POISSON SMOOTH STRUCTURES ON STRATIFIED SYMPLECTIC SPACES

HÔNG VÂN LÊ, PETR SOMBERG AND JIŘI VANŽURA

Abstract. In this note we introduce the notion of a Poisson smooth structure on a symplectic stratified space. We show that under a mild condition many properties of a symplectic manifold can be extended to a symplectic stratified space, e.g. the existence and uniqueness of a Hamiltonian flow, the isomorphism between the Brylinski-Poisson homology and the de Rham homology, the Hodge structure on a symplectic stratified space. We give many examples of symplectic stratified spaces satisfying these properties.

Contents

AMSC: 51H25, 53D05, 53D17
Key words: C^∞-ring, stratified space, symplectic form, Poisson structure

1. Introduction

Many classical problems on various classes of topological spaces reduce to the quest for its appropriate functional structure. Examples of topological spaces we are interested in comprise stratified spaces equipped with an additional structure of geometrical origin. Due to the lack of canonical notion of the sheaf of smooth (or analytic) functions on such spaces one is free to introduce such a structure with all derived smooth (or analytical) notions, e.g. the tangent space, the vector field or the de Rham complex.

In this note we continue the study of Poisson smooth structures on singular spaces called symplectically stratified spaces along the lines of ideas developed in [8], where smooth structure was generated by canonical smooth structure on its regular part together with controlled behavior on the singular locus.

The structure of this short note is as follows. First of all, we discuss the extension of a notion of (Poisson) smooth structure on conical pseudomanifolds to the category of stratified spaces. This extension then leads to a rather straightforward extension of variety of basic notions on symplectic manifolds, e.g. the existence and

Date: November 3, 2010.

uniqueness of a Hamiltonian flow, the isomorphism between the Brylinski-Poisson homology, the de Rham homology and the symplectic Hodge structure, to the wider framework of symplectic stratified spaces.

2. Stratified spaces and their smooth structures

In this section we introduce the notion of a stratified space following Goresky's and MacPherson's concept [6, p.36], see also [13, §1]. We introduce the notion of a smooth structure on a stratified space. Our concept of a smooth structure on a stratified space is a natural extension of our concept of a smooth structure on a pseudomanifold with conical singularities given in [8, §2]. We prove the existence of a locally smoothly contractible, resolvable smooth structure on pseudomanifolds with edges, see Lemma 2.8, and the infinite generatedness of a resolvable smooth structure satisfying a mild condition, see Proposition 2.10 for details.

Definition 2.1. ([6, p.36], [13, Definition 1.1]) Let X be a Hausdorff and paracompact topological space and let $\mathcal{S}$ be a partially ordered set with ordering denoted by $\le$. *An $\mathcal{S}$-decomposition of* X is a locally finite collection of disjoint locally closed manifolds $S_i \subset X$ (one for each $i \in \mathcal{S}$) called strata such that

1) $X = \cup_{i\in\mathcal{S}} S_i$;
2) $S_i \cap \bar{S}_j \neq \emptyset \Longleftrightarrow S_i \subset \bar{S}_j \Longleftrightarrow i \le j$.

We define the depth of a stratum S as follows

$$depth_X\, S := \sup\{n| \text{ there exist pieces } S = S_0 < S_1 < \cdots < S_n\}.$$

We define the depth of X to be the number $depth\, X := \sup_{i\in\mathcal{S}} depth\, S_i$. The dimension of X is defined to be the maximal dimension of its strata.

Given a space L a cone cL over L is the topological space $L \times [0,\infty)/L \times \{0\}$. If L has a $\mathcal{S}$-decomposition with depth n the cone cL has an induced decomposition with depth $(n+1)$ [13, p.379].

Definition 2.2. (cf. [5], [13, Definition 1.7]) A decomposed space X is called *a stratified space* if the pieces of X, called strata, satisfy the following condition defined recursively. Given a point x in a piece S there exists an open neighborhood U of x in X, an open ball B around x in S, a compact stratified space L, called the link of x, and a stratified diffeomorphism $\phi : B \times cL \to U$ that preserves the decomposition.

A homeomorphism $\phi : X \to Y$ from a stratified space X to a stratified space Y is called *a stratified diffeomorphism*, if ϕ maps a stratum of X onto a stratum of Y and the restriction of ϕ to each stratum is a diffeomorphism on its image.

Let X^n be a stratified space of dimension n. A stratum S is called *regular*, if $S \cap \overline{X^n \setminus S} = \emptyset$. Denote by X^{reg} the union of all regular strata. Then X^{reg} is an open subset of X and $X = \overline{X^{reg}}$. A point $x \in X^{reg}$ is called *a regular point.* Set $X^{sing} := X \setminus X^{reg}$. A point $x \in X^{sing}$ is called *a singular point.*

Example 2.3. A connected stratified space X of depth 1 is a disjoint union of a regular stratum X^{reg} and a countable number of strata S_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$, and $S_i \subset \overline{X^{reg}}$. We always assume that a stratum is connected space. Strata S_i are called *edges* of X, and X is also called *a pseudomanifold with edges.* A pseudomanifold X with edges is called *a pseudomanifold with conical singularities*, if its edges S_i are points s_i of X.

Let N be a smooth manifold which is fibered over a smooth base B. Then we say that B is a fiberwise contraction of N. A stratified space X of depth k can be constructed from a stratified space Y of depth $(k+1)$ by fiberwise contracting certain proper submanifolds N_{ij} of a stratum S_j of depth k. Here we assume that N_{ij} is a smooth fibration over some smooth base B_{ij}, and Y is obtained by gluing $X \setminus N_{ij}$ with B_{ij} with help of the smooth projection from N_{ij} to B_{ij}. In this case we say that M is *a covering* of X with depth k.

Now let us introduce the notion of a smooth structure on a stratified space, which is an immediate extension of our notion of a smooth structure on a pseudomanifold with conical singularities in [8]. We denote by $C^\infty(X^{reg})$ (resp. $C^\infty_0(X^{reg})$) the union of the spaces of smooth functions on each smooth stratum of X^{reg} (resp. the space of smooth functions with compact support).

