CLASSES OF EXTENSION MODULES BY SERRE SUBCATEGORIES

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Abstract. In [5], R. Takahashi showed an existence of isomorphism of lattices between the set of all Serre subcategories of the category consisting of all finitely generated modules and the set of all specialization closed subsets of the set of all prime ideals. In this paper, to find a way of constructing Serre subcategories of the modules category, we consider classes of extension modules of Serre subcategory by another one and study when these classes are Serre subcategory.

1. Introduction

Let R be a commutative noetherian ring, R-Mod be the category of all R-modules and R-mod be the full subcategory of all finitely generated R-modules.

In [4], A. Neeman showed that there exists an isomorphism of lattices between the set of all smashing subcategories of the derived category of R-Mod and the set of all specialization closed subsets of $\mathrm{Spec}(R)$. After of this, R. Takahashi constructed a module version of Neeman's theorem in [5]. Specifically, he showed that there exists an isomorphism of lattices between the set of all Serre subcategories of R-mod and the set of all specialization closed subsets of $\mathrm{Spec}(R)$.

A Serre subcategory is defined to be a full subcategory which is closed under submodules, quotients and extensions. Recently, many authors study the Serre subcategory not only in the category theory but also local cohomology theory. (For example see [1].) By the above Takahashi's result, we can give all Serre subcategories of R-mod. However, we want to find a way of constructing examples of Serre subcategory of R-Mod with a view of treating Serre subcategory in local cohomology theory. Therefore, one of the main purposes of this paper is to give a this way by considering the classes of extension modules of Serre subcategory by another one.

To be more precise, for two Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ of R-Mod, we consider a following class of extension modules

$$(\mathcal{S}_1, \mathcal{S}_2) = \left\{ M \in R\text{-Mod} \;\middle|\; \begin{array}{c} \text{there are } S_1 \in \mathcal{S}_1 \text{ and } S_2 \in \mathcal{S}_2 \text{ such that} \\ 0 \to S_1 \to M \to S_2 \to 0 \text{ is exact.} \end{array} \right\}.$$

For example, a class $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is known the set of all Minimax modules where $\mathcal{S}_{f.g.}$ is the set of all finitely generated R-modules and $\mathcal{S}_{Artin}$ is the set of all Artinian modules. In [2], K. Bahmanpour and R. Naghipour showed that this class is a Serre subcategory. However, in general, a class $(\mathcal{S}_1, \mathcal{S}_2)$ is not Serre subcategory. In fact, a class $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ is not it. In this paper, we shall give a necessary and sufficient condition that a class $(\mathcal{S}_1, \mathcal{S}_2)$ is Serre subcategory and several examples of Serre subcategory $(\mathcal{S}_1, \mathcal{S}_2)$. In particular, we shall see that the following classes are Serre subcategories:

(1) A class $(\mathcal{S}_1, \mathcal{S}_2)$ for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ of R-mod;

(2) A class $(\mathcal{S}_{f.g.}, \mathcal{S})$ for any Serre subcategory $\mathcal{S}$ of R-Mod;

(3) A class $(\mathcal{S}, \mathcal{S}_{Artin})$ for any Serre subcategory $\mathcal{S}$ of R-Mod.

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The organization of this paper is as follows.

In section 2, we shall give the definition of classes $(\mathcal{S}_1, \mathcal{S}_2)$ of extension modules related to Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ of R-Mod and study basic properties.

In section 3, we shall treat Serre subcategories of R-mod. In particular, we show the above example (1) is Serre subcategory of R-mod. (Theorem 3.3.)

The section 4 is a main part of this paper. We give a necessary and sufficient condition for a class $(\mathcal{S}_1, \mathcal{S}_2)$ is Serre subcategory. (Theorem 4.2.)

In section 5, we apply our argument to the local cohomology theory involving the condition C_I for an ideal I of R which is defined in [1].

2. The definition of classes of extension modules by Serre subcategories

Throughout this paper, all rings are commutative noetherian ring and all modules are unitary. We assume that all full subcategories $\mathcal{S}$ of R-Mod and R-mod are closed under isomorphisms, that is if $M \in \mathcal{S}$ and R-module N is isomorphic to M then $N \in \mathcal{S}$.

In this section, we shall give the definition of classes of extension modules by Serre subcategories and study basic properties.

