MATCHINGS IN 3-UNIFORM HYPERGRAPHS

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Abstract. We determine the minimum vertex degree that ensures a perfect matching in a 3-uniform hypergraph. More precisely, suppose that H is a sufficiently large 3-uniform hypergraph whose order n is divisible by 3. If the minimum vertex degree of H is greater than $\binom{n-1}{2} - \binom{2n/3}{2}$, then H contains a perfect matching. This bound is tight and answers a question of Hàn, Person and Schacht. More generally, we show that H contains a matching of size $d \leq n/3$ if its minimum vertex degree is greater than $\binom{n-1}{2} - \binom{n-d}{2}$, which is also best possible.

1. Introduction

A *perfect matching* in a hypergraph H is a collection of vertex-disjoint edges of H which cover the vertex set $V(H)$ of H. A theorem of Tutte [13] gives a characterisation of all those graphs which contain a perfect matching. On the other hand, the decision problem whether an r-uniform hypergraph contains a perfect matching is NP-complete for $r \geq 3$. (See, for example, [5] for complexity results in the area.) It is natural therefore to seek simple sufficient conditions that ensure a perfect matching in an r-uniform hypergraph.

Given an r-uniform hypergraph H and distinct vertices $v_1, \dots, v_\ell \in V(H)$ (where $1 \leq \ell \leq r-1$) we define $d_H(v_1, \dots, v_\ell)$ to be the number of edges containing each of $v_1, \dots, v_\ell$. The *minimum ℓ-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(v_1, \dots, v_\ell)$ over all ℓ-element sets of vertices in H. Of these parameters the two most natural to consider are the *minimum vertex degree* $\delta_1(H)$ and the *minimum collective degree* or *minimum codegree* $\delta_{r-1}(H)$. Rödl, Ruciński and Szemerédi [12] determined the minimum codegree that ensures a perfect matching in an r-uniform hypergraph. This improved bounds given in [6, 11]. An r-partite version was proved by Aharoni, Georgakopoulos and Sprüssel [1].

Much less is known about minimum vertex degree conditions for perfect matchings in r-uniform hypergraphs H. Hàn, Person and Schacht [4] showed that the threshold in the case when $r = 3$ is $(1 + o(1))\frac{5}{9}\binom{|H|}{2}$. This improved an earlier bound given by Daykin and Häggkvist [3]. Here we determine the threshold exactly, which answers a question from [4].

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Theorem 1. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-uniform hypergraph whose order $n \geq n_0$ is divisible by 3. If*

$$\delta_1(H) > \binom{n-1}{2} - \binom{2n/3}{2}$$

then H has a perfect matching.

While finalising the manuscript we learned from [10] that the same result was also announced recently by Szemerédi. The following example shows that the result is best possible: let H^* be the 3-uniform hypergraph whose vertex set is partitioned into two vertex classes V and W of sizes $2n/3+1$ and $n/3-1$ respectively and whose edge set consists precisely of all those edges with at least one endpoint in W. Then H^* does not have a perfect matching and $\delta_1(H) = \binom{n-1}{2} - \binom{2n/3}{2}$.

The example generalises in the obvious way to r-uniform hypergraphs. This leads to the following conjecture, which is implicit in several earlier papers (see e.g. [4, 7]). Partial results were proved by Hàn, Person and Schacht [4] as well as Markström and Ruciński [8].

Conjecture 2. *For each integer $r \geq 3$ there exists an integer $n_0 = n_0(r)$ such that the following holds. Suppose that H is an r-uniform hypergraph whose order $n \geq n_0$ is divisible by r. If*

$$\delta_1(H) > \binom{n-1}{r-1} - \binom{(r-1)n/r}{r-1},$$

then H has a perfect matching.

It is also natural to ask about the minimum (vertex) degree which guarantees a matching of given size d. Bollobás, Daykin and Erdős [2] solved this problem for the case when d is small compared to the order of H. We state the 3-uniform case of their result here. The above hypergraph H^* with W of size $d-1$ shows that the minimum degree bound is best possible.

Theorem 3 (Bollobás, Daykin and Erdős [2])**.** *Let $d \in \mathbb{N}$. If H is a 3-uniform hypergraph on $n > 54(d+1)$ vertices and*

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}$$

then H contains a matching of size at least d.

Here we extend this result to the entire range of d. Note that Theorem 4 generalises Theorem 1, so it suffices to prove Theorem 4.

Theorem 4. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-uniform hypergraph on $n \geq n_0$ vertices, that $n/3 \geq d \in \mathbb{N}$ and that*

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}.$$

Then H contains a matching of size at least d.

It would be interesting to obtain analogous results (i.e. minimum degree conditions which guarantee a matching of size d) for r-uniform hypergraphs and for r-partite hypergraphs (some bounds are given in [3]).

The situation for ℓ-degrees where $1 < \ell < r-1$ is also still open. Pikhurko [9] showed that if $\ell \geq r/2$ and H is an r-uniform hypergraph whose order n is divisible by r then H has a perfect matching provided that $\delta_\ell(H) \geq (1/2+o(1))\binom{n}{r-\ell}$. This result is best possible up to the $o(1)$-term. In [4], Hàn, Person and Schacht provided conditions on $\delta_\ell(H)$ that ensure a perfect matching in the case when $\ell < r/2$. These bounds were subsequently lowered by Markström and Ruciński [8]. See [10] for further results concerning perfect matchings in hypergraphs.

2. Notation

Given a hypergraph H and subsets V_1, V_2, V_3 of its vertex set $V(H)$, we say that an edge $v_1v_2v_3$ is of *type* $V_1V_2V_3$ if $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$.

Let $d \leq n/3$ and let V, W be a partition of a set of n vertices such that $|W| = d$. Define $H_{n,d}(V,W)$ to be the hypergraph with vertex set $V \cup W$ consisting of all those edges which have type VVW or VWW. Thus $H_{n,d}(V,W)$ has a matching of size d,

$$\delta_1(H_{n,d}(V,W)) = \binom{n-1}{2} - \binom{n-d-1}{2}$$

and $H_{n,d}(V,W)$ is very close to the extremal hypergraph which shows that the degree condition in Theorem 4 is best possible. V and W are the *vertex classes of* $H_{n,d}(V,W)$.

Given $\varepsilon > 0$, a 3-uniform hypergraph H on n vertices and a partition V, W of $V(H)$ with $|W| = d$, we say that H is *ε-close to* $H_{n,d}(V,W)$ if

$$|E(H_{n,d}(V,W)) \setminus E(H)| \leq \varepsilon n^3.$$

In this case we also call V and W *vertex classes of* H. (So H does not have unique vertex classes.) We say that H is *ε-close to* $H_{n,d}$ if there is a partition V, W of $V(H)$ such that $|W| = d$ and H is ε-close to $H_{n,d}(V,W)$.

