ON THE WEIGHTED FORWARD REDUCED ENTROPY OF RICCI FLOW

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ABSTRACT. In this paper, we first introduce the weighted forward reduced volume of Ricci flow. The weighted forward reduced volume, which related to expanders of Ricci flow, is well-defined on noncompact manifolds and monotone non-increasing under Ricci flow. For an application to weighted forward reduced volume, we show that if there exists sequence $\lambda_j \to 0$, $\lambda_j > 0$ and $x_j \in M^n$ such that the sequence $(M^n, g_j(0), x_j)$ defined as $g_j(0) = \lambda_j g(0)$, where $(M^n, g(t))$ is the Type III Ricci flow on a noncompact complete n-dimensional manifold, subconverges to $(M^n_\infty, g_\infty(0), x_\infty)$ in C^2 sense, then $(M^n, g(0))$ is isometric to $\mathbb{R}^n$.

1. INTRODUCTION

In [13], G.Perelman introduced the reduced entropy (i.e. reduced distance and reduced volume), which becomes one of powerful tools for studying Ricci flow. The reduced entropy enjoys very nice analytic and geometric properties, including in particular the monotonicity of the reduced volume. These properties can be used, as demonstrated by Perelman, to show the limit of the suitable rescaled Ricci flows is a gradient shrinking soliton.

Then M.Feldman, T.Ilmanen, L.Ni [4] observed that there is a dual version of G.Perelman's reduced entropy, which related to the expanders of Ricci flow. Let $g(t)$ solves the Ricci flow

$$\frac{\partial g}{\partial t} = -2Rc. \tag{1.1}$$

on $M \times [0, T]$. Fix $x \in M^n$ and let γ be a path $(x(\eta), \eta)$ joining $(x, 0)$ and (y, t). They define the forward $\mathcal{L}_+$-length as

$$\mathcal{L}_+(\gamma) = \int_0^t \sqrt{\eta}(R(\gamma(\eta)) + |\gamma'(\eta)|^2)d\eta. \tag{1.2}$$

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Denote $L_+(y,t)$ be the length of a shortest forward $\mathcal{L}_+$-length joining $(x,0)$ and (y,t). Let

$$l_+(y,t) = \frac{L_+(y,t)}{2\sqrt{t}} \tag{1.3}$$

be the forward l_+-length. Note that the forward reduced distance (1.3) is defined under the forward Ricci flow (1.1), which is the only difference from Perelman's reduced distance defined under the backward Ricci flow. The forward reduced volume is defined in [4] as

$$\theta_+(t) = \int_M (t)^{-\frac{n}{2}} e^{l_+(y,t)} dvol(y). \tag{1.4}$$

They also proved forward reduced volume defined in (1.4) is monotone non-increasing along the Ricci flow (1.1).

Unfortunately, the forward reduced volume defined in (1.4) may not well-defined on noncompact manifolds. In the first part of this paper, we introduce the weighted forward reduced volume in this paper based on the work in [4] and [13]. The weighted forward reduced volume is well-defined on noncompact manifolds and monotone non-increasing under the Ricci flow (1.1). Moreover, we show that, just the same as the Perelman's reduced volume, the weighted reduced volume entropy has the value $(4\pi)^{\frac{n}{2}}$ if and only if the Ricci flow is the trivial flow on flat Euclidean space.

We define the weighted forward reduced volume as follows. First, we define the forward $\mathcal{L}_+$-exponential map $\mathcal{L}_+exp(V,t) : T_xM \to M$ at time $t \in [0,T)$. For $V \in T_xM$, let γ_V denote the $\mathcal{L}_+$-geodesic such that $\gamma_V(0) = p$, $\lim_{t\to 0} \sqrt{t}\gamma'_V(t) = V$. If γ_V exists on $[0,t]$, we set

$$\mathcal{L}_+exp(V,t) = \gamma_V(t). \tag{1.5}$$

Denote τ_V be the first time the $\mathcal{L}_+$-geodesic γ_V stop minimizing. Define

