DETERMINISTIC SPECTRAL PROPERTIES OF ANDERSON-TYPE HAMILTONIANS

CONSTANZE LIAW

ABSTRACT. This paper concerns the deterministic spectral properties of the self-adjoint operator $A_\omega = A + V_\omega$ where $V_\omega = \sum_n \omega_n(\,\cdot\,,\varphi_n)\varphi_n$ on a separable Hilbert space $\mathcal{H}$ with a sequence $\{\varphi_n\} \subset \mathcal{H}$. Here $\omega = (\omega_1, \omega_2, \ldots)$ is a random variable corresponding to a probability measure $\mathbb{P}$ on $\mathbb{R}^\infty$ which is merely assumed to satisfy Kolmogorov's 0-1 law.

The main result - part 3 of Theorem 3.1 - states that under mild cyclicity conditions the essential parts of A_ω and A_η are almost surely with respect to the product measure $\mathbb{P} \times \mathbb{P}$ unitary equivalent modulo a rank one perturbation. Its proof is based on the techniques and results developed by A.G. Poltoratski in [8].

1. INTRODUCTION

In 1958 P.W. Anderson suggested that sufficiently large impurities in a semi-conductor could lead to spatial localization of electrons, see [1]. Since then the field has developed into a rich theory. A vast amount of progress has been made by both the mathematical and the physical community. Despite being one of the most studied problems in Mathematical Physics, the Anderson model, is far from being completely understood. Many problems with striking physical relevance remain unsolved, e.g. the so-called Anderson localization.

The deterministic dynamical properties of Anderson-type Hamiltonians, introduced below in subsection 1.1, are the focus of this article.

1.1. **Anderson-type Hamiltonian.**

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Let $\{\varphi_n\} \subset \mathcal{H}$ be a sequence of mutually linearly independent unit vectors in $\mathcal{H}$, and let $\omega = (\omega_1, \omega_2, \ldots)$ be a random variable corresponding to a probability measure $\mathbb{P}$ on $\mathbb{R}^\infty$. Assume that $\mathbb{P}$ satisfies Kolmogorov's 0-1 law (see subsection 2.2 below).

The Anderson-type Hamiltonian is formally given by

$$A_\omega = A + V_\omega \qquad \text{on } \mathcal{H}, \qquad V_\omega = \sum_n \omega_n(\,\cdot\,,\varphi_n)\varphi_n. \tag{1.1}$$

Assume that A_ω is almost surely essentially self-adjoint.

Remarks. (a) Literature provides many sufficient conditions for the formal expression (1.1) to yield an almost surely essentially self-adjoint operator. For example, if the vectors φ_n are mutually orthogonal, then it suffices to assume $\sum |\omega_n| < \infty$ almost surely.
(b) Let $\sum |\omega_n| < \infty$ almost surely. Then we have the almost sure self-adjointness, if the sequence $\{\varphi_n\}$ consists of linear combinations of vectors from an orthonormal basis $\{f_m\}$ where each f_m occurs only in finitely many φ_n's. In this case, the vectors in the sequence $\{\varphi_n\}$ are not mutually orthogonal (we still assume that they are mutually independent).
(c) It is possible to consider so-called singular form bounded perturbations, where the vectors φ_n come from a larger Hilbert space which is usually denoted by $\mathcal{H}_{-1}(A)$ (see e.g. [7]). This extension is necessary, in order to include the continuous Schrödinger operator with random potential, since the delta distributions do not belong to the underlying Hilbert space.

(d) For more singular perturbations, i.e. vectors $\varphi_n \notin \mathcal{H}_{-1}(A)$, the difficulty consists of defining the self–adjoint extension uniquely, see e.g. [6].
(e) In general, the perturbation V_ω is almost surely a non-compact operator. Therefore, many results from classical perturbation theory cannot immediately be applied here.

This definition slightly generalizes the Anderson-type Hamiltonians that were considered in [3] and [4] to the case of non-orthogonal sequences of vectors $\{\varphi_n\}$. The results in [4] provide a good picture of the deterministic structure and properties of the spectrum of the Anderson-type Hamiltonian.

Probably the most important special case of such Anderson-type Hamiltonians is the discrete Schrödinger operator with random potential on $l^2(\mathbb{Z}^d)$ given by

$$Af(x) = -\triangle f(x) = -\sum_{|n|=1} (f(x+n) - f(x)), \quad \varphi_n(x) = \delta_n(x) = \begin{cases} 1 & x = n, \\ 0 & \text{else.} \end{cases}$$

In fact, most Anderson models are special cases of the Anderson-type Hamiltonian given by (1.1).

1.2. **Outline.** We conclude this section with some notation used throughout this article.

In section 2, we review selected results from perturbation theory, and remind the reader of the definition of and a few facts about the Krein–Lifshits spectral shift function for rank one perturbations.

The main results are stated in section 3. Their proofs can be found in sections 4 and 5.

1.3. **Notation.** Briefly recall one of the standard ways to split up the spectrum of a normal, i.e. $T^*T = TT^*$, operator T. By the spectral theorem operator T is unitarily equivalent to M_z, multiplication by the independent variable z, in a direct sum of real Hilbert spaces $\mathcal{H} = \oplus \int \mathcal{H}(z)\, d\mu(z)$ with positive spectral measure $d\mu$ on $\mathbb{C}$.

By T_{ac} denote the restriction of T to its absolutely continuous part, i.e. T_{ac} is unitarily equivalent to $M_z\big|_{\oplus \int \mathcal{H}(z) d\mu_{\text{ac}}(z)}$. Similarly, define the singular, singular continuous and the pure point parts of operator T, denoted by T_{s}, T_{sc} and T_{pp}, respectively.