Recall that a class $\mathcal{S}$ of R-Mod is said to be a Serre subcategory of R-Mod if $\mathcal{S}$ is closed under submodules, quotients and extensions. We also say that a Serre subcategory $\mathcal{S}$ of R-Mod is a Serre subcategory of R-mod if $\mathcal{S}$ consists of finitely generated R-modules.

Definition 2.1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We denote by $(\mathcal{S}_1, \mathcal{S}_2)$ the class of all R-modules M with some R-modules $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$ such that a sequence $0 \to S_1 \to M \to S_2 \to 0$ is exact, that is

$$(\mathcal{S}_1, \mathcal{S}_2) = \left\{ M \in R\text{-Mod} \;\middle|\; \begin{array}{c} \text{there are } S_1 \in \mathcal{S}_1 \text{ and } S_2 \in \mathcal{S}_2 \text{ such that} \\ 0 \to S_1 \to M \to S_2 \to 0 \text{ is exact.} \end{array} \right\}.$$

We shall refer to $(\mathcal{S}_1, \mathcal{S}_2)$ as a class of extension modules of $\mathcal{S}_1$ by $\mathcal{S}_2$.

Remark 2.2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod.

(1) Since the zero module belongs to any Serre subcategory, it holds $\mathcal{S}_1 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$ and $\mathcal{S}_2 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

(2) It holds $\mathcal{S}_1 \supseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_1$.

(3) It holds $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_2$.

(4) A class $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under finite direct sums.

Example 2.3. We denote by $\mathcal{S}_{f.g.}$ the set of all finitely generated R-modules and by $\mathcal{S}_{Artin}$ the set of all Artinian modules. Then a class $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is the set of all Minimax R-modules and a class $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ is the set of all Maxmini R-modules.

Proposition 2.4. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. Then a class $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under submodules and quotients.*

Proof. Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules and assume that M is in $(\mathcal{S}_1, \mathcal{S}_2)$. We shall show that L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$.

There exists $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$ such that

$$0 \to S_1 \xrightarrow{\varphi} M \to S_2 \to 0$$

is an exact sequence. Then we can construct a following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_1 \cap L & \longrightarrow & S_1 & \longrightarrow & \dfrac{S_1}{S_1 \cap L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \overline{\varphi} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \dfrac{L}{S_1 \cap L} & \longrightarrow & S_2 & \longrightarrow & N' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns where $\overline{\varphi}$ is a natural map induced by φ and $N' = \mathrm{Coker}(\overline{\varphi})$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are Serre subcategories, we see that $S_1 \cap L$, $S_1/(S_1 \cap L) \in \mathcal{S}_1$ and $L/(S_1 \cap L)$, $N' \in \mathcal{S}_2$. Therefore, L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$. □

A natural question arises.

Question. For any Serre subcategory $\mathcal{S}_1$ and $\mathcal{S}_2$, is the class $(\mathcal{S}_1, \mathcal{S}_2)$ Serre subcategory?

For example, K. Bahmanpour and R. Naghipour showed that the class $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is a Serre subcategory in [2, Lemma 2.1]. The Proposition 2.4 says that a class $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory if this class is closed under extension. However, the conclusion in above question does not hold in general.

Example 2.5. We shall see the class $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ is not necessary closed under extension.

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension one with maximal ideal $\mathfrak{m}$. Then we have a minimal injective resolution

$$0 \to R \to \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(R),\\ \mathrm{ht}\mathfrak{p} = 0}} E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{m}) \to 0$$

of R. (For an R-module M, $E(M)$ denotes the injective hull of M.) We note R and $E_R(R/\mathfrak{m})$ are in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$.

Now, we suppose that $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ is closed under extension. Then $E_R(R) = \oplus_{\mathrm{ht}\mathfrak{p}=0} E_R(R/\mathfrak{p})$ is in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$, and so there exists an Artinian R-submodule N of $E_R(R)$ with $E_R(R)/N$ is a finitely generated R-module. But, since R is a Gorenstein local ring of dimension one, N must be zero module. Thus $E_R(R)$ is a finitely generated injective R-module. By the Bass formula, it holds $\dim R = \operatorname{depth} R = \operatorname{inj\,dim} E_R(R) = 0$. This is a contradiction.

3. Classes of extension modules by Serre subcategories of R-mod

In this section, we shall see that a class $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ of R-mod.

