) gives that $\nu_1 + \nu_2 \in \{\alpha^2 + \alpha \mid \alpha \in \mathbb{F}\}$. Conversely, suppose that $\nu_1, \nu_2 \in \mathbb{F}$ such that $x^2 + x + \nu_1 + \nu_2$ has a solution α in $\mathbb{F}$. Let A denote the automorphism represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $AS_1 = S_2$, and so two subspaces $\langle(1,0,0,1),(0,\nu_1,1,1)\rangle$ and $\langle(1,0,0,1),(0,\nu_2,1,1)\rangle$ are in the same $\mathrm{Aut}(L)$-orbit if and only if the polynomial $x^2 + x + \nu_1 + \nu_2$ has a root in $\mathbb{F}$.

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References

[1] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system I: The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997.

[2] Serena Cicalò, W. A. de Graaf, and Csaba Schneider. LieAlgDB – a GAP package, Version 2.1, November 2010, www.sztaki.hu/∼schneider/Research/LieAlgDB.

[3] P. M. Cohn. *Basic Algebra: Groups, Rings and Fields.* Springer, 2003.

[4] W. A. de Graaf. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2 (preprint), http://arxiv.org/abs/math.RA/0511668, 2005.

[5] W. A. de Graaf. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. *J. Algebra*, 309:640–653, 2007.

[6] B. Eick and E. A. O'Brien. Enumerating p-groups. *J. Austral. Math. Soc. Ser. A*, 67(2):191–205, 1999.

[7] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)

[8] M.-P. Gong. *Classification of Nilpotent Lie Algebras of Dimension* 7. Ph.D. thesis, University of Waterloo, Waterloo, Canada, 1998.

[9] Serge Lang. *Algebra.* Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.

[10] V. V. Morozov. Classification of nilpotent Lie algebras of sixth order. *Izv. Vysš. Učebn. Zaved. Matematika*, 1958(4 (5):161–171, 1958.

[11] M. F. Newman, E. A. O'Brien, and M. R. Vaughan-Lee. Groups and nilpotent Lie rings whose order is the sixth power of a prime. *J. Algebra*, 278(1):383–401, 2004.

[12] E. A. O'Brien. The p-group generation algorithm. *J. Symbolic Comput.*, 9(5-6):677–698, 1990.

[13] E. A. O'Brien and M. R. Vaughan-Lee. The groups with order p^7 for odd prime p. *J. Algebra*, 292(1):243–258, 2005.

[14] C. Schneider. A computer-based approach to the classification of nilpotent Lie algebras. *Experiment. Math.*, 14(2):153–160, 2005.

[15] Tor Skjelbred and Terje Sund. Sur la classification des algèbres de Lie nilpotentes. *C. R. Acad. Sci. Paris Sér. A-B*, 286(5), 1978.

[16] Markus Stroppel. The Klein quadric and the classification of nilpotent Lie algebras of class two. *J. Lie Theory*, 18:391–411, 2008.

[17] K. A. Umlauf. *Über die Zusammensetzung der endlichen continuierlichen Transformationsgruppen insbesondere der Gruppen vom Range Null.* Ph.D. thesis, University of Leipzig, Germany, 1891.

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