$$\Omega(t) = \{V \in T_xM^n : \tau_V > t\}.$$

Obviously, $\Omega(t_1) \subset \Omega(t_2)$ if $t_1 < t_2$. Let $J_i^V(t), i = 1,\cdots,n$, be $\mathcal{L}_+$-Jacobi fields along $\gamma_V(t)$ with $J_i^V(0) = 0, (\nabla_V J_i^V)(0) = E_i^0$, where $\{E_i^0\}_{i=1}^n$ is an orthonormal basis for T_xM with respect to $g(0)$. Then $D(\mathcal{L}_+exp(V,t))(E_i^0) = J_i^V(t)$. We define

$$\mathcal{L}_+J_V(t) = \sqrt{det(< J_i^V(t), J_j^V(t) >)}$$

and the weighted forward reduced volume as

$$\widetilde{\mathcal{V}}_+(t) = \int_{\Omega(t)} t^{-\frac{n}{2}} e^{l_+(\gamma_V(t),t)} e^{-2|V|^2_{g(0)}} \mathcal{L}_+J_V(t) dx_{g(0)}(V), \tag{1.6}$$

where $dx_{g(0)}$ is the standard Euclidean volume form on $(T_xM, g(x,0))$, i.e. we define the weighted forward reduced volume as

$$\widetilde{\mathcal{V}}_+(t) = \int_{\Omega(t)} t^{-\frac{n}{2}} e^{l_+(y,t)} e^{-2|\mathcal{L}_+exp^{-1}(y,t)|^2_{g(0)}} dvol(y), \tag{1.7}$$

We use the convention

$$\mathcal{L}_+J_V(t) \doteq 0 \text{ for } t \geq \tau_V.$$

Then we can write the weighted forward reduced volume as

$$\widetilde{\mathcal{V}}_+(t) = \int_{T_xM^n} t^{-\frac{n}{2}} e^{l_+(\gamma_V(t),t)} \mathcal{L}_+J_V(t) e^{-2|V|^2_{g(0)}} dx_{g(0)}(V). \tag{1.8}$$

We have the following properties for the weighted forward reduced volume.

Theorem 1.1. *The weighted forward reduced volume defined in (1.6) is monotone non-increasing under the Ricci flow (1.1) and* $\widetilde{\mathcal{V}}_+(t) \leq \lim_{t\to 0+} \widetilde{\mathcal{V}}_+(t) \leq (4\pi)^{\frac{n}{2}}$. *If* $\widetilde{\mathcal{V}}_+(t_1) = \widetilde{\mathcal{V}}_+(t_2)$ *for some* $0 < t_1 < t_2$*, then this flow is a gradient expanding soliton on* $0 \leq t < \infty$ *and hence is the trivial flow on flat Euclidean space. In particular, if* $\widetilde{\mathcal{V}}_+(\bar{t}) = (4\pi)^{\frac{n}{2}}$ *for some time* $\bar{t} > 0$*, then this flow is the trivial flow on flat Euclidean space.*

We also have the following rescaling property for the weighted forward reduced volume.

Theorem 1.2. *We have* $\widetilde{\mathcal{V}}^j_+(t) = \widetilde{\mathcal{V}}_+(\lambda_j^{-1}t)$ *under the rescaling* $g_j(t) = \lambda_j g(\lambda_j^{-1}t)$*, where* $\widetilde{\mathcal{V}}^j_+$ *and* $\widetilde{\mathcal{V}}_+$ *denote the weighted forward reduced volume with respect to metric* g_j *and* g *respectively.*

Finally, we give an application to the weighted forward reduced volume (1.6). Note that the flat Euclidean space $\mathbb{R}^n$, on which Ricci flow is of Type III, has the sequence $\lambda_j \to 0$, $\lambda_j > 0$ and $x_j \in \mathbb{R}^n$ such that the sequence $(\mathbb{R}^n, g_j(0), x_j)$ defined as $g_j(0) = \lambda_j g(0)$ subconverges to $(\mathbb{R}^n, g_\infty(0), x_\infty)$ in C^∞ sense. The following theorem shows that $\mathbb{R}^n$ is the only case which has such property.

Theorem 1.3. *Let* $(M^n, g(t))$ *be the Type III Ricci flow on a noncompact complete n-dimensional manifold. If there exists sequence* $\lambda_j \to 0$, $\lambda_j > 0$ *and* $x_j \in M^n$ *such that the sequence* $(M^n, g_j(0), x_j)$ *defined as* $g_j(0) = \lambda_j g(0)$ *subconverges to* $(M^n_\infty, g_\infty(0), x_\infty)$ *in* C^2 *sense, then* $(M^n, g(0))$ *is isometric to* $\mathbb{R}^n$.

