A note on Kuczek's argument for non nearest neighbor contact processes

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Abstract

We are concerned with the supercritical process on the integers. The extension of the argument of Kuczek [11] to this case is due to Mountford and Sweet [13]. Their approach is based on proving that there is a positive chance that the right endpoint descends for all times from the origin; we obtain a new proof of this as a consequence of the following new result: There is a positive chance that the processes started from all sites and from any finite set agree on this set for all times; notably this result extends for the process on $\mathbb{Z}^d$. Our approach allows us to give an independent proof of the existence of random points (termed break points) among which the behavior of the right endpoint stochastically replicates. Finally, an improved large deviations result than that of Lemma 3 in [13] is derived via the work of Durrett and Schonmann [8].

1 Introduction and Results

In the present paper $\mathbb{S}$ will be a set of spatial locations which will be taken to be either $\mathbb{Z}$, the integers, or $\mathbb{Z}^-$, the non positive integers. The *non nearest neighbors contact process* on $\mathbb{S}$, with parameter $\mu > 0$ and uniform range of interaction M, where $M \geq 1$ is some finite constant, is a continuous time Markov process ξ_t on the set of subsets of $\mathbb{S}$. The process is thought of as the evolution of particle's offspring; each site in ξ_t is regarded as being occupied by a particle, while all other sites are regarded as being vacant. In this interpretation, the process evolves according to the following local prescription: (i) Particles die at rate 1. (ii) A particle at site x gives birth to new ones at rate μ at each site of $\mathbb{S}$ within the interval $[x-M, x+M]$. (iii) There is at most one particle per site, i.e. particles being born at a site that is occupied coalesce for all subsequent times.

We denote by ξ_t^V the process with V as a starting set, i.e. ξ_t^V is thought of as the descendants of the particles initially placed in V. When $V \in \mathbb{R}$, where $\mathbb{R}$ denotes the real numbers, then the initial state should be taken to be $V \cap \mathbb{S}$, no confusion should arise from this common practice.

Since its introduction by Harris [9] in 1974 the contact process has been extensively studied in the literature, for a recent detailed account we refer to Part I of Liggett [12]. We note that the family of processes under consideration when $\mathbb{S}$ is taken to be the integers, $\mathbb{Z}$, is also named in the literature as the symmetric finite range contact processes, see e.g. [4].

We will say that the process is *supercritical* if $\mathbf{P}(\xi_t \text{ survives}) > 0$, where we use the shorthand $\{\xi_t \text{ survives}\}$ for $\{\xi_t \neq \emptyset \text{ for all } t \geq 0\}$. A fundamental known fact for the contact process is that if it is supercritical on $\mathbb{Z}$ then it is supercritical on $\mathbb{Z}^-$ as well (this was first established as a consequence of the construction in [6, 7], indeed, it is part of a general result, concerning orthands of $\mathbb{Z}^d$, see [1, 2]). This fact will be the key for connecting Proposition 1.1 below with the contact process on $\mathbb{Z}$, which we will subsequently consider.

The following is the first principle result of this paper.

Proposition 1.1. *Consider the supercritical non nearest neighbors contact processes on $\mathbb{Z}^-$, $(\xi_t^{\mathbb{Z}^-})_{t\geq 0}$ and $(\xi_t^F)_{t\geq 0}$, where F is any finite subset of $\mathbb{Z}^-$; we have that,* $\mathbf{P}\left(\xi_t^F \cap F = \xi_t^{\mathbb{Z}^-} \cap F, \text{ for all } t \geq 0\right) > 0.$

Proposition 1.1 can be intuitively interpreted as the result of coalescence among the descendants of particles initially placed at F and those placed at $\mathbb{Z}^-\backslash F$ on the event $\{\xi_t^F \text{survives}\}$. Notably, the proof given below provides us the extension of this result for the process on all graphs for which what is known as a shape theorem holds (most prominently, $\mathbb{Z}^d$, see e.g. [1]).

Consider the supercritical non nearest neighbors contact processes on $\mathbb{Z}$, $(\xi_t^0)_{t\geq 0}$ and also $(\xi_t^{(-\infty,0]})_{t\geq 0}$; further, define their respective right endpoints to be $r_t = \sup \xi_t^0$ and $R_t = \sup \xi_t^{(-\infty,0]}, t \geq 0$. The following statement is Theorem 3 in [13], the proof there goes through a sophisticated block construction argument. We provide an elementary proof of this as a consequence of Proposition 1.1.

