ENUMERATION OF STANDARD YOUNG TABLEAUX OF CERTAIN TRUNCATED SHAPES

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Abstract. Product formulas for the number of standard Young tableaux of truncated shifted staircase and truncated rectangular shapes are given.

1. Introduction

A truncated shape is obtained from a Ferrers diagram (where parts decrease from top to bottom) by deleting cells from the northeast corner. A renewed interest in enumeration of standard Young tableaux of truncated shapes was motivated by a recent result [1, Prop. 9.7]: The number of geodesics between distinguished pairs of antipodes in the flip graph of triangle-free triangulations is equal to twice the number of Young tableaux of a truncated shifted staircase shape. Motivated by this result, computer experiments were carried out to enumerate the number of standard Young tableaux of these and other truncated shapes. It was found that, in many cases (but not all), all prime factors are "small", hinting on an existence of product formulas. In this paper, hook-like product formulas for truncated shifted staircase and truncated rectangular shapes are proved.

Note added in proof: We were informed by Richard Stanley, that a different method to derive formulas for truncated shapes was developed by Greta Panova [7].

2. Preliminaries and Basic Concepts

A partition λ of positive integer n is a sequence of non-negative integers $(\lambda_1, \lambda_2, \cdots, \lambda_m)$, such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. The Ferrers diagram $[\lambda]$ is the left-justified array of n cells with row i (read from top to bottom) containing λ_i cells. A standard Young tableau (SYT) T of shape λ is a labelling by $\{1, 2, \cdots, n\}$ of the cells in the shape λ such that every row is increasing, when read from left to right, and every column is increasing from top to bottom. The number of SYT of shape λ is denoted by f^λ.

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Proposition 2.1. (The Frobenius-Young Formula) [4, 10] *The number of SYT of shape λ is*

$$f^{\lambda} = \frac{(|\lambda|)!}{\prod_{i=1}^{k}(\lambda_i + k - i)!} \cdot \prod_{1 \le i < j \le k} (\lambda_i - \lambda_j - i + j).$$

Note that trailing zeros in λ (with k appropriately increased) do not change the right-hand side of the above formula.

A partition $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ of n is strict if $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$. The (shifted) shape λ is the array of n cells with row i (read from top to bottom) containing λ_i cells and indented $i-1$ spaces. A standard Young tableau (SYT) T of shifted shape λ is a labelling by $\{1, 2, \cdots, n\}$ of the cells in the shape λ such that every row and column are increasing. The number of SYT of shifted shape λ is denoted by g^{λ}.

Proposition 2.2. (Schur's Formula)[8][6, p. 267 (2)] *The number of SYT of shifted shape λ is*

$$g^{\lambda} = \frac{(|\lambda|)!}{\prod_i \lambda_i!} \cdot \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

For any nonnegative integer m, let $[m] := (m, m-1, \ldots, 1)$ be the corresponding shifted staircase shape. Consider the truncated shifted staircase shape $[m] \setminus \lambda$ where a partition $\lambda = (\lambda_1, ..., \lambda_k) \subseteq [m]$ is deleted from the northeast corner; namely, λ_1 cells are deleted from right end of first row, λ_2 cells are deleted from right end of second row, etc. Let n be the size of $[m] \setminus \lambda$. A standard Young tableau (SYT) of truncated shifted staircase shape $[m] \setminus \lambda$ is a labelling of the cells of this shape by $\{1, \ldots, n\}$, such that labels are increasing along rows (from left to right) and columns (from top to bottom).