Definition 2.4. (cf. [8, Definition 2.3]) *A smooth structure* on a stratified space X^n of dimension n is a choice of a subalgebra $C^\infty(X)$ of the algebra $C^0(X)$ (over $\mathbb{R}$) of all continuous functions on X satisfying the following five properties.

1. $C^\infty(X)$ is a C^∞-ring.
2. For any stratum $S \subset X$ we have $C^\infty_0(S) \subset C^\infty(X)_{|S} \subset C^\infty(S)$. Here $C^\infty_0(S)$ denotes the space of smooth functions with compact support in S.
3. $C^\infty_0(X^{reg}) \subset C^\infty(X)$.
4. $C^\infty(X)$ is complete in the following sense. If $f_i \in C^\infty(X)$, and a family $\{sppt\, f_i\}$ is a locally finite open sets (i.e. for any $x \in X$ there is only a finite number of f_i such that $x \in sppt\, f_i$), then $\sum f_i \in C^\infty(X)$.
5. C^∞ is partially invertible in the following sense. If $f \in C^\infty(X)$ is nowhere vanishing, then $1/f \in C^\infty(X)$.

Since $X \subset \overline{X^{reg}}$ the condition 2 in Definition 2.4 allows us to regard $C^\infty(X)$ as a subalgebra of $C^\infty(X^{reg})$. Furthermore any function $f \in C^\infty_0(X^{reg})$ has a unique extension to a continuous function on X by setting $f(x) = 0$ if $x \in X \setminus X^{reg}$. The condition 2 implies that $C^\infty_0(X^{reg})$ is a subalgebra of $C^\infty(X)$. Thus we have inclusions $C^\infty_0(X^{reg}) \subset C^\infty(X) \subset C^\infty(X^{reg})$.

A continuous map f between smooth stratified spaces $(X, C^\infty(X))$ and $(Y, C^\infty(Y))$ is called *a smooth map*, if $f^*(C^\infty(Y)) \subset C^\infty(X)$.

The following Lemma is a generalization of [8, Lemma 2.8].

Lemma 2.5. *For any locally finite open covering $\{U_i\}$ of X there exists a smooth partition of unity subordinate to U_i (i.e. there are nonnegative smooth functions $f_i \in C^\infty(X)$ with support in U_i satisfying $\sum f_i = 1$).*

Proof. This Lemma is proved in the same way as in [8]. Mimicking the proof of the existence of a partition of unity on a smooth manifold, it suffices to show that for any open neighborhood $U(s)$, $s \in (X \setminus X^{reg})$, and for any open set $V(s) \subset U(s)$ containing s such that $\overline{V(s)} \subset U(s)$ there exists a nonnegative function $f \in C^\infty(X)$ with support in $U(s)$ such that $f_{|V(s)} = 1$. Let χ be the smooth constant function on X taking value 1. Here we use essentially conditions 4 and 5 in our Definition 2.4. Since $C^\infty_0(X^{reg}) \subset C^\infty(X)$, there exists a function $\chi_{U(s)} \in C^\infty_0(X^{reg}) \subset C^\infty(X)$ with support in $X^{reg} \setminus \overline{V(s)}$ such that $0 \le \chi_{U(s)}(x) \le 1$ and $\chi_{U(s)}(x) = 1$ if $x \in X^{reg} \setminus U(s)$. Since $U(s)$ is open and $\overline{X^{reg}} = X$ we get $\overline{X^{reg} \setminus U(s)} = X \setminus U(s)$. Hence $\chi_{U(s)}(x) = 1$ if $x \in X \setminus U(s)$. The function $f := \chi - \chi_{U(s)}$ is the required smooth function. □

Let D be an open subset of X. A continuous function $f : D \to \mathbb{R}$ is called *smooth*, if for each $x \in D$ there is an open neighborhood $U(x) \subset D$ and a function $f_U \in C^\infty(X)$ such that $f_{|U(x)} = (f_U)_{|U(x)}$. The set of all smooth functions on D is denoted by $C^\infty(D)$. It is easy to see that the collection $U \mapsto C^\infty(U)$ over open sets $U \subset X$ is a sheaf. We call it *the sheaf of smooth functions on* X. Denote by $C^\infty_x(X)$ the germs of smooth functions on X at x. For $x \in X^{reg}$ conditions 2 and 3 in Definition 2.4 imply that $C^\infty_x(X) = C^\infty_x(X^{reg})$.

Corollary 2.6. *1.(cf.* [8, Corollary 2.9]*) The sheaf of smooth functions on X defined by letting $D = X$ in our definition above coincides with the original algebra $C^\infty(X)$, that is our smooth structure is germ determined.*
2. Smooth functions on X separate points on X.

Proof. 1. The proof of the first assertion of Corollary 2.6 is identical with the proof of Corollary 2.9 in [8]. For the convenience of the reader we rewrite the proof here. Let us prove that a function $f \in C^0(X)$ belongs to $C^\infty(X)$ if and only if for all x there is a neighborhood $V(x) \ni x$ such that $f_{|V(x)} \in C^\infty(V(x))$. The "only if" assertion follows from the definition of $C^\infty(V(x))$. Now let us prove the "if" assertion, that is if $f \in C^0(X)$ and for any x there exists $V(x) \ni x$ such that $f_{|V(x)} = G_{|V(x)}$ for some $G \in C^\infty(X)$ then $f \in C^\infty(X)$. Without loss of generality we can assume that there is a locally finite open covering of X such that for each V_i there exists $G_i \in C^\infty(X)$ satisfying $f_{|V_i} = (G_i)_{|V_i}$. Let λ_i be a smooth partition of unity subordinate to V_i, whose existence follows from Lemma 2.5. Note that $f(x) = \sum \lambda_i f(x)$ (this sum is well-defined for each x since V_i is a locally finite open covering). Since $\lambda_i \in C^\infty(X)$ we have $\lambda_i G_i \in C^\infty(X)$ such that for each V_i there exists $G_i \in C^\infty(X)$ satisfying $f_{|V_i} = (G_i)_{|V_i}$. Clearly we have $\lambda_i f_{|V_i} = (\lambda_i G_i)_{|V_i}$. Since $sppt\lambda_i \subset V_i$ we get $\lambda_i f(x) = \lambda_i G_i(x)$ for all $x \in X$. Hence $f = \sum \lambda_i G_i$ belongs to $C^\infty(X)$, since our ring of smooth functions is a complete C^∞-ring.

