THE SEMICLASSICAL LIMIT OF EIGENFUNCTIONS OF THE SCHRÖDINGER EQUATION AND THE BOHR-SOMMERFELD QUANTIZATION CONDITION, REVISITED

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To Vasilij Mikhailovich Babich on his 80-th birthday

Abstract. Consider the semiclassical limit, as the Planck constant $\hbar \to 0$, of bound states of a quantum particle in a one-dimensional potential well. We justify the semiclassical asymptotics of eigenfunctions and recover the Bohr-Sommerfeld quantization condition.

1. Introduction

1.1. We study the limit as $\hbar \to 0$ of eigenfunctions $\psi(x) = \psi(x; \lambda, \hbar)$ of the Schrödinger equation

$$-\hbar^2 \psi''(x) + v(x)\psi(x) = \lambda\psi(x), \quad v(x) = \overline{v(x)}, \quad \psi \in L^2(\mathbb{R}), \tag{1.1}$$

for λ close to some non-critical energy λ_0 (that is $v'(x) \neq 0$ for x such that $v(x) = \lambda_0$). We assume that the equation $v(x) = \lambda$ has exactly two solutions (the turning points) $x_\pm = x_\pm(\lambda)$ and that $v(x) < \lambda$ for $x \in (x_-, x_+)$. Thus, (x_-, x_+) is a potential well and the energy λ is separated from its bottom. We suppose that eigenfunctions $\psi(x)$ are real and normalized, that is

$$\int_{-\infty}^{\infty} \psi^2(x)dx = 1.$$

It is a common wisdom that the limit of $\psi(x) = \psi(x; \lambda, \hbar)$ as $\hbar \to 0$ is described by the Green-Liouville approximation away from the turning points $x_\pm$. In neighborhoods of the turning points the asymptotics of $\psi(x)$ is more complicated and is given in terms of an Airy function. Surprisingly, we have not found a precise formulation and a proof of this result in the literature. Our goal is to fill in this gap. We follow here the scheme suggested by R. E. Langer and thoroughly exposed by F. W. Olver in his book [7].

The detailed asymptotics of $\psi(x)$ described in Theorems 2.5 and 4.4 allows one to recover the classical Bohr-Sommerfeld quantization condition on λ (see Theorem 4.1). Actually, we prove somewhat more establishing a one-to-one correspondence between eigenvalues of the Schrödinger operator $H_\hbar = -\hbar^2 d^2/dx^2 + v(x)$ from a neighborhood of a non-critical energy and points $(n + 1/2)\hbar$ where n is an integer. This implies the semiclassical Weyl formula for the distribution of eigenvalues of the operator $H_\hbar$ as $\hbar \to 0$ with a strong estimate of the remainder. It turns out (see Corollary 4.2) that this remainder never exceeds 1. We also obtain

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in Theorem 5.2 the quantization condition for discontinuous functions $v(x)$. This formula generalizes that of Bohr and Sommerfeld and is probably new.

We note that the Bohr-Sommerfeld quantization condition is also well known in much more difficult multidimensional problems. In this context we mention book [1] by V. M. Babich and V. S. Buldyrev (where the ray approximation is used), book [3] by M. V. Fedoryuk and V. P. Maslov (where the Maslov canonical operator is used) as well as papers [5] by B. Helffer et D. Robert and [4] by B. Helffer, A. Martinez and D. Robert (where the methods of microlocal analysis are used).

However, in the one-dimensional problem it is more natural to rely on methods of ordinary differential equations. Such an approach was developed by M. V. Fedoryuk (see his book [2]) for analytic potentials. In this case one can avoid a study of turning points so that the Airy function does not appear.

1.2. The asymptotics of eigenfunctions yields (see Proposition 4.5) asymptotics of observables

$$\int_{-\infty}^{\infty} w(x)\psi^2(x;\lambda,\hbar)dx \tag{1.2}$$

for sufficiently arbitrary functions $w(x)$. For example, we can take for $w(x)$ characteristic functions of Borel subsets of $\mathbb{R}$ or choose $w(x) = v(x)$. This gives the asymptotics of the kinetic energy

$$K(\lambda,\hbar) := \hbar^2 \int_{-\infty}^{\infty} \psi'(x;\lambda,\hbar)^2 dx = K_{cl}(\lambda) + O(\hbar^{1/3}) \tag{1.3}$$

as $\hbar \to 0$ uniformly for λ from a neighborhood of the point λ_0. The leading term $K_{cl}(\lambda)$ (the index "*cl*" stands of course for the corresponding classical object) is given by the expression

$$K_{cl}(\lambda) = \int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(x))^{1/2} dx \Big(\int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(x))^{-1/2} dx \Big)^{-1}. \tag{1.4}$$

Note that the integrals here are taken over the classically allowed region and that $K_{cl}(\lambda)$ coincides (see subsection 4.3) with the averaged value of the kinetic energy of a particle of energy λ in classical mechanics.

We emphasize that our derivation of the quantization condition and of asymptotic formulas for observables (1.2) requires Airy functions although they do not enter into the final answer. However, we do not know how to avoid Airy functions without additional assumptions on $v(x)$.

2. Semiclassical solutions of the Schrödinger equation

2.1. It is convenient to rewrite equation (1.1) as

$$- u_\hbar''(x) + \hbar^{-2} q(x) u_\hbar(x) = 0, \tag{2.1}$$

where

$$q(x) = q(x;\lambda) = v(x) - \lambda.$$

We need some regularity of the function $v(x)$ and a weak condition on its behavior at infinity.

Assumption 2.1. The function $v \in C^2(\mathbb{R})$ and, for some $\rho_0 > 1$, the function

$$\big(|q(x)|^{-3} q'(x)^2 + q(x)^{-2} |q''(x)|\big) \Big| \int_0^x |q(y)|^{1/2} dy \Big|^{\rho_0}$$

is bounded for sufficiently large $|x|$.

The last condition is satisfied in all reasonable cases. For example, if $v(x) \to v_0 > \lambda$, it is sufficient to require that

$$v'(x)^2 + |v''(x)| = O(|x|^{-\rho_0}), \quad \rho_0 > 1, \quad |x| \to \infty.$$

It is also satisfied if $v(x)$ behaves at infinity as $|x|^\alpha$ or $e^{\alpha|x|}$ where $\alpha > 0$; in these cases $\rho_0 = 2$.

We consider the case of one potential well. To be more precise, we make the following

Assumption 2.2. The equation $v(x) = \lambda$ has two solutions $x_+ = x_+(\lambda)$ and $x_- = x_-(\lambda)$. We suppose that $x_- < x_+$, $v(x) < \lambda$ for $x \in (x_-, x_+)$, $v(x) > \lambda$ for $x \notin [x_-, x_+]$ and

$$\liminf_{|x|\to\infty} v(x) > \lambda.$$

Moreover, the function v belongs to the class C^3 in some neighborhoods of the points $x_\pm$ and $\pm v'(x_\pm) > 0$.

Note that if Assumption 2.2 is satisfied for some λ_0, then it is also satisfied for all λ from some neighborhood of λ_0.

Our goal in this section is to describe asymptotics as $\hbar \to 0$ of solutions $u_+(x) = u_+(x; \lambda, \hbar)$ and $u_-(x) = u_-(x; \lambda, \hbar)$ of equation (2.1) exponentially decaying as $x \to +\infty$ and $x \to -\infty$, respectively. These asymptotics will be given in terms of an Airy function and are uniform with respect to $x \in [x_1, \infty)$ or $x \in (-\infty, x_1]$ where x_1 is an arbitrary point from the interval (x_-, x_+).

2.2. Let us recall the definition of Airy functions and their necessary properties (see, e.g., [7]), for details). Consider the equation

$$-w''(t) + tw(t) = 0 \tag{2.2}$$

and denote by $\mathrm{Ai}\,(t)$ its solution with asymptotics

$$\mathrm{Ai}\,(t) = 2^{-1}\pi^{-1/2}t^{-1/4}\exp(-2t^{3/2}/3)(1 + O(t^{-3/2})), \quad t \to +\infty. \tag{2.3}$$

Then

$$\mathrm{Ai}\,(t) = \pi^{-1/2}|t|^{-1/4}\sin(2|t|^{3/2}/3 + \pi/4) + O(|t|^{-7/4}), \quad t \to -\infty. \tag{2.4}$$

Note that $\mathrm{Ai}\,(t) > 0$ for all $t \geq 0$.