Recall that an operator T is said to be cyclic, if for some φ the span of the orbit $T^n\varphi$, $n \in \mathbb{N}$, is dense in the Hilbert space $\mathcal{H}$; or equivalently, if $L^2(\mu) = \oplus \int \mathcal{H}(z) d\mu(z)$, that is if there is only one fiber in this direct sum of Hilbert spaces for some real-valued Borel measure μ on the real line. We say that T_{ac} is cyclic, if $L^2(\mu_{\text{ac}}) = \oplus \int \mathcal{H}(z) d\mu_{\text{ac}}(z)$.

Let $\sigma(T)$ denote the spectrum of a (closed) operator T. We use

$$\sigma_{\text{ess}}(T) = \sigma(T) \backslash \{\text{isolated point spectrum of finite multiplicity}\}$$

to denote the essential spectrum of T.

The essential support of the absolutely continuous part of a measure τ (on $\mathbb{R}$) is given by

$$\text{ess-supp}\, \tau_{\text{ac}} = \left\{ x \in \mathbb{R} : \overline{\lim_{\varepsilon \to 0}} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} d\tau > 0 \text{ and } < \infty \right\}.$$

At this point it is worth mentioning that $\text{ess-supp}\, \tau_{\text{ac}} \subsetneq \text{supp}\, \tau_{\text{ac}}$ may happen. For example, let τ_{ac} be given by the Lebesgue measure on intervals that have all rational points of $[0,3]$ as centers and with width 2^{-n}. Outside those intervals, τ_{ac} is the zero measure. For the Lebesgue measures we have $|\,\text{ess-supp}\, \tau_{\text{ac}}| = 2 \neq 3 = |\,\text{supp}\, \tau_{\text{ac}}|$.

Further, we write $A \sim B$ for two operators A and B, if the operators are unitary equivalent, i.e. $UAU^{-1} = B$ for some unitary operator U. The notation $A \sim B (\text{mod}\, \text{Class}\, X)$ is used, if there exists a unitary operator U such that $UAU^{-1} - B$ is an element of Class X. At this Class X can be any class of operators, e.g. compact, trace class, or finite rank operators.

Acknowledgments. I would like to thank my postdoctoral advisor Alexei Poltoratski for suggesting the problems which led to this paper, for the many insightful discussions and comments along the way as well as for proof reading this final version.

2. Preliminaries

2.1. **Perturbation Theory.** Perturbation theory is concerned with the general question: Given some information about the spectrum of an operator A, what can be said about the spectrum of the operator $A + B$ for B in some operator class? Depending on which class of operators the perturbation B is taken from, we obtain different results of spectral stability, i.e. preserving of parts of the spectrum under such perturbations.

For self-adjoint operators A and B let us recall the following well-known theorems that will be used in the proof of Theorem 3.1 below.

Theorem 2.1 (Weyl–von Neumann, see e.g. [5])**.** *The essential spectrum of two self-adjoint operators A and B satisfies $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ if and only if $A \sim B (\text{mod } \textit{compact operators})$.*

Theorem 2.2 (Kato–Rosenblum, see e.g. [5])**.** *If for two self-adjoint operators we have $A \sim B (\text{mod } \textit{trace class})$, then their absolutely continuous parts are equivalent, i.e. $A_{\text{ac}} \sim B_{\text{ac}}$.*

Remarks. (a) In [2] Carey and Pincus found a complete characterization of when we have $A \sim B (\text{mod trace class})$ in terms of the operators' spectrum.
(b) Two operators satisfy $A_{\text{ac}} \sim B_{\text{ac}}$ if and only if the essential support of their spectral measures are equal up to a set of measure zero.

2.2. **Kolmogorov's 0-1 law.** An event $A \subset \mathbb{R}^\infty$ is said to be a tail event, if its occurrence or failure is independent of finitely many values of the random variable. In other words, A is a tail event, if $\omega = (\omega_1, \omega_2, \ldots) \in A$, implies that all $\eta = (\eta_1, \eta_2, \ldots)$ with $\eta_k = \omega_k$ for $k > N$ and some N, satisfy $\eta \in A$.

We say that a probability distribution $\mathbb{P}$ on $\mathbb{R}^\infty$ satisfies Kolmogorov's 0-1 law, if any event is a tail event.

Theorem 2.3 (Kolmogorov's 0-1 law)**.** *If A is a tail event, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.*

Consider the Anderson-type Hamiltonian A_ω given by (1.1). As an immediate consequence of Kolmogorov's 0-1 law we mention the following observation.

Observation 2.4 (Kolmogorov's 0-1 law applied to Anderson-type Hamiltonians)**.** *Assume that the probability distribution $\mathbb{P}$ satisfies the 0-1 law. Then those spectral properties that are invariant under finite rank perturbations are enjoyed by A_ω almost surely or almost never.*

2.3. **Cauchy transform.** The deep connection between operator theory and the Cauchy transform

$$K\tau(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\tau(t)}{t - z}, \qquad z \in \mathbb{C}_+,$$

of an operator's spectral measure τ poses - although well studied - still a wonderful source of information. This connection is frequently used to learn about the spectral properties of the operator under investigation.

It is well-known that the density of the absolutely continuous part of the measure can be recovered via

$$d\tau_{\text{ac}}(x) = \lim_{y \downarrow 0} \Im K\tau(x + iy)\, dx, \quad x \in \mathbb{R}, \tag{2.1}$$

where $\Im$ denotes the imaginary part.