Corollary 1.2. $\mathbf{P}(r_t = R_t,$ *for all* $t \geq 0) > 0.$

By use of Corollary 1.2 we shall also obtain an independent proof of the following result.

Theorem 1.3. *There exist almost surely finite random (but not stopping) times $0 := \psi_{-1} < \psi_0 < \psi_1 < \ldots$ such that on $\{\xi_t^0 \text{ survives}\}$, $(r_{\psi_n} - r_{\psi_{n-1}}, \psi_n - \psi_{n-1})$, $n \geq 1$, are i.i.d. random vectors.*

A result stronger than Theorem 1.3 is provided by Mountford and Sweet [13]; however, our proof is an improvement in the sense that it is simpler and shorter.

In the next section of this paper we introduce the graphical representation. In section 3 we prove Proposition 1.1, while Corollary 1.2 and Theorem 1.3 are proved in section 4. In the final section, as a consequence of a result in [8], we establish a large deviations result for the density of occupied sites in oriented percolation which is an improvement of Lemma 3 in [13].

2 Preliminaries

The graphical representation (introduced by Harris [10]) is a way to embed the process in a graphical construction over $\mathbb{S} \times [0, \infty)$, where $\times$ will denote Cartesian product. The space $\mathbb{S} \times [0, \infty)$ should be thought of as giving a time line to each site of $\mathbb{S}$.

Recall that we denote by $M \geq 1$ the (constant) range of interaction. To carry out the construction, assign a Poisson process $N_{(x,y)}$ of rate μ to each ordered pair of sites of $\mathbb{S}$, (x, y), such that $|y - x| \leq M$; assign also a Poisson process N_x of rate 1 to each site $x \in \mathbb{S}$. For the arrival times t of $N_{(x,y)}$ we place a directed arrow from $x \times t$ to $y \times t$, this will indicate that a birth will occur at y if x is occupied at time t. For the arrival times t of N_x we place a δ-symbol at $x \times t$, this will indicate that a death will occur at x if it is occupied at time t.

Given any realization of the graphical representation, which we will typically denote below by ω, we say that a path exists from $V \times s$ to $U \times t$, $t \geq s$, and write $V \times s \to U \times t$ if for some $n \geq 0$ there exists an increasing sequence of times $(s_k)_{k=0}^{n+1}$, $s_0 = s$, $s_{n+1} = t$ and a sequence of sites $(x_k)_{k=0}^{n}$, for some $x_0 \in V$ and some $x_n \in U$, such that: (i) there is an arrow from x_{k-1} to x_k at time s_k, for all $k = 1, \ldots, n$, and (ii) there is no δ-symbol on the vertical segments $x_k \times (s_k, s_{k+1})$, for all $k = 0, 1, \ldots, n$. Letting $\xi_t^V = \{x : V \times 0 \to x \times t\}$, $t \geq 0$, we have that $(\xi_t^V)_{t\geq 0}$ is a version of the process on $\mathbb{S}$ with V as a starting set.

The graphical representation provides a joint coupling of processes with various starting sets two consequences of which are the following. The property that for any set of sites such that $U \subseteq V$ we have that $\xi_t^U \subseteq \xi_t^V$ is known as monotonicity; while the property that $\xi_t^F = \bigcup_{x\in F} \xi_t^x$, for any finite $F \in \mathbb{S}$, is known as additivity.

3 The process on $\mathbb{Z}^-$

The purpose of this section is to prove Proposition 1.1. Consider the graphical representation for the supercritical non nearest neighbors contact process on $\mathbb{Z}^-$.

Theorem 3.1, which we demonstrate below, is known as a shape theorem; it is a consequence of the construction provided in [7], or in [1], due to the general theorem in [5] and additivity. We introduce some necessary notation. For two events A, B we write that for all $\omega \in A$, $\omega \in B$ *a.e.* whenever $\mathbf{P}(\{\omega : \omega \in A, \omega \notin B\}) = 0$, where a.e. stands for almost everywhere (on A). We also denote by 1_A the indicator function of A.