The solution $\mathrm{Bi}\,(t)$ of equation (2.2) is defined by its asymptotics as $t \to -\infty$ which differs from (2.4) only by the phase shift:

$$\mathrm{Bi}\,(t) = -\pi^{-1/2}|t|^{-1/4}\sin(2|t|^{3/2}/3 - \pi/4) + O(|t|^{-7/4}), \quad t \to -\infty. \tag{2.5}$$

For $t \geq 0$, this function is positive and satisfies the estimate

$$\mathrm{Bi}\,(t) \leq C(1+t)^{-1/4}\exp(2t^{3/2}/3). \tag{2.6}$$

Here and below we denote by C and c different positive constants whose precise values are of no importance.

We also use that all asymptotics (2.3), (2.4) and (2.5) can be differentiated in t. In particular, the Wronskian

$$\{\mathrm{Ai}\,(t), \mathrm{Bi}\,_-(t)\} := \mathrm{Ai}\,'(t)\,\mathrm{Bi}\,(t) - \mathrm{Ai}\,(t)\,\mathrm{Bi}\,'(t) = -\pi^{-1}.$$

It follows that

$$\operatorname{Bi}(s)\operatorname{Ai}^{-1}(s)-\operatorname{Bi}(t)\operatorname{Ai}^{-1}(t)=\pi^{-1}\int_t^s \operatorname{Ai}^{-2}(\tau)d\tau,\quad s\geq t\geq 0. \tag{2.7}$$

2.3. To formulate results, we need the following auxiliary functions $\xi_\pm(x)=\xi_\pm(x;\lambda)$:

$$\begin{aligned}\xi_+(x)&=\Big(\frac{3}{2}\int_{x_+}^x q(y)^{1/2}dy\Big)^{2/3},\quad x\geq x_+,\\ \xi_+(x)&=-\Big(\frac{3}{2}\int_x^{x_+}|q(y)|^{1/2}dy\Big)^{2/3},\quad x_-<x\leq x_+,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\xi_-(x)&=\Big(\frac{3}{2}\int_x^{x_-} q(y)^{1/2}dy\Big)^{2/3},\quad x\leq x_-,\\ \xi_-(x)&=-\Big(\frac{3}{2}\int_{x_-}^{x}|q(y)|^{1/2}dy\Big)^{2/3},\quad x_-\leq x<x_+.\end{aligned}$$

Here is a list of properties of these functions. The following result is practically the same as Lemma 3.1 from Chapter 11 of [7].

Lemma 2.3. *Let* $x_1\in(x_-,x_+)$. *Then* $\xi_+\in C^3(x_1,\infty)$, $\xi_-\in C^3(-\infty,x_1)$ *and* $\xi_\pm(x)\to+\infty$ *as* $x\to\pm\infty$. *The derivatives*

$$\pm\,\xi'_\pm(x)>0,\quad \xi'_\pm(x_\pm)=\pm|v'(x_\pm)|^{1/3} \tag{2.9}$$

and the functions $\xi_\pm(x)$ *satisfy the equation*

$$\xi'_\pm(x)^2\xi_\pm(x)=q(x). \tag{2.10}$$

It follows from this lemma that the function

$$p_\pm(x)=(|\xi'_\pm(x)|^{-1/2})''|\xi'_\pm(x)|^{-3/2} \tag{2.11}$$

is continuous. Moreover, using identity (2.10), we see that

$$-16p_\pm(x)=5\xi_\pm(x)^{-2}+\xi_\pm(x)\big(4q(x)^{-2}q''(x)-5q(x)^{-3}q'(x)^2\big),\quad x\neq x_\pm, \tag{2.12}$$

and hence according to Assumption 2.1

$$|p_\pm(x)|\leq C|\xi_\pm(x)|^{-1/2-\rho},\qquad \rho=3\min\{\rho_0-1,1\}/2>0. \tag{2.13}$$

2.4. Let us construct solutions $u_\pm(x)=u_\pm(x;\lambda,\hbar)$ of equation (2.1) with semiclassical asymptotics as $\hbar\to0$ or (and) $x\to\pm\infty$. We define these solutions by their asymptotics as $x\to\pm\infty$. Below all asymptotic relations are supposed to be differentiable with respect to x. In this subsection, we only formulate results.

Proposition 2.4. *Under Assumption* 2.1 *for every fixed* $\hbar>0$, *equation* (2.1) *has a* (*unique*) *solution* $u_\pm(x)$ *such that*

$$\begin{aligned}u_\pm(x)=&2^{-1}\pi^{1/2}\hbar^{1/6}q(x)^{-1/4}\exp\Big(\mp\hbar^{-1}\int_{x_\pm}^x q(y)^{1/2}dy\Big)\\ &\times\Big(1+O\big(\big|\int_{x_\pm}^x q(y)^{1/2}dy\big|^{-\rho_1}\big)\Big)\end{aligned}$$

where $\rho_1=\min\{\rho_0-1,1\}>0$ *as* $x\to\pm\infty$.

Uniform asymptotic formulas for $u_\pm(x)$ are given in the following assertion.

Theorem 2.5. *Let Assumptions* 2.1 *and* 2.2 *hold. If* $\pm x \geq \pm x_\pm$, *then the solutions* $u_\pm(x) = u_\pm(x;\lambda,\hbar)$ *admit the representations*

$$u_\pm(x) = |\xi'_\pm(x)|^{-1/2}\,\mathrm{Ai}\,(\hbar^{-2/3}\xi_\pm(x))\big(1+\varepsilon_\pm(x;\hbar)\big) \tag{2.14}$$

where the remainder satisfies the estimate

$$|\varepsilon_\pm(x;\lambda,\hbar)| \leq C\hbar(1+|\xi_\pm(x)|)^{-\rho}, \quad \rho = 3\min\{\rho_0-1,1\}/2 > 0. \tag{2.15}$$

Let $x_1 \in (x_-,x_+)$. *On the interval* $[x_1,x_+]$ (*on the interval* $[x_-,x_1]$) *the function* u_+ (*the function* u_-) *admits the representation*

$$u_\pm(x) = |\xi'_\pm(x)|^{-1/2}\,\mathrm{Ai}\,(\hbar^{-2/3}\xi_\pm(x)) + O(\hbar^{7/6}(\hbar^{2/3}+|x-x_\pm|)^{-1/4}). \tag{2.16}$$

Away from the points $x_\pm$, we can replace the Airy function $\mathrm{Ai}\,(t)$ by its asymptotics (2.3) or (2.4). Indeed, in view of (2.9), we see that

$$|\xi_\pm(x)| \geq c|x-x_\pm|, \quad c>0, \tag{2.17}$$

and hence $\hbar^{-2/3}\xi_+(x) \to \pm\infty$ if $\hbar^{-2/3}(x-x_+) \to \pm\infty$ and $\hbar^{-2/3}\xi_-(x) \to \pm\infty$ if $\hbar^{-2/3}(x-x_-) \to \mp\infty$. This leads to the following result.

Corollary 2.6. *Suppose that* $\delta_\hbar \hbar^{-2/3} \geq c > 0$ (*in particular,* $\delta_\hbar$ *may be fixed*). *Then the functions* $u_\pm(x)$ *have asymptotics*

$$u_\pm(x) = 2^{-1}\pi^{1/2}\hbar^{1/6}q(x)^{-1/4}\exp\Big(\mp\hbar^{-1}\int_{x_\pm}^{x} q(y)^{1/2}dy\Big)\big(1+O(\hbar|\xi_\pm(x)|^{-3/2})\big) \tag{2.18}$$

as $\hbar \to 0$ *uniformly in* $x \geq x_+ + \delta_\hbar$ *for* $u_+(x)$ *and in* $x \leq x_- - \delta_\hbar$ *for* $u_-(x)$. *Let* $x_1 \in (x_-,x_+)$. *Then the functions* $u_\pm(x)$ *have asymptotics*

$$\begin{aligned} u_\pm(x) = \pi^{1/2}\hbar^{1/6}|q(x)|^{-1/4}\sin\Big(\pm\hbar^{-1}\int_x^{x_\pm}|q(y)|^{1/2}dy + \pi/4\Big) \\ + O(\hbar^{7/6}|x-x_\pm|^{-7/4}) \end{aligned} \tag{2.19}$$

as $\hbar \to 0$ *uniformly in* $x \in [x_1, x_+ - \delta_\hbar]$ *for* $u_+(x)$ *and uniformly in* $x \in [x_- + \delta_\hbar, x_1]$ *for* $u_-(x)$.