Given a vertex v of a 3-uniform hypergraph H, we write $N_H(v)$ for the *neighbourhood of* v, i.e. the set of all those (unordered) tuples of vertices which form an edge together with v. Given two disjoint sets $A, B \subseteq V(H)$, we define the *link graph* $L_v(A,B)$ *of* v *with respect to* A, B to be the bipartite graph whose vertex classes are A and B and in which $a \in A$ is joined to $b \in B$ if and only if $ab \in N_H(v)$. Similarly, given a set $A \subseteq V(H)$, we define the *link graph* $L_v(A)$ *of* v *with respect to* A to be the graph whose vertex set is A and in which $a, a' \in A$ are joined if and only if $aa' \in N_H(v)$. Also, given disjoint sets $A, B, C, D, E \subseteq V(H)$, we write $L_v(ABCD)$ for $L_v(A,B) \cup L_v(B,C) \cup L_v(C,D)$. We define $L_v(ABCDE)$ similarly. If M is a matching in H and E, F are two edges in M with $v \notin E, F$, we write $L_v(EF)$ for $L_v(V(E), V(F))$. If $E_1, \dots, E_5$ are matching edges avoiding v, we define $L_v(E_1 \dots E_4)$ and $L_v(E_1 \dots E_5)$ similarly. If $e = uw$ is an edge in the link graph of v, then we write ve for the edge vuw of H. A matching in H of size d is called a *d-matching.*

Given a set M and $k \geq 2$, we write $\binom{M}{k}$ for the set of all k-element subsets of M. Given sets M and M', we write MM' for the set of all pairs mm' with $m \in M$ and $m' \in M'$.

Given two graphs G and G', we write $G \cong G'$ if they are isomorphic. A bipartite graph is called *balanced* if its vertex classes have equal size. By a *directed graph* we mean a graph whose edges are directed, but we only allow at most two edges between any pair of vertices: at most one edge in each direction. We write vw for the edge directed from v to w. Given disjoint vertex sets V and W of a directed graph, we write $e(V, W)$ for the number of all those edges which are directed from some vertex in V to some vertex in W. A directed graph G is an *oriented graph* if it has at most one edge between any pair of vertices (i.e. if G has no directed cycle of length 2).

We will often write $0 < a_1 \ll a_2 \ll a_3$ to mean that we can choose the constants a_1, a_2, a_3 from right to left. More precisely, there are increasing functions f and g such that, given a_3, whenever we choose some $a_2 \leq f(a_3)$ and $a_1 \leq g(a_2)$, all calculations needed in our proof are valid. Hierarchies with more constants are defined in the obvious way.

3. Preliminaries and outline of proof

Our approach towards Theorem 4 follows the so-called *stability approach*: we prove an approximate version of the desired result which states that the minimum degree condition implies that either (i) H contains a d-matching or (ii) H is 'close' to the extremal hypergraph. The latter implies that H is 'close' to the hypergraph $H_{n,d}$ defined in the previous section. This extremal situation (ii) is then dealt with separately. We do this in Section 4, where we prove Lemma 7. The proof of Lemma 7 makes use of Theorem 3.

The non-extremal case is proved in Section 5. As mentioned earlier, an approximate version of Theorem 1 was proved in [4]. However, we need to proceed somewhat differently as the argument in [4] fails to guarantee the 'closeness' of H to the extremal hypergraph in case (ii). (But we do use the same general approach and a number of ideas from [4].)

We begin by considering a matching M of maximum size and suppose that $|M| < d$. We then carry out a sequence of steps, where in each step we show that we can either find a larger matching (and thus obtain a contradiction), or show that H is successively 'closer' to $H_{n,d}$. Amongst others, the following fact from [4] will be used to achieve this (see Figure 1 for the definitions of $B_{033}, B_{023}, B_{113}$).

Fact 5. *Let B be a balanced bipartite graph on 6 vertices.*

- *If $e(B) \geq 7$ then B contains a perfect matching.*
- *If $e(B) = 6$ then either B contains a perfect matching or $B \cong B_{033}$.*
- *If $e(B) = 5$ then either B contains a perfect matching or $B \cong B_{023}, B_{113}$.*

We call the vertices of degree 3 in B_{113} the *base vertices* of B_{113} and the edge between them the *base edge* of B_{113}.

To see how the above fact can be used, suppose for example that x_1, x_2 and x_3 are unmatched vertices, that E and F are edges in M and that the link graphs $L_{x_i}(EF)$

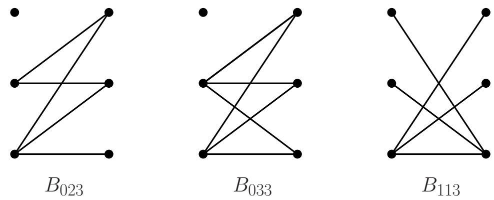

FIGURE 1. The graphs B with $e(B) \geq 5$ and no perfect matching

are identical (call this graph B). The minimum degree condition implies that, for almost all unmatched vertices x, we have $e(L_x(EF)) \geq 5$. So let us assume this holds for x_1, x_2, x_3. If B contains a perfect matching, it is easy to see that we can transform M into a (larger) matching which also covers the x_i. If $B = B_{113}$, we can use this to prove that we are 'closer' to $H_{n,d}$. In particular, note that if $H = H_{n,d}$, then in the above example we have $B = B_{113}$. If $B \cong B_{023}, B_{033}$, we need to consider link graphs involving more than 2 edges from M in order to gain further information.

To find a matching which is larger than M, we will often need several vertices whose link graphs with respect to some set of matching edges are identical (as in the above example). We can usually achieve this with a simple application of the pigeonhole principle. But for this to work, we need to be able to assume that the number of vertices not covered by M is fairly large. This may not be true if e.g. we are seeking a perfect matching. To overcome this problem, we apply the 'absorbing method' which was first introduced in [12]. The method (as used in [4]) guarantees the existence of a small matching M^* which can 'absorb' any (very) small set of leftover vertices V' into a matching covering all of $V' \cup V(M^*)$. (The existence of M^* is shown using a probabilistic argument.) So if we are seeking e.g. a perfect matching, it suffices to prove the existence of an almost perfect one outside M^*. In particular, we can always assume that the set of vertices not covered by M is reasonably large, as otherwise we are done by the following lemma.

Lemma 6 (Hàn, Person and Schacht [4]). *Given any $\gamma > 0$ there exists an integer $n_0 = n_0(\gamma)$ such that the following holds. Suppose that H is a 3-uniform hypergraph on $n \geq n_0$ vertices such that $\delta_1(H) \geq (1/2 + 2\gamma)\binom{n}{2}$. Then there is a matching M^* in H of size $|M^*| \leq \gamma^3 n/3$ such that for every set $V' \subseteq V(H) \backslash V(M^*)$ with $\gamma^6 n \geq |V'| \in 3\mathbb{Z}$ there is a matching in H covering precisely the vertices in $V(M^*) \cup V'$.*

4. EXTREMAL CASE

The aim of this section is to show that hypergraphs which satisfy the degree condition in Theorem 4 and are close to $H_{n,d}$ contain a d-matching.

Lemma 7. *There exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-uniform hypergraph on $n \geq n_0$ vertices and $d \leq n/3$ is an integer. If*

- *$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}$ and*
- *H is ε-close to $H_{n,d}$,*

then H contains a d-matching.

We will first prove the lemma in the case when H is not only close to $H_{n,d}$, but when for every vertex v most of the edges of $H_{n,d}$ incident to v also lie in H. More precisely, given $\alpha > 0$ and a 3-uniform hypergraph H on the same vertex set $V(H)$ as $H_{n,d}$, we say that a vertex $v \in V(H)$ is *α-bad* if $|N_{H_{n,d}}(v) \backslash N_H(v)| > \alpha n^2$. Otherwise we say that v is *α-good*. So if v is α-good then all but at most αn^2 of the edges incident to v in $H_{n,d}$ also lie in H. We will now show that if $d \geq n/150$ then any such H contains a d-matching.