Recall that a subset W of $\mathrm{Spec}(R)$ is said to be a specialization closed subset if $\mathfrak{p} \in W$ and $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(R)$ imply $\mathfrak{q} \in W$. Γ_W denotes the section functor with support in a specialization closed subset W of $\mathrm{Spec}(R)$, that is

$$\Gamma_W(M) = \{x \in M \mid \mathrm{Supp}(Rx) \subseteq W\}$$

for each R-module M.

Let us start to prove following two lemmas.

Lemma 3.1. *Let M be an R-module and W be a specialization closed subset of $\mathrm{Spec}(R)$. Then $\mathrm{Ass}(\Gamma_W(M))$ and $\mathrm{Ass}(M/\Gamma_W(M))$ are disjoint, and that*

$$\mathrm{Ass}(M) = \mathrm{Ass}(\Gamma_W(M)) \cup \mathrm{Ass}(M/\Gamma_W(M)).$$

Proof. It is clear that $\mathrm{Ass}(\Gamma_W(M)) \subseteq W$ and $\mathrm{Ass}(M/\Gamma_W(M)) \cap W = \emptyset$. Thus $\mathrm{Ass}(\Gamma_W(M)) \cup \mathrm{Ass}(M/\Gamma_W(M))$ is disjoint union.

To see the equality, it is enough to show that $\mathrm{Ass}(M/\Gamma_W(M)) \subseteq \mathrm{Ass}(M)$. Let $E_R(M)$ be an injective hull of M. Then $\Gamma_W(E_R(M))$ is also an injective R-module. (Also see [6, Theorem 2.7].) We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_W(M) & \longrightarrow & M & \longrightarrow & M/\Gamma_W(M) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \Gamma_W(\varphi)=\varphi|_{\Gamma_W(M)}} & & \Big\downarrow {\scriptstyle \varphi} & & \Big\downarrow {\scriptstyle \overline{\varphi}} & & \\ 0 & \longrightarrow & \Gamma_W(E_R(M)) & \longrightarrow & E_R(M) & \longrightarrow & E_R(M)/\Gamma_W(E_R(M)) & \longrightarrow & 0 \end{array}$$

with exact rows where φ is an inclusion map from M to $E_R(M)$ and $\overline{\varphi}$ is a homomorphism induced by φ. It follows from the injectivity of $\Gamma_W(E_R(M))$ that the second row is a split exact sequence, so that it holds

$$E_R(M) = \Gamma_W(E_R(M)) \oplus E_R(M)/\Gamma_W(E_R(M))$$

and $E_R(M)/\Gamma_W(E_R(M))$ is an injective R-module.

Here, we note that $\overline{\varphi}$ is monomorphism. Actually, we assume $m \in M$ such that $\overline{\varphi}(m + \Gamma_W(M)) = 0 \in E_R(M)/\Gamma_W(E_R(M))$. Then $m \in M \cap \Gamma_W(E_R(M)) = \Gamma_W(M)$. Thus $m + \Gamma_W(M) = 0 \in M/\Gamma_W(M)$.

By properties of injective hulls, it holds

$$\begin{aligned} \mathrm{Ass}(M/\Gamma_W(M)) &= \mathrm{Ass}\big(E_R(M/\Gamma_W(M))\big) \\ &\subseteq \mathrm{Ass}\big(E_R(M)/\Gamma_W(E_R(M))\big) \\ &\subseteq \mathrm{Ass}(E_R(M)) \\ &= \mathrm{Ass}(M). \end{aligned}$$

The proof is completed. □

If Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ are related to specialization closed subsets, then the structure of a class $(\mathcal{S}_1, \mathcal{S}_2)$ is clear.

Lemma 3.2. *For specialization closed subsets W_1 and W_2 of $\mathrm{Spec}(R)$, the following assertions hold.*

(1) *We set $\mathcal{S}_i = \{M \in R\text{-Mod} \mid \mathrm{Supp}(M) \subseteq W_i\}$ for $i = 1, 2$. Then it holds*

$$(\mathcal{S}_1, \mathcal{S}_2) = \{M \in R\text{-Mod} \mid \mathrm{Supp}(M) \subseteq W_1 \cup W_2\}.$$

(2) *We set $\mathcal{S}_i = \{M \in R\text{-mod} \mid \mathrm{Supp}(M) \subseteq W_i\}$ for $i = 1, 2$. Then it holds*

$$(\mathcal{S}_1, \mathcal{S}_2) = \{M \in R\text{-mod} \mid \mathrm{Supp}(M) \subseteq W_1 \cup W_2\}.$$

In particular, $(\mathcal{S}_1, \mathcal{S}_2)$ in (1) and (2) are Serre subcategories.