On the other hand, using estimates (2.17) and $|\,\mathrm{Ai}\,(t)| \leq C(1+|t|)^{-1/4}$, we obtain uniform in $\hbar$ estimates of the functions $u_\pm(x)$ in neighborhoods of the turning points.

Corollary 2.7. *For sufficiently small* $|x-x_\pm|$, *the estimate*

$$|u_\pm(x)| \leq C(1+\hbar^{-2/3}|x-x_\pm|)^{-1/4} \tag{2.20}$$

holds with a constant C *which does not depend on* $\hbar$.

We note that all asymptotic relations (2.14), (2.16), (2.18) and (2.19) can be differentiated with respect to x. In particular, we have asymptotics

$$\begin{aligned} u'_\pm(x) = \mp\pi^{1/2}\hbar^{-5/6}|q(x)|^{1/4}\cos\Big(\pm\hbar^{-1}\int_x^{x_\pm}|q(y)|^{1/2}dy + \pi/4\Big) \\ + O(\hbar^{1/6}|x-x_\pm|^{-7/4}) \end{aligned} \tag{2.21}$$

as $\hbar\to 0$ uniformly in $x\in[x_1,x_+-\delta_\hbar]$ for $u_+(x)$ and uniformly in $x\in[x_-+\delta_\hbar,x_1]$ for $u_-(x)$. All these relations can also be differentiated with respect to λ. For example, we have

$$\partial u_\pm(x;\lambda,\hbar)/\partial\lambda=\pm 2^{-1}\pi^{1/2}\hbar^{-5/6}|q(x;\lambda)|^{-1/4}\int_x^{x_\pm}|q(y;\lambda)|^{-1/2}dy$$
$$\times\cos\Big(\pm\hbar^{-1}\int_x^{x_\pm}|q(y;\lambda)|^{1/2}dy+\pi/4\Big)+O(\hbar^{1/6}|x-x_\pm|^{-7/4})$$

as $\hbar\to 0$ uniformly in $x\in[x_1,x_+-\delta_\hbar]$ for $u_+(x)$ and uniformly in $x\in[x_-+\delta_\hbar,x_1]$ for $u_-(x)$.

2.5. Let us now calculate the norm of the function $u_\pm(x)$ in the space $L^2(x_1,\pm\infty)$ where $x_1\in(x_-,x_+)$. Actually, we will obtain a more general result.

Proposition 2.8. *Let a function $w(x)$ be differentiable on the interval (x_-,x_+) except a finite number of points. Suppose that $w(x)$and $w'(x)$ are locally bounded functions and that, for some N,*

$$|w(x)|\leq Cq(x)\Big|\int_0^x|q(y)|^{1/2}dy\Big|^N \tag{2.22}$$

if $|x|$ is large. Then under Assumptions 2.1 *and* 2.2 *we have the asymptotic relation*

$$\int_{x_1}^{\pm\infty}w(x)u_\pm^2(x)dx=2^{-1}\pi\hbar^{1/3}\int_{x_1}^{x_\pm}w(x)(\lambda-v(x))^{-1/2}dx+O(\hbar^{2/3}). \tag{2.23}$$

Proof. We will prove (2.23) for the sign " + ", omit this index and add $\hbar$. Using asymptotics (2.14), (2.15) and the estimate $\xi'(x)\geq c>0$, we see that

$$\int_{x_+}^{x_++1}w(x)u_\hbar^2(x)dx\leq C\int_{x_+}^{\infty}\xi'(x)\,\mathrm{Ai}^2(\hbar^{-2/3}\xi(x))dx=C_1\hbar^{2/3}.$$

Similarly, using identity (2.10) and condition (2.22) we find that

$$\int_{x_++1}^{\infty}w(x)u_\hbar^2(x)dx\leq C\int_{x_++1}^{\infty}\xi'(x)\xi^{3N/2+1}(x)\,\mathrm{Ai}^2(\hbar^{-2/3}\xi(x))dx=O(\hbar^\infty).$$

Suppose that $\delta_\hbar\to 0$ as $\hbar\to 0$ but $\delta_\hbar\hbar^{-2/3}\geq c>0$. The integral of $u_\hbar^2(x)$ over $(x_+-\delta_\hbar,x_+)$ is estimated by $C\delta_\hbar$ because according to (2.20) the functions $u_\hbar(x)$ are uniformly bounded in a neighborhood of the point x_+. On the interval $(x_1,x_+-\delta_\hbar)$, we have a relation

$$\int_{x_1}^{x_+-\delta_\hbar}w(x)u_\hbar^2(x)dx=\pi\hbar^{1/3}\int_{x_1}^{x_+-\delta_\hbar}w(x)|q(x)|^{-1/2}$$
$$\times\sin^2\Big(\hbar^{-1}\int_x^{x_+}|q(y)|^{1/2}dy+\pi/4\Big)dx+O(\hbar^{4/3}\delta_\hbar^{-1}). \tag{2.24}$$

Indeed, in view of asymptotics (2.19) we have to show that the integrals

$$\hbar^{7/3}\int_{x_1}^{x_+-\delta_\hbar}|x-x_+|^{-7/2}dx\quad\text{and}\quad\hbar^{4/3}\int_{x_1}^{x_+-\delta_\hbar}|q(x)|^{-1/4}|x-x_+|^{-7/4}dx$$

are $O(\hbar^{4/3}\delta_\hbar^{-1})$. The first of them equals $C\hbar^{7/3}\delta_\hbar^{-5/2}$ which is $O(\hbar^{4/3}\delta_\hbar^{-1})$ because $\hbar=O(\delta_\hbar^{3/2})$. To estimate the second integral, we have to additionally take into account that

$$|q(x)|\geq c(x_+-x),\quad c>0. \tag{2.25}$$

Next, we replace $sin^2(\cdot)$ in the right-hand side of (2.24) by $1/2$. Let us estimate the error. Integrating by parts separately on every interval where $w(x)$ is differentiable, we see that

$$\int_{x_1}^{x_+-\delta_\hbar} w(x)|q(x)|^{-1/2}\exp\Big(2i\hbar^{-1}\int_x^{x_+}|q(y)|^{1/2}dy\Big)dx$$
$$= -2^{-1}i\hbar w(x_+-\delta_\hbar)q(x_+-\delta_\hbar)^{-1}\exp\Big(2i\hbar^{-1}\int_{x_+-\delta_\hbar}^{x_+}|q(y)|^{1/2}dy\Big)$$
$$+2^{-1}i\hbar\int_{x_1}^{x_+-\delta_\hbar}\big(w'(x)q(x)^{-1}-v'(x)q(x)^{-2}w(x)\big)$$
$$\times\exp\Big(2i\hbar^{-1}\int_x^{x_+}|q(y)|^{1/2}dy\Big)dx+O(\hbar).$$

The right-hand side here is bounded by

$$C\hbar\Big(1+|q(x_+-\delta_\hbar)|^{-1}+\int_{x_1}^{x_+-\delta_\hbar}q(x)^{-2}dx\Big)$$

which in view of estimate (2.25) does not exceed $C\hbar\delta_\hbar^{-1}$. Thus, it follows from (2.24) that

$$\int_{x_1}^{x_+-\delta_\hbar} w(x)u_\hbar^2(x)dx = 2^{-1}\pi\hbar^{1/3}\int_{x_1}^{x_+-\delta_\hbar}w(x)(\lambda-v(x))^{-1/2}dx+O(\hbar^{4/3}\delta_\hbar^{-1}).$$

Finally, making an error of order $O(\hbar^{1/3}\delta_\hbar^{1/2})$, we can extend the integral in the right-hand side to the whole interval (x_1,x_+). Setting $\delta_\hbar=\hbar^{2/3}$ and putting the results obtained together, we arrive at asymptotic relation (2.23). □

Of course (2.22) is a very mild restriction. It is satisfied for $v(x)=w(x)$. It is also true for all functions $v(x)$ if $w(x)$ if bounded by some power of $|x|$ at infinity and is even less restrictive if $v(x)\to\infty$ as $|x|\to\infty$. In particular, setting $w(x)=1$, we obtain

Corollary 2.9. *The asymptotic relation holds:*

$$\int_{x_1}^{\pm\infty}u_\pm^2(x)dx = 2^{-1}\pi\hbar^{1/3}\int_{x_1}^{x_\pm}(\lambda-v(x))^{-1/2}dx+O(\hbar^{2/3}). \tag{2.26}$$

3. Proof of Theorem 2.5

3.1. We will prove Theorem 2.5 for the sign " + " and omit this index. On the contrary, we add the index $\hbar$ to emphasize the dependence on it of various objects. Let $x_1\in(x_-,x_+)$, $x\in(x_1,\infty)$ and let the function $\xi(x)$ be defined by formulas (2.8). According to Lemma 2.3 $\xi(x)\in(\xi_1,\infty)$ where $\xi_1=\xi(x_1)$ and x can be considered as a function of ξ if $\xi\in(\xi_1,\infty)$.