Lemma 8. *Let $0 < \alpha < 10^{-6}$ and let $n, d \in \mathbb{N}$ be such that $n/150 \leq d \leq n/3$. Suppose that H is a 3-uniform hypergraph on the same vertex set as $H_{n,d}$ and every vertex of H is α-good. Then H contains a d-matching.*

Proof. Let V and W denote the vertex classes of $H_{n,d}$ of sizes $n-d$ and d respectively. Consider the largest matching M in H which consists entirely of edges of type VVW. Let V' denote the set of vertices in V uncovered by M. Define W' similarly. For a contradiction we assume that $|M| < d$. First note that $|M| \geq n/4$. Indeed, to see this consider any vertex $w \in W'$. Since w is α-good but $N_H(w) \cap \binom{V'}{2} = \emptyset$, it follows that $|V'| \leq 2\sqrt{\alpha} n$. Thus $|M| = |V \setminus V'|/2 \geq (n - d - 2\sqrt{\alpha} n)/2 \geq n/4$.

Consider $v_1, v_2 \in V'$ and $w \in W'$ where $v_1 \neq v_2$. Given a pair $e_1 e_2$ of distinct matching edges from M, we say that $e_1 e_2$ is *good for $v_1 v_2 w$* if there are all possible edges e in H which take the following form: e has type VVW and contains one vertex from $\{v_1, v_2, w\}$, one vertex from e_1 and one vertex from e_2. Note that if $e_1 e_2$ is good for $v_1 v_2 w$ then H has a 3-matching which consists of edges of type VVW and contains precisely the vertices in e_1, e_2 and $\{v_1, v_2, w\}$. So if such a pair $e_1 e_2$ exists, we obtain a matching in H that is larger than M, yielding a contradiction.

Since $|M| \geq n/4$ we have at least $\binom{n/4}{2} > n^2/40$ pairs of distinct matching edges $e_1, e_2 \in M$. Since v_1, v_2 and w are α-good there are at most $3\alpha n^2 < n^2/40$ such pairs $e_1 e_2$ that are not good for $v_1 v_2 w$. So one such pair must be good for $v_1 v_2 w$, a contradiction. □

We now use Lemma 8 to prove Lemma 7. Our strategy is to obtain a 'small' matching M in H that covers all 'bad' vertices in H. We will construct M in stages so as to ensure that $H - V(M)$ satisfies the hypothesis of Lemma 8. Thus we obtain a $(d - |M|)$-matching M' of $H - V(M)$, and hence a d-matching $M \cup M'$ of H.

Proof of Lemma 7. Let $0 < 1/n_0 \ll \varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \varepsilon''' \ll 1$. By Theorem 3 we may assume that $d \geq n/100$. Suppose that H is as in the statement of the lemma and let V and W denote the vertex classes of H of sizes $n-d$ and d respectively. Since H is ε-close to $H_{n,d}$, all but at most $3\sqrt{\varepsilon} n$ vertices in H are $\sqrt{\varepsilon}$-good. Let V^{bad} denote the set of $\sqrt{\varepsilon}$-bad vertices in V. Define W^{bad} similarly. So $|V^{bad}|, |W^{bad}| \leq 3\sqrt{\varepsilon} n$.

Define $c := |W^{bad}|$, $V_1 := V \cup W^{bad}$ and $W_1 := W \backslash W^{bad}$. Thus $a := |V_1| = n - d + c$ and $b := |W_1| = d - c$. Moreover,

$$\delta_1(H[V_1]) \geq \delta_1(H) - \binom{b}{2} - (a-1)b > \binom{n-1}{2} - \binom{n-d}{2} - \binom{b}{2} - (a-1)b.$$

But $\binom{n-1}{2} = \binom{a-1}{2} + (a-1)b + \binom{b}{2}$ and so

$$\delta_1(H[V_1]) > \binom{a-1}{2} - \binom{n-d}{2} = \binom{a-1}{2} - \binom{a-c}{2}.$$

Since $c \leq 3\sqrt{\varepsilon}n$ we can apply Theorem 3 to obtain a matching M_1 of size c in $H[V_1]$.

Let $H_1 := H - V(M_1)$ and $V_2 := V_1 \backslash V(M_1)$. (Note that if $W^{bad} = \emptyset$ then $H_1 = H$.) So H_1 has vertex classes V_2 and W_1 where $|V_2| = a - 3c$. Since H is ε-close to $H_{n,d}(V, W)$ and $3c \leq 9\sqrt{\varepsilon}n \ll \varepsilon' n$ we have that H_1 is ε'-close to $H_{|H_1|,b}(V_2, W_1)$. By definition of W_1 all vertices in W_1 are ε'-good in H_1. Furthermore, if a vertex $v \in V(H_1)$ is ε'-bad in H_1 then $v \in V_2$ and $v \in V^{bad} \cup W^{bad}$. Let V_2^{bad} denote the set of such vertices. So $|V_2^{bad}| \leq 3\sqrt{\varepsilon}n$. If $V_2^{bad} = \emptyset$ then we can apply Lemma 8 to obtain a b-matching M_2 in H_1. We thus obtain a matching $M_1 \cup M_2$ of size $b + c = d$ in H . So we may assume that $V_2^{bad} \neq \emptyset$.

We say that a vertex $v \in V_2^{bad}$ is *useful* if there are at least $\varepsilon' n^2$ pairs of vertices $v'w \in V_2 W_1$ such that $vv'w$ is an edge in H_1. Clearly we can greedily select a matching M_2 in H_1 such that $m_2 := |M_2| \leq |V_2^{bad}|$ where M_2 covers all useful vertices and consists entirely of edges of type $V_2 V_2 W_1$. Let $H_2 := H_1 - V(M_2)$, $V_3 := V_2 \backslash V(M_2)$ and $W_2 := W_1 \backslash V(M_2)$. Then $|V_3| = |V_2| - 2m_2 = a - 3c - 2m_2$ and $|W_2| = b - m_2$. Note that

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2} \geq (1-\varepsilon)\left(1 - \left(1 - \frac{d}{n}\right)^2\right)\frac{n^2}{2}$$

$$= (1-\varepsilon)\left(\frac{2d}{n} - \frac{d^2}{n^2}\right)\frac{n^2}{2} = (1-\varepsilon)d\left(n - \frac{d}{2}\right). \tag{1}$$

Consider any vertex $v \in V_2^{bad} \backslash V(M_2)$. Since v is not useful, it must lie in more than

$$\delta_1(H) - n|V(H) \setminus V(H_2)| - \varepsilon' n^2 - \binom{|W_2|}{2} \overset{(1)}{\geq} (1-\varepsilon)d\left(n - \frac{d}{2}\right) - \varepsilon' n^2 - \varepsilon' n^2 - \frac{d^2}{2}$$

$$\geq d(n-d) - \varepsilon dn - 2\varepsilon' n^2 \geq \frac{2dn}{3} - 3\varepsilon' n^2 \geq 2\varepsilon' n^2$$

edges of $H_2[V_3]$. Since $|V_2^{bad}| \leq 3\sqrt{\varepsilon}n$ we can greedily select a matching M_3 in $H_2[V_3]$ of size $m_3 := |M_3| \leq |V_2^{bad}|$ which covers all the vertices in H_2 which lie in V_2^{bad}.