Example 2.3. There are four SYT of shape $[4] \setminus (1)$

$$\begin{matrix} 1 & 2 & 3 & * \\ & 4 & 5 & 6 \\ & & 7 & 8 \\ & & & 9 \end{matrix}\ , \quad \begin{matrix} 1 & 2 & 4 & * \\ & 3 & 5 & 6 \\ & & 7 & 8 \\ & & & 9 \end{matrix}\ , \quad \begin{matrix} 1 & 2 & 3 & * \\ & 4 & 5 & 7 \\ & & 6 & 8 \\ & & & 9 \end{matrix}\ , \quad \begin{matrix} 1 & 2 & 4 & * \\ & 3 & 5 & 7 \\ & & 6 & 8 \\ & & & 9 \end{matrix}$$

Similarly, for any nonnegative integers, m and k, let $(m^k) = (m, \ldots, m)$ (k parts) be the corresponding rectangular shape and consider the truncated rectangular shape $(m^k) \setminus \lambda$ where $\lambda \subseteq (m^k)$ is deleted from the northeast corner; namely, λ_1 cells are deleted from right end of first row, λ_2 cells are deleted from right end of second row, etc. Letting n be the size of $(m^k) \setminus \lambda$, a SYT of truncated shape $(m^k) \setminus \lambda$ is a labelling of the cells of this shape by $\{1, \ldots, n\}$, such that labels are increasing along rows (from left to right) and columns (from top to bottom).

Preliminary computer experiments hinted that that a nice phenomenon occurs when a square is truncated from a staircase shape: while the number of SYT of truncated staircase shape $[m] \setminus (d^d)$ increase exponentially with respect to the size of the shape, all prime factors of this number are smaller than the size. A similar phenomenon occurs for squares truncated from rectangular shapes [1]. In this paper, product formulas for truncated shapes, which explain the above factorization phenomenon, will be proved (see, e.g., Corollaries 4.7, 4.8, 5.6, 5.8 below).

3. Main Idea: Pivoting

Consider, for example, the shifted shape $\nu = (m{+}2k{+}2, m{+}2k{+}1, \ldots, 1) \setminus (k^k)$, a shifted staircase shape of size $\binom{m+2k+3}{2}$ minus a $k \times k$ square at the northeast (NE) corner. Define the *pivot cell* of this shape to be the cell with coordinates $(k+1, k+1)$, measured from the NE corner. Let T be a SYT of shape ν, and assume that the pivot cell of T contains the value t. Clearly, all the cells of T which are weakly NW of the pivot cell contain values smaller or equal to t. Similarly, all the cells of T which are weakly SE of the pivot cell contain values larger or equal to t. All the other cells of T belong to a SW shifted staircase sub-shape of size $\binom{m+1}{2}$. They are naturally partitioned into two subsets: those with values less than t form a shifted shape λ, and those with values more than t form the transpose of a shifted shape λ^c. λ clearly determines λ^c (see Lemma 4.2 below for the details). Thus overall partition of the cells of T according to the threshold t is now clear, and it follows that the number of SYT of shape ν is

$$\sum_{\lambda \subseteq (m,\ldots,1)} g^{\mu\cup\lambda} g^{\mu\cup\lambda^c},$$

where $\mu = (m+k+2, m+k+1, \ldots, m+3, m+1)$ and g^λ is the number of SYT of shifted shape λ. Using the explicit formula for g^λ one can, in principle, compute this quantity. It turns out that a multiplicative formula also exists in this case.

In the following sections we shall first develop the relevant theory for truncated shifted shapes, and then for the (somewhat more complicated) regular (non-shifted) shapes.

4. Truncated Shifted Staircase Shapes

In this section, $\lambda = (\lambda_1, \ldots, \lambda_k)$ (with $\lambda_1 > \ldots > \lambda_k > 0$ integers) will be a strict partition of $|\lambda| := \lambda_1 + \ldots + \lambda_k$. Denote by g^λ the number of SYT of shifted shape λ.

[1] It should be noted that this factorization phenomenon does not hold in the general case. For example, the number of SYT of truncated rectangular shape $(7^6) \setminus (2)$ has a prime factor 5333.

For any nonnegative integer m, let $[m] := (m, m-1, \ldots, 1)$ be the corresponding shifted staircase shape. Schur's formula (Proposition 2.2) implies the following.