Let us make the change of variables $x\mapsto\xi$ in equation (2.1) and set

$$u_\hbar(x)=\xi'(x)^{-1/2}f_\hbar(\hbar^{-2/3}\xi(x)). \tag{3.1}$$

Then using identity (2.10), we obtain that

$$-f_\hbar''(\hbar^{-2/3}\xi)+\hbar^{-2/3}\xi f_\hbar(\hbar^{-2/3}\xi)=\hbar^{4/3}r(\xi)f_\hbar(\hbar^{-2/3}\xi), \tag{3.2}$$

where

$$r(\xi)=p(x(\xi)) \tag{3.3}$$

and $p(x)$ is defined by formula (2.11). In view of (2.13) we have the estimate

$$|r(\xi)| \leq C(1+|\xi|)^{-1/2-\rho}, \quad \rho = 3\min\{\rho_0 - 1, 1\}/2 > 0. \tag{3.4}$$

Setting in (3.2) $t = \hbar^{-2/3}\xi$, we get the following intermediary result.

Lemma 3.1. *Let $t = \hbar^{-2/3}\xi(x)$, and let the functions $u_\hbar(x)$ and $f_\hbar(t)$ be related by formula* (3.1)*. Then equation* (2.1) *for $x \geq x_1$ is equivalent to the equation*

$$-f_\hbar''(t) + tf_\hbar(t) = R_\hbar(t)f_\hbar(t) \quad \textit{for} \quad t \geq \xi_1\hbar^{-2/3} \tag{3.5}$$

where

$$R_\hbar(t) = \hbar^{4/3}r(\hbar^{2/3}t). \tag{3.6}$$

3.2. Let us reduce differential equation (3.5) to a Volterra integral equation. Set

$$K_\hbar(t,s) = -\pi\big(\operatorname{Ai}(t)\operatorname{Bi}(s) - \operatorname{Ai}(s)\operatorname{Bi}(t)\big)R_\hbar(s), \quad s \geq t, \tag{3.7}$$

and consider the equation

$$f_\hbar(t) = \operatorname{Ai}(t) + \int_t^\infty K_\hbar(t,s)f_\hbar(s)ds. \tag{3.8}$$

Differentiating it twice, we see that its solution satisfies also differential equation (3.5). We will study equations (3.5) or (3.8) separately for $t \geq 0$ and $t \leq 0$.

Lemma 3.2. *For $t \geq 0$, equation* (3.5) *has a solution $f_\hbar(t)$ such that*

$$f_\hbar(t) = \operatorname{Ai}(t)\big(1 + \eta_\hbar(t)\big) \tag{3.9}$$

where

$$|\eta_\hbar(t)| \leq C\hbar(1+\hbar^{2/3}t)^{-\rho}, \quad \rho = 3\min\{\rho_0 - 1, 1\}/2 > 0. \tag{3.10}$$

Proof. Making the multiplicative change of variables

$$f_\hbar(t) = \operatorname{Ai}(t)g_\hbar(t) \tag{3.11}$$

and using (2.7), we rewrite equation (3.8) as

$$g_\hbar(t) = 1 - \int_t^\infty L_\hbar(t,s)g_\hbar(s)ds, \tag{3.12}$$

where

$$L_\hbar(t,s) = \operatorname{Ai}(t)^{-1}K_\hbar(t,s)\operatorname{Ai}(s) = \int_t^s \operatorname{Ai}^{-2}(\tau)d\tau \operatorname{Ai}^2(s)R_\hbar(s), \quad s \geq t.$$

It follows from (2.3) that

$$\int_t^s \operatorname{Ai}^{-2}(\tau)d\tau \leq C\exp(4s^{3/2}/3)$$

so that according to (3.4) and (3.6)

$$|L_\hbar(t,s)| \leq C\hbar^{4/3}s^{-1/2}(1+\hbar^{2/3}s)^{-1/2-\rho}, \quad 0 \leq t \leq s. \tag{3.13}$$

This estimate allows us to solve equation (3.12) by iterations. In particular, the solution of (3.12) satisfies the estimate

$$|g_\hbar(t) - 1| \leq C\int_t^\infty |L_\hbar(t,s)|ds.$$

Now estimate (3.10) on the remainder $\eta_\hbar(t) = g_\hbar(t) - 1$ follows again from (3.13). □

Putting together formulas (3.1) and (3.9), we obtain representation (2.14) with $\varepsilon_\hbar(x) = \eta_\hbar(\hbar^{-2/3}\xi(x))$. Estimate (3.10) implies estimate (2.15). This leads to the assertion of Theorem 2.5 for $x \geq x_+$. In particular, for a fixed $\hbar$, we get Proposition 2.4.

Next, we consider the case $t \leq 0$.

Lemma 3.3. *For $t \in [\hbar^{-2/3}\xi_1, 0]$, the solution $f_\hbar(t)$ of equation* (3.5) *satisfies the estimate*

$$|f_\hbar(t) - \mathrm{Ai}\,(t)| \leq C\hbar(1 + |t|)^{-1/4}. \tag{3.14}$$

Proof. Let us rewrite equation (3.8) as

$$f_\hbar(t) = f_\hbar^{(0)}(t) + \int_t^0 K_\hbar(t,s) f_\hbar(s)ds, \tag{3.15}$$

where the new "free" term

$$f_\hbar^{(0)}(t) = \mathrm{Ai}\,(t) + f_\hbar^{(1)}(t), \quad f_\hbar^{(1)}(t) = \int_0^\infty K_\hbar(t,s) f_\hbar(s)ds. \tag{3.16}$$

It follows from (2.3) and (2.6) that

$$\mathrm{Ai}^2(t) + \mathrm{Ai}\,(t)\,\mathrm{Bi}\,(t) \leq C(1+t)^{-1/2}, \quad t \geq 0,$$

and from (2.4) and (2.5) that

$$|\,\mathrm{Ai}\,(t)| + |\,\mathrm{Bi}\,(t)| \leq C(1 + |t|)^{-1/4}, \quad t \leq 0. \tag{3.17}$$

Therefore using (3.4), (3.6) and (3.7), we find that

$$\begin{aligned} |f_\hbar^{(1)}(t)| &\leq C\Big(|\,\mathrm{Ai}\,(t)| \int_0^\infty \mathrm{Ai}\,(s)\,\mathrm{Bi}\,(s)|R_\hbar(s)|ds + |\,\mathrm{Bi}\,(t)| \int_0^\infty \mathrm{Ai}^2(s)|R_\hbar(s)|ds\Big) \\ &\leq C_1(1+|t|)^{-1/4}\hbar^{4/3} \int_0^\infty s^{-1/2}(1+\hbar^{2/3}s)^{-1/2-\rho}ds \\ &\leq C_2(1+|t|)^{-1/4}\hbar. \end{aligned} \tag{3.18}$$

Let us now consider equation (3.15). By virtue of estimates (3.4) and (3.17) its kernel satisfies the bound

$$|K_\hbar(t,s)| \leq C\hbar^{4/3}(1+|t|)^{-1/4}(1+|s|)^{-1/4} r(\hbar^{2/3}s), \quad t \leq s \leq 0, \tag{3.19}$$

where the function $r(\hbar^{2/3}s)$ can be estimated by a constant. Thus, solving (3.15) again by iterations, we obtain the estimate

$$\begin{aligned} |f_\hbar(t) - f_\hbar^{(0)}(t)| &\leq C_1 \int_t^0 |K_\hbar(t,s)|(1+|s|)^{-1/4}ds \leq C_2\hbar^{4/3} \int_t^0 (1+|s|)^{-3/4}ds \\ &\leq C_3\hbar^{4/3}(1+|t|)^{1/4}. \end{aligned} \tag{3.20}$$

If $t \in [\hbar^{-2/3}\xi_1, 0]$, then combining definition (3.16) with estimates (3.18) and (3.20), we get estimate (3.14). □

In view of formula (3.1), this lemma yields the result of Theorem 2.5 for $x \in [x_1, x_+]$.