Let $H_3 := H_2 - V(M_3)$ and $V_4 := V_3 \backslash V(M_3)$. So H_3 has vertex classes V_4 and W_2 where $|V_4| = |V_3| - 3m_3 = a - 3c - 2m_2 - 3m_3$. Recall that every vertex in $V(H_1) \backslash V_2^{bad}$ is ε'-good in H_1. Since $V_2^{bad} \subseteq V(M_2 \cup M_3)$ and $|H_1| - |H_3| = 3(|M_2| + |M_3|) \ll \varepsilon' n$, it follows that every vertex of H_3 is ε''-good. So certainly for every vertex $w \in W_2$ there are at least $|V_4||W_2|/2$ pairs $vw' \in V_4 W_2$ such that vww' is an edge in H_3. Thus we can greedily find a matching M_4 of size m_3 such that each edge in M_4 has type $V_4 W_2 W_2$.

Let $H_4 := H_3 - V(M_4)$, $V_5 := V_4 \backslash V(M_4)$ and $W_3 := W_2 \backslash V(M_4)$. So H_4 has vertex classes V_5 and W_3 of sizes $|V_5| = |V_4| - m_3 = a - 3c - 2m_2 - 4m_3 = n - d - 2c - 2m_2 - 4m_3$ and $|W_3| = |W_2| - 2m_3 = b - m_2 - 2m_3 = d - c - m_2 - 2m_3$. Moreover, every vertex of H_4 is ε'''-good. Thus we can apply Lemma 8 to H_4 to obtain a $|W_3|$-matching M_5 in

H_4. But then $M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$ is a matching of size $c+m_2+m_3+m_3+|W_3| = d$ in H, as desired. $\square$

We remark that the only point in the proof of Theorem 4 where we need the full strength of the minimum degree condition is when we apply Theorem 3 to find the matching M_1 in the proof of Lemma 7.

5. Proof of Theorem 4

5.1. **Preliminaries.** We first define constants satisfying

(2)
$$0 < 1/n_0 \ll 1/C \ll \gamma'' \ll \gamma' \ll \gamma \ll \varepsilon' \ll \varepsilon \ll \eta' \ll \eta \ll \alpha' \ll \alpha \ll \rho' \ll \rho \ll \tau \ll 1.$$

Let H be a 3-uniform hypergraph on $n \geq n_0$ vertices such that

$$\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2} \geq (1-\gamma')d(n-d/2), \tag{3}$$

where d is an integer such that $1 \leq d \leq n/3$. (Note that the second inequality in (3) follows from the same argument as (1).) We wish to find a d-matching in H. Note that Theorem 3 covers the case when $d \leq n/100$. So we may assume that $n/100 \leq d \leq n/3$.

Suppose $d \geq n/3 - \tau n$. Since $\tau \ll 1$, (3) gives us that $\delta_1(H) \geq (1/2 + 2\gamma'')\binom{n}{2}$. So by Lemma 6 there is a matching M^* in H of size $|M^*| \leq (\gamma'')^3 n/3$ such that for every set $V' \subseteq V(H)\backslash V(M^*)$ with $(\gamma'')^6 n \geq |V'| \in 3\mathbb{Z}$ there is a matching in H covering precisely the vertices in $V(M^*) \cup V'$. If $n/100 \leq d < n/3 - \tau n$ we set $M^* := \emptyset$.

In both cases we define $H' := H - V(M^*)$. (So $H' = H$ if $n/100 \leq d < n/3 - \tau n$.) Thus

$$\delta_1(H') \geq \delta_1(H) - \gamma' n^2. \tag{4}$$

Let M be the largest matching in H'. Clearly we may assume that $|M| < d$. Theorem 3 implies that

$$n/200 \leq |M| < d. \tag{5}$$

Let $V_M := V(M)$ and $V_0 := V(H')\backslash V_M$. So $|V_0| \leq n - |V_M|$. If $n/100 \leq d < n/3 - \tau n$ then $|V_0| > n - 3d > 3\tau n$. Suppose $d \geq n/3 - \tau n$. If $|V_0| \leq (\gamma'')^6 n$, then by definition of M^*, there is a matching M' in H containing all but at most two vertices from $V(M^*) \cup V_0$. But then $M \cup M'$ is a matching in H of size $\lfloor n/3 \rfloor \geq d$, as desired. So in both cases we may assume that

$$(\gamma'')^6 n \leq |V_0| \leq n - |V_M|. \tag{6}$$

5.2. **Finding structure in the link graphs.** In this section we show that 'most' of our link graphs $L_v(EF)$ with $v \in V_0$ and $EF \in \binom{M}{2}$ are copies of B_{113} (recall that B_{113} was defined after Fact 5).

Claim 9. *There does not exist $v_1v_2v_3 \in \binom{V_0}{3}$ and $EF \in \binom{M}{2}$ such that*

- $L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ *and*
- $L_{v_1}(EF)$ *contains a perfect matching.*

Proof. The proof is identical to the proof of Fact 17 in [4]. We include it here for completeness. Let $E = \{x_1, x_2, x_3\}$ and $F = \{y_1, y_2, y_3\}$ and suppose x_1y_1, x_2y_2 and x_3y_3 is a perfect matching in $L_{v_1}(EF)$. Since these edges lie in $L_{v_i}(EF)$ for each $1 \leq i \leq 3$ the edges $v_1x_1y_1$, $v_2x_2y_2$ and $v_3x_3y_3$ lie in H'. Replacing E and F in M with these edges we obtain a larger matching in H', a contradiction. □

We will now use Claim 9 to show that only a constant number of vertices $v \in V_0$ have 'many' link graphs $L_v(EF)$ containing perfect matchings.

Claim 10. *Let V_0' denote the set of all those vertices $v \in V_0$ for which there are at least εn^2 pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains a perfect matching. Then $|V_0'| \leq C$.*

Proof. Let G be the bipartite graph with vertex classes V_0' and $\binom{M}{2}$ where $\{v, EF\}$ is an edge in G precisely when $L_v(EF)$ contains a perfect matching. So G contains at least $|V_0'|\varepsilon n^2$ edges. If $|V_0'| \geq C$ then there is a pair $EF \in \binom{M}{2}$ such that $d_G(EF) \geq C\varepsilon \geq 3 \cdot 2^9$ (since $1/C \ll \varepsilon$). Since there are 2^9 labelled bipartite graphs with vertex classes E and F, there are 3 vertices $v_1, v_2, v_3 \in V_0'$ such that $L_{v_1}(EF) = L_{v_2}(EF) = L_{v_3}(EF)$ and $L_{v_1}(EF)$ contains a perfect matching. This contradicts Claim 9, as required. □

Claim 11. *Let V_0'' denote the set of all those vertices $v \in V_0$ for which there are at least εn^2 pairs $EF \in \binom{M}{2}$ such that $L_v(EF) \cong B_{023}, B_{033}$. Then $|V_0''| \leq C$.*

Proof. Suppose for a contradiction that $|V_0''| > C$. Given any $v \in V_0''$, define an auxiliary oriented graph G_v as follows: The vertex set of G_v is M and given $EF \in \binom{M}{2}$ there is an edge directed from E to F precisely when $L_v(EF) \cong B_{023}, B_{033}$ where E is the vertex class that contains the isolated vertex in $L_v(EF)$. Since $v \in V_0''$, we have that $e(G_v) \geq \varepsilon n^2$.