Proof. (1) If M is in $(\mathcal{S}_1, \mathcal{S}_2)$, then there exists a short exact sequence $0 \to S_1 \to M \to S_2 \to 0$ with $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$. Then $\mathrm{Supp}(M) = \mathrm{Supp}(S_1) \cup \mathrm{Supp}(S_2) \subseteq W_1 \cup W_2$.

Conversely, let M be an R-module with $\mathrm{Supp}(M) \subseteq W_1 \cup W_2$. We consider a short exact sequence

$$0 \to \varGamma_{W_1}(M) \to M \to M/\varGamma_{W_1}(M) \to 0.$$

We note $\mathrm{Supp}(\varGamma_{W_1}(M)) \subseteq W_1$. Therefore, to prove our assertion, it is enough to show that it holds $\mathrm{Ass}(M/\varGamma_{W_1}(M)) \subseteq W_2$. It follows from Lemma 3.1 that we have

$$\mathrm{Ass}(M/\varGamma_{W_1}(M)) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Supp}(M) \subseteq W_1 \cup W_2.$$

Furthermore, we note $\mathrm{Ass}(M/\varGamma_{W_1}(M)) \cap W_1 = \emptyset$. Consequently, it holds $\mathrm{Ass}(M/\varGamma_{W_1}(M)) \subseteq W_2$, so M is in $(\mathcal{S}_1, \mathcal{S}_2)$.
(2) We can show the assertion by the same argument in (1). □

Now, we can prove the purpose of this section.

Theorem 3.3. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-mod. Then a class $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory of R-mod.*

Proof. By [5, Theorem 4.1], there is a bijection between the set of all Serre subcategories of R-mod and the set of all specialization closed subsets of $\mathrm{Spec}(R)$. Thus, there exists a specialization closed subset W_1 (resp. W_2) of $\mathrm{Spec}(R)$ corresponding to the Serre subcategory $\mathcal{S}_1$ (resp. $\mathcal{S}_2$). In particular, we can denote

$$\mathcal{S}_i = \{M \in R\text{-mod} \mid \mathrm{Supp}(M) \subseteq W_i\} \text{ and } W_i = \bigcup_{M \in \mathcal{S}_i} \mathrm{Supp}(M)$$

for each i. By lemma 3.2, it holds

$$(\mathcal{S}_1, \mathcal{S}_2) = \{M \in R\text{-mod} \mid \mathrm{Supp}(M) \subseteq W_1 \cup W_2\}$$

and this is a Serre subcategory of R-mod. □

4. The condition of closed under extension for $(\mathcal{S}_1, \mathcal{S}_2)$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. In this section, we shall give a necessary and sufficient condition that a class $(\mathcal{S}_1, \mathcal{S}_2)$ is Serre subcategory and several examples of Serre subcategory $(\mathcal{S}_1, \mathcal{S}_2)$.

We start to prove the following lemma. If a class $(\mathcal{S}_1, \mathcal{S}_2)$ is Serre subcategory, then we have already seen $(\mathcal{S}_1, (\mathcal{S}_1, \mathcal{S}_2)) = ((\mathcal{S}_1, \mathcal{S}_2), \mathcal{S}_2) = (\mathcal{S}_1, \mathcal{S}_2)$ in Remark 2.2. However, we can see that the following assertion holds without such an assumption.

Lemma 4.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We suppose that a sequence $0 \to L \to M \to N \to 0$ of R-modules is exact. Then the following assertions hold.*

(1) *If $L \in \mathcal{S}_1$ and $N \in (\mathcal{S}_1, \mathcal{S}_2)$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