Differentiating integral equation (3.8) with respect to t, we obtain asymptotic relations for $f_\hbar'(t)$ and then for $u_\hbar'(x)$. This concludes the proof of Theorem 2.5.

4. Semiclassical asymptotics of eigenfunctions

4.1. Let $\lambda = \lambda(\hbar)$ be an eigenvalue of the Schrödinger operator $H_\hbar = -\hbar^2 d^2/dx^2 + v(x)$ from a neighborhood of a non-critical point λ_0. Then the solutions $u_\pm(x)$ are proportional:

$$u_-(x;\lambda,\hbar) = a(\lambda,\hbar) u_+(x;\lambda,\hbar). \tag{4.1}$$

Choose an arbitrary interior point x of the interval $(x_-(\lambda), x_+(\lambda))$. To calculate the Wronskian of $u_+(x)$ and $u_-(x)$, we use asymptotic relations (2.19) and (2.21). Setting

$$\varphi_\pm(x;\lambda) = \pm \int_x^{x_\pm(\lambda)} (\lambda - v(y))^{1/2} dy, \quad x \in (x_-(\lambda), x_+(\lambda)), \tag{4.2}$$

we find that

$$\begin{aligned} w(\lambda,\hbar) &= u_+(x;\lambda,\hbar) u_-'(x;\lambda,\hbar) - u_-(x;\lambda,\hbar) u_+'(x;\lambda,\hbar) \\ &= \pi \hbar^{-2/3} \Big(\sin(\hbar^{-1}\varphi_+(x;\lambda) + \pi/4) \cos(\hbar^{-1}\varphi_-(x;\lambda) + \pi/4) \\ &\quad + \cos(\hbar^{-1}\varphi_+(x;\lambda) + \pi/4) \sin(\hbar^{-1}\varphi_-(x;\lambda + \pi/4)\Big) + O(\hbar^{1/3}) \\ &= \pi \hbar^{-2/3} \sin(\hbar^{-1}\Phi(\lambda) + \pi/2) + O(\hbar^{1/3}) \end{aligned} \tag{4.3}$$

where $\Phi(\lambda) = \varphi_+(x;\lambda) + \varphi_-(x;\lambda)$ so that

$$\Phi(\lambda) = \int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(y))^{1/2} dy. \tag{4.4}$$

Since $w(\lambda,\hbar) = 0$, we see that

$$\sin(\hbar^{-1}\Phi(\lambda) + \pi/2) = O(\hbar)$$

and hence

$$\int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(x))^{1/2} dx = \pi(n + 1/2)\hbar + O(\hbar^2) \tag{4.5}$$

for some integer number $n = n(\lambda,\hbar)$. This gives us the famous Bohr-Sommerfeld quantization condition.

Suppose now that a number $\pi(n+1/2)\hbar$ belongs to a neighborhood of λ_0. Let us check that there exists an eigenvalue $\lambda_n(\hbar)$ of the operator $H_\hbar$ satisfying the estimate

$$|\Phi(\lambda_n(\hbar)) - \pi(n + 1/2)\hbar| \leq C\hbar^2. \tag{4.6}$$

Since $u_\pm \in L^2(\mathbb{R}_\pm)$, it suffices to show that $w(\lambda,\hbar) = 0$ for some $\lambda = \lambda_n(\hbar)$ satisfying estimate (4.6). Using the equality $\lambda - v(x_\pm(\lambda)) = 0$, we find that

$$\Phi'(\lambda) = 2^{-1} \int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(y))^{-1/2} dy > 0. \tag{4.7}$$

Hence Φ is a one-to-one mapping of a neighborhood of λ_0 on a neighborhood of $\mu_0 = \Phi(\lambda_0)$. Set $\mu = \Phi(\lambda)$ and

$$\epsilon(\mu,\hbar) = \pi^{-1}\hbar^{2/3} w(\Phi^{-1}(\mu),\hbar) - \sin(\hbar^{-1}\mu + \pi/2). \tag{4.8}$$

In view of (4.3) this function satisfies the estimate $|\epsilon(\mu,\hbar)| \leq C\hbar$ with a constant C which does not depend on $\hbar$ and μ from a neighborhood of μ_0. We have to show that the equation

$$\sin(\hbar^{-1}\mu + \pi/2) + \epsilon(\mu,\hbar) = 0$$

has a solution $\mu_n(\hbar)$ obeying the estimate

$$|\mu_n(\hbar) - \pi(n + 1/2)\hbar| \le C\hbar^2.$$

Setting $s = \hbar^{-1}\mu + \pi/2$, we see that this assertion is equivalent to the existence of a solution $s = s_n(\hbar)$ of the equation

$$\sin s + \epsilon(\hbar(s - \pi/2), \hbar) = 0 \tag{4.9}$$

obeying the estimate

$$|s_n(\hbar) - \pi(n + 1)| \le C\hbar. \tag{4.10}$$

The last fact is obvious because $\epsilon(\hbar(s - \pi/2), \hbar) = O(\hbar)$.

Next, we will show that for every n there is only one eigenvalue of the operator $H_\hbar$ satisfying (4.6). To that end, we have to check that equation (4.9) cannot have two solutions satisfying (4.10). Supposing the contrary, we find a point $\tilde{s} = \tilde{s}_n(\hbar)$ such that

$$\cos\tilde{s} = -\hbar\frac{\partial\epsilon}{\partial\mu}(\hbar(\tilde{s} - \pi/2), \hbar) \tag{4.11}$$

and $\tilde{s}_n(\hbar) = \pi(n + 1) + O(\hbar)$. Observe that relation (4.3) can be differentiated in λ which yields

$$\partial w(\lambda, \hbar)/d\lambda = \pi\Phi'(\lambda)\hbar^{-5/3}\cos(\hbar^{-1}\Phi(\lambda) + \pi/2) + O(\hbar^{-2/3}).$$

It follows that function (4.8) obeys the estimate $\partial\varepsilon(\mu, \hbar)/d\mu = O(1)$. Thus, the right-hand side of equation (4.11) is $O(\hbar)$ while its left-hand side tends to $(-1)^{n+1}$ as $\hbar \to 0$.

Finally, plugging asymptotics (2.19) and (2.21) into (4.1), we find that

$$\begin{aligned}\sin(\hbar^{-1}\varphi_-(x; \lambda) + \pi/4) + O(\hbar)&\\ =a(\lambda, \hbar)\big(\sin(\hbar^{-1}\varphi_+(x; \lambda) + \pi/4) + O(\hbar)\big)&\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\cos(\hbar^{-1}\varphi_-(x; \lambda) + \pi/4) + O(\hbar)&\\ = - a(\lambda, \hbar)\big(\cos(\hbar^{-1}\varphi_+(x; \lambda) + \pi/4) + O(\hbar)\big).&\end{aligned} \tag{4.13}$$

Together, these two relations imply that $|a(\lambda, \hbar)| = 1 + O(\hbar)$. Moreover, since

$$\varphi_+(x; \lambda) + \varphi_-(x; \lambda) = \pi(n + 1/2)\hbar + O(\hbar^2),$$

it follows from (4.12) and (4.13) that

$$a(\lambda, \hbar) = (-1)^n + O(\hbar). \tag{4.14}$$

Thus, we have obtained the following result.