Corollary 4.1. *The number of SYT of shifted staircase shape $[m]$ is*

$$g^{[m]} = M! \cdot \prod_{i=0}^{m-1} \frac{i!}{(2i+1)!},$$

where $M := |[m]| = \binom{m+1}{2}$.

Lemma 4.2. *Let m and t be nonnegative integers with $t \le \binom{m+1}{2}$. Let T be a SYT of shifted staircase shape $[m]$, let T_1 be the set of all cells in T with values at most t, and let T_2 be obtained from $T \setminus T_1$ by transposing the shape and replacing each entry i by $M - i + 1$. Then:*

(1) *T_1 and T_2 are shifted SYT.*
(2) *Treating strict partitions as sets, $[m]$ is the disjoint union of the shape of T_1 and the the shape of T_2.*

Proof. (1) is clear. In order to prove (2), denote the shifted shapes of T_1 and T_2 by λ_1 and λ_2, respectively. The border between T_1 and $T \setminus T_1$ is a lattice path of length exactly m, starting at the NW corner of the staircase shape $[m]$ and using only S and W steps. If the first step is S then the first part of λ_1 is m, and the rest (of both λ_1 and λ_2) corresponds to a lattice path in $[m-1]$. Similarly, if the first step is W then the first part of λ_2 is m, and the rest corresponds to a lattice path in $[m-1]$. Thus exactly one of λ_1, λ_2 has a part equal to m, and the whole result follows by induction on m.

$\square$

Corollary 4.3. *For any nonnegative integers m and t with $t \le \binom{m+1}{2}$,*

$$\sum_{\substack{\lambda \subseteq [m] \\ |\lambda| = t}} g^{\lambda} g^{\lambda^c} = g^{[m]}.$$

Here summation is over all strict partitions λ with the prescribed restrictions, and λ^c is the complement of λ in $[m]$ (where strict partitions are treated as sets). In particular, the LHS is independent of t.

Lemma 4.4. *Let λ and λ^c be strict partitions whose disjoint union (as sets) is $[m]$, and let $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_1 > \ldots > \mu_k > m$. Then*

$$g^{\mu \cup \lambda} g^{\mu \cup \lambda^c} = c(\mu, |\lambda|, |\lambda^c|) \cdot g^{\lambda} g^{\lambda^c},$$

where

$$c(\mu, |\lambda|, |\lambda^c|) = (g^{\mu})^2 \binom{|\mu| + |\lambda|}{|\mu|} \binom{|\mu| + |\lambda^c|}{|\mu|} \prod_{i=1}^{k} \prod_{j=1}^{m} \frac{\mu_i - j}{\mu_i + j}.$$

Proof. By Proposition 2.2,

$$\frac{g^{\mu\cup\lambda}g^{\mu\cup\lambda^c}}{g^{\lambda}g^{\lambda^c}} = \frac{(|\mu|+|\lambda|)!(|\mu|+|\lambda^c|)!}{(|\lambda|)!(|\lambda^c|)!}\left(\prod_i \frac{1}{\mu_i!}\prod_{i<j}\frac{\mu_i-\mu_j}{\mu_i+\mu_j}\right)^2 \cdot$$
$$\cdot \prod_{i,j}\frac{\mu_i-\lambda_j}{\mu_i+\lambda_j}\prod_{i,j}\frac{\mu_i-\lambda^c_j}{\mu_i+\lambda^c_j}.$$

By the assumption on λ and λ^c,

$$\prod_j \frac{\mu_i-\lambda_j}{\mu_i+\lambda_j}\prod_j\frac{\mu_i-\lambda^c_j}{\mu_i+\lambda^c_j} = \prod_{j=1}^{m}\frac{\mu_i-j}{\mu_i+j} \qquad (\forall i).$$

One more application (to g^{μ}) of Proposition 2.2 gives the desired result. □

A technical lemma, that will be used to prove Theorems 4.6 and 5.5, is the following.