(2) *If $L \in (\mathcal{S}_1, \mathcal{S}_2)$ and $N \in \mathcal{S}_2$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) We assume $L \in \mathcal{S}_1$ and $N \in (\mathcal{S}_1, \mathcal{S}_2)$. Since N is in $(\mathcal{S}_1, \mathcal{S}_2)$, there exists an exact sequence $0 \to S \to N \to T \to 0$ with $S \in \mathcal{S}_1$ and $T \in \mathcal{S}_2$. Then we have a pull buck diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & S' & \longrightarrow & S & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T & = & T & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Since $\mathcal{S}_1$ is a Serre subcategory, it follows from the first row that S' is in $\mathcal{S}_1$. Thus, by the middle column, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$.
(2) We assume $L \in (\mathcal{S}_1, \mathcal{S}_2)$ and $N \in \mathcal{S}_2$. Since L is in $(\mathcal{S}_1, S_2)$, there exists an exact sequence $0 \to S \to L \to T \to 0$ with $S \in \mathcal{S}_1$ and $T \in \mathcal{S}_2$. Then we have a push out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T & \longrightarrow & T' & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns. Since $\mathcal{S}_2$ is a Serre subcategory, it follows from the third row that T' is in $\mathcal{S}_2$. Thus, by the middle column, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$. $\square$

Now, we can show the main purpose of this paper.

Theorem 4.2. *For Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ of R-*Mod*, the following conditions are equivalent:*

(1) *A class $(\mathcal{S}_1, \mathcal{S}_2)$ is Serre subcategory;*
(2) *It holds $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) $\Rightarrow$ (2) We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$. By the definition of the class $(\mathcal{S}_2, \mathcal{S}_1)$, there exists an exact sequence

$$0 \to S_2 \to M \to S_1 \to 0$$

where $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$. We note $S_1 \in \mathcal{S}_1 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$ and $S_2 \in \mathcal{S}_2 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$. Since a class $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extension by the assumption (1), we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$.
(2) $\Rightarrow$ (1) We only have to prove that a class $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extension by Proposition 2.4. Let $0 \to L \to M \to N \to 0$ be an exact sequence such that L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$. We shall show that M is also in $(\mathcal{S}_1, \mathcal{S}_2)$.

It follows from $L \in (\mathcal{S}_1, \mathcal{S}_2)$ that there exists a short exact sequence

$$0 \to S \to L \to L/S \to 0$$

where $S \in S_1$ with $L/S \in \mathcal{S}_2$. We have a push out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L/S & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns. Next, it follows from $N \in (\mathcal{S}_1, \mathcal{S}_2)$ that there exists a short exact sequence

$$0 \to T \to N \to N/T \to 0$$

where $T \in S_1$ with $N/T \in \mathcal{S}_2$. We have a pull back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/S & \longrightarrow & P' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/S & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N/T & = & N/T & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns.

In the first row of the second diagram, it follows from $L/S \in \mathcal{S}_2$ and $T \in \mathcal{S}_1$ that we see $P' \in (\mathcal{S}_2, \mathcal{S}_1)$. Now here, by the assumption (2), P' is in $(\mathcal{S}_1, \mathcal{S}_2)$. Next, in the middle column of the second diagram, we have the short exact sequence with $P' \in (\mathcal{S}_1, \mathcal{S}_2)$ and $N/T \in \mathcal{S}_2$. Therefore, it follows from Lemma 4.1 that P is in $(\mathcal{S}_1, \mathcal{S}_2)$. Finally, in the middle column of the first diagram, we have the short exact sequence with $S \in \mathcal{S}_1$ and $P \in (\mathcal{S}_1, \mathcal{S}_2)$. Thus, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ by Lemma 4.1.

The proof is completed. □

In the rest of this section, we shall give several examples of Serre subcategory $(\mathcal{S}_1, \mathcal{S}_2)$. The first example is a generalization of [2, Lemma 2.1] which states that $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is a Serre subcategory.

Corollary 4.3. *A class $(\mathcal{S}_{f.g.}, \mathcal{S})$ is a Serre subcategory for any Serre subcategory $\mathcal{S}$ of R-*Mod.

Proof. Let $\mathcal{S}$ be a Serre subcategory of R-Mod. To prove our assertion, it is enough to show $(\mathcal{S}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S})$ by Theorem 4.2. Let M be in $(\mathcal{S}, \mathcal{S}_{f.g.})$. Then there exists an R-submodule $L \in \mathcal{S}$ of M with $M/L \in \mathcal{S}_{f.g.}$. It is easy to see that there exists a finitely generated R-submodule K of M such that $M = K + L$. Since $K \oplus L \in (\mathcal{S}_{f.g.}, \mathcal{S})$ and M is a homomorphic image of $K \oplus L$, M is in $(\mathcal{S}_{f.g.}, \mathcal{S})$. $\square$

Example 4.4. Let R be a domain but not a field, and Q be a field of fractions of R. We set $\mathcal{S}_{Tor}$ the set of all torsion R-modules, that is $\mathcal{S}_{Tor} = \{M \in R\text{-Mod} \mid M \otimes_R Q = 0\}$. Then we shall see that it holds

$$(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneqq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor}) = \{M \in R\text{-Mod} \mid \dim_Q M \otimes_R Q < \infty\}.$$

So $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$ is a Serre subcategory, but $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$ is not closed under extension by theorem 4.2.