Theorem 4.1. *Let Assumptions* 2.1 *and* 2.2 *hold for a point* λ_0. *Suppose that an eigenvalue* $\lambda = \lambda(\hbar)$ *of the operator* $H_\hbar$ *belongs to a neighborhood of* λ_0. *Then necessarily condition* (4.5) *is satisfied with some integer number* $n = n(\lambda, \hbar)$. *Conversely, for every* n *such that* $\pi(n+1/2)\hbar$ *belongs to a neighborhood of* $\Phi(\lambda_0)$, *there exists an eigenvalue* $\lambda_n(\hbar)$ *of the operator* $H_\hbar$ *satisfying estimate* (4.6) *with a constant* C *not depending on* n *and* $\hbar$. *Such an eigenvalue* $\lambda_n(\hbar)$ *is unique. Moreover, the coefficient* $a(\lambda, \hbar)$ *in* (4.1) *has asymptotics* (4.14) *where* n *is the same number as in* (4.5).

Corollary 4.2. *Let an interval (a_1, a_2) belong to a neighborhood of a point λ_0 satisfying Assumptions* 2.1 *and* 2.2*. Then the total number $N_\hbar$ of eigenvalues of the operator $H_\hbar$ in this interval equals*

$$N_\hbar = \pi^{-1}(\Phi(a_2) - \Phi(a_1))\hbar^{-1} + \epsilon(\hbar) \tag{4.15}$$

where $|\epsilon(\hbar)| \leq 1$ for sufficiently small $\hbar$.

Proof. According to Theorem 4.1 there is exactly one eigenvalue of the operator $H_\hbar$ in a neighborhood of size $C\hbar^2$ of every point $\Phi^{-1}(\pi(n+1/2)\hbar)$. These neighborhoods have empty intersections for sufficiently small $\hbar$. Thus, $N_\hbar$ equals the number of points $\pi(n+1/2)\hbar$ lying in the interval $(\Phi(a_1), \Phi(a_2))$. Clearly, this number equals the right-hand side of (4.15). □

Remark 4.3. Suppose that Assumptions 2.1 and 2.2 hold true for all $\lambda \in [a_1, a_2]$. Then remainders in different asymptotic formulas of this paper can be estimated uniformly in $\lambda \in [a_1, a_2]$. Formula (4.15) also remains true for such (a_1, a_2).

Note that definition (4.4) can be rewritten as

$$\Phi(\lambda) = 2^{-1} \int\int_{p^2+v(x)\leq\lambda} dpdx. \tag{4.16}$$

Indeed, integrating in the right-hand side first over p we obtain the right-hand side of (4.4). It follows that the asymptotic coefficient in (4.15) is the volume of a part of the phase space:

$$\Phi(a_2) - \Phi(a_1) = 2^{-1}\mathrm{mes}\{(x,p) \in \mathbb{R}^2 : a_1 \leq p^2 + v(x) \leq a_2\}.$$

Thus, relation (4.15) is the semiclassical Weyl formula with a strong estimate of the remainder.

4.2. Let us denote by $\psi(x) = \psi(x; \lambda, \hbar)$ the eigenfunction of the operator $H_\hbar$ corresponding to its eigenvalue λ. We suppose that $\psi = \overline{\psi} \in L^2(\mathbb{R}_+)$ and $\|\psi\| = 1$ which fixes ψ up to a sign. Clearly,

$$\psi(x) = c_\pm u_\pm(x), \quad c_\pm = c_\pm(\lambda, \hbar), \tag{4.17}$$

where according to (4.14)

$$|c_+(\lambda, \hbar)| = |c_-(\lambda, \hbar)|(1 + O(\hbar)).$$

Therefore it follows from Corollary 2.9 that

$$|c_\pm(\lambda, \hbar)| = 2^{1/2}\pi^{-1/2}\hbar^{-1/6}\Big(\int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(x))^{-1/2}dx\Big)^{-1/2} + O(\hbar^{1/6}), \tag{4.18}$$

which in view of Theorem 2.5 yields the following result.

Theorem 4.4. *Under the assumptions of Theorem* 4.1*, let us denote by $\psi(\lambda, \hbar)$ the real normalized eigenfunction (defined up to a sign) of the operator $H_\hbar$ corresponding to its eigenvalue $\lambda = \lambda(\hbar)$. Let x_1 be an arbitrary point from the interval $(x_-(\lambda), x_+(\lambda))$. Then, for $x \in (x_1, \infty)$, asymptotics of $\psi(x; \lambda, \hbar)$ as $\hbar \to 0$ is given by formulas* (4.17), (4.18) *for the sign "+" and asymptotic relations of Theorem* 2.5 *for the function $u_+(x; \lambda, \hbar)$. Similarly, for $x \in (-\infty, x_1)$, asymptotics of $\psi(x; \lambda, \hbar)$ as $\hbar \to 0$ is given by formulas* (4.17), (4.18) *for the sign "$-$" and asymptotic relations of Theorem* 2.5 *for the function $u_-(x; \lambda, \hbar)$. In neighborhoods of the turning points, the estimate*

$$|\psi(x; \lambda, \hbar)| \leq C(\hbar^{2/3} + |x - x_\pm|)^{-1/4} \tag{4.19}$$

holds with a constant C which does not depend on $\hbar$.

In view of formula (2.23), this result can be supplemented by the following

Proposition 4.5. *Let a function w satisfy the assumptions of Proposition 2.8. Then under the assumptions of Theorem 4.4 we have*

$$\int_{-\infty}^{\infty} w(x)\psi^2(x;\lambda,\hbar)dx = \int_{x_-(\lambda)}^{x_+(\lambda)} w(x)(\lambda - v(x))^{-1/2}dx \times \Big(\int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(x))^{-1/2}dx\Big)^{-1} + O(\hbar^{1/3}). \tag{4.20}$$

In particular, this relation applies to the potential energy

$$V(\lambda,\hbar) = \int_{-\infty}^{\infty} v(x)\psi^2(x;\lambda,\hbar)dx$$

and by virtue of the energy conservation $K(\lambda,\hbar) + V(\lambda,\hbar) = \lambda$, we also obtain the asymptotics of the kinetic energy.

Corollary 4.6. *Under the assumptions of Theorem 4.4 asymptotic relation (1.3) holds with the leading term $K_{cl}(\lambda)$ given by (1.4).*

Since $K_{cl}(\lambda) > 0$, for small $\hbar$ the kinetic energy $K(\lambda,\hbar) \geq c > 0$ or, equivalently, the potential energy $V(\lambda,\hbar) \leq \lambda - c$. This implies that the eigenfunctions $\psi(x;\lambda,\hbar)$ are not too strongly localized in neighborhoods of the turning points $x_\pm(\lambda)$. In view of estimate (4.19), this statement can be reinforced.

Proposition 4.7. *Let the assumptions of Theorem 4.1 hold, and let $\|\psi(\lambda,\hbar)\| = 1$. Then*

$$\int_{x_\pm(\lambda)-\delta}^{x_\pm(\lambda)+\delta} \psi^2(x;\lambda,\hbar)dx \leq C\delta^{1/2}$$

where the constant C does not depend on $\hbar$.

Remark 4.8. It follows from asymptotics (2.14) and (4.18) that

$$|\psi(x_\pm(\lambda);\lambda,\hbar)| = \alpha_\pm(\lambda)\hbar^{-1/6}(1 + O(\hbar^{1/3})),$$

where the coefficient

$$\alpha_\pm(\lambda) = 2^{1/2}\pi^{-1/2}\Big(\int_{x_-(\lambda)}^{x_+(\lambda)} (\lambda - v(x))^{-1/2}dx\Big)^{-1/2} |v'(x_\pm(\lambda))|^{-1/6}\,\mathrm{Ai}\,(0) \neq 0.$$

This contradicts the assertion of Theorem 7.1 of [8] that normalized eigenfunctions are uniformly bounded in neighborhoods of turning points.