Lemma 4.5. *Let t_1, t_2 and N be nonnegative integers. Then*

$$\sum_{i=0}^{N}\binom{t_1+i}{t_1}\binom{t_2+N-i}{t_2} = \binom{t_1+t_2+N+1}{t_1+t_2+1}.$$

Proof. This is a classical binomial identity, which follows for example from computation of the coefficients of x^N on both sides of the identity

$$(1-x)^{-(1+t_1)}\cdot(1-x)^{-(1+t_2)} = (1-x)^{-(2+t_1+t_2)}.$$

□

Theorem 4.6. *Let m be a nonnegative integer, $M := \binom{m+1}{2}$, and $\mu = (\mu_1,\ldots,\mu_k)$ a strict partition with $\mu_1 > \ldots > \mu_k > m$. Then*

$$\sum_{\lambda\subseteq[m]} g^{\mu\cup\lambda}g^{\mu\cup\lambda^c} = (g^{\mu})^2 g^{[m]}\cdot\binom{M+2|\mu|+1}{2|\mu|+1}\prod_{i=1}^{k}\prod_{j=1}^{m}\frac{\mu_i-j}{\mu_i+j}$$
$$= g^{\mu}g^{\mu\cup[m]}\cdot\frac{(M+2|\mu|+1)!(|\mu|)!}{(M+|\mu|)!(2|\mu|+1)!}.$$

Proof. Restrict the summation on the LHS to strict partitions λ of a fixed size $|\lambda| = t$. By Lemma 4.4 and Corollary 4.3,

$$\sum_{\substack{\lambda\subseteq[m]\\|\lambda|=t}} g^{\mu\cup\lambda}g^{\mu\cup\lambda^c} = c(\mu,t,M-t)\cdot\sum_{\substack{\lambda\subseteq[m]\\|\lambda|=t}} g^{\lambda}g^{\lambda^c} = c(\mu,t,M-t)\cdot g^{[m]}.$$

Now sum over all t and use the explicit formula for $c(\mu,t,M-t)$ (from Lemma 4.4) together with Lemma 4.5, to obtain the first explicit formula above. The second formula then follows by use of Proposition 2.2.

□

We shall apply this theorem in several special cases.

First, consider the case of $k = 2$ and $\mu = (m+3, m+1)$. This corresponds to truncating the NE corner of a shifted staircase shape $[m+4]$.

Corollary 4.7. *The number of SYT of truncated shifted staircase shape $[m+4] \setminus (1)$ is*

$$g^{[m]} \cdot \binom{N}{4m+9} \frac{4(2m+3)}{m+3} = \frac{N! \cdot 4(2m+3)}{(4m+9)! \cdot (m+3)} \cdot \prod_{i=0}^{m-1} \frac{i!}{(2i+1)!},$$

where $N = (m+3)(m+6)/2$ is the size of the shape.

This result was the initial motivation for the current paper, as it answers a question raised in [1].

More generally, taking $\mu = (m+k+1, \dots, m+3, m+1)$ (k parts) corresponds to truncating a $(k-1) \times (k-1)$ square from the NE corner of a shifted staircase shape $[m+2k]$. The computation is left to the interested reader.

Now take $\mu = (m+k, \dots, m+1)$ (k parts). This corresponds to truncating a $k \times k$ square from the NE corner of a shifted staircase shape $[m+2k]$, but adding back the SW corner of this square.

Corollary 4.8. *The number of SYT of truncated shifted staircase shape $[m+2k] \setminus (k^{k-1}, k-1)$ is*

$$g^{[m]} \cdot \binom{N}{2|\mu|+1} (|\mu|)!^2 \cdot \prod_{i=1}^{k} \frac{2 \cdot (i-1)!}{(2m+k+i)!},$$

where $|\mu| = k(2m+k+1)/2$ and $N = \binom{m+2k+1}{2} - k^2 + 1$ is the size of the shape.