The organization of the paper is as follows. In section 2, we first recall some basic formulas and properties about forward reduced entropy in [4]. Then we study the properties of forward reduced volume density which defined by forward $\mathcal{L}_+$-exponential map. Finally, we give the proofs of

Theorem 1.1 and Theorem 1.2. In section 3, we study the singularities of Type III Ricci flow. Then we give the proof of Theorem 1.3.

2. Weighted Forward Reduced volume and Expanders

Before we present the proofs of Theorem 1.1 and Theorem 1.2, we recall some basic formulas and properties about forward reduced entropy in [4]. Clearly, one can show that $l_+(y,t)$ is locally lipschitz function and the cut-Locus of $\mathcal{L}_+exp(V,t)$ is a closed set of measure zero by using the similar methods in [16].

We need the following two theorems due to M.Feldman, T.Ilmanen, L.Ni [4], which state the following adapted form.

Theorem 2.1. [4] *let γ be a path $(x(\eta),\eta)$ joining $(x,0)$ and (y,t). Set $X=\gamma'(t)$ and Y be a variational vector along γ such that $Y(0)=0$. The first variation of $\mathcal{L}_+$ is that*

$$\delta\mathcal{L}_+ = 2\sqrt{t} < X, Y > (t) + \int_0^t \sqrt{\eta} < Y, \nabla R - 2\nabla_X X + 4Rc(X,\cdot) - \frac{1}{\eta}X > d\eta. \tag{2.1}$$

If $\gamma(t)$ is the mimimal $\mathcal{L}_+$-geodesic, then

$$\nabla L_+ = 2\sqrt{t}X, \tag{2.2}$$

$$t^{\frac{3}{2}}(R+|X|^2) = K + \frac{1}{2}L_+, \tag{2.3}$$

where $K=\int_0^t \eta^{\frac{3}{2}}H(X)d\eta$, $H(X)=\frac{\partial R}{\partial t}+2<\nabla R, X> +2Rc(X,X)+\frac{R}{t}$. The second variation of $\mathcal{L}_+$ is that

$$\begin{aligned}\delta_Y^2\mathcal{L}_+ =& 2\sqrt{t} < X, Y > (t) + \int_0^t \sqrt{\eta}(HessR(Y,Y) - 2R(X,Y,X,Y) \\ &+ 2|\nabla_X Y|^2 + 4\nabla_Y Rc(Y,X) - 2\nabla_X Rc(Y,Y))d\eta.\end{aligned} \tag{2.4}$$

Let $\widetilde{Y}$ be a vector field along γ satisfies the ODE

$$\begin{cases} \nabla_X\widetilde{Y}(\eta) = Rc(\widetilde{Y}(\eta),\cdot) + \frac{1}{2\eta}\widetilde{Y}(\eta), \eta\in[0,t] \\ \widetilde{Y}(0) = Y(0) = 0. \end{cases} \tag{2.5}$$

Then

$$HessL_+(\widetilde{Y},\widetilde{Y}) \le \frac{|\widetilde{Y}|^2}{\sqrt{t}} + 2\sqrt{t}Rc(\widetilde{Y},\widetilde{Y}) - \int_0^t \sqrt{\eta}H(X,\widetilde{Y})d\eta, \tag{2.6}$$

where $H(X,\widetilde{Y}) = -HessR(X,\widetilde{Y}) + 2R(X,\widetilde{Y},X,\widetilde{Y}) + 2|Rc(X,\cdot)|^2 + \frac{Rc(\widetilde{Y},\widetilde{Y})}{t} + 2\frac{\partial Rc}{\partial t}(\widetilde{Y},\widetilde{Y}) - 4\nabla_{\widetilde{Y}}Rc(\widetilde{Y},X) + 4\nabla_X Rc(\widetilde{Y},\widetilde{Y})$. The equality holds in (2.6) if and only if the vector filed $\widetilde{Y}$ satisfying (2.16) is an $\mathcal{L}_+$-Jacobi field.