In order to avoid difficulties with convergence, we also introduce an alternative definition of the Cauchy transform

$$K_1\tau(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t-z} - \frac{t}{t^2+1}\, d\tau(t), \qquad z \in \mathbb{C}_+.$$

It is worth mentioning that (for τ such that $K\tau$ is defined on $\mathbb{C}_+$) the real part of $K_1\tau$ differs from the conjugate Poisson integral by a finite additive constant. The advantage of introducing this alternative definition is that it is possible to define $K_1\tau$ for more general measures τ (because the kernel decays faster at infinity). Further notice that locally the $K\tau$ and $K_1\tau$ behave alike.

We use both $K\tau$ and $K_1\tau$ below.

2.4. **Rank One Perturbations.** The connection between operator theory and the Cauchy transform also plays a central role in the spectral theory of rank one perturbations. Due to space limitations, we merely recall the results that are applied later in this article. An accessible exposition of the material, except Theorem 2.6, can be found e.g. in [10].

Let

$$A_\alpha = A + \alpha(\cdot,\varphi)\varphi, \qquad \alpha \in \mathbb{R}, \tag{2.2}$$

be the rank one perturbation of a self-adjoint operator A with cyclic vector φ. It is well-known that φ is then also a cyclic vector of the operator A_α for all $\alpha \in \mathbb{R}$. By μ_α denote the spectral measure of A_α with respect to φ, i.e. $((A_\alpha - z\mathbf{I})^{-1}\varphi,\varphi)_{\mathcal{H}} = \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t-z}$ for $z \in \mathbb{C}\backslash\mathbb{R}$. We use the notation $\mu = \mu_0$.

With the resolvent formula, it is easy to see that the measures μ and μ_α of the rank one perturbation (2.2) are related via the Aronszajn–Krein formula

$$K\mu_\alpha = \frac{K\mu}{1+\alpha K\mu}\,. \tag{2.3}$$

The Aronszajn–Donoghue theory gives a good picture of the spectrum of the perturbed operator for rank one perturbations. One of its results says that the singular part of rank one perturbations must move when we change the perturbation parameter α.

Theorem 2.5 (Aronszajn–Donoghue)**.** *For non-equal coupling constants $\alpha \neq \beta$, the singular parts $(\mu_\alpha)_{\mathrm{s}}$ and $(\mu_\beta)_{\mathrm{s}}$ are mutually singular.*

2.5. **Krein–Lifshits Spectral Shift for Rank One Perturbations.** We present the Krein–Lifshits spectral shift function in the case of rank one perturbations. For more detailed explanations, examples and proofs we refer to [9] and the references within.

Consider the rank one perturbations A_α given by (2.2) and their spectral measures μ_α corresponding to the cyclic vector φ.

Since the spectral measure μ is non-negative, the imaginary part of its Cauchy transform $K\mu(z)$ is non-negative for $z \in \mathbb{C}_+$. Recall that the angular boundary values of the Cauchy transform exits almost everywhere with respect to the Lebesgue measure. For every $\alpha \in \mathbb{R}$ it is hence possible to find an essentially bounded by $\pi < u(t) \leq \pi$, $t \in \mathbb{R}$, function and a constant $c \in \mathbb{R}$ such that

$$1 + \pi\alpha K\mu = e^{K_1 u + c}\,. \tag{2.4}$$

Function u is called the Krein–Lifshits spectral shift of the rank one perturbation A_α.

Using the Aronszajn–Krein formula (2.3) we obtain a relation between the shift function and the measure μ_α

$$1 - \pi\alpha K\mu_\alpha = e^{-K_1 u - c}\,. \tag{2.5}$$

We can label A and A_α so that $\alpha > 0$.

For such α, it is possible to define u via the principal argument

$$u = \arg(1 + \pi\alpha K\mu) \qquad \in [0, \pi]. \tag{2.6}$$

Indeed, consider the logarithm of (2.4), take its imaginary part and recall the relation (2.1).

Now by breaking $K\mu$ into real and imaginary part $K\mu = iP\mu - Q\mu$ (where P denotes the Poisson integral and Q denotes the conjugate Poisson integral), one can see that u jumps from 0 to π at isolated points of $\operatorname{supp}\mu_{\mathrm{s}}$. Similarly, the analog

$$u = -\arg(1 - \pi\alpha K\mu_\alpha)$$

of (2.6) for μ_α implies that u jumps from π to 0 at isolated points of $\operatorname{supp}(\mu_\alpha)_{\mathrm{s}}$.

Further the set where $u \in (0, \pi)$ is equal (up to a set of Lebesgue measure zero) to $\text{ess-supp}(\mu)_{\mathrm{ac}} = \text{ess-supp}(\mu_\alpha)_{\mathrm{ac}}$.

Remark. These observations about the relationship between the spectrum of A and A_α, and the behavior of u give an alternative proof for the fact that the discrete spectrum of two purely singular operators in the same family of rank one perturbations must be interlacing (the Krein–Lifshits spectral shift must jump from 0 to π and then back).

Vice versa, it is well-known that for fixed $\alpha > 0$ any measurable function u which is essentially bounded by $0 \le u \le \pi$ is the Krein–Lifshits spectral shift of the rank one perturbation $M_\mu + \alpha(\cdot, \mathbf{1})\mathbf{1}$ of the multiplication operator M_μ by the independent variable on $L^2(\mu)$. In fact, given such a function u and $\alpha > 0$ we obtain a unique pair of measures μ and $\nu = \mu_\alpha$, if we impose a normalization condition on the measures. For $\alpha = 1$, we say that the measures μ and ν correspond to u.