We call a path $E_1 \dots E_5$ of length 4 in G_v *suitable* if its (directed) edges are E_1E_2, E_3E_2, E_3E_4 and E_5E_4. Our first aim is to find at least $\varepsilon' n^5$ suitable paths in G_v. Choose a partition V_1, V_2 of $V(G_v)$ such that $e_{G_v}(V_1, V_2) \geq e(G_v)/5 \geq \varepsilon n^2/5$. (To see the existence of such a partition, consider the expected number of edges from V_1 to V_2 in a random partition of $V(G_v)$.) Let G_v' denote the undirected bipartite graph with vertex classes V_1 and V_2 whose edges are all those edges in G_v that are oriented from V_1 to V_2. Since $e(G_v') \geq \varepsilon n^2/5$, G_v' contains a subgraph G_v'' with $\delta(G_v'') \geq d(G_v')/2 \geq \varepsilon n/5$. Thus we can greedily find at least

$$\frac{1}{2} \cdot \frac{\varepsilon n}{5} \left(\frac{\varepsilon n}{5} - 1\right) \dots \left(\frac{\varepsilon n}{5} - 4\right) \geq \varepsilon' n^5$$

paths of length 4 in G_v'' whose endpoints both lie in V_1. By definition of G_v'', each of these paths corresponds to a suitable path in G_v.

Consider a suitable path $E_1 \dots E_5$ in G_v. So $L_v(E_2E_3), L_v(E_3E_4) \cong B_{023}, B_{033}$ with the isolated vertex in both graphs lying in E_3. Choose edges e_1 of $L_v(E_2E_3)$ and e_2 of $L_v(E_3E_4)$ such that e_1 and e_2 are disjoint. Since $L_v(E_1E_2) \cong B_{023}, B_{033}$ and E_1 contains the isolated vertex in this graph, there is a 2-matching $\{e_3, e_4\}$ in $L_v(E_1E_2)$ that is disjoint from e_1. Similarly since $L_v(E_4E_5) \cong B_{023}, B_{033}$ and E_5 contains the isolated vertex in this graph, there is a 2-matching $\{e_5, e_6\}$ in $L_v(E_4E_5)$ that is disjoint from e_2. Hence $L_v(E_1E_2E_3E_4E_5)$ contains a 6-matching $\{e_1, e_2, e_3, e_4, e_5, e_6\}$.

Let G be the bipartite graph with vertex classes V_0'' and the set $(M)^5$ of all ordered 5-tuples of elements of M where $\{v, E_1E_2E_3E_4E_5\}$ is an edge in G precisely when $E_1 \dots E_5$ is a suitable path in G_v. So G contains at least $|V_0''|\varepsilon' n^5$ edges.

Since $|V_0''| > C$ there exists $E_1E_2E_3E_4E_5 \in (M)^5$ such that $d_G(E_1E_2E_3E_4E_5) \geq C\varepsilon' \geq 6{\cdot}2^{36}$. Further, there are at most 2^{36} distinct graphs in the collection of all those graphs $L_v(E_1E_2E_3E_4E_5)$ for which $v \in N_G(E_1E_2E_3E_4E_5)$. Thus there are 6 vertices $v_1, \dots, v_6 \in V_0''$ such that $v_1, \dots, v_6 \in N_G(E_1E_2E_3E_4E_5)$ and $L_{v_1}(E_1E_2E_3E_4E_5) = \dots = L_{v_6}(E_1E_2E_3E_4E_5)$. Let $\{x_1y_1, \dots, x_6y_6\}$ be a 6-matching in $L_{v_1}(E_1E_2E_3E_4E_5)$. So $\{v_1x_1y_1, \dots, v_6x_6y_6\}$ is a 6-matching in H'. Replacing the edges $E_1, \dots, E_5$ in M with $\{v_1x_1y_1, \dots, v_6x_6y_6\}$ we obtain a larger matching, a contradiction. □

Claim 12. *Let V_0''' denote the set of all those vertices $v \in V_0$ which fail to satisfy*

$$e(L_v(V_0, V_M)) \leq (1 + \sqrt{\gamma'})|V_0||M|. \tag{7}$$

Then $|V_0'''| \leq C$.

Proof. Suppose for a contradiction that $|V_0'''| > C \geq 2/\gamma'$. Given an edge E in M, we say that E is *good for* $v \in V_0'''$ if at least two vertices in E have degree at least 3 in $L_v(E, V_0)$. For every $v \in V_0'''$, there are at least $\gamma'|M|$ edges in M which are good for v. (To see this, suppose there are fewer edges which are good for v. Then

$$\begin{aligned} e(L_v(V_0, V_M)) &< (1-\gamma')|M|(4 + |V_0|) + \gamma'|M| \cdot 3|V_0| \\ &\leq |M||V_0| \left((1-\gamma')(1+\gamma') + 3\gamma'\right) \leq (1 + \sqrt{\gamma'})|V_0||M|, \end{aligned}$$

a contradiction to the fact that $v \in V_0'''$.) This in turn implies that there are $v_1, v_2 \in V_0'''$ and an edge E in M which is good for both v_1 and v_2. Then the definition of 'good' implies that are disjoint edges $e_1 \in L_{v_1}(E, V_0)$ and $e_2 \in L_{v_2}(E, V_0)$ which do not contain v_1 or v_2. Now we can enlarge M by removing E and adding v_1e_1 and v_2e_2. This contradiction to the maximality of M proves the claim. □

Claim 13. *Every vertex $v \in V_0 \backslash V_0'''$ satisfies*

$$e(L_v(V_M)) \geq (5 - \gamma)\binom{|M|}{2}.$$

Proof. Suppose $v \in V_0 \backslash V_0'''$. Then as $e(L_v(V_0)) = 0$

$$\begin{aligned} e(L_v(V_M)) &\overset{(4)}{\geq} \delta_1(H) - e(L_v(V_0, V_M)) - \gamma' n^2 \\ &\overset{(3),(7)}{\geq} (1-\gamma')d(n - d/2) - \left(1 + \sqrt{\gamma'}\right)|V_0||M| - \gamma' n^2. \end{aligned}$$

Now note that the function $d(n-d/2)$ is increasing in d for $d \le n/3$. So

$$\begin{aligned} e(L_v(V_M)) &\ge (1-\gamma')|M|\left(n - \frac{|M|}{2}\right) - \left(1+\sqrt{\gamma'}\right)(n-3|M|)|M| - \gamma' n^2 \\ &\ge \left(n|M| - \frac{|M|^2}{2} - \gamma' n|M|\right) - \left(n|M| - 3|M|^2 + \sqrt{\gamma'}n|M|\right) - \gamma' n^2 \\ &\overset{(5)}{\ge} \frac{5|M|^2}{2} - 400\sqrt{\gamma'}|M|^2 \ge (5-\gamma)\binom{|M|}{2}, \end{aligned}$$

which completes the proof of the claim. □

Claim 14. *Let V_0'''' denote the set of all those vertices $v \in V_0 \backslash V_0'''$ for which there are at least ηn^2 pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains at most 4 edges. Then $|V_0''''| \le 2C$.*

Proof. Suppose for a contradiction that $|V_0''''| > 2C$. Let $v \in V_0''''$. At most $3|M|$ edges $e = vv_1v_2$ in H containing v are such that v_1 and v_2 lie in the same edge $E \in M$. Thus Claim 13 implies that

$$\sum_{EF \in \binom{M}{2}} e(L_v(EF)) \ge (5-\gamma)\binom{|M|}{2} - 3|M| \ge 5\binom{|M|}{2} - \gamma n^2. \tag{8}$$

Let c denote the number of pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains at most 4 edges. Then $c \ge \eta n^2$ and so (8) implies that there are at least $\eta' n^2$ pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains at least 6 edges. Indeed, suppose that this is not the case. Then

$$\begin{aligned} \sum_{EF \in \binom{M}{2}} e(L_v(EF)) &\le 4c + 9\eta' n^2 + 5\left[\binom{|M|}{2} - c\right] = 5\binom{|M|}{2} - c + 9\eta' n^2 \\ &< 5\binom{|M|}{2} - \gamma n^2 \end{aligned}$$

since $\gamma \ll \eta' \ll \eta$. This contradicts (8), as desired.