The special case $k = 1$ (with $\mu = (m+1)$) gives back the number $g^{[m+2]}$ of SYT of shifted staircase shape $[m+2]$:

$$g^{[m+2]} = g^{[m]} \cdot \binom{N}{2m+3} \frac{m!(m+1)!}{(2m+1)!} = N! \cdot \prod_{i=0}^{m+1} \frac{i!}{(2i+1)!},$$

where $N = (m+2)(m+3)/2$ is the size of the shape. This agrees, of course, with Corollary 4.1.

The special case $k = 2$ (with $\mu = (m+2, m+1)$) corresponds to truncating a small shifted staircase shape $[2] = (2,1)$ from the shifted staircase shape $[m+4]$.

Corollary 4.9. *The number of SYT of truncated shifted staircase shape $[m+4] \setminus (2,1)$ is*

$$g^{[m]} \cdot \binom{N}{4m+7} \frac{2}{m+2} = \frac{N! \cdot 2}{(4m+7)! \cdot (m+2)} \cdot \prod_{i=0}^{m-1} \frac{i!}{(2i+1)!},$$

where $N = (m+2)(m+7)/2$ is the size of the shape.

5. Truncated Rectangular Shapes

In this section, $\lambda = (\lambda_1, \ldots, \lambda_k)$ (with $\lambda_1 \ge \ldots \ge \lambda_k \ge 0$ integers) will be a partition of $|\lambda| := \lambda_1 + \ldots + \lambda_k$. Note that some of the parts may be zeros. Two partitions which differ only in trailing zeros will be considered equal. Denote by f^λ the number of SYT of regular (non-shifted) shape λ.

For any nonnegative integers m and n, let $(n^m) := (n, \ldots, n)$ (m times) be the corresponding rectangular shape. Frobenius-Young formula (Proposition 2.1) implies the following.

Corollary 5.1. *The number of SYT of rectangular shape (n^m) is*

$$f^{(n^m)} = (mn)! \cdot \frac{F_m F_n}{F_{m+n}},$$

where

$$F_m := \prod_{i=0}^{m-1} i!.$$

For two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ (with trailing zeros added in order to get the same number of parts) define

$$\lambda + \mu := (\lambda_1 + \mu_1, \ldots, \lambda_k + \mu_k).$$

Note that if either λ or μ is a strict partition then $\lambda + \mu$ is also strict.

Lemma 5.2. *Let m, n and t be nonnegative integers with $t \le mn$. Let T be a SYT of rectangular shape (n^m), let T_1 be the set of all cells in T with values at most t, and let T_2 be obtained from $T \setminus T_1$ by transposing the shape and replacing each entry i by $mn - i + 1$. Then:*

1. *T_1 and T_2 are SYT.*
2. *Denote by λ_1 and λ_2 the shapes of T_1 and T_2, respectively, and treat strict partitions as sets. Then the strict partition $[m+n]$ is the disjoint union of $\lambda_1 + [m]$ and $\lambda_2 + [n]$.*

Proof. (1) is clear; let us prove (2). The border between T_1 and $T \setminus T_1$ is a lattice path of length exactly $m+n$, starting at the NW corner of the rectangular shape (n^m), using only S and W steps, and ending at the SW corner of the shape. If the first step is S then the first part of $\lambda_1 + [m]$ is $m+n$, and the rest (of both $\lambda_1 + [m]$ and $\lambda_2 + [n]$) corresponds to a lattice path in n^{m-1}. Similarly, if the first step is W then the first part of $\lambda_2 + [n]$ is $m+n$, and the rest corresponds to a lattice path in $(n-1)^m$. Thus exactly one of the strict partitions $\lambda_1 + [m]$ and $\lambda_2 + [n]$ has a part equal to $m+n$, and the whole result follows by induction on $m+n$.