2.6. **Equivalence up to rank one perturbation.** The next theorem can be seen as an inverse spectral problem in the following sense: It gives conditions on the spectrum of two purely singular operators which guarantee that the operators are unitary equivalent modulo a rank one perturbation.

Recall that two operators A and B are said to be completely non-equivalent, if there are no non-trivial closed invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$ such that $A|_{\mathcal{H}_1} \sim A|_{\mathcal{H}_2}$. Clearly, two operators are completely non-equivalent, if and only if their spectral measures are mutually singular.

Theorem 2.6 (Poltoratski [8])**.** *Let $K \subset \mathbb{R}$ be closed. By $I_1 = (x_1; y_1), I_2 = (x_2; y_2), \dots$ denote disjoint open intervals such that $K = \mathbb{R} \setminus \bigcup I_n$. Let A and B be two cyclic self-adjoint completely non-equivalent operators with purely singular spectrum. Suppose $\sigma(A) = \sigma(B) = K$ and assume that for the pure point spectra of A and B we have $\sigma_{\mathrm{pp}}(A) \cap \{x_1, y_1, x_2, y_2, \dots\} = \sigma_{\mathrm{pp}}(B) \cap \{x_1, y_1, x_2, y_2, \dots\} = \varnothing$. Then we have $A \sim B (\mathrm{mod}\, \textit{rank one})$.*

The idea of the proof of our main result (part 3 of Theorem 3.1, below) is rooted in the proof of Theorem 2.6. In fact, we will need the version of this proof that is explained in the last paragraph of this subsection. First, let us take a brief moment to recall its main steps.

The goal of the proof is to show that for any pair of mutually singular spectral measures μ and ν on $\mathbb{R}$ there exist measures μ_0 and ν_0 which are equivalent (respectively) and which possess a Krein–Lifshits spectral shift u_0. This is done by constructing u_0 as a certain limit of

Krein–Lifshits spectral shift functions. The main part of the construction consists of recursive applications of the following lemma.

Lemma 2.7 (Poltoratski [8])**.** *Let μ and ν be singular measures on $\mathbb{R}$, $I \subset \mathbb{R}$ be an open interval, $E \subset I$ be a closed set, $|E| = 0$. Suppose that $I \subset \operatorname{supp}\mu$ and $I \subset \operatorname{supp}\nu$. Then for any $\varepsilon > 0$ there exist closed subsets F and G of I , and measures μ' and ν' satisfying the following conditions*

1) *$|F| = |G| = 0$, $\mu(G) = \nu(F) = 0$ and $E \subset F$,*
2) *$\mu' \sim \mu|_F$ and $\nu' \sim \nu|_G$,*
3) *the measures μ' and ν' possess a Krein–Lifshits spectral shift u,*
4) *$u = \pi$ on $\mathbb{R}\backslash I$ and $\int_I (\pi - u(t))dt < \varepsilon$.*

Alternatively, condition 4 can be replaced by

5) *$u = \pi$ on $\mathbb{R}\backslash I$ and $\int_I u(t)dt < \varepsilon$, or by*
6) *$u = 0$ on $\mathbb{R}\backslash I$ and $\int_I (\pi - u(t))dt < \varepsilon$, or by*
7) *$u = 0$ on $\mathbb{R}\backslash I$ and $\int_I u(t)dt < \varepsilon$.*

Conditions 4 and 7 are used in the proof of Theorem 2.6 to control the impact that introducing a non-constant spectral shift function (and hence better approximating spectral measures) on an interval where the spectral shift function in the previous step was constant has on the parts of the previous measures (that already have the desired behavior). In the first step of the proof of Theorem 2.6, Lemma 2.7 is applied without assuming condition 4 through 7 on the resulting Krein–Lifshits spectral shift.

Consider the interval I_1. Fix $0 \leq c \leq \pi$. We can easily alter the proof of Theorem 2.6 such that $\left|\int_{I_1}(c - u(t))dt\right|$ is arbitrarily small by imposing condition 4 on some part of I_1 and condition 7 on the other part in the first step of the proof of Theorem 2.6. Changing the size of those parts gives us the desired result.

3. The Main Theorem

Let A_ω be given by (1.1) and assume the hypotheses of section 1.1.

Theorem 3.1. *In this setting, we have almost surely with respect to the product measure $\mathbb{P} \times \mathbb{P}$:*

1) *$(A_\omega)_{\mathrm{ac}} \sim (A_\eta)_{\mathrm{ac}}$,*
2) *$A_\omega \sim A_\eta (\mathrm{mod}$ compact operator$)$ and*
3) *If $(A_\omega)_{\mathrm{ac}}$ is cyclic $\omega \in \mathbb{P}$ almost surely, then $(A_\omega)_{\mathrm{ess}} \sim (A_\eta)_{\mathrm{ess}} (\mathrm{mod}$ rank one$)$.*

Parts 1 and 2 of Theorem 3.1 are proven in section 4. The proof of part 3 of Theorem 3.1 can be found in section 5.

Remarks. (a) Part 1 is essentially proven in [4] where the sequence $\{\varphi_n\}$ was assumed to form an orthonormal basis. Note that spectral properties are invariant under a change of basis. We decided to repeat the statement here for completeness.
(b) Part 3 cannot be concluded trivially, e.g., by using Theorem 2.6 and part 1 and a separation of the singular from the absolutely continuous part, because embedded singular spectrum can possibly occur. In particular, the singular spectrum of one operator may be interlaced with the absolutely continuous spectrum of the other operator. For a more precise explanation we refer to the first remark in section 5.
(c) In the conclusion of part 3 it is necessary to restrict to the essential parts of the operators. The statement $A_\omega \sim A_\eta (\mathrm{mod}\,\mathrm{rank\ one})$ is not true. Indeed, because on the finite isolated

point spectra of A_ω and A_η might not interlace - one of the necessary conditions for two operators to be unitarily equivalent up to rank one perturbation. In fact, between two points in the discrete spectrum of A_ω there may be any number of points from the discrete spectrum of A_η (almost surely).