Recall from Fact 5 that a balanced bipartite graph B on 6 vertices that contains at least 6 edges either has a perfect matching or $B \cong B_{033}$. Thus, given any $v \in V_0''''$ there are at least $r \ge \eta' n^2/2 \ge \varepsilon n^2$ pairs $E_1F_1, \dots, E_rF_r \in \binom{M}{2}$ such that either

- $L_v(E_iF_i)$ contains a perfect matching for all $1 \le i \le r$ or,
- $L_v(E_iF_i) \cong B_{033}$ for all $1 \le i \le r$.

So since $|V_0''''| > 2C$ one of the following holds:

(α_1) There are more than C vertices $v \in V_0''''$ for which there are at least εn^2 pairs $EF \in \binom{M}{2}$ such that $L_v(EF)$ contains a perfect matching.

(α_2) There are more than C vertices $v \in V_0''''$ for which there are at least εn^2 pairs $EF \in \binom{M}{2}$ such that $L_v(EF) \cong B_{033}$.

In either case we get a contradiction: (α_1) contradicts Claim 10 and (α_2) contradicts Claim 11. □

Recall from Fact 5 that if B is a balanced bipartite graph on 6 vertices with $e(B)=5$ then either B contains a perfect matching or $B\cong B_{023},B_{113}$. If $e(B)\geq 6$ then either B contains a perfect matching or $B\cong B_{033}$. Thus Claims 10, 11, 12 and 14 together imply that all vertices $v\in V_0\setminus(V_0'\cup V_0''\cup V_0'''\cup V_0'''')$ satisfy

(β) $L_v(EF)\cong B_{113}$ for at least $\binom{|M|}{2}-2\varepsilon n^2-\eta n^2\geq(1-\alpha')\binom{|M|}{2}$ pairs $EF\in\binom{M}{2}$.

Let $V_0^*:=V_0\setminus(V_0'\cup V_0''\cup V_0'''\cup V_0'''')$. Thus

$$|V_0\setminus V_0^*|\leq 5C.$$

Moreover, each $v\in V_0^*$ satisfies

$$e(L_v(V_M))\leq 5(1-\alpha')\binom{|M|}{2}+9\alpha'\binom{|M|}{2}+3|M|\leq 5(1+\alpha')\binom{|M|}{2}. \tag{9}$$

Here the term $3|M|$ accounts for the edges which have both endpoints in the same matching edge of M.

We can now show that M has almost the required size. This will be used in Section 5.3 to prove that H is close to $H_{n,d}$.

Claim 15. $|M|>d-\alpha n$.

Proof. Assume for a contradiction that $|M|\leq d-\alpha n$. Consider any $v\in V_0^*$. Then

$$d_{H'}(v)\overset{(3),(4)}{\geq}(1-\gamma')d(n-d/2)-\gamma' n^2\geq d(n-d/2)-2\gamma' n^2. \tag{10}$$

Also $e(L_v(V_0))=0$ since M is maximal. Thus

$$\begin{aligned} d_{H'}(v) &= e(L_v(V_M))+e(L_v(V_0,V_M))\overset{(7),(9)}{\leq} 5(1+\alpha')\binom{|M|}{2}+(1+\sqrt{\gamma'})|V_0||M| \\ &\leq 5(1+\alpha')\binom{|M|}{2}+\left(|M|(n-3|M|)+\sqrt{\gamma'}n^2\right) \\ &\leq |M|(n-|M|/2)+\sqrt{\alpha'}n^2<(d-\alpha n)(n-d/2+\alpha n/2)+\sqrt{\alpha'}n^2 \\ &< d(n-d/2)-2\gamma' n^2, \end{aligned}$$

a contradiction to (10), as desired. (In the third line we again used that the function $d(n-d/2)$ is increasing in d for $d\leq n/3$.) □

In the next sequence of claims, we will show that there are vertices $v_1,\dots,v_{10}\in V_0^*$ whose link graphs $L_{v_i}(V_M)$ are very similar to each other (see Claim 19 for the precise statement).

Claim 16. *Suppose $v_1,\dots,v_{10}\in V_0^*$ are distinct vertices such that for some $EF\in\binom{M}{2}$, $L_{v_1}(EF),\dots,L_{v_{10}}(EF)\cong B_{113}$. Then $L_{v_1}(EF)=\dots=L_{v_{10}}(EF)$.*

Proof. We suppose for a contradiction that the claim does not hold. Since there are 9 labelled bipartite graphs with vertex classes E and F which are isomorphic to B_{113}, two of the $L_{v_i}(EF)$ must be the same. So we may assume that $L_{v_1}(EF)=L_{v_2}(EF)$ but $L_{v_1}(EF)\neq L_{v_3}(EF)$. Let $E=\{x_1,x_2,x_3\}$ and $F=\{y_1,y_2,y_3\}$. Suppose $E(L_{v_1}(EF))=E(L_{v_2}(EF))=\{x_1y_1,x_1y_2,x_1y_3,x_2y_1,x_3y_1\}$. (So x_1y_1 is the base edge of $L_{v_1}(EF)$ and $L_{v_2}(EF)$ as defined after Fact 5.) Since $L_{v_1}(EF)\neq L_{v_3}(EF)$

there is an edge $e \in L_{v_3}(EF) \backslash L_{v_1}(EF)$. We may assume $e = x_3y_3$. Replacing E and F with $v_1x_1y_2, v_2x_2y_1$ and $v_3x_3y_3$ in M we obtain a larger matching, a contradiction. □

Choose distinct $v_1, \dots, v_{10} \in V_0^*$ which will be fixed throughout the remainder of the proof.

Claim 17. *There is a set $\mathcal{E}$ of at least $(1-\alpha)|M|$ matching edges $E \in M$ such that for each $E \in \mathcal{E}$ there are at least $(1-\alpha)|M|$ edges $F \in M$ for which*

$$L_{v_1}(EF) = \cdots = L_{v_{10}}(EF) \cong B_{113}.$$

Proof. By (β) and Claim 16 there are at least $(1-10\alpha')\binom{|M|}{2}$ pairs $EF \in \binom{M}{2}$ such that $L_{v_1}(EF) = \cdots = L_{v_{10}}(EF) \cong B_{113}$. This in turn immediately implies the claim. □

Claim 18. *For every $E \in \mathcal{E}$ there is a set $\mathcal{F}_E$ of at least $(1-2\alpha)|M|$ edges in M such that*

(δ_1) $L_{v_1}(EF) = \cdots = L_{v_{10}}(EF) \cong B_{113}$ *for each* $F \in \mathcal{F}_E$ *and*
(δ_2) *in each of the* $L_{v_1}(EF)$ *with* $F \in \mathcal{F}_E$ *the same vertex* x *plays the role of the base vertex in* E.

Proof. Since $E \in \mathcal{E}$ there is a set $\mathcal{F}'_E$ of at least $(1-\alpha)|M|$ edges in M such that $L_{v_1}(EF) = \cdots = L_{v_{10}}(EF) \cong B_{113}$ for each $F \in \mathcal{F}'_E$. Let $\mathcal{F}_E := \mathcal{F}'_E \cap \mathcal{E}$. Then $|\mathcal{F}_E| \geq (1-2\alpha)|M|$ and for each $F \in \mathcal{F}_E$ there are at least $(1-\alpha)|M|$ edges $F' \in M$ for which $L_{v_1}(FF') = \cdots = L_{v_5}(FF') \cong B_{113}$.