First of all, we shall show that the above equality holds. We suppose that M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Then there exists a short exact sequence

$$0 \to L \to M \to N \to 0$$

with $L \in \mathcal{S}_{f.g.}$ and $N \in \mathcal{S}_{Tor}$. We apply an exact functor $- \otimes_R Q$ to this sequence, then we see that $M \otimes_R Q \cong L \otimes_R Q$ is a finite Q-vector space. Conversely, let M be an R-module with $\dim_Q M \otimes_R Q < \infty$. Then we can denote $M \otimes_R Q = \sum_{i=1}^n Q(m_i \otimes 1_Q)$ with $m_i \in M$ and the unit element 1_Q of Q. We consider a short exact sequence

$$0 \to \textstyle\sum_{i=1}^n Rm_i \to M \to M/\sum_{i=1}^n Rm_i \to 0.$$

It is clear that $\sum_{i=1}^n Rm_i \in \mathcal{S}_{f.g.}$ and $M/\sum_{i=1}^n Rm_i \in \mathcal{S}_{Tor}$. So M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$.

Next, it is clear that $M \otimes_R Q$ has a finite dimension as Q-vector space for $M \in (\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Thus it holds $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$.

Finally, we shall see $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneqq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. We fix a non-inverse element $r \neq 0$ of R and consider $M = R + \sum_{n \in \mathbb{N}} R\frac{1}{r^n}$. Then it holds $\dim_Q M \otimes_R Q = 1$, so M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. We suppose that M is in $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Since R is domain and M is R-submodule of Q, torsion R-submodule of M is only zero module. This means that M must be a finitely generated R-module. But, this is a contradiction. Consequently, it holds $M \in (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor}) \setminus (\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$.

We note that $\mathcal{S}_{Artin}$ is a Serre subcategory with closed under injective hulls. Therefore we can see that a class $(\mathcal{S}, \mathcal{S}_{Artin})$ is also Serre subcategory for any Serre subcategory of R-Mod by a following assertion.

Corollary 4.5. *Let $\mathcal{S}_2$ be a Serre subcategory of* R-Mod *with closed under injective hulls. Then a class $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory for any Serre subcategory $\mathcal{S}_1$ of* R-Mod.

Proof. By Theorem 4.2, it is enough to show $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$ and shall show that M is in $(\mathcal{S}_1, \mathcal{S}_2)$. Then there exists a short exact sequence

$$0 \to S_2 \xrightarrow{i} M \to S_1 \to 0$$

with $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$. Since $\mathcal{S}_2$ is closed under injective hulls, the injective hull $E_R(S_2)$ of S_2 is in $\mathcal{S}_2$. It follows from the injectivity of $E_R(S_2)$ that there exists a homomorphism $\varphi : M \to E_R(S_2)$ such that $j = \varphi \circ i$ where j is the inclusion map from S_2 to $E_R(S_2)$. Here, we consider a push out diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Ker}(\varphi) & \xrightarrow{\eta=\psi|_{\mathrm{Ker}(\varphi)}} & \mathrm{Ker}(\beta) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_2 & \xrightarrow{i} & M & \xrightarrow{\psi} & S_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j=\varphi|_{S_2}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \mathrm{Ker}(\alpha) & \longrightarrow & \mathrm{Im}(\varphi) & \xrightarrow{\alpha} & T & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

with exact rows and columns where

$$T = (\mathrm{Im}(\varphi) \oplus S_1)/\{\big(\varphi(m), -\psi(m)\big) \mid m \in M\},$$

$\alpha(t) = \overline{(t, 0)}$ for $t \in \mathrm{Im}(\varphi)$ and $\beta(s) = \overline{(0, s)}$ for $s \in S_1$.