4.3. Recall that a classical particle (of mass m and energy λ) moves periodically (see, e.g., [6]) in a potential well bounded by the points $x_- = x_-(\lambda)$ and $x_+ = x_+(\lambda)$ such that $v(x_\pm) = \lambda$. Let us check that the asymptotic coefficient K_{cl} in (1.3) coincides with the averaged over the period $T = T(\lambda)$ value

$$K_{av} = T^{-1}\int_0^T K(t)dt$$

of the classical kinetic energy

$$K(t) = mx'(t)^2/2 = \lambda - v(x(t)).$$

Since

$$dt = x'(t)^{-1}dx = (m/2)^{1/2}(\lambda - v(x))^{-1/2}dx,$$

the period is given by the formula

$$T = 2\int_{x_-}^{x_+} \frac{dt}{dx}dx = (2m)^{1/2}\int_{x_-}^{x_+}(\lambda - v(x))^{-1/2}dx$$

and

$$K_{av} = T^{-1}\int_0^T (\lambda - v(x(t)))dt = 2(m/2)^{1/2}T^{-1}\int_{x_-}^{x_+}(\lambda - v(x))^{1/2}dx.$$

Putting these two relations together, we obtain for K_{av} the same expression (1.4) as for K_{cl}. This proves the equality

$$K_{cl}(\lambda) = K_{av}(\lambda).$$

We also note that

$$K_{cl}(\lambda) = \big(2d\ln\Phi(\lambda)/d\lambda\big)^{-1}$$

where the function $\Phi(\lambda)$ is defined by formulas (4.4) or, equivalently, (4.16). For the proof of this equality, it suffices to plug representation (4.7) for the function $\Phi'(\lambda)$ into formula (1.4).

5. Discontinuous potentials

5.1. Away from the turning points, assumptions on $v(x)$ can be somewhat relaxed. Consider, for example, an interval $(x_- + \delta, x_+ - \delta)$ where $\delta > 0$. There, it suffices to require that $v \in C^1$ and that v' be absolutely continuous so that $v'' \in L^1$ (instead of $v \in C^2$). In this case the function $r(\xi)$ defined by formulas (2.11) and (3.3) belongs to L^1 only so that the factor $r(\hbar^{2/3}s)$ in the right-hand side of estimate (3.19) cannot be neglected. Therefore (cf. (3.20)) we have the estimate

$$\begin{aligned}\int_t^0 |K_\hbar(t,s)|(1+|s|)^{-1/4}ds &\le C\hbar^{4/3}(1+|t|)^{-1/4}\int_t^0 |r(\hbar^{2/3}s)|ds\\ &\le C_1\hbar^{2/3}(1+|t|)^{-1/4}\int_{\xi(x_-+\delta)}^0 |r(s)|ds.\end{aligned}$$

It follows that instead of (3.14) we have a slightly weaker estimate with $\hbar^{2/3}$ in place of $\hbar$ in the right-hand side. All other estimates remain unchanged. Thus, Theorem 2.5 is true with a little bit weaker estimates of remainders in asymptotic formulas for $u_\pm(x;\lambda,\hbar)$ inside the interval $(x_-+\delta, x_+-\delta)$. Repeating the arguments of Section 4, we get the following result.

Proposition 5.1. *Under the assumptions above, all results of Theorem* 4.1 (*and of Corollary* 4.2) *about eigenvalues of the operators $H_\hbar$ remain true with the remainders $O(\hbar^{5/3})$ in* (4.5), *$O(\hbar^{2/3})$ in* (4.14) *and $C\hbar^{5/3}$ in the right-hand side of* (4.6). *Theorem* 4.4 *about corresponding eigenfunctions remains also true with the remainders $O(\hbar^{5/6}|x - x_\pm|^{-7/4})$ in* (2.19), *$O(\hbar^{-1/6}|x - x_\pm|^{-7/4})$ in* (2.21) *and $O(\hbar^{1/18})$ in* (4.18).

5.2. Our goal in this subsection is to extend the results of Section 4 to functions $v(x)$ with a singular point x_0 inside a potential well.

We suppose that Assumption 2.1 holds everywhere except a point x_0 and that Assumption 2.2 holds for some λ_0 such that x_0 is an interior point of the interval

$(x_-(\lambda_0), x_+(\lambda_0))$. We assume that $v(x)$ has finite limits at x_0 but the left and right limits might be different. Finally, we require that $v' \in L^2(x_0, x_0 \pm \delta)$ and $v'' \in L^1(x_0, x_0 \pm \delta)$ for some $\delta > 0$.

Now we can construct solutions $u_+(x)$ and $u_-(x)$ of equation (1.1) on the intervals (x_0, ∞) and $(-\infty, x_0)$, respectively. Define, as usual, the function $r(\xi)$ by formulas (2.11) and (3.3). Since $r \in L^1(x_0, x_0 \pm \delta)$, the limits $u_\pm(x_0 \pm 0)$ and $u'_\pm(x_0 \pm 0)$ exist, and we can use formulas (2.19) and (2.21) for these limits (with slightly weaker estimates of the remainders – see subs. 5.1). It follows that the Wronskian $w(\lambda, \hbar)$ of u_+ and u_- calculated at the point x_0 is given by the expression (cf. (4.3))

$$\pi\hbar^{-2/3}\Big(p(x_0,\lambda)\sin(\hbar^{-1}\varphi_+(x_0;\lambda)+\pi/4)\cos(\hbar^{-1}\varphi_-(x_0;\lambda)+\pi/4)$$
$$+p(x_0,\lambda)^{-1}\cos(\hbar^{-1}\varphi_+(x_0;\lambda)+\pi/4)\sin(\hbar^{-1}\varphi_-(x_0;\lambda)+\pi/4)\Big)+O(1)$$

where

$$p(x_0,\lambda) = \big(\lambda - v(x_0-0)\big)^{1/4}\big(\lambda - v(x_0+0)\big)^{-1/4}. \tag{5.1}$$

Let an eigenvalue λ of the operator $H_\hbar$ be close to λ_0. Since $w(\lambda,\hbar)=0$, we see that

$$\begin{aligned} &p(x_0,\lambda)\sin(\hbar^{-1}\varphi_+(x_0;\lambda)+\pi/4)\cos(\hbar^{-1}\varphi_-(x_0;\lambda)+\pi/4)\\ &+p(x_0,\lambda)^{-1}\sin(\hbar^{-1}\varphi_+(x_0;\lambda)+\pi/4)\cos(\hbar^{-1}\varphi_-(x_0;\lambda)+\pi/4) = O(\hbar^{2/3}). \end{aligned} \tag{5.2}$$

Formula (5.2) yields a generalization of the Bohr-Sommerfeld quantization condition (4.5) and reduces to it if $v(x_0+0) = v(x_0-0)$.

Let the coefficient $a(\lambda,\hbar)$ be defined by equality (4.1). To calculate $|a(\lambda,\hbar)|$, we use again relations (4.12) and (4.13). However, additional factors $|q(x_0-0)|^{-1/4}$ and $|q(x_0-0)|^{1/4}$ appear now in their left-hand sides. Similarly, additional factors $|q(x_0+0)|^{-1/4}$ and $|q(x_0+0)|^{1/4}$ appear in their right-hand sides. This implies that

$$\begin{aligned} a^2(\lambda,\hbar) =& p^2(x_0,\lambda)\cos^2(\hbar^{-1}\varphi_-(x_0;\lambda)+\pi/4)\\ &+p^{-2}(x_0,\lambda)\sin^2(\hbar^{-1}\varphi_-(x_0;\lambda)+\pi/4)+O(\hbar^{2/3})\\ =&\big(p^2(x_0,\lambda)\sin^2(\hbar^{-1}\varphi_+(x_0;\lambda)+\pi/4)\\ &+p^{-2}(x_0,\lambda)\cos^2(\hbar^{-1}\varphi_+(x_0;\lambda)+\pi/4)\big)^{-1}+O(\hbar^{2/3}). \end{aligned} \tag{5.3}$$

As before, using formula (2.26) (where $x_1 = x_0$) and the normalization condition $\|\psi\| = 1$, we obtain explicit expressions for the absolute values of constants $c_\pm(\lambda,\hbar)$ in (4.17):

$$\begin{aligned} |c_+(\lambda,\hbar)| = 2^{1/2}\pi^{-1/2}\hbar^{-1/6}\Big(&\int_{x_0}^{x_+(\lambda)}(\lambda - v(x))^{-1/2}dx\\ &+a^{-2}(\lambda,\hbar)\int_{x_-(\lambda)}^{x_0}(\lambda - v(x))^{-1/2}dx\Big)^{-1/2}+O(\hbar^{1/18}) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} |c_-(\lambda,\hbar)| = 2^{1/2}\pi^{-1/2}\hbar^{-1/6}\Big(&a^2(\lambda,\hbar)\int_{x_0}^{x_+(\lambda)}(\lambda - v(x))^{-1/2}dx\\ &+\int_{x_-(\lambda)}^{x_0}(\lambda - v(x))^{-1/2}dx\Big)^{-1/2}+O(\hbar^{1/18}). \end{aligned} \tag{5.5}$$

Thus, Theorems 4.1 and 4.4 can be supplemented by the following result.