We claim that $\mathcal{F}_E$ satisfies the claim. Certainly $\mathcal{F}_E$ satisfies (δ_1). Suppose for a contradiction that there are $F_1, F_2 \in \mathcal{F}_E$ such that the vertex $x_1 \in E$ that plays the role of a base vertex in $L_{v_1}(EF_1)$ is different from the vertex $x_2 \in E$ that plays the role of a base vertex in $L_{v_1}(EF_2)$. Let $F' \in M$ be such that $L_{v_1}(F_2F') = \cdots = L_{v_5}(F_2F') \cong B_{113}$, and $F' \neq E, F_1$.

Since $L_{v_1}(EF_1) \cong B_{113}$ and $x_1 \neq x_2$, there exists a 2-matching $\{e_1, e_2\}$ in $L_{v_1}(EF_1)$ that is disjoint from x_2. Similarly since $L_{v_1}(F_2F') \cong B_{113}$ there exists a 2-matching $\{e_3, e_4\}$ in $L_{v_1}(F_2F')$. Since $x_2 \in E$ is a base vertex in $L_{v_1}(EF_2)$, there is an edge e_5 from x_2 to the vertex in F_2 that is uncovered by $\{e_3, e_4\}$. So $\{e_1, e_2, e_3, e_4, e_5\}$ is a 5-matching in $L_{v_1}(F_1EF_2F')$. We have chosen F_1, F_2 and F' so that $L_{v_1}(F_1EF_2F') = L_{v_2}(F_1EF_2F') = \cdots = L_{v_5}(F_1EF_2F')$. Thus $M' := \{v_1e_1, v_2e_2, v_3e_3, v_4e_4, v_5e_5\}$ is a 5-matching in H' that contains only vertices from $E \cup F' \cup F_1 \cup F_2 \cup \{v_1, v_2, v_3, v_4, v_5\}$. Replacing E, F', F_1 and F_2 in M with the edges in M' yields a larger matching, a contradiction. □

Given $E \in \mathcal{E}$, we call the unique vertex $x \in V(E)$ satisfying (δ_2) a *bottom vertex*. If $y \in E$ is such that $y \neq x$ then we say that y is a *top vertex*. So each $E \in \mathcal{E}$ contains one bottom vertex and two top vertices whereas none of the at most $\alpha|M|$ edges in $M \setminus \mathcal{E}$ contains a top or bottom vertex.

Claim 19. *There are at least $(1-6\alpha)|M|^2/2$ pairs $EF \in \binom{M}{2}$ such that*

(ε_1) $L_{v_1}(EF) = \cdots = L_{v_{10}}(EF) \cong B_{113}$;

(ε_2) *both E and F contain a bottom vertex w and z respectively;*
(ε_3) *wz is the base edge of $L_{v_1}(EF)$.*

Proof. Consider the directed graph G whose vertex set is M and in which there is a directed edge from E to F if $E \in \mathcal{E}$ and $F \in \mathcal{F}_E$. Claims 17 and 18 together imply that G has at least $(1-3\alpha)|M|^2$ edges and thus at least $(1-6\alpha)|M|^2/2$ pairs EF of vertices in G must be joined by a double edge. But each such pair EF satisfies the claim. □

5.3. **Showing that H is $\sqrt{\rho}$-close to $H_{n,d}$.** We have now collected all the information we need for showing that H is close to $H_{n,d}(V,W)$, where W will be constructed from the set of bottom vertices in M. More precisely, let W' denote the set of all the bottom vertices. So Claims 15 and 17 together imply that

$$d - 2\alpha n \leq (1-\alpha)|M| \leq |\mathcal{E}| = |W'| \leq |M| \leq d. \tag{11}$$

Let V' denote the set of all the top vertices in H. Thus

$$2d - 4\alpha n \leq 2(1-\alpha)|M| \leq |V'| = 2|W'| \leq 2d. \tag{12}$$

Choose a partition V, W of $V(H)$ such that $|W| = d$, $W' \subseteq W$, $V' \subseteq V$. Note that since (11) implies that $|W \setminus W'| \leq 2\alpha n$, all but at most $2\alpha n$ vertices of V_0 lie in V. Our aim is to show that H is $\sqrt{\rho}$-close to $H_{n,d}(V,W)$. Note that showing this proves Theorem 4 as we can apply Lemma 7 since we chose $\rho \ll 1$ in (2).

Claim 20. *H does not contain an edge of type $V'V_0V_0$.*

Proof. Suppose that the claim is false and let $v'vv_0$ be an edge of H with $v' \in V'$ and $v, v_0 \in V_0$. Let $E \in \mathcal{E}$ be the matching edge containing v'. Take any $F \in \mathcal{F}_E$. Take any 2 vertices from $v_1, \dots, v_{10}$ which are not equal to v_0 or v, call them x and y. Since v' is a top vertex of E, it follows that $L_x(EF)$ contains a 2-matching e_1, e_2 avoiding v'. Note that this is also a 2-matching in $L_y(EF)$. Now we can enlarge M by removing E, F and adding $v'vv_0$, xe_1 and ye_2. This contradicts the maximality of M and proves the claim. □

Claim 21.

- *H contains at least $(1-\rho')|W'||V'||V_0|$ edges of type $W'V'V_0$.*
- *H contains at least $(1-\rho')|V_0|\binom{|W'|}{2}$ edges of type $W'W'V_0$.*
- *H contains at most $\rho'|V_0|\binom{|V'|}{2}$ edges of type $V'V'V_0$.*

Proof. To see the first part of the claim, consider any $v \in V_0^*$ and any pair w', v' with $w' \in W'$ and $v' \in V'$. Both w', v' could lie in the same matching edge from M, but there are at most $3|M|$ such pairs. Also, w', v' could lie in a pair E, F of matching edges from M for which either $L_v(EF) \not\cong B_{113}$ or which does not satisfy (ε_1)–(ε_3) in Claim 19. But (β) and Claim 19 together imply that there are at most $\sqrt{\alpha}n^2$ such pairs E, F. So suppose next that w', v' lie in a pair E, F satisfying $L_v(EF) \cong B_{113}$ and (ε_1)–(ε_3). Then $L_v(EF), L_{v_1}(EF), \dots, L_{v_9}(EF) \cong B_{113}$ and so

$L_v(EF) = L_{v_1}(EF) = \cdots = L_{v_9}(EF)$ by Claim 16. Conditions (ε_2) and (ε_3) now imply that $w'v' \in E(L_v(W', V'))$. So

$$e(L_v(V', W')) \geq |V'||W'| - 2\sqrt{\alpha}n^2 \geq (1 - \rho'/2)|V'||W'|.$$

Summing over all vertices $v \in V_0^*$ and using that $|V_0 \setminus V_0^*| \leq 5C$ implies the first part of the claim. The remaining parts of the claim can be proved similarly. □

Claim 22. *H contains at least $|W'|\binom{|V_0|}{2} - \rho n^3$ edges of type $W'V_0V_0$.*