2. The second assertion of Corollary 2.6 follows directly from the proof of Lemma 2.5. □

There are many ways to provide a stratified space X with a smooth structure, most notably using the notion of the quotient smooth structure, i.e. the existence of a continuous map $M \to X$ from a smooth manifold M to X (e.g. X is the quotient of a manifold M under an action of a compact Lie group G), or using an embedding of X into a smooth manifold M (e.g. the notion of Whitney smooth functions see e.g. [13, Example 1.10].)

Definition 2.7. Assume that we have a continuous projection $M \stackrel{\pi}{\to} X$ from a smooth manifold M with corner to a stratified space X such that for each stratum $S_i \subset X$ the triple $(\pi^{-1}(S_i), \pi_i, S_i)$ is a differentiable fibration, moreover for each $x \in X^{reg}$ the preimage $\pi^{-1}(x)$ consists of a single point. The smooth structure $C^\infty(X) := \{f \in C^0(N) | \, \pi^* f \in C^\infty(M)\}$ is called *a resolvable smooth structure.* The space M is called *a resolution* of M.

We say that $C^\infty(M)$ is *locally smoothly contractible*, if for any $x \in M$ there exists an open neighborhood $U(x) \ni x$ together with a smooth homotopy $\sigma : U(x) \times [0, 1] \to U(x)$ joining the identity map with the constant map $U(x) \mapsto x$ [12, §5]. Note that there is a natural smooth structure $C^\infty(U(x) \times [0, 1])$ generated by $C^\infty(U(x))$ and $C^\infty([0, 1])$ [12, §3], so that the projections from $U(x) \times [0, 1]$ to $U(x)$ and to $[0, 1]$ are smooth maps.

Lemma 2.8. *Every pseudomanifold X with edges has a resolvable smooth structure, which is locally smoothly contractible.*

Proof. Let S_i be a singular stratum. By assumption there is an open neighborhood $U(S_i)$ of S in X such that $U(S_i)$ is a topological fibration over S whose fiber is a cone cL_i, where L_i is a compact smooth manifold. Now we consider a new space $M := X \setminus \cup_i U(S_i)$. Clearly M is a smooth manifold provided with a projection π onto X contracting the boundary $\partial U(S_i)$ to S_i such that the restriction of π to $M \setminus \cup_i \partial U_i(S_i)$ is a diffeomorphism on its image. Let us consider the following commutative diagram

$$\begin{array}{ccc} I \times V(\partial U(S_i)) & \xrightarrow{\tilde{F}} & V(\partial U(S_i)) \\ \Big\downarrow{\scriptstyle (Id\times\pi)} & & \Big\downarrow{\scriptstyle \pi} \\ I \times U(S_i) & \xrightarrow{F} & U(S_i) \end{array}$$

where $V(\partial U(S_i))$ is an open normal neighborhood of $\partial U(S_i)$ in M, and $\tilde{F}$ is a smooth deformation retraction from $V(\partial U(S_i))$ to $\partial U(S_i)$, constructed using the fibration $[0,1) \to V(\partial U(S_i)) \to \partial U(S_i)$. We set

$$F(t,x) := \pi(\tilde{F}(t, \pi^{-1}(x))).$$

Since $\tilde{F}_{|\partial U(S_i)} = Id$, the map F is well-defined. This proves Proposition 2.8. □

Next we introduce the notion of the cotangent bundle of a stratified space X, which is identical with the notion we introduced in [8] and similar to the notions introduced in [13], [15]. Note that the germs of smooth functions $C^\infty_x(X)$ is a local ring with the unique maximal ideal $\mathfrak{m}_x$ consisting of functions vanishing at x. Set $T^*_x(X) := \mathfrak{m}_x/\mathfrak{m}_x^2$. Since we have a direct sum $C^\infty_x(X) = \mathfrak{m}_x \oplus \mathbb{R}$, it is known that the space T^*_xX can be identified with the space of Kähler differentials $d : C^\infty_x(X) \to T^*_xX$, see e.g. [9, 26.1], or [15, Proposition B.1.2]. We call T^*_xX *the cotangent space* of X at x. Its dual space $T^Z_xX := Hom(T^*_xX, \mathbb{R})$ is called *the Zariski tangent space* of X at x. The union $\cup_{x\in X} T^*_xX$ is called *the cotangent bundle* of X. The union $\cup_{x\in X} T^Z_xX$ is called *the Zariski tangent bundle* of X.

Let us denote by $\Omega^1_x(X)$ the $C^\infty_x(X)$-module $C^\infty_x(X) \otimes_{\mathbb{R}} \mathfrak{m}_x/\mathfrak{m}_x^2$. We called $\Omega^1_x(X)$ *the germs of 1-forms at x*. Set $\Omega^k_x(X) := C^\infty_x(X) \otimes_{\mathbb{R}} \Lambda^k(\mathfrak{m}_x/\mathfrak{m}_x^2)$. The Kähler differential $d : C^\infty_0(X) := \Omega^0_x(X) \to \Omega^1_x(X)$ induces the differential $d : \Omega^k_x(X) \to \Omega^{k+1}_x(X)$.

Definition 2.9. A section $\alpha : X \to \Lambda^k T^*(X)$ is called *a smooth differential k-form*, if for each $x \in X$ there exists $U(x) \subset X$ such that $\alpha(x)$ can be represented as $\sum_{i_0 i_1 \cdots i_k} f_{i_0} df_{i_1} \wedge \cdots \wedge df_{i_k}$ for some $f_{i_0}, \cdots, f_{i_k} \in C^\infty(X)$.

Denote by $\Omega(X)$ the space of all smooth differential forms on X. By Definition 2.4 we can regard $\Omega(X)$ as a subspace in $\Omega(X^{reg})$. The Kähler differential d extends to a differential also denoted by d mapping $\Omega(X)$ to $\Omega(X)$.