Theorem 2.2. [4] *Let $l_+ \doteq l_+(y,t)$ be the minimal $\mathcal{L}_+$-geodesic from $(x,0)$ to (y,t). If (y,t) is not in the cut-Locus of $\mathcal{L}_+exp$, then at (y,t)*

$$\frac{\partial l_+}{\partial t} = R - \frac{l_+}{t} - \frac{K}{2t^{\frac{3}{2}}}, \tag{2.7}$$

$$|\nabla l_+|^2 = \frac{l_+}{t} - R + \frac{K}{t^{\frac{3}{2}}}, \tag{2.8}$$

$$\Delta l_+ \le R + \frac{n}{2t} - \frac{K}{2t^{\frac{3}{2}}}, \tag{2.9}$$

$$\frac{\partial l_+}{\partial t} + \Delta l_+ + |\nabla l_+|^2 - R - \frac{n}{2t} \le 0, \tag{2.10}$$

$$2\Delta l_+ + |\nabla l_+|^2 - R - \frac{l_+ + n}{t} \le 0, \tag{2.11}$$

We first study properties of the forward reduced volume density defined as

$$d\mathcal{V}_+ = t^{-\frac{n}{2}} e^{l_+(\gamma_V(t),t)} \mathcal{L}_+ J_V(t) dx_{g(0)}(V), \tag{2.12}$$

Note that the weighted forward reduced volume

$$\widetilde{\mathcal{V}}_+(t) = \int_{T_xM^n} e^{-2|V|^2_{g(0)}} d\mathcal{V}_+.$$

Analogous to [13], we have the following theorem.

Theorem 2.3. *The forward reduced volume density $d\mathcal{V}_+$ defined in (2.12) is monotone non-increasing along the Ricci flow (1.1). Moreover, if $d\mathcal{V}_+(t_1) = d\mathcal{V}_+(t_2)$ for some $0 < t_1 < t_2$, then this flow is a gradient expanding soliton.*

Proof. Let $\gamma_V(t)$ be the minimal $\mathcal{L}_+$-geodesic defined in (1.5) and $y = \gamma_V(t)$. We consider (y,t) in the cut-Locus of $\mathcal{L}_+exp(V,t)$. Recall that $\nabla l_+(y,t) = \gamma_V'(t) = X(t)$. Then by (2.3) and (2.7), we get

$$\begin{aligned} \frac{\partial l_+(\gamma_V(t),t)}{\partial t} &= \frac{\partial l_+(y,t)}{\partial t} + \nabla l \cdot X \\ &= R - \frac{l_+(y,t)}{t} - \frac{K}{2t^{\frac{3}{2}}} + |X|^2 \\ &= \frac{1}{2} t^{-\frac{3}{2}} K. \end{aligned} \tag{2.13}$$

For any fixed t, we choose an orthonormal basis $\{E_i(t)\}$ of $T_{\gamma_V(t)}M$. We extend $E_i(\eta)$, $\eta \in [0,t]$ to an $\mathcal{L}_+$-Jacobi field along γ_V with $E_i(0) = 0$. We write $J_i^V(t) = \sum_i^n A_i^j E_j(t)$ for same matrix $(A_i^j) \in GL(n,\mathbb{R})$. Then $J_i^V(\eta) = \sum_i^n A_i^j E_j(\eta)$ for all $\eta \in [0,t]$.

Hence, by (2.6), we calculate at time t

$$\begin{aligned}
\frac{d}{d\eta}|_{\eta=t}\ln\mathcal{L}_+J_V &= \frac{d}{d\eta}|_{\eta=t}\ln\sqrt{det(<\sum_{k=1}^n A_i^kE_k,\sum_{l=1}^n A_i^lE_l>)}\\
&= \frac{1}{2}\frac{d}{d\eta}|_{\eta=t}\sum_i|E_i|^2\\
&= \sum_i(-Rc(E_i,E_i)+<\nabla_{E_i}X,E_i>)\\
&= \sum_i(-Rc(E_i,E_i)+\frac{1}{2\sqrt{t}}HessL_+(E_i,E_i)) \qquad (2.14)\\
&\leq \sum_i(\frac{1}{2t}-\frac{1}{2\sqrt{t}}\int_0^t\sqrt{\eta}H(X,\widetilde{E}_i)d\eta) \qquad (2.15)
\end{aligned}$$

where $\widetilde{E}_i(\eta)$ are the vector fields along γ_V satisfying

$$\begin{cases}\nabla_X\widetilde{E}_i(\eta)=Rc(\widetilde{E}_i(\eta),\cdot)+\frac{1}{2\eta}\widetilde{E}_i(\eta),\eta\in[0,t]\\ \widetilde{E}_i(t)=E_i(t),\end{cases} \qquad (2.16)$$

which in particular implies that

$$<\widetilde{E}_i,\widetilde{E}_j>(\eta)=\frac{\eta}{t}<E_i,E_j>(t)=\frac{\eta}{t}\delta_{ij}. \qquad (2.17)$$