(d) If we make the mild assumption that the vectors φ_n are a cyclic family for A_ω almost surely, then the almost sure cyclicity of $(A_\omega)_{\mathrm{ac}}$ implies the almost sure cyclicity of $(A_\omega)_{\mathrm{ess}}$. Indeed, for orthonormal sequences $\{\varphi_n\}$, it was proven in [4] that the restricted operator $(A_\omega)_{\mathrm{s}}$ is cyclic almost surely. Also recall that that cyclicity of an operator is independent of the choice of basis, and observe that the Anderson-type Hamiltonians (1.1), where the sequence $\{\varphi_n\}$ is not required to be orthogonal, can be obtained from those with orthonormal sequences (considered in [4]) by a change of basis. Finally note that the operators $(A_\omega)_{\mathrm{ac}}$ and $(A_\omega)_{\mathrm{s}}$ are completely non-equivalent, because the essential supports of their spectral measures are mutually singular. So the cyclicity of $(A_\omega)_{\mathrm{ac}}$ and $(A_\omega)_{\mathrm{s}}$ implies the cyclicity of $(A_\omega)_{\mathrm{ess}}$.

(e) In [11] B. Simon gives another set of sufficient conditions for cyclicity in the case of discrete Anderson-type Hamiltonians.

4. Proof of parts 1 and 2 of Theorem 3.1

Proof of Theorem 3.1, parts 1 and 2. The words 'almost surely' in this proof refer to almost surely with respect to the product measure $\mathbb{P} \times \mathbb{P}$, unless otherwise stated.

Let $A_{\widetilde{\omega}}$ denote finite rank perturbations of A, i.e. $\widetilde{\omega} \neq 0$ only in finitely many components. In particular, $A_{\widetilde{\omega}}$ are compact and trace class perturbations of A.

To show part 1, fix point $x \in \mathbb{R}$. Without loss of generality, let μ_ω denote the fiber of the spectral measure of A_ω for which ess-supp μ_ω is maximal (with respect to the inclusion of sets). Let $\mu_{\widetilde{\omega}}$ be the analog measure for $A_{\widetilde{\omega}}$. By the Kato–Rosenblum theorem, Theorem 2.2, for almost every $x \in \mathbb{R}$ we have $x \in \text{ess-supp}(\mu_{(0,0,0,\ldots)})_{\mathrm{ac}}$ if and only if $x \in \text{ess-supp}(\mu_{\widetilde{\omega}})_{\mathrm{ac}}$. In virtue of observation 2.4, we have that $x \in \text{ess-supp}(\mu_\omega)_{\mathrm{ac}}$ almost surely or almost never. The set (up to a set of measure zero) of points x for which the latter is almost surely true is hence deterministic and part 1 is proven.

Part 2 follows in analogy from the Weyl–von Neumann theorem 2.1. □

5. Proof of part 3 of Theorem 3.1

Proof of Theorem 3.1, part 3. Most of this proof is to be understood almost surely with respect to the product measure $\mathbb{P} \times \mathbb{P}$, although this might not be stated everywhere explicitly.

Fix generic ω and η. By μ and ν denote the spectral measures of the operators $(A_\omega)_{\mathrm{ess}}$ and $(A_\eta)_{\mathrm{ess}}$ with respect to some cyclic vectors, respectively. (It is worth mentioning here that the spectral measures of an operator corresponding to any two cyclic vectors are equivalent.)

Consider the measure τ on $\mathbb{R}$ given by $d\tau(t) = (t^2+1)^{-1}dt$.

The goal is to produce a sequence of spectral shift functions $\{u_n\}$ which converges in measure with respect to τ to a spectral shift function u_0. The spectral measures μ_0 and ν_0 corresponding to this limit function u_0 are then proven to be equivalent to the spectral measures μ and ν, respectively.

First recall, by part 1 of Theorem 3.1, the symmetric difference ess-supp $\mu_{\mathrm{ac}} \triangle$ ess-supp ν_{ac} is a set of measure zero (almost surely with respect to the product measure). Let us denote the intersection of these sets by $A = \text{ess-supp}\,\mu_{\mathrm{ac}} \cap \text{ess-supp}\,\nu_{\mathrm{ac}}$. Further, by part 2 of Theorem 3.1 and the Weyl–von Neumann theorem, Theorem 2.1, their essential spectra satisfy $\sigma_{\mathrm{ess}}(A_\omega) = \operatorname{supp}\mu = \operatorname{supp}\nu$. We use E to refer to this set.

Notice that μ and ν are regular measures. Take open sets O_n such that

$$A \subset O_{n+1} \subset O_n, \tag{5.1}$$

$$\tau(O_n \backslash A) < n^{-1}2^{-n}, \tag{5.2}$$

$$|O_n \backslash A| < 1/n, \text{ and} \tag{5.3}$$

$$\mu_{\mathrm{s}}(\bigcap O_n) = \nu_{\mathrm{s}}(\bigcap O_n) = 0. \tag{5.4}$$

Being open, each O_n consists of countably many open intervals.