Using the notion of cotangent space we will prove the following

Proposition 2.10. *A resolvable smooth structure on X obtained from a smooth manifold M is infinitely generated, if there exists $x \in X$ such that $\dim \pi^{-1}(x) \geq 1$, where $\pi : M \to X$ is the associated projection.*

Proof. Assume the opposite i.e. $C^\infty(X)$ is generated by $g_1, \cdots, g_n$. Then $G = (g_1, \cdots, g_n)$ defines a smooth embedding $X \to \mathbb{R}^n$, so $C^\infty(X) = C^\infty(\mathbb{R}^n)/I$, where I is an ideal of $C^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ vanishing on $G(X)$ [10, p.21]. In particular, the cotangent space $T^*_x X$ is a finite dimensional linear space for all $x \in X$.

Let S be a stratum of X such that $\dim(\pi^{-1}(S)) \ge \dim S + 1$. Let $x \in S$ and $U(x)$ a small open neighborhood of x in X. Let $f \in C^\infty(U((x)))$, so $\pi^*(f) \in C^\infty(\pi^{-1}U((x)))$, and by definition $\pi^*(f) \in C^\infty(\tilde{U})$ for some open set $\tilde{U} \subset \mathbb{R}^n$ containing $\chi(\pi^{-1}(U((x))))$, where $\chi : \pi^{-1}(U((x))) \to \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$ is a coordinate map on $\pi^{-1}(U((x)))$. Denote by $\tilde{S}$ the preimage $\chi \circ \pi^{-1}(S \cap U(x))$, which is a submanifold of $\tilde{U}(x)$. Then $\tilde{\pi} = \pi \circ \xi^{-1} : \tilde{S} \to S \cap U(x)$ is a smooth fibration. We note that $(\chi^{-1})^*\pi^*(f)$ belongs to the subalgebra $C^\infty(\tilde{U}, \tilde{S}, \tilde{\pi})$ consisting of smooth functions on $\tilde{U}$ which are constant along fiber $\tilde{\pi}^{-1}(x')$ for all $x' \in U(x)$. Shrinking $U(x)$ we can assume that $\tilde{S} = \tilde{U} \cap \mathbb{R}^k$ and $\tilde{\pi}$ is the restriction of a projection $\bar{\pi} : \mathbb{R}^k \to \mathbb{R}^l \subset \mathbb{R}^k$. Let $\mathbb{R}^{n-k}$ with coordinate $\tilde{x} = (\tilde{x}^1, \cdots, \tilde{x}^{n-k})$ be a complement to $\mathbb{R}^k$ in $\mathbb{R}^n$, and let $\mathbb{R}^{k-l} \subset \mathbb{R}^k$ with coordinate $\tilde{y}$ be the kernel of $\bar{\pi}$. We also equip the subspace $\mathbb{R}^l$ with coordinate $\tilde{z} = (\tilde{z}^1, \cdots, \tilde{z}^l)$. The condition $\dim \pi^{-1}(x) \ge 1$ in Proposition 2.8 is equivalent to the equality $k - l \ge 1$.

Lemma 2.11. *A function $g \in C^\infty(\tilde{U})$ belongs to $C^\infty(\tilde{U}, \tilde{S}, \tilde{\pi})$ if and only if g has the form*

$$g(\tilde{x}^1, \cdots, \tilde{x}^{n-k}, \tilde{y}, \tilde{z}) = \tilde{x}^1 g_1(\tilde{x}, \tilde{y}, \tilde{z}) + \cdots + \tilde{x}^{n-k} g_{n-k}(\tilde{x}, \tilde{y}, \tilde{z}) + c(\tilde{z}),$$

where $g_i \in C^\infty(\tilde{U})$ and $c(\tilde{z}) \in C^\infty(\mathbb{R}^l)$.

Proof. We write for $g \in C^\infty(\tilde{U}, \tilde{S}, \tilde{\pi})$

$$g(\tilde{x}, \tilde{y}, \tilde{z}) - g(0, \tilde{y}, \tilde{z}) = \int_0^1 \frac{dg(t\tilde{x}, \tilde{y}, \tilde{z})}{dt} dt = \int_0^1 \sum_{i=1}^{n-k} \frac{\partial g(t\tilde{x}^1, \cdots, t\tilde{x}^{n-k}, \tilde{y}, \tilde{z})}{\partial \tilde{x}^i}.$$

Setting

$$g_i = \int_0^1 \frac{\partial g(t\tilde{x}^1, \cdots, t\tilde{x}^{n-k}, \tilde{y}, \tilde{z})}{\partial \tilde{x}^i},$$

we get $g(\tilde{x}, \tilde{y}, \tilde{z}) = \sum_{i=1}^{n-k} \tilde{x}^i g_i(\tilde{x}, \tilde{y}, \tilde{z}) + g(0, \tilde{y}, \tilde{z})$. Since $g(0, \tilde{y}, \tilde{z})$ depends only on $\tilde{z}$, we obtain Lemma 2.11 immediately. □

Now let us complete the proof of Proposition 2.10. Take a point $s \in S$ and $\tilde{s} \in \tilde{\pi}^{-1}(s)$ such that $x(\tilde{s}) = y(s) = z(s) = 0$. By Lemma 2.11 the maximal ideal $\mathfrak{m}_s$ is a linear space generated by functions of the form $\tilde{x}^i g_{i,\alpha}(\tilde{x}, \tilde{y}, \tilde{z}), i = \overline{1, n-k}$, $\tilde{z}^j f_{j,\beta}(\tilde{z}), j = \overline{1, l}$. Let us consider the sequence $S := \{\tilde{x}^1\tilde{y}_1, \cdots, \tilde{x}^1\tilde{y}_1^m \in \mathfrak{m}_s\}$, $m \to \infty$. If $\dim T^*_s X = n$, there exists a subsequence $\tilde{x}^1\tilde{y}^{k_1}, \cdots, \tilde{x}^1\tilde{y}^{k_n}$ of S such that $\tilde{x}^1\tilde{y}^m$ is a linear combination of $\tilde{x}^1\tilde{y}^{k_j}$ for any m, which is a contradiction. □

Proposition 2.10 answers question 2 we posed in [8, §5]. We observe that there are many quotient smooth structures which are finitely generated, i.e. a quotient by a smooth group action. In this case the dimension of the fiber over singular strata is smaller than or equal to the dimension of the generic fiber.