It follows that

$$\sum_{i=1}^n H(X,\widetilde{E}_i)(\eta)=\frac{\eta}{t}H(X).$$

Hence

$$\frac{d}{d\eta}|_{\eta=t}\ln\mathcal{L}_+J_V\leq\frac{n}{2t}-\frac{1}{2}t^{-\frac{3}{2}}K,$$

and

$$\frac{d}{dt}\ln d\mathcal{V}_+=-\frac{n}{2t}+\frac{\partial l+}{\partial t}+\frac{dln\mathcal{L}_+J_V}{dt}\leq 0. \qquad (2.18)$$

If equality in (2.18) holds, then we have equality in (2.15) holds. By Theorem 2.1, we conclude that each $\widetilde{E}_i(\eta)$ is an $\mathcal{L}_+$-Jacobi field. Hence

$$\frac{d}{d\eta}|_{\eta=t}|E_i|^2=\frac{d}{d\eta}|_{\eta=t}|\widetilde{E}_i|^2=\frac{|E_i(t)|^2}{t}. \qquad (2.19)$$

Combining with (2.14) and (2.19), we get

$$Rc(E_i,E_i)-\frac{1}{2\sqrt{t}}HessL_+(E_i,E_i)=-\frac{|E_i|^2}{2t}.$$

□

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem 2.3, we know that

$$\frac{d}{dt}(t^{-\frac{n}{2}}e^{l_+(\gamma_V(t),t)}\mathcal{L}_+J_V(t)) \leq 0.$$

It follows that

$$\frac{d}{dt}(t^{-\frac{n}{2}}e^{l_+(\gamma_V(t),t)}\mathcal{L}_+J_V(t)e^{-2|V|^2_{g(0)}}) \leq 0.$$

Hence $\widetilde{\mathcal{V}}_+(t_1) \leq \widetilde{\mathcal{V}}_+(t_1)$ for $t_1 < t_2$ since we have $\Omega(t_1) \subset \Omega(t_2)$. Since

$$\begin{aligned}\lim_{t\to 0+} l_+(\gamma_V(t),t) &= \lim_{t\to 0+}\frac{1}{2\sqrt{t}}\int_0^t \sqrt{\eta}(R(\gamma_V(\eta),\eta) + |\frac{d\gamma_V}{d\eta}|^2)d\eta \\ &= |V|^2_{g(0)},\end{aligned}$$

and

$$\lim_{t\to 0^+}\frac{\mathcal{L}_+J_V(t)}{t^{\frac{n}{2}}} = \lim_{t\to o^+}\frac{\sqrt{det(< 2\sqrt{t}E_i(t), 2\sqrt{t}E_j(t) >_{g(0)})}}{t^{\frac{n}{2}}} = 2^n,$$

we conclude that

$$\lim_{t\to 0^+} t^{-\frac{n}{2}}e^{l_+(\gamma_V(t),t)}\mathcal{L}_+J_V(t) = 2^n e^{|V|^2_{g(0)}}.$$

Hence

$$\lim_{t\to 0^+}\widetilde{\mathcal{V}}_+(t) \leq \int_{T_pM} 2^n e^{-|V|^2_{g(0)}}dx(V) = (4\pi)^{\frac{n}{2}}.$$