To prove our assertion, we shall show that $\mathrm{Ker}(\varphi)$ is in $\mathcal{S}_1$ and $\mathrm{Im}(\varphi)$ is in $\mathcal{S}_2$. Since $\mathrm{Im}(\varphi)$ is R-submodule of $E_R(S_2)$, $\mathrm{Im}(\varphi)$ is in $\mathcal{S}_2$. To prove the first, all we have to see that η is injective. Indeed, it follows from the injectivity of η that we see $\mathrm{Ker}(\varphi) \subseteq \mathrm{Ker}(\beta) \subseteq S_1 \in \mathcal{S}_1$.

We shall see that η is injective. Let $m \in \mathrm{Ker}(\varphi)$ such that $\eta(m) = 0$. Then $\varphi(m) = 0 \in \mathrm{Im}(\varphi)$ and $\psi(m) = 0 \in S_1$, and so $m \in S_2$ and $0 = \varphi(m) = \varphi \circ i(m) = j(m)$. It follows from the injectivity of j that we have $m = 0$. Consequently, η is injective.

The proof is completed. □

Remark 4.6. If $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod with closed under injective hulls, then we can see that a class ($\mathcal{S}_1$, $\mathcal{S}_2$) is also a Serre subcategory with closed under injective hulls as following.

Let M be in $(\mathcal{S}_1, \mathcal{S}_2)$ and we shall prove that $E_R(M)$ is also in $(\mathcal{S}_1, \mathcal{S}_2)$. There exists a short exact sequence $0 \to L \to M \to N \to 0$ with $L \in \mathcal{S}_1$ and $N \in \mathcal{S}_2$. We consider a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \tau} & & \\
0 & \longrightarrow & E_R(L) & \longrightarrow & E_R(L) \oplus E_R(N) & \longrightarrow & E_R(N) & \longrightarrow & 0
\end{array}
$$

with exact rows and columns. (For $m \in M$, we define $\eta(m) = (\mu(m), \tau \circ \psi(m))$ where $\mu : M \to E_R(L)$ is a homomorphism induced by the injectivity of $E_R(L)$ such that $\sigma = \mu \circ \varphi$.) Therefore, $E_R(M)$ is a direct summand of $E_R(L) \oplus E_R(N) \in (\mathcal{S}_1, \mathcal{S}_2)$, and so $E_R(M)$ is in $(\mathcal{S}_1, \mathcal{S}_2)$ by Proposition 2.4.

5. On conditions C_I for $(\mathcal{S}_1, S_2)$

Let I be an ideal of R. The following conditions C_I were defined for Serre subcategories of R-Mod by M. Aghapouranahr and L. Melkersson in [1]. Here, we shall apply this conditions to classes of R-modules.

Definition 5.1. Let $\mathcal{C}$ be a class of R-modules and I be an ideal of R. We say that $\mathcal{C}$ satisfies the condition C_I if the following condition satisfied:

$$(C_I) \quad \text{If } M = \Gamma_I(M) \text{ and } (0 :_M I) \text{ is in } \mathcal{C}, \text{ then } M \text{ is in } \mathcal{C}.$$

If the class $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory, we have already seen that it holds $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$. In this section, we shall see that the condition C_I for a class $(\mathcal{S}_2, \mathcal{S}_1)$ induces the condition C_I for a Serre subcategory $(\mathcal{S}_1, \mathcal{S}_2)$.

Lemma 5.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subcategories of R-Mod. We suppose that N is a finitely generated R-module and M is in $(\mathcal{S}_1, \mathcal{S}_2)$. Then $\operatorname{Ext}^i_R(N, M) \in (\mathcal{S}_1, \mathcal{S}_2)$ for all integer i.*

Proof. Since $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under finite direct sums, submodules and quotients, this is clear. □

Theorem 5.3. *Let I be an ideal of R and $\mathcal{S}_1, \mathcal{S}_2$ be Serre subcategories of R-Mod. We suppose that $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory of R-Mod. If a class $(\mathcal{S}_2, \mathcal{S}_1)$ satisfies the condition C_I, then $(\mathcal{S}_1, \mathcal{S}_2)$ also satisfies the condition C_I.*

Proof. Suppose that $M = \Gamma_I(M)$ and $(0 :_M I) \in (\mathcal{S}_1, \mathcal{S}_2)$. Then we have to show $M \in (\mathcal{S}_1, \mathcal{S}_2)$.