Theorem 3.1. *Consider the processes* $(\xi_t^{\mathbb{Z}^-})_{t\geq 0}$ *and also* $(\xi_t^F)_{t\geq 0}$*, for* F *any finite set in* $\mathbb{Z}^-$*. Define* $l_t = \inf \xi_t^F$ *and consider the set of sites,*

$$C_t = \left\{y \geq \inf_{s\leq t} l_s : 1_{\{y\in\xi_t^F\}} = 1_{\{y\in\xi_t^{\mathbb{Z}^-}\}}\right\},$$

$t \geq 0$. *There exists an* $a > 0$ *such that for all* $\omega \in \{\xi_t^F \text{ survives}\}$ *there exists* $t' = t'(\omega, a)$ *such that* $\omega \in \left\{[-at, 0] \cap \mathbb{Z}^- \subset C_t, \text{ for all } t \geq t'\right\}$ *a.e..*

The preceding theorem is necessary in the following proof.

Proof of Proposition 1.1. Let F be any fixed finite subset of $\mathbb{Z}^-$, consider the events,

$$B_n = \{\xi_s^F \cap F = \xi_s^{\mathbb{Z}^-} \cap F, \text{ for all } s \in [n, \infty)\},$$

for all integer times $n \geq 0$; we want to prove that $\mathbf{P}(B_0) > 0$.

By Theorem 3.1 we have that for all $\omega \in \{\xi_t^F \text{ survives}\}$ there exists a $t_0 = t_0(\omega, a)$ such that $\omega \in \{\xi_s^{\mathbb{Z}^-} \cap [-at, 0] = \xi_s^F \cap [-at, 0], \text{ for all } s \geq t_0\}$ a.e., and, further, there exists $t_1 = t_1(\omega, a, F)$ such that $\omega \in B_{\lceil t_1 \rceil}$ a.e. as well, since we have that, $[-at, 0] \supset F$ for all t sufficiently large. Thus, $\mathbf{P}\left(\bigcup_{n\geq 0} B_n\right) = \mathbf{P}(\xi_t^F \text{ survives})$ and, due to the process being supercritical, we have (e.g. by contradiction) that:

$$\text{there exists some } n_0 \geq 0 \text{ such that } \mathbf{P}(B_{n_0}) > 0. \tag{3.1}$$

We show that the last statement implies that $\mathbf{P}(B_0) > 0$, hence, completing the proof. Let n_0 be as in (3.1) and define the event B'_{n_0} as follows: $\omega' \in B'_{n_0}$ if and only if there exists $\omega \in B_{n_0}$ such that ω and ω' are identical realizations except perhaps from any δ-symbols in $F \times (0, n_0]$. We further define the event $E = \{\text{no } \delta\text{-symbols exists in } F \times (0, n_0]\}$. By independence of the Poisson processes used in the graphical representation and then because $B'_{n_0} \supseteq B_{n_0}$, we have that,

$$\begin{aligned} \mathbf{P}(B'_{n_0} \cap E) &= \mathbf{P}(B'_{n_0})\mathbf{P}(E) \\ &\geq \mathbf{P}(B_{n_0})e^{-|F|n_0} > 0. \end{aligned} \tag{3.2}$$

However, we also have that

$$B_0 \supseteq B'_{n_0} \cap E, \tag{3.3}$$

to prove (3.3), note that if ω and ω' are identical realizations, except that ω' does not contain any δ-symbols that possibly exist for ω in $F \times (0, n_0]$, then $\omega \in B_{n_0}$ implies that $\omega' \in B_{n_0}$ and, indeed $\omega' \in B_0$. From (3.2) and (3.3) the proof is complete. □

4 The process on $\mathbb{Z}$

The purpose of this section is to prove Corollary 1.2 and then Theorem 1.3. Consider the graphical representation for the supercritical non nearest neighbors contact process on $\mathbb{Z}$. In this section we shall be concerned with $r_t = \sup\{x : 0 \times 0 \to x \times t\}$, $t \geq 0$, i.e. the right endpoint of (ξ_t^0).

In the next proof we will need to use the following definition. Let $\mathbb{S}'$ be a subset of $\mathbb{Z}$, we write $V \times s \overset{\mathbb{S}'}{\to} U \times t$ to denote the existence of a path in our graphical representation which is restricted over $\mathbb{S}'$, i.e. in the definition of paths, we impose the additional restriction that all elements of $(x_k)_{k=0}^n$ there are in $\mathbb{S}'$.