3. Symplectic stratified spaces and compatible Poisson smooth structures

In this section we introduce the notion of a symplectic stratified space (X, ω) and a smooth structure compatible with ω. We also introduce the notion of a Poisson smooth structure on X whose restriction to each symplectic stratum agrees with the given Poisson structure defined by ω. We give new examples of symplectic stratified spaces with compatible (Poisson) smooth structures, see Examples 3.4, 3.6 and Lemma 3.7. We prove that the Brylinski-Poisson homology of a symplectic stratified space X with a compatible Poisson smooth structure is isomorphic to the de Rham cohomology of X, see Theorem 3.9.

Definition 3.1. A stratified space X is called *symplectic*, if every stratum S_i is provided with a symplectic form ω_i. The collection $\{\omega_i\}$ is called *a stratified symplectic form* or simply *a symplectic form* if no misunderstanding can occur.

Note that on each symplectic stratum (S_i, ω_i) there is a Poisson structure G_{ω_i} which is a section of bundle $\Lambda^2 TS_i$, such that $G_{\omega_i}(x) = \partial y_1 \wedge \partial x_1 + \cdots + \partial y_n \wedge \partial x_n$ if $\omega_i(x) = \sum_{i=1}^n dx^i \wedge dy^i$ [2, §1.1]. If we regard ω_i as an element in $End(TS_i, T^*S_i)$ and G_{ω_i} as an element in $End(T^*S_i, TS_i)$, then G_{ω_i} is the inverse of ω_i.

Definition 3.2. (cf. [8, Remark 4.8]) Let (X, ω) be a symplectic stratified space and $C^\infty(X)$ be a smooth structure on X.

1. A smooth structure $C^\infty(X)$ is said to be *weakly symplectic*, if there is a smooth 2-form $\tilde{\omega} \in \Omega^2(X)$ such that the restriction of Ω to each stratum S_i coincides with ω_i. In this case we also say that ω is compatible with $C^\infty(X)$ and $C^\infty(X)$ is compatible with ω.

2. A smooth structure $C^\infty(X)$ is called *Poisson*, if $G(\omega)$ extends to a smooth section $\tilde{G}_\omega$ of $\Lambda^2 T^Z(X)$ (i.e. the action $\tilde{G}_\omega : \Omega(X) \to \Omega(X)$, $\tilde{G}_\omega(\alpha)(V_1 \wedge \cdots \wedge V_k) := \alpha(G_\omega \wedge V_1 \wedge \cdots \wedge V_k)$ sends smooth differential forms to smooth differential forms).

Remark 3.3. The condition 2 in Definition 3.2 is equivalent to the condition that there is the Poisson structure $\{,\}_\omega$ on $C^\infty(X)$ such that its restriction to each subalgebra $C_0^\infty(S)$ is equal to the Poisson structure defined by ω_i on S_i. A typical example of a symplectic stratified singular space equipped with a Poisson smooth structure is a disjoint union X of symplectic leaves of a Poisson manifold M. The Poisson smooth structure on X is induced by the embedding $X \to M$.

Example 3.4. We assume that a Lie group G is compact and M is a Hamiltonian G-space with moment map $J : M \to \mathfrak{g}^*$. For a subgroup H of G denote by $M_{(H)}$ the set of all points whose stabilizer is conjugate to H, the stratum of M of orbit type (H). Let $Z = J^{-1}(0)$. The quotient space $M_0 = Z/G$ is called a symplectic reduction of M. If 0 is a singular value of J then Z is not a manifold and M_0 is called *a singular symplectic reduction*. Recall that the canonical smooth structure on M_0 is defined as follows $C^\infty(M_0)_{can} := C^\infty(M)^G/I^G$, where I^G is the ideal of G-invariant functions vanishing on Z [13, Example 1.11]. Denote by π the natural projection $Z \to Z/G$. Denote by $C^\infty(Z)$ the space of smooth functions on Z defined by the natural embedding of Z to M. Since Z is closed, $C^\infty(Z) = C^\infty(M)/I_Z$, where I_Z is the ideal of smooth functions on M vanishing on Z. We claim that the space $C^\infty(Z)^G$ of G-invariant smooth functions on Z can be identified with the space $C^\infty(M_0)_{can} = C^\infty(M)^G/I^G$. Clearly $C^\infty(M)^G/I^G$ is a subspace of G-invariant smooth functions on Z. On the other hand any smooth function f on G

can be modified to a G-invariant smooth function $f_G \in C^\infty(M)$ by setting

$$f_G(x) = \int_G f(g \cdot x)\mu_g$$

for a G-invariant measure μ_g on G normalizing by the condition $vol(G) = 1$. So if $g \in C^\infty(Z)^G$, then g is the restriction of a G-invariant function on M. In other words we have an injective map $C^\infty(Z)^G \to C^\infty(M)^G/I^G$. Hence follows the identity $C^\infty(Z)^G = C^\infty(M_0)_{can}$. It s follows that $C^\infty(M_0)_{can}$ is the quotient of the smooth structure obtained from $C^\infty(Z)$ via the projection $\pi : Z \to M_0$.

The symplectic form ω_0 on M_0 is defined by Sjamaar and Lerman as follows.

Proposition 3.5. [13, Theorem 2.1] *Let (M, ω) be a Hamiltonian G-space with moment map $J : M \to \mathfrak{g}^*$. The intersection of the stratum $M_{(H)}$ of orbit type (H) with the zero level set Z of the moment map is a manifold, and the orbit space*

$$(M_0)_{(H)} = (M_{(H)} \cap Z)/G$$

has a natural symplectic structure $(\omega_0)_{(H)}$ whose pullback to $Z_{(H)} := M_{(H)} \cap Z$ coincides with the restriction to $Z_{(H)}$ of the symplectic form on M. Consequently the stratification of M by orbit types induces a decomposition of the reduced space $M_0 = Z/G$ into a disjoint union of symplectic manifolds $M_0 = \cup_{H<G}(M_0)_{(H)}$.

The smooth structure $C^\infty(M_0)_{can}$ is known to inherit the Poisson structure from M, see [13, Proposition 3.1]. We observe that $C^\infty(M_0)_{can}$ is also weakly symplectic, since by Proposition 3.5 the pull back $\pi^{-1}(\omega_0)$ is equal to the restriction of the symplectic form ω to Z.