Let $A_n = \bigcup_{l=1}^{L_n} A_{n,l}$ be the union of finitely many disjoint finite open intervals such that

$$A_n \subset O_n \text{ and} \tag{5.5}$$

$$\tau(O_n \backslash A_n) < Cn^{-1}2^{-n}. \tag{5.6}$$

Being open, each A_n consists of countably many open intervals. Without loss of generality, we can assume that μ and ν have no singular parts on the boundaries ∂A_n for all n. Indeed, the set $\bigcup_n \partial A_n$ is countable.

We define the set $\mathcal{A} = \liminf_n A_n$, i.e. a point $x \in \mathcal{A}$ if and only if there exists $N \in \mathbb{N}$ such that $x \in A_n$ for all $n \geq N$. With this we have

$$A \subset \mathcal{A} \subset \bigcap O_n. \tag{5.7}$$

By (5.1) and (5.3) we have

$$|\bigcap O_n \backslash A| = 0. \tag{5.8}$$

Finally, it is worth mentioning that the sets $A_{n+1}\backslash A_n$, $A\backslash A_n$ and $A_n\backslash E$ may be non-empty.

Next, we define the sequence $\{u_n\}$ of Krein–Lifshits spectral shift functions recursively.

In order to define u_1, first observe that, by definition of $E = \sigma_{\mathrm{ess}}(A_\omega)$, operators $(A_\omega)_{\mathrm{ess}}$ and $(A_\eta)_{\mathrm{ess}}$ have dense singular spectrum on the set $E\backslash \operatorname{clos}(A)$. Hence it is possible to choose two purely singular measures μ' and ν' on $\mathbb{R}$ such that μ' and ν' are mutually singular ($\mu' \perp \nu'$), $\mu'|_{\mathbb{R}\backslash A} = \nu'|_{\mathbb{R}\backslash A} = 0$ and so that $\mu_1 = \mu_{\mathrm{s}} + \mu'$ and $\nu_1 = \nu_{\mathrm{s}} + \nu'$ have dense alternating spectrum on E. By Theorem 2.6, the measures μ_1 and ν_1 possess a spectral shift function u_1, i.e. there exists a function u_1 which is essentially bounded by $0 \leq u_1 \leq \pi$ and such that

$$u_1 = \arg(1 + K\mu_1) = -\arg(1 - K\nu_1).$$

In what follows the idea is to destroy the artificially created (by adding μ' and ν') singular spectrum and produce appropriate absolutely continuous spectrum. This is accomplished by carefully defining a sequence of spectral shift functions, the corresponding spectral measures of which are 'closer and closer' to μ and ν.

The shift function u_2 is defined by

$$u_2(x) = \begin{cases} u_1(x), & \text{if } x \in E\backslash A_1, \\ \operatorname{dist}(\mathbb{R}\backslash A_1, x), & \text{if } x \in A_1. \end{cases}$$

The spectral measures corresponding to u_1 are denoted by μ_1 and ν_1.

Recall the definition of the sets $A_n = \bigcup_{l=1}^{L_n} A_{n,l}$ such that (5.5) and (5.6) are satisfied. Let $\triangle$ denote the symmetric difference of sets. From u_{n-1} we recursively define

$$u_n(x) = \begin{cases} u_{n-1}(x), & \text{if } x \in E\backslash[A_n \triangle A_{n-1}], \\ \left|u_{n-1}(x) - \max\{2^{-n}\operatorname{dist}(\mathbb{R}\backslash A_{n-1}, x), \pi/2\}\right|, & \text{if } x \in A_n\backslash A_{n-1}, \\ 0, & \text{if } x \in \mathbb{R}\backslash(A_n \cup E), \\ u_n^*(x), & \text{if } x \in E \cap (A_{n-1}\backslash A_n), \end{cases}$$

where u_n^* - while corresponding to the singular measures μ_1 and ν_1 (on $E\cap(A_{n-1}\setminus \operatorname{clos}(A_n))$) - is defined to be close to u_{n-1} in the sense of averages over certain intervals. The details of the construction of u_n^* can be found below.

Let μ_n and ν_n be the measures corresponding to u_n.

By the requirements (5.2) and (5.6), we have

$$\tau(\{u_n \neq u_{n-1}\}) \leq Cn^{-1}2^{-n}. \tag{5.9}$$

Hence the sequence $\{u_n\}$ converges in measure with respect to τ to some measurable Krein–Lifshits shift function denoted by u_0. Indeed, we clearly have $0 \leq u_0 \leq \pi$. Let μ_0 and ν_0 be the spectral measures corresponding to the shift function u_0 (for $\alpha = 1$).

Next, let us prove the $*$-weak convergence of the sequences $\{\mu_n\}$ and $\{\nu_n\}$ to μ_0 and ν_0, respectively. Observe that for any sequence of measures κ_n (for which $K\kappa_n$ is defined) the measures $d\kappa_n$ converge $*$-weakly if and only if $K\kappa_n$ converges pointwise everywhere on $\mathbb{R}$ (due to the density of kernels for $z \in \mathbb{C}_+$). And similarly for $K_1\kappa_n$. In virtue of (2.4), it suffices to show the $*$-weak convergence of $u_n dx$, or that for all compactly supported continuous functions f we have $\int f(u_{n-1} - u_n)dx \to 0$ fast enough. The latter follows easily from estimate (5.9) and the boundedness of the functions u_n.

In the remainder of this proof, we show the equivalencies $\mu \sim \mu_0$ and $\nu \sim \nu_0$. Recall properties (5.7) and (5.8).

First consider the absolutely continuous parts.