Proof of Corollary 1.2. Define $R_t = \sup\{y : (-\infty, 0] \times 0 \to y \times t\}$, $t \geq 0$, we want to prove that $\mathbf{P}(r_t = R_t, \text{ for all } t \geq 0) > 0$. Recall that M denotes the range of interaction; considering the event,

$$\{\xi_1^0 \supseteq [-M, 0]\} \cap \{0 \in \xi_s^0 \text{ and } R_s \leq 0, \text{ for all } s \in (0, 1]\},$$

from the Markov property and monotonicity, we have that it is sufficient to prove that

$$\mathbf{P}(r_t^{[-M,0]} = R_t, \text{ for all } t \geq 0) > 0, \tag{4.1}$$

where $r_t^{[-M,0]} := \sup\{y : [-M, 0] \times 0 \to y \times t\}$, $t \geq 0$.

We prove (4.1). Define the event,

$$A = \{(-\infty, -M-1] \times 0 \overset{\mathbb{Z}^-}{\to} [-M, 0] \times t \text{ and } [-M, 0] \times 0 \to [-M, 0] \times t, \text{ for all } t \geq 0\},$$

and also the event,

$$B = \{(-\infty, -M-1] \times 0 \overset{\mathbb{Z}^-}{\to} [-M, 0] \times t \text{ and } [-M, 0] \times 0 \overset{\mathbb{Z}^-}{\to} [-M, 0] \times t, \text{ for all } t \geq 0\}.$$

Because the process on $\mathbb{Z}^-$ is also supercritical, by Proposition 1.1 we have that $\mathbf{P}(B) > 0$. From this and noting that $B \subseteq A \subseteq \{r_t^{[-M,0]} = R_t, \text{ for all } t \geq 0\}$, the proof is complete.

□

For proving Theorem 1.3 below we will need the following definitions. For any given space time point $x \times s$ define

$$R_t^{x\times s} = \sup\{y : (-\infty, x] \times s \to y \times t + s\},$$

and also

$$r_t^{x\times s} = \sup\{y : x \times s \to y \times t + s\},$$

for $t \geq 0$; we also define the events,

$$\{x \times s \text{ c.s.e.}\} := \{R_t^{x\times s} = r_t^{x\times s}, \text{ for all } t \geq 0\},$$

where c.s.e. stands for "controls subsequent edges". Define also,

$$p := \mathbf{P}(0 \times 0 \text{ c.s.e.}) > 0, \tag{4.2}$$

where $p > 0$ due to Corollary 1.2.

Proof of Theorem 1.3. Consider the non stopping time,

$$\psi = \inf\{t \geq 1 : r_t \times t \text{ c.s.e.}\}$$

we will prove that, conditional on either $\{\xi_t^0 \text{ survives}\}$ or $\{0 \times 0 \text{ c.s.e.}\}$, ψ and r_ψ are a.s. finite. This is sufficient to complete the proof because along with (4.2) gives us that the proof of Lemma 7 in [13] applies as it is and implies that: Conditional on the event $\{\xi_t^0 \text{ survives}\}$, defining $\psi_n = \inf\{t \geq 1 + \psi_{n-1} : r_t \times t \text{ c.s.e.}\}$, $n \geq 0$, and $\psi_{-1} := 0$, we have that $(r_{\psi_n} - r_{\psi_{n-1}}, \psi_n - \psi_{n-1})$, $n \geq 1$, are i.i.d. vectors.
We use what is known as a restart argument. Define $T_0 = \inf\{t : \xi_t^0 = \emptyset\}$ and then, inductively, for all $n \geq 0$, on $\{T_n < \infty\}$, define $\xi_t^{n+1} = \{x : 0 \times T_n \to x \times t\}$ and $r_t^{n+1} = \sup \xi_t^{n+1}$, $t \geq T_n$, define also, $T_{n+1} = \inf\{t \geq T_n : \xi_t^{n+1} = \emptyset\}$.
Consider the process $r'_t := r_t^n$ for all $t \in [T_{n-1}, T_n)$, $n \geq 0$, where $r_t^0 := r_t$, for all $t \in [0, T_0)$. Define $\tau_1 := 1$ and then, inductively, for all $n \geq 1$, on $\{\tau_n < \infty\}$, let $\sigma_n := \sum_{k=1}^n \tau_k$ and define

$$\tau_{n+1} = \inf\{t \geq 0 : R_t^{r'_{\sigma_n} \times \sigma_n} > r_t^{r'_{\sigma_n} \times \sigma_n}\};$$