Theorem 5.2. *Under the assumptions above, let an eigenvalue $\lambda = \lambda(\hbar)$ of the operator $H_\hbar$ belong to a neighborhood of λ_0. Then necessarily condition* (5.2) *is satisfied with the numbers $\varphi_\pm(x_0;\lambda)$ and $p(x_0,\lambda)$ defined by* (4.2) *and* (5.1), *respectively. All assertions (for $x_1 = x_0$) of Theorem* 4.4 *about the corresponding normalized eigenfunction $\psi(x;\lambda,\hbar)$ are true with the constants $c_\pm(\lambda,\hbar)$ whose absolute values are determined by formulas* (5.3), (5.4) *and* (5.5).

Remark 5.3. If the functions $v'(x)$ and $v''(x)$ are bounded in a neighborhood of the point x_0, then even in the case $v(x_0+0) \neq v(x_0-0)$ estimates of all remainders are the same as in Section 4. Thus, we have $O(\hbar)$ in (5.2) and $O(\hbar^{1/6})$ in (5.3) – (5.5).

Remark 5.4. Let under the assumptions above $v(x_0+0) = v(x_0-0)$. Then all conclusions of Proposition 5.1 remain true although the function $v'(x)$ is not required to be continuous at the point x_0. In particular, we see that jumps of derivatives of the function $v(x)$ at the point x_0 are inessential.

5.3. Let us consider an explicit example:

$$v(x) = a_+ + v_+ x^{\alpha_+} \quad \text{for} \quad x > 0 \quad \text{and} \quad v(x) = a_- + v_-|x|^{\alpha_-} \quad \text{for} \quad x < 0, \tag{5.6}$$

where $v_\pm > 0$ and $\alpha_\pm > 0$. Then all $\lambda > \max\{a_+, a_-\}$ are non-critical, the equation $v(x) = \lambda$ has two solutions $x_+ > 0$, $x_- < 0$ and (x_-, x_+) is a potential well. The point $x_0 = 0$ might be singular and $p(0,\lambda) = (\lambda - a_-)^{1/4}(\lambda - a_+)^{-1/4}$.

For potentials (5.6), the integrals in formulas (4.4) and (4.18) can be calculated in terms of the beta function B. Observe that $x_+ = (\lambda v^{-1})^{1/\alpha}$ if $v(x) = vx^\alpha$ for $x > 0$. For the integrals over $(0, x_+)$, we have

$$\int_0^{x_+} (\lambda - vx^\alpha)^{1/2} dx = \lambda^{1/2}(\lambda/v)^{1/\alpha}\alpha^{-1}\mathrm{B}(3/2, 1/\alpha) \tag{5.7}$$

and

$$\int_0^{x_+} (\lambda - vx^\alpha)^{-1/2} dx = \lambda^{-1/2}(\lambda/v)^{1/\alpha}\alpha^{-1}\mathrm{B}(1/2, 1/\alpha). \tag{5.8}$$

The integrals over $(x_-, 0)$ can be calculated quite similarly.

It follows that the quantization condition (5.2) holds with

$$\varphi_+(0,\lambda) = \lambda_+^{1/2+1/\alpha_+} v_+^{-1/\alpha_+} \alpha_+^{-1}\mathrm{B}(3/2, 1/\alpha_+),$$
$$\varphi_-(0,\lambda) = \lambda_-^{1/2+1/\alpha_-} v_-^{-1/\alpha_-} \alpha_-^{-1}\mathrm{B}(3/2, 1/\alpha_-),$$

where $\lambda_\pm = \lambda - a_\pm$. In particular, in the case $a_+ = a_- =: a$ the Bohr-Sommerfeld quantization condition reads as

$$(\lambda - a)^{1/2+1/\alpha_+} v_+^{-1/\alpha_+} \alpha_+^{-1}\mathrm{B}(3/2, 1/\alpha_+) + (\lambda - a)^{1/2+1/\alpha_-} v_-^{-1/\alpha_-} \alpha_-^{-1}\mathrm{B}(3/2, 1/\alpha_-) = \pi\hbar(n + 1/2) + O(\hbar^2). \tag{5.9}$$

Plugging expressions (5.7) and (5.8) into (1.4) we also find that

$$K_{cl}(\lambda) = \frac{\lambda_+^{1/2+1/\alpha_+} v_+^{-\alpha_+} \alpha_+^{-1}\mathrm{B}(3/2, 1/\alpha_+) + \lambda_-^{1/2+1/\alpha_-} v_-^{-\alpha_-} \alpha_-^{-1}\mathrm{B}(3/2, 1/\alpha_-)}{\lambda_+^{-1/2+1/\alpha_+} v_+^{-\alpha_+} \alpha_+^{-1}\mathrm{B}(1/2, 1/\alpha_+) + \lambda_-^{-1/2+1/\alpha_-} v_-^{-\alpha_-} \alpha_-^{-1}\mathrm{B}(1/2, 1/\alpha_-)}.$$

Observe that Theorems 4.1 and 4.4 can be applied to potential (5.6) if $a_+ = a_-$ and $\alpha_\pm \geq 2$. If $a_+ = a_-$ but $\alpha_\pm \in [1, 2)$, then we have to use Remark 5.4; in this case $O(\hbar^2)$ in (5.9) should be replaced by $O(\hbar^{5/3})$. If $a_\pm$ are arbitrary and $\alpha_\pm \geq 1$, then the conditions of Theorem 5.2 are satisfied. Moreover, according to Remark 5.3 in the case $\alpha_\pm \geq 2$, the estimates of the remainders can be improved. Finally, we note that if $\alpha_j < 1$, then $v'' \notin L^1(-\delta,\delta)$ so that the semiclassical approximation does not directly work (even for $a_+ = a_-$) although all formulas above remain meaningful.

5.4. Let us briefly consider the problem on the half-axis. We now suppose that equation (1.1) is satisfied for $x \geq 0$, $\psi \in L^2(\mathbb{R}_+)$ and $\psi(0) = 0$. Assumptions 2.1 and 2.2 should be slightly modified. Namely, we assume that the equation $v(x) = \lambda$ has only one solution $x_+ = x_+(\lambda)$ and $v'(x_+) > 0$ so that $(0, x_+)$ is a potential well. We suppose that the limit of $v(x)$ as $x \to 0$ exists and that the functions $v'(x)$ and $v''(x)$ are bounded in a neighborhood of $x = 0$. Then the results of Theorem 2.5 on the solution $u_+(x)$ of equation (1.1) are true for all $x \geq 0$. In particular, it follows from formula (2.19) that

$$u_+(0;\lambda,\hbar) = \pi^{1/2}\hbar^{1/6}(\lambda - v(0))^{-1/4}\sin\big(\hbar^{-1}\int_0^{x_+(\lambda)}(\lambda - v(x))^{1/2}dx + \pi/4\big) + O(\hbar^{7/6}). \tag{5.10}$$

Since $\psi(x;\lambda,\hbar) = c_+u_+(x;\lambda,\hbar)$, this yields the quantization condition

$$\int_0^{x_+(\lambda)}(\lambda - v(x))^{1/2}dx = \pi\hbar(n + 3/4) + O(\hbar^2)$$

where $n = n(\lambda,\hbar)$ is an integer.

Consider now the boundary condition $\psi'(0) = b\psi(0)$, $b = \bar{b}$. It follows from (2.21) that

$$u_+'(0;\lambda,\hbar) = -\pi^{1/2}\hbar^{-5/6}(\lambda - v(0))^{1/4}\cos\big(\hbar^{-1}\int_0^{x_+(\lambda)}(\lambda - v(x))^{1/2}dx + \pi/4\big) + O(\hbar^{1/6}).$$

Comparing this formula with (5.10), we see that the value of $u_+(0;\lambda,\hbar)$ is inessential so that the quantization condition looks like

$$\int_0^{x_+(\lambda)}(\lambda - v(x))^{1/2}dx = \pi\hbar(n + 1/4) + O(\hbar^2).$$

It does not depend on b.

Other results of Section 4 can also be naturally extended to the problem on the half-axis.

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