Example 3.6. Let us consider the closure of a nilpotent orbit in a complex semisimple Lie algebra $\mathfrak{g}$. For $x \in \mathfrak{g}$ let $x = x_s + x_n$ be the Jordan decomposition of x, where $x_n \neq 0$ is a nilpotent element, x_s is a semisimple and $[x_s, x_n] = 0$. Denote by G the adjoint group of $\mathfrak{g}$. The adjoint orbit $G(x)$ is a fibration over $G(x_s)$ whose fiber is $\mathcal{Z}_G(x_s)$-orbit of x_n and $\mathcal{Z}_G(x_s)$ denotes the centralizer of x_s in G. It is well-known that the closure $\overline{\mathcal{Z}_G(x_s)(x_n)}$ is a finite union of $\mathcal{Z}_G(x_s)$-orbits of nilpotent elements in the Lie subalgebra $\mathcal{Z}_{\mathfrak{g}}(x_s)$ [3, chapter 6], so the closure $\overline{G(x)}$ is a finite union of adjoint orbits in $\mathfrak{g}$. It is a stratified symplectic space provided with the Kostant-Kirillov symplectic structure. The embedding $\overline{G(x)} \to \mathfrak{g}$ provides $\overline{G(x)}$ with a natural finitely generated smooth structure $C^\infty_1(\overline{G(x)})$. This smooth structure is Poisson, inherited from the Poisson structure on $\mathfrak{g}$. It is also compatible with the symplectic structure on $\overline{G(x)}$, since the symplectic structure on $\overline{G(x)}$ is the restriction of the smooth 2-form $\omega_x(v, w) = \langle x, [v, w]\rangle$ on $\mathfrak{g}$. In [14, Lemma 2] Panyushev showed that $\overline{G(x)}$ possesses also a resolvable smooth structure which is compatible with ω. The corresponding resolution is constructed as follows. Let h be a characteristic of x and $\mathfrak{g}(i) = \{s \in \mathfrak{g} | [h, s] = is\}$, $\mathfrak{n}_2 = \oplus_{i\geq 2}\mathfrak{g}(i)$ and P a parabolic subgroup with the Lie algebra $lP = \oplus_{i\geq 0}\mathfrak{g}(i)$ and N_- the connected Lie subgroup of G with Lie algebra $lN_- = \oplus_{i<0}\mathfrak{g}(i)$. It is known that $\overline{Px} = \mathfrak{n}_2$ and $G_x = P$. Hence there exists a natural map $\tau : G *_P \mathfrak{n}_2 \to \overline{G(x)} : g * n \mapsto gn$ is a resolution of the singularity of $\overline{G(x)}$. Moreover it is shown in [14] that this resolvable smooth structure is compatible with the given symplectic structure on $\overline{G(x)}$. The form $\tau^*(\omega)$ is symplectic if and only if x is even. If x is minimal, the preimage $\tau^{-1}(\overline{G(x)} \setminus G(x))$ is a Lagrangian submanifold in $G *_P \mathfrak{n}_2 = T^*(G/P)$, see [1, §2], [14], thus this resolvable smooth structure is Poisson by the following Lemma 3.7.

Lemma 3.7. *Assume that X is a symplectic stratified space of depth 1 and $(\tilde{X}, \tilde{\omega}, \pi : \tilde{X} \to X)$ a symplectic smooth resolution of X. If for each singular point $x \in X$ the preimage $\pi^{-1}(x)$ is a coisotropic submanifold in $\tilde{X}$, then the resolvable smooth structure is Poisson.*

Proof. We define the Poisson bracket by $\{g, f\}_\omega(x) := \{\pi^* g, \pi^* f\}_{\tilde{\omega}}(\tilde{x})$, for $\tilde{x} \in \pi^{-1}(x)$. We will show that this definition does not depend on the choice of a particular $\tilde{x}$. By definition $\{\pi^* g, \pi^* f\}_{\tilde{\omega}}(\tilde{x}) := \tilde{\omega}(X_{\pi^* g}, X_{\pi^* f})(\tilde{x})$. Since $\pi^* f$ and $\pi^* g$ are constant along the same coisotropic submanifold, we get $\tilde{\omega}(X_{\pi^* g}, X_{\pi^* f})(\tilde{x}) = 0$. This proves Lemma 3.7. □

Let us study the Brylinski-Poisson homology of a stratified symplectic manifold M equipped with a Poisson smooth structure. Set $\Omega(M) := \oplus_{p=0}^m \Omega^p(M)$. By Lemma 2.9 we can regard $\Omega(M)$ as a linear subspace in $\Omega(M^{reg})$. Assume that $C^\infty(M)$ is a Poisson smooth structure.

We consider the *canonical complex*

$$\to \Omega^{n+1}(M) \stackrel{\delta}{\to} \Omega^n(M) \to ...$$

where δ is a linear operator defined as follows. Let $\alpha \in \Omega(M)$ and $\alpha = \sum_j f_0^j df_1^j \wedge df_p^j$ be a local representation of α as in Lemma 2.9. Then we set (see [7], [2]):

$$\delta(f_0 df_1 \wedge \cdots \wedge df_n) = \sum_{i=1}^{n} (-1)^{i+1} \{f_0, f_i\}_\omega df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_n$$

$$+ \sum_{1 \le i < j \le n} (-1)^{i+j} f_0 d\{f_i, f_j\}_\omega \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_n.$$

Lemma 3.8. *The boundary operator δ fulfills*

(1) *$\delta = i(G_\omega) \circ d - d \circ i(G_\omega)$. In particular, δ is well-defined.*
(2) *$\delta^2 = 0$.*

Proof. 1) This assertion has been proved in the case of a smooth Poisson manifold M by Brylinski, [2, Lemma 1.2.1]. Since we also have $\{f, g\}_\omega = G_\omega(df \wedge dg)$ on a singular pseudomanifold M, the proof in [2] can be repeated word-by-word, so we omit it. This proves the first assertion.

2) To prove the second assertion we note that $\delta^2(\alpha)(x) = 0$ at all $x \in M^{reg}$, since δ is local operator by the first assertion. Hence $\delta^2(\alpha)(x) = 0$ for all $x \in M$. □

In general it is very difficult to compute the Poisson homology of a Poisson manifold M unless it is a symplectic manifold. The following theorem shows that the isomorphism between the Poisson homology and the de Rham homology on M is a consequence of ω-compatibility of the Poisson smooth structure $C^\infty(M)$.