while on $\{\tau_n = \infty\}$ define $\tau_l = \infty$ for all $l \geq n$. Define further the associated random variable, $N = \inf\{n \geq 1 : \tau_n = \infty\}$.
We will prove that σ_N, r'_{σ_N} are a.s. finite; from the restart argument and the definitions above, this completes the proof because on $\{\xi_t^0 \text{ survives}\}$, and on its subset $\{0 \times 0 \text{ c.s.e.}\}$, we have that $\psi = \sigma_N$ and $r'_{\sigma_N} = r_\psi$.
By translation invariance and the Markov property, we have that, for all $n \geq 1$, the event $\{\tau_{n+1} = \infty\}$ has probability $p > 0$, as in (4.2), and is independent of the graphical representation up to time σ_n. Hence, because

$$\{N = n\} = \{\tau_k < \infty \text{ for all } k = 1, \ldots, n \text{ and } \tau_{n+1} = \infty\}, \tag{4.3}$$

we have that $\mathbf{P}(N = n) = p(1-p)^{n-1}, n \geq 1$, and thus N is a.s. finite. From this and (4.3), we further have that σ_N is a.s. finite. However, because $|r'_t|$ is bounded above in distribution by the number of events by time t of a Poisson process at rate $M\mu$, we also have that r'_{σ_N} is a.s. finite.

□

5 Large deviations result

In this section derive a large deviations result which is an improvement of Lemma 3 of Mountford and Sweet [13].

Define the set sites $\mathbb{L} = \{(y, n) \in \mathbb{Z}^2 : n \geq 0 \text{ and } y + n \text{ is even}\}$. We shall refer to a collection of Bernoulli random variables $\{w(y, n); (y, n) \in \mathbb{L}\}$ as 1-dependent (site) percolation with density at least $1 - \epsilon$ whenever it satisfies the following condition: For any $(y_i)_{i=1}^I$, $I \geq 1$ any finite integer, such that $|y_i - y_{i'}| > 2$ for all $i, i' \in \{1, \ldots, I\}$ we have that

$$\mathbf{P}\big(w(y_i, n+1) = 0 \ \forall i = 1, \ldots, I | \{w(y, m); m \leq n\}\big) \leq \epsilon^I,$$

for all $n \geq 1$.

Consider a realization of 1-dependent site percolation. A site (y, n) is regarded as *open* if $w(y, n) = 1$, otherwise, it is regarded as closed. We shall write $(x, m) \to (y, n)$ whenever there exists a sequence of open sites in $\mathbb{L}$, $(x, m) \equiv (y_0, m), \ldots, (y_{n-m-1}, n)$ such that and $|y_i - y_{i-1}| = 1$ for all $i = 1, \ldots, n - m - 1$ and $|y_{n-m-1} - y| = 1$.

We will need the following definition. Given ℓ an interval in $\mathbb{R}$, and $n \geq 1$ an integer, we will denote by $\mathcal{V}(\ell \times n)$ the set of sites of $\mathbb{L}$ within the region $\ell \times n$, i.e., $\mathcal{V}(\ell \times n) := \{\ell \times n\} \cap \mathbb{L}$.

Define the process started with all sites closed, $\overline{W}_n = \{y : (\mathcal{V}(\mathbb{R} \times 0), 0) \to (y, n)\}$, $n \geq 1$. The next lemma will be used in the proof of Proposition 5.2, following below, which is the aim of this section.

Lemma 5.1. *For all $\rho < 1$ we can choose $\epsilon > 0$ sufficiently small such that for any $n \geq 1$ and any $Y \subseteq \mathcal{V}(\mathbb{R} \times n)$,*

$$\mathbf{P}\left(\sum_{y \in Y} 1_{\{y \in \overline{W}_n\}} < \rho |Y|\right) \leq C e^{-\gamma |Y|} \tag{5.1}$$

where C, γ are strictly positive and finite constants independent of n.

Proof. Consider supercritical independent bond percolation process on $\mathbb{L}$ with parameter $p > \rho$. For general information and standard terminology concerning this well studied process we refer to [3]. Define the process $(\bar{B}_n)_{n \geq 0}$, started from $\bar{B}_0 = \mathcal{V}(\mathbb{R} \times 0)$, and the process $(\tilde{B}_n)_{n \geq 0}$, started from $\tilde{B}_0$, a random subset of $\mathcal{V}(\mathbb{R} \times 0)$ distributed according to the upper invariant measure with density p, independently of the bond percolation.