Proof. Consider any $v \in V_0$. By Claim 20 there are no edges in $L_v(V(H))$ with one endpoint in V' and the other in V_0. By (11) there are at most $3\alpha|M|n \leq 3\alpha n^2$ edges in $L_v(V(H))$ with one endpoint in $V_M \backslash (V' \cup W')$ and the other in V_0. Furthermore, $L_v(V_0)$ contains no edges. Thus,

$$\begin{aligned} e(L_v(W', V_0)) &\geq \delta_1(H') - e(L_v(V_M)) - 3\alpha n^2 \\ &\overset{(3),(4),(9)}{\geq} (1-\gamma')d\left(n - \frac{d}{2}\right) - \gamma' n^2 - 5(1+\alpha')\binom{|M|}{2} - 3\alpha n^2 \\ &\overset{(5)}{\geq} (1-\gamma')|M|\left(n - \frac{|M|}{2}\right) - (5+\sqrt{\alpha})\frac{|M|^2}{2} \\ &\geq |M|(n - 3|M|) - \sqrt{\alpha}|M|n \geq |W'||V_0| - \rho' n^2. \end{aligned}$$

As earlier, here we use the fact that the function $d(n - d/2)$ is increasing in d for $d \leq n/3$. Summing over all vertices $v \in V_0^*$ and using the fact that $|V_0 \setminus V_0^*| \leq 5C$ now proves the claim. □

Claim 23.

- *H contains at least $(1-\rho)|W'|\binom{|V'|}{2}$ edges of type $W'V'V'$.*
- *H contains at least $(1-\rho)|V'|\binom{|W'|}{2}$ edges of type $W'W'V'$.*

Proof. First note that the last part of Claim 21 implies that all but at most $2\sqrt{\rho'}n$ vertices $x \in V'$ lie in at most $\sqrt{\rho'}|V'||V_0|$ edges of type $V'V'V_0$. Call such vertices x *useful*. Consider any useful x. Then $x \in E'$ for some $E' \in \mathcal{E} \subseteq M$. Further, since x is a top vertex in E', certainly there exists an edge $F' \in M$ such that $L_{v_1}(E'F') = L_{v_2}(E'F') \cong B_{113}$, where x is not a base vertex in $L_{v_1}(E'F')$. So $L_{v_1}(E'F')$ contains a 2-matching $\{e_1, e_2\}$ which avoids x.

Consider any pair $EF \in \binom{M \setminus \{E', F'\}}{2}$ satisfying (ε_1)–(ε_3). We claim that $L_x(EF) \subseteq L_{v_1}(EF)$. Indeed, if not then there exist disjoint edges e_3, e_4 and e_5 such that $e_3 \in E(L_x(EF))$ and $e_4, e_5 \in E(L_{v_1}(EF))$. Since $L_{v_1}(E'F') = L_{v_2}(E'F')$ and since EF satisfies (ε_1) we have that $v_1e_1, v_2e_2, xe_3, v_3e_4$ and v_4e_5 are edges in H'. Replacing E, F, E', F' with $v_1e_1, v_2e_2, xe_3, v_3e_4$ and v_4e_5 in M yields a larger matching in H', a contradiction. So indeed $L_x(EF) \subseteq L_{v_1}(EF)$.

There are at least $(1-6\alpha)|M|^2/2 - 2|M| \geq (1-7\alpha)|M|^2/2$ pairs $EF \in \binom{M \setminus \{E', F'\}}{2}$ satisfying (ε_1)–(ε_3). We claim that at most $\rho^2|M|^2/2$ of these pairs EF are such that $L_x(EF)$ contains fewer than 5 edges. Indeed, suppose not. Since for such EF,

$L_x(EF) \subseteq L_{v_1}(EF) \cong B_{113}$, the number of edges of H which contain x and have no endpoint outside V_M is at most

$$4 \cdot \rho^2|M|^2/2 + 5 \cdot (1 - 7\alpha - \rho^2)|M|^2/2 + 9 \cdot 7\alpha|M|^2/2 + 3|M| \le (5 + 30\alpha - \rho^2)|M|^2/2.$$

Here the third term accounts for edges between pairs not satisfying (ε_1)–(ε_3) and the final term for edges with 2 vertices in the same matching edge from M. Let us now bound the number of edges containing x which have an endpoint outside V_M. There are at most $|W'|(n - 3|M|) \le |M|(n - 3|M|)$ such edges having an endpoint in W' and at most $\sqrt{\alpha}n^2$ such edges having an endpoint outside $V' \cup W' \cup V_0$. Since H has no edge of type $V'V_0V_0$ by Claim 20, the only other such edges consist of x, one vertex in V' and one vertex in V_0. But since x is useful the number of such edges is at most $\sqrt{\rho'}|V'||V_0|$. Thus in total there are at most $|M|(n - 3|M|) + 2\sqrt{\rho'}n^2$ edges which contain x and have an endpoint outside V_M. So the degree of x in H is at most

$$\begin{aligned}(5 + 30\alpha - \rho^2)|M|^2/2 + |M|(n - 3|M|) + 2\sqrt{\rho'}n^2 &\le |M|(n - |M|/2) - \rho^3 n^2 \\ &\le d(n - d/2) - \rho^3 n^2 \overset{(5),(3)}{<} \delta_1(H),\end{aligned}$$

a contradiction. Thus there are at least $(1 - 7\alpha - \rho^2)|M|^2/2$ pairs $EF \in \binom{M \setminus \{E',F'\}}{2}$ satisfying (ε_1)–(ε_3) such that $L_x(EF) = L_{v_1}(EF) \cong B_{113}$. Let $\mathcal{P}$ denote the set of such pairs.

Now consider any pair w', v' with $w' \in W'$ and $v' \in V' \setminus \{x\}$. Both w', v' could lie in the same matching edge from M, but there are at most $3|M|$ such pairs. Also, w', v' could lie in a pair E, F of matching edges which does not belong to $\mathcal{P}$. But there at most $5\rho^2|M|^2$ such pairs w', v'. So suppose next that w', v' lies in a pair E, F belonging to $\mathcal{P}$. Since $L_x(EF) = L_{v_1}(EF) \cong B_{113}$ and EF satisfies (ε_2) and (ε_3) it follows that $w'v' \in E(L_x(EF))$. Thus $e(L_x(W', V')) \ge (1 - 6\rho^2)|W'||V'|$. Summing over all useful vertices $x \in V'$ proves the first part of the claim. The second part follows similarly (the only change is that we consider a pair $w'_1, w'_2 \in W'$ in the final paragraph). □

Claims 21–23 together with (11) and (12) now show that H contains all but at most $\sqrt{\rho}n^3$ edges of type WVV and WWV and thus H is $\sqrt{\rho}$-close to $H_{n,d}(V, W)$. Hence H contains a perfect matching by Lemma 7.

Remark. One can also obtain Theorem 4 by proving the result only in the case when $d = \lfloor n/3 \rfloor$. Indeed, suppose that H is as in the theorem. Let $a := \lfloor (n-3d)/2 \rfloor$. Obtain a new 3-uniform hypergraph H' from H by adding a new vertices to H such that each of these vertices forms an edge with all pairs of vertices in H'. It is not hard to check that $\delta_1(H') > \binom{|H'|-1}{2} - \binom{|H'|-\lfloor |H'|/3 \rfloor}{2}$ and so H' has a matching M' of size $\lfloor |H'|/3 \rfloor$. One can then show that M' contains at least d edges from H, as desired. (We thank Peter Allen for suggesting this trick.)

However, the proof of Theorem 4 is only slightly simpler in the case when $d = \lfloor n/3 \rfloor$ (we do not need Claims 20–22 in this case) and to show that the above trick works, one requires some extra calculations.

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