Theorem 3.9. *Suppose (X, ω) is a stratified symplectic space equipped with a Poisson smooth structure $C^\infty(M)$ compatible with symplectic form ω. Then the symplectic homology of the complex $(\Omega(X), \delta)$ is isomorphic to the de Rham cohomology with inverse grading : $H_k(\Omega(X), \delta) = H^{m-k}(\Omega, d)$. If the smooth structure is locally smoothly contractible, $H_k(\Omega(X), \delta)$ is equal to the singular cohomology $H^{m-k}(X, \mathbb{R})$.*

We denote by $*_\omega$ the symplectic star operator

$$*_\omega : \Lambda^p(\mathbb{R}^{2n}) \to \Lambda^{2n-p}(\mathbb{R}^{2n})$$

satisfying

$$\beta \wedge *_\omega \alpha = G^k(\beta, \alpha) vol,$$

where $vol = \omega^n/n!$. Now let us consider a stratified symplectic space (M^{2n}, ω) with a Poisson smooth structure $C^\infty(M)$. The operator $*_\omega : \Lambda^p T^*_x M^{reg} \to \Lambda^{2n-p} T^*_x M^{reg}$ extends to a linear operator $*_\omega : \Omega^p(M^{reg}) \to \Omega^{2n-p}(M^{reg})$. By Definition 2.9 we can regard the space $\Omega^p(M)$ as a subspace of $\Omega^p(M^{reg})$. In particular, we have $*_\omega(\Omega^p(M)) \subset \Omega^{2n-p}(M^{reg})$.

Proposition 3.10. *If ω is compatible with a Poisson smooth structure $C^\infty(M)$, then $*_\omega(\Omega^k(M)) = \Omega^{2n-k}(M)$.*

Proof. We set $\Omega_A(M) := \{\gamma \in \Omega(M) |\ *_\omega \gamma \in \Omega(M)\}$. To prove the claim it suffices to show that $\Omega_A(M) = \Omega(M)$. Using Lemma 2.9 and taking into account that ω is smooth w.r.t. $C^\infty(M)$ we conclude that the $C^\infty(M)$-module $\Omega^{2n}(M)$ is generated by ω^n. Thus $*_\omega f = f vol$. This proves $*_\omega(C^\infty(M)) = \Omega^{2n}(M)$. In particular $\Omega^0(M) \subset \Omega_A(cL)$, and $\Omega^n(M) \subset \Omega_A(M)$.

Lemma 3.11. *We have*

$$*_\omega(\Omega_A(M)) = \Omega_A(M).$$

Proof. Let $\gamma \in \Omega_A(M)$. By definition $*_\omega \gamma = \beta \in \Omega(M)$. Using the identity $*^2_\omega = Id$, see [2, Lemma 2.1.2], we get $*_\omega \beta = \gamma$. It follows $\beta \in \Omega_A(M)$. This proves that $*_\omega(\Omega_A(M)) \subset \Omega_A(M)$. Taking into account $*^2_\omega = Id$, this proves Lemma(3.11). □

Lemma 3.12. *$\Omega_A(M)$ has the following properties:*

(1) *$\Omega_A(M)$ is a $C^\infty(M)$-module.*
(2) *$d(\Omega_A(M)) \subset \Omega_A(M)$.*

Proof. 1. The first assertion follows from the identity $*_\omega(f(x)\phi(x)) = f(x) \cdot *_\omega \phi(x)$ and the fact that $\Omega(M)$ is a $C^\infty(M)$-module.

2. We need to show that for any $\gamma \in \Omega_A(M)$ we have $*_\omega(d\gamma) \in \Omega(M)$. Using Lemma 3.11 we can write $\gamma = *_\omega \beta$ for some $\beta \in \Omega_A(M)$. Regarding β as an element in $\Omega(M^{reg})$, where we can apply the identity $\delta\beta = (-1)^{deg\,\beta+1} *_\omega d *_\omega$ [2, Theorem 2.2.1], we have

$$*_\omega(d\gamma) = *_\omega d *_\omega \beta = (-1)^{deg\,\beta+1} \delta(\beta) \in \Omega(M).$$

Hence $(d\gamma) \in \Omega(M)$. This proves the second assertion. □

Let us complete the proof of Proposition 3.10. Since $\Omega^1(M)$ is a $C^\infty(M)$-module whose generators are differentials df, $f \in C^\infty(M)$, using Lemma 3.12 we obtain that $\Omega^1(M) \subset \Omega_A(M)$. Inductively, we observe that $\Omega^k(M)$ is a $C^\infty(M)$-module whose generators are the k-forms $d(f(x)\phi(x))$, where $\phi(x) \in \Omega^{k-1}(M)$. By Lemma 3.12, $\Omega^k(M) \subset \Omega_A(M)$ if $\Omega^{k-1}(M) \subset \Omega_A(M)$. This completes the proof of Proposition 3.10.

□

4. The existence of Hamiltonian flows

In this section we prove the existence and uniqueness of a Hamiltonian flow associated with a smooth function H on a symplectic stratified space M equipped with a Poisson smooth structure, see Theorem 4.2. This Theorem generalizes a result by Sjamaar and Lerman in [13, §3].

Let (M, ω) be a stratified symplectic space and $C^\infty(M)$ a Poisson smooth structure on M.

Lemma 4.1. *For any $H \in C^\infty(M)$ the associated Hamiltonian vector field X_H defined on M by setting*

$$X_H(f) := \{H, f\}_\omega \text{ for any } f \in C^\infty(M)$$

is a smooth Zariski vector field on M. If x is a point in a stratum S, then $X_H(x) \in T_xS$.

Proof. By definition of a Poisson structure, the function $X_H(f)$ is smooth for all $f \in C^\infty(M)$. Hence X_H is a smooth Zariski vector field. This proves the first assertion of Lemma 4.1. To prove the second assertion it suffices to show that, if the restriction of a function $f \in C^\infty(M)$ to a neighborhood of $x \in S$ is zero, then $X_H(f)(x) = 0$. But by definition $X_H(f)(x)$ is equal to the Poisson bracket of the restriction of H and f to S. This completes the proof. □

The following theorem generalizes a result by Sjamaar and Lerman [13, §3].