If $\widetilde{\mathcal{V}}_+(t_1) = \widetilde{\mathcal{V}}_+(t_2)$ for any $0 < t_1 < t_2$, then $d\mathcal{V}_+(t_1) = d\mathcal{V}_+(t_2)$ for any $0 < t_1 < t_2$. So $(M^n, g(t))$ must be a gradient expanding soliton by Theorem 2.3, i.e. we have

$$Rc + Hess(-l_+) = -\frac{g}{2t}$$

for some smooth function l_+ on M^n. Let $\phi_t : M \to M, 0 < t \leq \bar{t}$ be the one-parameter family of diffeomorphisms obtained by

$$\frac{d\phi_t}{dt} = \nabla l_+ \quad \text{and} \quad \phi_{\bar{t}} = Id.$$

We consider $h(t) = \frac{\bar{t}}{t}\phi_t^* g(t)$ and calculate

$$\begin{aligned}\frac{dh}{dt} &= -\frac{\bar{t}}{t^2}\phi_t^* g(t) + \frac{\bar{t}}{t}\phi_t^*\mathcal{L}_{\frac{d\phi_t}{dt}}(g(t)) - 2\frac{\bar{t}}{t}\phi_t^* Rc(g(t)) \\ &= -\frac{\bar{t}}{t^2}\phi_t^* g(t) + \frac{\bar{t}}{t}2Hess(l_+) + \frac{\bar{t}}{t}\phi_t^*(\frac{g}{t} - 2Hess(l_+)) = 0.\end{aligned}$$

It follows that

$$g(t) = \frac{t}{\bar{t}}(\phi_t^{-1})^* g(\bar{t}).$$

Suppose that there is some $(y, \bar{t})$ with $|Rm|(y, \bar{t}) = K > 0$, we have $|Rm|(\phi_t^{-1}(y), t) = \frac{K\bar{t}}{t}$, and these curvatures are not bounded as $t \to 0$, which is a contradiction. Then we have

$$Hess(l_+) = \frac{1}{2t}g.$$

Thus l_+ is strictly convex function. The similar arguments to Lemma 2.3 in [16] can show that

$$l_+(y,t) \geq e^{-2ct}\frac{d_{g(0)}(x,y)}{4t} - \frac{nc}{3}t,$$

if $Rc \geq -cg$ on $[0,t]$, so that $l_+(y,t)$ have the only minimum point in M^n. Hence M^n is diffeomorphic to $\mathbb{R}^n$.

Since $\widetilde{\mathcal{V}}_+(t)$ is monotone non-increasing, $\widetilde{\mathcal{V}}_+(t)$ is independent of t if $\widetilde{\mathcal{V}}_+(\bar{t}) = (4\pi)^{\frac{n}{2}}$ for some time $\bar{t} > 0$. Then we derive that M^n is isometric to $\mathbb{R}^n$. □

Finally, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. We denote $\gamma_v^j(t)$ (resp. $\gamma_v^\infty(t)$) be the minimal $\mathcal{L}_+$-geodesic with respect to $g_j(t)$ (resp. $g_\infty(t)$) which starting from $(x_j, 0)$ (resp. $(x_\infty, 0)$) and satisfying $\lim_{t\to 0}\sqrt{t}\frac{d\gamma_V^j(t)}{dt} = V$ (resp. $\lim_{t\to 0}\sqrt{t}\frac{d\gamma_V^\infty(t)}{dt} = V$).

We have that $\gamma^j_{\sqrt{\lambda_j^{-1}}V}(t) = \gamma_V(\lambda_j^{-1}t)$, $l_+^j(y,t) = l_+(y, \lambda_j^{-1}t)$ and $\mathcal{L}_+J_V^j(t)dx_{g_j(0)}(V) = (\lambda_j^{-1})^{-\frac{n}{2}}\mathcal{L}_+J_{\sqrt{\lambda_j}V}(\lambda_j^{-1}t)dx_{g(0)}(\sqrt{\lambda_j}V)$. Hence

$$\begin{aligned}
\widetilde{\mathcal{V}}_+^j(t) &= \int_{T_xM^n}(t)^{-\frac{n}{2}}e^{l_+^j(\gamma_V(t),t)}\mathcal{L}_+J_V^j(t)e^{-2|V|^2_{g_j(0)}}dx_{g_j(0)}(V)\\
&= \int_{T_xM^n}(\lambda_j^{-1}t)^{-\frac{n}{2}}e^{l_+(\gamma_{\sqrt{\lambda_j}V}(\lambda_j^{-1}t),\lambda_j^{-1}t)}\\
&\qquad\qquad \times\mathcal{L}_+J_{\sqrt{\lambda_j}V}(\lambda_j^{-1}t)e^{-2|\sqrt{\lambda_j}V|^2_{g(0)}}dx_{g(0)}(\sqrt{\lambda_j}V)\\
&= \widetilde{\mathcal{V}}_+(\lambda_j^{-1}t).
\end{aligned}$$