∎

Corollary 5.3. *For any nonnegative integers m, n and t with $t \le mn$,*

$$\sum_{\substack{\lambda \subseteq (n^m) \\ |\lambda| = t}} f^\lambda f^{\lambda^c} = f^{(n^m)}.$$

Here summation is over all partitions λ with the prescribed restrictions, and λ^c is such that $\lambda^c+[n]$ is the complement of $\lambda+[m]$ in $[m+n]$ (where strict partitions are treated as sets). In particular, the LHS is independent of t.

Lemma 5.4. *Let λ and λ^c be partitions such that $[m+n]$ is the disjoint union of $\lambda+[m]$ and $\lambda^c+[n]$, and let $\mu=(\mu_1,\ldots,\mu_k)$ be an arbitrary partition ($\mu_1 \geq \ldots \geq \mu_k \geq 0$). Then*

$$f^{(\mu+(n^k))\cup\lambda} f^{(\mu+(m^k))\cup\lambda^c} = d(\mu,|\lambda|,|\lambda^c|)\cdot f^\lambda f^{\lambda^c},$$

where

$$d(\mu,|\lambda|,|\lambda^c|) = f^\mu f^{\mu+((m+n)^k)} \cdot \frac{(|\mu|+nk+|\lambda|)!(|\mu|+mk+|\lambda^c|)!}{(|\lambda|)!(|\lambda^c|)!(|\mu|)!(|\mu|+(m+n)k)!}.$$

Proof. From the assumptions it follows that λ is contained in (n^m). We may thus assume that it has m (nonnegative) parts. Similarly, λ^c is contained in (m^n) and we may assume that it has n (nonnegative) parts. Thus $((\mu+(n^k))\cup\lambda$ has $k+m$ parts and $(\mu+(m^k))\cup\lambda^c$ has $k+n$ parts. By Proposition 2.1,

$$\begin{aligned}\frac{f^{(\mu+(n^k))\cup\lambda} f^{(\mu+(m^k))\cup\lambda^c}}{f^\lambda f^{\lambda^c}} &= \frac{(|\mu|+nk+|\lambda|)!(|\mu|+mk+|\lambda^c|)!}{(|\lambda|)!(|\lambda^c|)!}\cdot \\ &\cdot\left(\prod_{i=1}^k \frac{1}{(\mu_i+m+n+k-i)!}\prod_{i<j}(\mu_i-\mu_j-i+j)\right)^2\cdot \\ &\cdot\prod_{i,j}(\mu_i+n-i-\lambda_j+k+j)\prod_{i,j}(\mu_i+m-i-\lambda^c_j+k+j).\end{aligned}$$

By the assumption on λ and λ^c,

$$\prod_j(\mu_i+n-i-\lambda_j+k+j)\prod_j(\mu_i+m-i-\lambda^c_j+k+j). = \prod_{j=1}^{m+n}(\mu_i-i+k+j) \qquad (\forall i).$$

Since

$$\frac{1}{(\mu_i+m+n+k-i)!}\cdot\prod_{j=1}^{m+n}(\mu_i-i+k+j) = \frac{1}{(\mu_i+k-i)!} \qquad (\forall i),$$

an application of Proposition 2.1 to f^μ and to $f^{\mu+((m+n)^k)}$ gives the desired result.