Since $(0 :_M I) \in (\mathcal{S}_1, \mathcal{S}_2)$, there exists an R-submodule $L \in \mathcal{S}_1$ of $(0 :_M I)$ such that $(0 :_M I)/L \in \mathcal{S}_2$. We consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & (0 :_M I) & \longrightarrow & (0 :_M I)/L & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & M/L & \longrightarrow & 0 \end{array}$$

with exact rows. To prove our assertion, it is enough to show that

$$\Gamma_I(M/L) = M/L \quad \text{and} \quad (0 :_{M/L} I) \in (\mathcal{S}_2, \mathcal{S}_1).$$

Indeed, if we can show these, then M/L is in $(\mathcal{S}_2, \mathcal{S}_1)$ by the condition C_I for $(\mathcal{S}_2, \mathcal{S}_1)$. Furthermore, since $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory, it holds $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$ by Theorem 4.2. Therefore it follows from $L \in \mathcal{S}_1$ and $M/L \in (\mathcal{S}_1, \mathcal{S}_2)$ that M is in $(\mathcal{S}_1, \mathcal{S}_2)$.

The first is clear by $\operatorname{Supp}(M/L) \subseteq \operatorname{Supp}(M) \subseteq V(I)$. To see the second, we apply a functor $\operatorname{Hom}_R(R/I, -)$ to the short exact sequence

$$0 \to L \to M \to M/L \to 0.$$

Then there exists an exact sequence

$$0 \longrightarrow (0 :_L I) \longrightarrow (0 :_M I) \stackrel{\varphi}{\longrightarrow} (0 :_{M/L} I) \longrightarrow \operatorname{Ext}^1_R(R/I, L).$$

It follows from $L \subseteq (0 :_M I)$ that we have $(0 :_L I) = L$, so that $\operatorname{Im}(\varphi) \cong (0 :_M I)/L \in \mathcal{S}_2$. Moreover, we have $\operatorname{Ext}^1_R(R/I, L) \in S_1$ by Lemma 5.2. Consequently, we see $(0 :_{M/L} I) \in (\mathcal{S}_2, \mathcal{S}_1)$.

The proof is completed. □

Finally, we try to apply the notion of $(\mathcal{S}_1, \mathcal{S}_2)$ to the local cohomology theory. It seems that we can rewrite several results of it concerned with Serre subcategories. However, we shall only rewrite [3, 7.1.6 Theorem] as demonstration here.

Proposition 5.4. *Let $(R, \mathfrak{m})$ be a local ring and I be an ideal of R. For non-zero Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$ of R-*Mod*, we suppose that a class $(\mathcal{S}_1, \mathcal{S}_2)$ satisfies the condition C_I. Then $H_I^{\dim M}(M) \in (\mathcal{S}_1, \mathcal{S}_2)$ for any finitely generated R-module M.*

Proof. We use induction on $n = \dim M$. If $n = 0$, M has finite length. Since it is clear that finite length R-modules are in any non-zero Serre subcategory, it holds $\Gamma_I(M) \in \mathcal{S}_1 \cup \mathcal{S}_2 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

Now suppose that $n > 0$ and we have established the result for finitely generated R-modules of dimension smaller than n. It is clear that $H_I^i(M) \cong H_I^i(M/\Gamma_I(M))$ for all $i > 0$. Thus we may assume that $\Gamma_I(M) = 0$. Then the ideal I contains an M-regular element x. Therefore there exists a short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0,$$

and this sequence induces an exact sequence

$$H_I^{n-1}(M/xM) \longrightarrow H_I^n(M) \xrightarrow{x} H_I^n(M) \longrightarrow 0.$$

By the induction hypothesis, $H_I^{n-1}(M/xM) \in (\mathcal{S}_1, \mathcal{S}_2)$. Here, since $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under quotients, $(0 :_{H_I^n(M)} x)$ is in $(\mathcal{S}_1, \mathcal{S}_2)$. Furthermore, since $(0 :_{H_I^n(M)} I) \subseteq (0 :_{H_I^n(M)} x)$ and $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under submodules, $(0 :_{H_I^n(M)} I)$ is in $(\mathcal{S}_1, \mathcal{S}_2)$. It follows from the condition C_I for $(\mathcal{S}_1, \mathcal{S}_2)$ that $H_I^n(M)$ is in $(\mathcal{S}_1, \mathcal{S}_2)$.

The proof is completed. □

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