By coupling of the two processes and monotonicity we have that $\tilde{B}_n \subseteq \bar{B}_n$, for all $n \geq 0$; using this and then invariance of the distribution of $(\tilde{B}_n)$, the large deviation result (concerning the density of the upper invariant measure) of Theorem 1 in [8] gives that, for any $n \geq 1$ and any $Y \subseteq \mathcal{V}(\mathbb{R} \times n)$,

$$\begin{aligned} \mathbf{P}\left(\sum_{y \in Y} 1_{\{y \in \bar{B}_n\}} < p|Y|\right) &\leq \mathbf{P}\left(\sum_{y \in Y} 1_{\{y \in \tilde{B}_n\}} < p|Y|\right) \\ &\leq C e^{-\gamma |Y|}, \end{aligned} \tag{5.2}$$

for some C, γ strictly positive and finite constants independent of n. The result follows from (5.2) because for all $p < 1$ we can choose $\epsilon > 0$ sufficiently small such that $\overline{W}_n \supseteq \bar{B}_n$ for all $n \geq 1$, a.s.; this comparison is a consequence of the comparison of independent bond percolation with independent site percolation, see Theorem B24 in [12], and of the comparison of the latter with 1-dependent percolation, see Theorem B26 in the aforementioned reference. □

Define $W_n := \{y : (0,0) \to (y,n)\}$, $n \geq 1$, we are concerned with $(W_n)_{n\geq 0}$. Define the endpoints $R_n = \sup\{y : y \in W_n\}$ and $L_n = \inf\{y : y \in W_n\}$, $n \geq 0$; define also the stopping time $\tau = \inf\{n : W_n = \emptyset\}$. In the proof of the next proposition we will need two well known results, see e.g. [3],[12], regarding (W_n), which we now state. For any $\beta < 1$, we can choose $\epsilon > 0$ sufficiently small such that,

$$\mathbf{P}\big([L_n, R_n] \subseteq [-\beta n, \beta n]\big) \leq Ce^{-\gamma n}, \tag{5.3}$$

and that,

$$\mathbf{P}(n \leq \tau < \infty) \leq Ce^{-\gamma n} \tag{5.4}$$

$n \geq 1$, for some C, γ strictly positive and finite constants.

We now give our final result.

Proposition 5.2. *For any $\rho, \beta < 1$ we can choose $\epsilon > 0$ sufficiently small such that for all $b \in (0, \beta]$, considering the events,*

$$F_n = \left\{ \sum_{k=1}^{bn} 1_{\{y_k \in W_n\}} < \rho bn \ \text{ for some } (y_k)_{k=1}^{bn} \in \mathcal{V}([-\beta n, \beta n] \times n) \right.$$
$$\left. \text{such that } |y_{k+1} - y_k| = 2 \ \forall k = 1, \ldots, bn-1 \right\}, \tag{5.5}$$

$n \geq 1$, we have that

$$\mathbf{P}\big(F_n, W_n \neq \emptyset\big) \leq Ce^{-\gamma bn} \tag{5.6}$$

for all $n \geq 1$, where C, γ are strictly positive and finite constants independent of n and b.

Proof. Choose $\epsilon > 0$ sufficiently small such that (5.1), (5.3), and (5.4) are all satisfied. Because for any $b \leq \beta'$ the number of $(y_k)_{k=1}^{bn}$, as in (5.5), is of polynomial order in both n and b, it is sufficient to prove that for any set of sites $Y \subseteq \mathcal{V}([-\beta n, \beta n] \times n)$,

$$\mathbf{P}\left(\sum_{y \in Y} 1_{\{y \in W_n\}} < \rho |Y|, \ \tau \geq n \right) \leq C(e^{-\gamma n} + e^{-\gamma |Y|}), \tag{5.7}$$

$n \geq 1$, for some C, γ strictly positive and finite constants. From this, using (5.3) and (5.4), we have that it is sufficient to prove that the probability of the event $\{\sum_{y\in Y} 1_{\{y \in W_n\}} < \rho|Y|\}$ on $\{[L_n, R_n] \supseteq [-\beta n, \beta n]\} \cap \{\tau = \infty\}$ decays exponentially in $|Y| \geq 1$; however, this is immediate by (5.1) because by coupling we have that, $W_n = \overline{W}_n \cap [L_n, R_n]$ on $\{\tau = \infty\}$. □

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