□

3. The proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. We argue by contradiction. If $(M^n, g(0))$ is not isometric to $\mathbb{R}^n$, we define the rescaled Ricci flows $(M^n, g_j(t), x_j)$ by $g_j(t) = \lambda_j g(\lambda_j^{-1}t)$. First, we have at any $y \in M$ that

$$|Rm_{g_j(t)}|_{g_j(t)}(y) = \frac{|Rm_{g(\frac{t}{\lambda_j})}|_{g(\frac{t}{\lambda_j})}(y)}{\lambda_j} \leq \frac{C}{\lambda_j\cdot\frac{t}{\lambda_j}} = \frac{C}{t}, \tag{3.1}$$

which gives an uniformly curvature bound on any compact subsets of $(0,\infty)$.

Since $(M^n, g_j(0), x_j)$ subconverges to $(M^n_\infty, g_\infty(0), x_\infty)$ in C^2 sense, we conclude that $(M^n, g_j(t), x_j)$ subconverges to $(M^n_\infty, g_\infty(t), x_\infty)$ in C^∞ sense for any $t \in (0,\infty)$ by R.Hamilton's precompactness theorem. We consider the forward reduced distance based on $(x_j, 0)$ with respect to the metric $g_j(t)$, which defined as $l^j_{+(x_j,0)}$. We denote $\gamma^j_v(t)$ (resp. $\gamma^\infty_v(t)$) be the minimal $\mathcal{L}_+$-geodesic with respect to $g_j(t)$ (resp. $g_\infty(t)$) which starting from $(x_j, 0)$ (resp. $(x_\infty, 0)$) and satisfying $\lim_{t\to 0}\sqrt{t}\frac{d\gamma^j_V(t)}{dt} = V$ (resp. $\lim_{t\to 0}\sqrt{t}\frac{d\gamma^\infty_V(t)}{dt} = V$). It follows that $\gamma^j_V(t) \to \gamma^\infty_V(t)$, $l^j_+(\gamma^j_V(t), t) \to l^\infty_+(\gamma^\infty_V(t), t)$ and $\mathcal{L}_+ J^j_V(t) \to \mathcal{L}_+ J^\infty_V(t)$.

Hence

$$\begin{aligned}
\widetilde{\mathcal{V}}^\infty_+(t) &= \int_{T_{x_\infty}M^n} (t)^{-\frac{n}{2}} e^{l^\infty_+(\gamma^\infty_V(t),t)} \mathcal{L}_+ J^\infty_V(t) e^{-2|V|^2_{g_\infty(0)}} dx_{g_\infty(0)}(V) \\
&= \lim_{j\to\infty} \int_{T_{x_j}M^n} (t)^{-\frac{n}{2}} e^{l^j_+(\gamma^j_V(t),t)} \mathcal{L}_+ J^j_V(t) e^{-2|V|^2_{g_j(0)}} dx_{g_j(0)}(V) \\
&= \lim_{j\to\infty} \widetilde{\mathcal{V}}^j_+(t), \\
&= \lim_{j\to\infty} \widetilde{\mathcal{V}}_+(\lambda_j^{-1} t),
\end{aligned}$$

where $\lambda_j \to 0$. Since $\widetilde{\mathcal{V}}_+$ is a monotone decreasing function and $(M^n, g(0))$ is not isometric to $\mathbb{R}^n$, we have

$$\widetilde{\mathcal{V}}^\infty_+ = \lim_{t\to\infty} \widetilde{\mathcal{V}}_+(t) \equiv c < (4\pi)^{\frac{n}{2}}. \tag{3.2}$$

In particular, $\widetilde{\mathcal{V}}^\infty_+$ is independent of t. Hence $(M^n_\infty, g_\infty(0))$ is isometric to $\mathbb{R}^n$ by Theorem 1.1. Then $\widetilde{\mathcal{V}}^\infty_+ = (4\pi)^{\frac{n}{2}}$ by Theorem 1.1 which contradicts to (3.2). □

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