Take $x \in \mathbb{R}\setminus(\bigcap O_n)$. By (5.1), it follows that $x \notin \text{ess-supp}\,\mu_{\mathrm{ac}}$ Lebesgue almost surely. On the other hand, because of (5.7), we easily see that $u_0(x) \in \{0, \pi\}$ Lebesgue almost surely. So Lebesgue almost surely $x \notin \text{ess-supp}(\mu_0)_{\mathrm{ac}}$. By (5.7) and (5.8) we have $|\mathcal{A}\setminus A| = 0$ and to obtain the equivalence of the absolutely continuous parts, it suffices to prove that μ_{ac} is absolutely continuous with respect to $(\mu_0)_{\mathrm{ac}}$ on $\mathcal{A}$. Indeed, we clearly have $\mu_{\mathrm{ac}} \sim dx$ on $\mathcal{A}$.

Let us prove that μ_{ac} is absolutely continuous with respect to $(\mu_0)_{\mathrm{ac}}$ on $\mathcal{A}$.

By (5.9) we have $\tau(\{u_n \neq u_0\}) \leq C/n$. Assume that $\tau(\{u_0 = 0\} \cap A) = \varepsilon > 0$ or $\tau(\{u_0 = \pi\} \cap A) = \varepsilon > 0$. Then for n such that $\varepsilon > C/n$ we have

$$\tau(\{u_n = 0\}) \geq \varepsilon - C/n > 0 \qquad \text{or} \qquad \tau(\{u_n = \pi\}) \geq \varepsilon - C/n > 0. \tag{5.10}$$

The claim is that we can define u_n^* so that (5.10) is contradicted. If the claim is true - by defining u_n^* in this way - assumption $\tau(\{u_0 = 0\} \cap A) = \varepsilon > 0$ or $\tau(\{u_0 = \pi\} \cap A) = \varepsilon > 0$ cannot be maintained. By (2.1), measure μ_{ac} is absolutely continuous with respect to $(\mu_0)_{\mathrm{ac}}$ on $\mathcal{A}$.

We now prove the claim, i.e. we define u_n^* so that $\tau(\{u_n = 0 \cap A\}) = 0$ and $\tau(\{u_n = 0 \cap A\}) = 0$ for all n. By definition, measure μ_2 is equivalent to the Lebesgue measure on A_2 and μ_n is equivalent to the Lebesgue measure on $A_n\setminus A_{n-1}$. So the claim follows, if the choice of u_n^* on $X = E \cap (A_{n-1}\setminus A_n)$ does not destroy the absolutely continuous spectrum on $A_n \cap A_{n-1}$ for all n.

To see this take $x \in A_n$. It suffices to prove that

$$\left| \int_{t\in X} \frac{u_n(t) - u_{n-1}(t)}{x - t} dt \right| < \infty. \tag{5.11}$$

Indeed, the latter, together with (2.4), implies $\frac{d\mu_n}{d\mu_{n-1}}(x) < \infty$ (Lebesgue a.e.). Further we have $\frac{d\mu_{n-1}}{dx}(x) < \infty$.

Recall that for fixed n each $|A_{n-1,l}| < \infty$. Let us subdivide each of the finitely many open intervals $A_{n-1,l} \cap X$, $l = 1, \ldots, L_{n-1}$, into intervals $I^k_{n-1,l}$ such that $|I^k_{n-1,l}| = 2^{-k-1}|A_{n-1,l}|$ and arrange them in within $A_{n-1,l}$ such that their length is decreasing as one gets closer

and closer to the boundary $\partial(A_{n-1,l} \cap X)$. Without loss of generality, we may assume that $\frac{d\mu}{dt}(t) < \infty$ or does not exist for all $t \in \partial I^k_{n-1,l}$, for all n, l and k, and similarly for $\frac{d\nu}{dt}(t)$. Let $u^k_{n-1,l}$ denote the average of u_{n-1} on $I^k_{n-1,l}$, i.e. $u^k_{n-1,l} = |I^k_{n-1,l}|^{-1} \int_{I^k_{n-1,l}} u_{n-1}(t)dt$. Recall that $x \in A_n$. Then we have

$$\left|\int_{t\in X} \frac{u_n(t)-u_{n-1}(t)}{x-t}dt\right| \le \sum_{l=1}^{L_n}\sum_{k\in\mathbb{N}} \left|\int_{I^k_{n-1,l}} \frac{u_n(t)-u_{n-1}(t)}{x-t}dt\right|$$

$$\le \sum_l\sum_k \left|\int_{I^k_{n-1,l}} \frac{u_n(t)-u^k_{n-1,l}}{x-t}dt\right| + C \le \sum_l\sum_k \frac{1}{\operatorname{dist}(I^k_{n-1,l},x)} \left|\int_{I^k_{n-1,l}} u_n(t)-u^k_{n-1,l}dt\right| + C.$$

As it was explained in subsection 2.6, on each $I^k_{n-1,l}$ we can obtain a spectral shift function u^*_n that corresponds to measures μ^*_n and ν^*_n which are equivalent to μ_1 and ν_1, respectively, on those intervals and such that for all n, l and k

$$\left|\int_{I^k_{n-1,l}} u_n(t)-u^k_{n-1,l}dt\right| < \varepsilon. \tag{5.12}$$

The choice of $\varepsilon = \varepsilon^k_{n-1,l} = 2^{-l-k}\operatorname{dist}(I^k_{n-1,l},x)$ now implies (5.11).

We have proven that $\mu_{\mathrm{ac}} \sim (\mu_0)_{\mathrm{ac}}$. Similarly we obtain $\nu_{\mathrm{ac}} \sim (\nu_0)_{\mathrm{ac}}$.