Theorem 4.2. *Given a Hamiltonian function $H \in C^\infty(M)$ and a point $x \in M$ there exists a unique smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that for any $f \in C^\infty(M)$ we have*

$$\frac{d}{dt} f(\gamma(t)) = \{H, f\}.$$

The decomposition of M is defined by the Poisson algebra of smooth functions.

Proof. For $x \in S$ we define $\gamma(t)$ to be the Hamiltonian flow on S defined by X_H, which is by Lemma 4.1 a smooth vector field on S. This proves the existence of the required Hamiltonian flow. Now let us prove the uniqueness of the Hamiltonian flow using Sjamaar's and Lerman's argument in [13, §3]. Denote by Φ_t the Hamiltonian flow whose existence we just proved. Clearly for any $x \in M$ and a compact neighborhood $U(x)$ of $x \in M$ there exists $\varepsilon > 0$ such that $\Phi_t(x')$ is defined for all $t \le \varepsilon$ and for all $x' \in U(x)$. Let $x \in M$ and $\gamma(t)$, $t \in (-\varepsilon_1, \varepsilon_1)$ be an integral curve of X_H with $\gamma_0(0) = x$. We will show that $\Phi_t(\gamma(t)) = x_0$ for all $0 \le t \le \min(\varepsilon, \varepsilon_1)$. By Corollary 2.6.2 smooth functions on M separate points. Therefore it suffices to show that for all $t \le \min(\varepsilon, \varepsilon_1)$ and all $f \in C^\infty(M)$ we have

$$f(\Phi_t(\gamma_t(t))) = f(x).$$

Now we compute

$$\frac{d}{dt} f(\Phi_t(\gamma(t))) = \{H, f\}_\omega(\gamma(t)) + \{f, H\}_\omega(\gamma(t)) = 0.$$

This completes the proof of the uniqueness of the Hamiltonian flow. The last assertion of Theorem 4.2 follows from the inclusion $X_H(x) \in S$, if $x \in S$. □

5. Hodge structure on a compact stratified symplectic space

A stratified symplectic space (M, ω) equipped with a Poisson smooth structure $C^\infty(M, \omega)$ is said to satisfy the hard Lefschetz condition, if the cup product

$$[\omega^k] : H^{m-k}(M) \to H^{m+k}(M)$$

is an isomorphism for any $k \le m = \frac{1}{2} \dim M$. A differential form $\alpha \in \Omega(M)$ is called *harmonic*, if $d\alpha = 0 = \delta\alpha$.

Theorem 5.1. *Let (M, ω) is a compact stratified symplectic space and $C^\infty(M)$ a locally smoothly contractible Poisson smooth structure which is also compatible with ω. Then the following two assertions are equivalent:*

(1) *Any cohomology class contains a harmonic cocycle.*
(2) *$C^\infty(M, \omega)$ satisfies the hard Lefschetz condition.*

Proof. The proof of Theorem 5.1 for smooth symplectic manifold by Yan in [16, Theorem 0.1] can be repeated word-by-word. In the proof we use the structure of $sl_2(\tilde{G}_\omega, L, E)$ on $\Omega(M)$, where L is the wedge multiplication operator by ω and $E = [\tilde{G}_\omega, L]$ while taking into account the validity of Poicare Lemma as a consequence of locally smoothly contractibility of $C^\infty(M)$. □

Remark 5.2. The equivalence between the hard Lefschetz property and the formality of the de Rham complex under the condition of Theorem 5.1 can be also proved in the same way as in [11].

Acknowledgement H.V.L. and J.V. are supported in part by Grant of ASCR Nr100190701. P.S. is supported in part by MSM 0021620839 and GACR 20108/0397.

References

[1] A. Beauville, Symplectic singularities, Invent. Math. 139 (2000), 541-549.
[2] J.C. Brylinksi, A differential complex for Poisson manifolds, JDG 28 (1988), 93-114.
[3] D. H. Collingwood and W. M. McGovern: Nilpotent Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold, 1993, New York.
[4] B. Fu, A survey on symplectic singularities and symplectic resolutions, Ann. Math. Blaise Pascal 13 (2006), no. 2, 209236.
[5] M. Goresky and R. MacPherson, Intersection homology theory, Topology 19 (1980), 135-162.
[6] M. Goresky and R. MacPherson, Stratified Morse theory, Springer Verlag, New York 1988.
[7] J.L. Kozsul, Crochet de Schouten-Nijenhuis et cohomologie, in "Elie Cartan et les Math. d'Aujour d'Hui", Asterisque hors-serie, 1985, 251-171.
[8] H. V. Le, P. Somberg and J. Vanzura, Smooth structures on conical pseudomanifolds, arXiv:1006.5707.
[9] H. Matsumura, Commutative algebra, Benjamin/Cummings, London 1980.
[10] I. Moerdijk and G. E. Reyes, Models for smooth infinitesimal analysis, Springer-Verlag, New York 1991.
[11] S. A. Merkulov, Formality of canonical symplectic complexes and Frobenius manifolds, Internat. Math. Res. Notices (1998), no. 14, 727-733.
[12] M. Mostow, The differentiable structure of Milnor classifying spaces, simplicial complexes, and geometric realizations, J. D. G. 14 (1979), 255-293.
[13] R. Sjamaar, and E. Lerman, Stratified Spaces and Reduction, Ann. of Math., 134 (1991), p.375-422.
[14] D. Panyushev, Rationality of singularities and the Gorenstein properties of nilpotent orbits, Functional Anal. Appl. 25 (1991), 225226.

[15] M. J. Pflaum, Ein Beitrag zur Geometrie und Analysis auf stratifizierten Raumen, Habilitationschrift, Humboldt Univ. 2000.
[16] D. Yan, Hodge Structure on Symplectic Manifolds, Advances in Math. 120 (1996), 143-154.

Hông Vân Lê, Institute of Mathematics of ASCR, Zitna 25, 11567 Praha 1, Czech Republic
Petr Somberg, Mathematical Institute, Charles University, Sokolovska 83, 180 00 Praha 8, Czech Republic.
Jiři Vanžura, Institute of Mathematics of ASCR, Zizkova 22, 61662 Brno, Czech Republic