□

Theorem 5.5. *Let m and n be nonnegative integers, and let $\mu=(\mu_1,\ldots,\mu_k)$ be a partition ($\mu_1\geq\ldots\geq\mu_k\geq 0$). Then*

$$\begin{aligned}\sum_{\lambda\subseteq(n^m)} f^{(\mu+(n^k))\cup\lambda} f^{(\mu+(m^k))\cup\lambda^c} &= f^\mu f^{\mu+((m+n)^k)} f^{(n^m)}\cdot \\ &\cdot\binom{2|\mu|+mk+nk+mn+1}{mn}\cdot\frac{(|\mu|+mk)!(|\mu|+nk)!}{(|\mu|)!(|\mu|+mk+nk)!}.\end{aligned}$$

Proof. Restrict the summation to partitions λ of a fixed size $|\lambda| = t$. By Lemma 5.4 and Corollary 5.3,

$$\sum_{\substack{\lambda\subseteq(n^m)\\|\lambda|=t}} f^{(\mu+(n^k))\cup\lambda} f^{(\mu+(m^k))\cup\lambda^c} = d(\mu,t,M-t)\cdot \sum_{\substack{\lambda\subseteq(n^m)\\|\lambda|=t}} f^\lambda f^{\lambda^c} = d(\mu,t,M-t)\cdot f^{(n^m)}.$$

Now sum over all t and use the explicit formula for $d(\mu,t,M-t)$ (from Lemma 5.4) together with Lemma 4.5, to obtain the explicit formula above.

$\square$

Corollary 5.6. ($\mu = (0,\ldots,0)$, k parts) *The number of SYT of truncated rectangular shape $((n+k)^{m+k}) \setminus (k^{k-1}, k-1)$ is*

$$f^{((m+n)^k)} f^{(n^m)} \cdot \binom{mk+nk+mn+1}{mn} \cdot \frac{(mk)!(nk)!}{(mk+nk)!} = \frac{N!(mk)!(nk)!}{(mk+nk+1)!} \cdot \frac{F_m F_n F_k}{F_{m+n+k}},$$

where $N = mk+nk+mn+1 = (m+k)(n+k) - k^2 + 1$ is the size of the shape and F_n is as in Corollary 5.1.

For $k = 1$ we obtain

$$f^{(n+1)^{m+1}} = \frac{N!m!n!}{(m+n+1)!} \cdot \frac{F_m F_n}{F_{m+n+1}} = N! \cdot \frac{F_{m+1}F_{n+1}}{F_{m+n+2}},$$

in accordance with Corollary 5.1.

For $k = 2$ we obtain

Corollary 5.7. ($\mu = (0,0)$) *The number of SYT of truncated rectangular shape $((n+2)^{m+2}) \setminus (2,1)$ is*

$$\frac{N!(2m)!(2n)!}{(2m+2n+1)!} \cdot \frac{F_m F_n}{F_{m+n+2}},$$

where $N = mn + 2m + 2n + 1 = (m+2)(n+2) - 3$ is the size of the shape.

Corollary 5.8. ($\mu = (1,\ldots,1,0)$, k parts) *The number of SYT of truncated rectangular shape $((n+k)^{m+k}) \setminus ((k-1)^{k-1})$ is*

$$f^{((m+n+1)^{k-1},m+n)} f^{(n^m)} \cdot \binom{mk+nk+mn+2k-1}{mn} \cdot \frac{(mk+k-1)!(nk+k-1)!}{(k-1)!(mk+nk+k-1)!} =$$

$$= \frac{N!(mk+k-1)!(nk+k-1)!(m+n)!}{(mk+nk+k-1)!(mk+nk+2k-1)!} \cdot \frac{F_m F_n F_{k-1}}{F_{m+n+k+1}},$$

where $N = (m+k)(n+k) - (k-1)^2$ is the size of the shape and F_n is as in Corollary 5.1.

In particular, for $k = 2$:

Corollary 5.9. ($k = 2$, $\mu = (1,0)$) *The number of SYT of truncated rectangular shape $((n+2)^{m+2}) \setminus (1)$ is*

$$\frac{N!(2m+1)!(2n+1)!(m+n)!}{(2m+2n+1)!(2m+2n+3)!} \cdot \frac{F_m F_n}{F_{m+n+2}},$$

where $N = mn + 2m + 2n + 3 = (m+2)(n+2) - 1$ *is the size of the shape and* F_n *is as in Corollary 5.1.*

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