Now consider the singular parts. Recall condition (5.4), that $\frac{d\mu}{dx}(x) = \infty$ for a dense set of $x \in E\backslash \operatorname{clos}(A)$, and similarly for ν, and that have no singular parts on the boundaries ∂A_n for all $n \in \mathbb{N}$.

We clearly have $\mu_{\mathrm{s}}(\mathbb{R}\backslash E) = (\mu_0)_{\mathrm{s}}(\mathbb{R}\backslash E) = 0$.

Assume that μ_{s} is not equivalent to $(\mu_0)_{\mathrm{s}}$ on $E\backslash\bigcap O_n$. Then there exists a compact set $K \subset E\backslash\bigcap O_n$ such that the two measures are not equivalent on K. But by (5.1) in the definition of the sets O_n there exists N such that $K \subset E\backslash\bigcap_{n\ge N} O_n$. Choose such N minimal (i.e. such that $K \cap O_{N-1} \neq \varnothing$). On $E\backslash\bigcap_{n\ge N} O_n$ we have $(\mu_0)_{\mathrm{s}} \sim (\mu_N)_{\mathrm{s}} \sim \mu_{\mathrm{s}}$. Indeed, the second equivalence follows immediately by the construction. To see the first equivalence, it suffices to mention that by the definition of u_n the expression $\left|\int_{A_n} \frac{u_n(t)-u_{n-1}(t)}{t-x}dt\right|$ is bounded by $C2^{-n}$ for fixed n and all $x \in E\backslash A_n$, and that $K \subset E\backslash A_n$ for all $n \ge N$. In particular, the introduction of new intervals in A_n does not influence the behavior of μ_n outside of $\bigcap O_n$.

Recall that $\mu_{\mathrm{s}}(\bigcap O_n) = 0$. Let us prove that the same holds for the singular part of μ_0. As $A \subset \mathcal{A}$ we have $\bigcap O_n = (\bigcap O_n\backslash A) \cup \mathcal{A}$.

On the set $\bigcap O_n\backslash A$ we have defined $\mu_1 = (\mu_1)_{\mathrm{s}} \equiv 0$. (Indeed, $\mu_1 = \mu_{\mathrm{s}} + \mu'$ where $\frac{d\mu_{\mathrm{s}}}{dx}(x) = 0$ for all $x \in \bigcap O_n$ by (5.4) and $\frac{d\mu'}{dx}(x) = 0$ for all $x \in \mathbb{R}\backslash A$ by definition.) From there on we have maintained $(\mu_1)_{\mathrm{s}}|_{\bigcap O_n\backslash A_n} \sim (\mu_n)_{\mathrm{s}}|_{\bigcap O_n\backslash A_n}$ and we have $u_n \in (0,\pi)$ on A_n. Note that, in particular, $(\mu_n)_{\mathrm{s}}(\partial A_n) = 0$ because of the way the A_n's were chosen and by the above argument.

By the definition of $\mathcal{A}$, for every point $x \in \mathcal{A}$ there exists N such that $x \in A_n$ for all $n \ge N$. Therefore, we have $u_n(x) \in (0,\pi)$ for all $n \ge N$ and it follows that $(\mu_0)_{\mathrm{s}}(\mathcal{A}) = 0$.

Last but not least, if suffices to mention that by (5.4) and by the definition of the u_n's the singular parts of μ and μ_0 are equivalent on $E\backslash\mathcal{A}$.

In analogy we obtain $\nu_{\mathrm{s}} \sim (\nu_0)_{\mathrm{s}}$. □

References

[1] P. W. Anderson, *Absence of Diffusion in Certain Random Lattices*, Phys. Rev., **109** (1958), 1492–1505.

[2] R. W. Carey and J. D. Pincus, *Unitary equivalence modulo the trace class for self-adjoint operators*, Amer. J. Math. **98** (1976), no. 2, 481–514. MR 0420323 (54 #8337)
[3] Vojkan Jakšić and Yoram Last, *Spectral structure of Anderson type Hamiltonians*, Invent. Math. **141** (2000), no. 3, 561–577. MR 1779620 (2001g:47069)
[4] ———, *Simplicity of singular spectrum in Anderson-type Hamiltonians*, Duke Math. J. **133** (2006), no. 1, 185–204. MR 2219273 (2007g:47062)
[5] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition. MR 1335452 (96a:47025)
[6] P. Kurasov, *Singular and supersingular perturbations: Hilbert space methods*, Spectral Theory of Schrödinger Operators (2004).
[7] C. Liaw and S. Treil, *Rank one perturbations and singular integral operators*, Journal of Functional Analysis **257** (2009), no. 6, 1947–1975.
[8] Alexei G. Poltoratski, *Equivalence up to a rank one perturbation*, Pacific J. Math. **194** (2000), no. 1, 175–188. MR 1756633 (2001j:47013)
[9] A. Poltoratskiĭ, *Kreĭn's spectral shift and perturbations of spectra of rank one*, Algebra i Analiz **10** (1998), no. 5, 143–183. MR 1659988 (2000d:47028)
[10] B. Simon, *Spectral analysis of rank one perturbations and applications*, Mathematical Quantum Theory I: Field Theory and Many-Body Theory (1994).
[11] Barry Simon, *Cyclic vectors in the Anderson model*, Rev. Math. Phys. **6** (1994), no. 5A, 1183–1185, Special issue dedicated to Elliott H. Lieb. MR 1301372 (95i:82058)

Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843, USA
E-mail address: conni@math.tamu.edu
URL: http://www.math.tamu.edu/~conni