A COMMUTATOR METHOD FOR THE DIAGONALIZATION OF HANKEL OPERATORS

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To the memory of Mikhail Shlëmovich Birman

Abstract. We present a method for the explicit diagonalization of some Hankel operators. This method allows us to recover classical results on the diagonalization of Hankel operators with the absolutely continuous spectrum. It leads also to new results. Our approach relies on the commutation of a Hankel operator with some differential operator of second order.

1. Introduction

1.1. Hankel operators can be defined (see, e.g., book [8]) as integral operators in the space $L^2(\mathbb{R}_+)$ whose kernels depend on the sum of variables only. Thus, a Hankel operator A is defined by the formula

$$(Af)(x) = \int_0^\infty a(x+y)f(y)dy. \tag{1.1}$$

Of course, A is self-adjoint if $a = \bar{a}$. If

$$\int_0^\infty |a(x)|^2 x dx < \infty,$$

then A belongs to the Hilbert-Schmidt class. This condition is satisfied if, for example, the function a is continuous, it is not too singular at $x = 0$ and decays sufficiently rapidly as $x \to \infty$. On the contrary, if $a(x) \sim a_0 x^{-1}$ as $x \to 0$ or (and) $a(x) \sim a_\infty x^{-1}$ as $x \to \infty$, then the operator A is no longer compact although it remains bounded. A general philosophy (see paper [4] by J. S. Howland) is that each of these singularities gives rise to the branch $[0, a_0\pi]$ or (and) $[0, a_\infty \pi]$ of the simple absolutely continuous spectrum.

There are very few examples where the operator A can be explicitly diagonalized, that is its exact eigenfunctions can be found. The first result is due to F. Mehler [6] who considered the case $a(x) = (x+2)^{-1}$. He has shown that functions

$$\psi_k(x) = \big(k \tanh \pi k\big)^{1/2} P_{-1/2+ik}(x+1), \quad \lambda = \pi/\cosh \pi k, \quad k > 0, \tag{1.2}$$

where $P_{-1/2+ik}$ is the Legendre function (see [3], Chapter 3), satisfy equations $A\psi_k = \lambda \psi_k$. The functions ψ_k are usually parametrized by the quasimomentum k related to $\lambda = \lambda(k)$ by formula (1.2). The operator $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ defined[1]

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[1]A precise definition of the operator U can be given in terms of the corresponding sesquilinear form.

by the equality

$$(Uf)(k) = \int_0^\infty \psi_k(x) f(x) dx \tag{1.3}$$

is unitary. Observe that $\lambda(k)$ is a one-to-one mapping of $\mathbb{R}_+$ on $(0, \pi)$ and that $(ULf)(k) = \lambda(k) f(k)$ which implies that the spectrum of the operator A is simple, absolutely continuous and coincides with the interval $[0, \pi]$.

Below we use the term "eigenfunction" for ψ_k (although $\psi_k \notin L^2(\mathbb{R}_+)$) such that $A\psi_k = \lambda \psi_k$ for the spectral parameter λ from the continuous spectrum of the operator A. *By definition*, we also say that eigenfunctions ψ_k of the continuous spectrum are orthogonal, normalized and the set of all ψ_k is complete if the corresponding operator (1.3) is unitary (if A has no point spectrum).

The next result is due to W. Magnus [5] who considered the case $a(x) = x^{-1} e^{-x/2}$. A more general result of the same type was obtained by M. Rosenblum [9] who has diagonalized the operator A with kernel

$$a(x) = \Gamma(1 + \beta) x^{-1} W_{-\beta, 1/2}(x), \quad \beta \in \mathbb{R}, \quad \beta \neq -1, -2, \ldots, \tag{1.4}$$

where $W_{-\beta,1/2}$ is the Whittaker function (see [3], Chapter 6) and Γ is the gamma function. Note that $W_{0,1/2}(x) = e^{-x/2}$. The spectrum of the operator A with such kernel is again simple and, up to a finite number of eigenvalues, it is absolutely continuous and coincides with the interval $[0, \pi]$. Its "normalized eigenfunctions" are expressed in terms of the Whittaker functions

$$\psi_k(x) = (2\pi)^{-1} \sqrt{k |\Gamma(1/2 - ik + \beta)| \sinh 2\pi k}\, x^{-1} W_{-\beta, ik}(x), \quad k > 0. \tag{1.5}$$

Observe that the function $a(x) = (x + 2)^{-1}$ is singular at $x = \infty$ and eigenfunctions (1.2) decay as linear combinations of $x^{-1/2 \pm ik}$ as $x \to \infty$ while function (1.4) is singular at $x = 0$ and eigenfunctions (1.5) behave as linear combinations of the same functions $x^{-1/2 \pm ik}$ as $x \to 0$.

We note also a simple case $a(x) = x^{-1}$ where the operator A is directly diagonalized (see paper [2] by T. Carleman) by the Mellin transform. In this case the spectrum of A has multiplicity 2 (because of the singularities of $a(x)$ both at $x = 0$ and at $x = \infty$), it is absolutely continuous and coincides with the interval $[0, \pi]$. The eigenfunctions of the Carleman operator equal $x^{-1/2 \pm ik}$ (up to a normalization).

We emphasize a parallelism of theories of singular differential operators and Hankel operators with singular kernels. Thus, the functions $x^{-1/2 \pm ik}$ play (both for $x \to \infty$ and $x \to 0$) for Hankel operators the role of exponential functions $e^{\pm ikx}$ for differential operators of second order. From this point of view, the Carleman operator plays the role of the operator $-d^2/dx^2$ in the space $L^2(\mathbb{R})$.

1.2. In the author's opinion, the reason why in the cases described above eigenfunctions of a Hankel operator can be found explicitly remained unclarified. Our approach shows that all diagonalizable Hankel operators A commute with differential operators

$$L = -\frac{d}{dx}(x^2 + \gamma x)\frac{d}{dx} + \alpha x^2 + \beta x \tag{1.6}$$

for suitably chosen parameters $\alpha \geq 0, \beta \in \mathbb{R}$ and $\gamma \geq 0$. Thus, eigenfunctions of the operators A and L are the same which allows us to diagonalize the operator A.

Hopefully the commutator method will be applied to other kernels a. In this paper we use the commutator method to find in subs. 4.4 eigenfunctions of a new

Hankel operator with kernel

$$a(x)=\sqrt{\frac{8}{x}}K_1(\sqrt{8x}), \tag{1.7}$$

where K_1 is the MacDonald function (see [3], Chapter 7). Similarly to (1.4), this function decays exponentially as $x\to\infty$ and $a(x)\sim x^{-1}$ as $x\to 0$. An example of a different nature are Hankel operators with regular kernels; such operators are compact.

Note that operator (1.6) for $\gamma=0$ and $\alpha>0$ appeared already in [9]. Actually, M. Rosenblum proceeded from the identity

$$\Gamma(1+\beta)\int_0^\infty (x+y)^{-1}W_{-\beta,1/2}(x+y)y^{-1}W_{-\beta,ik}(y)dy=\frac{\pi}{\cosh \pi k}x^{-1}W_{-\beta,ik}(x) \tag{1.8}$$

found earlier by H. Shanker in [10]. This identity shows that functions (1.5) are eigenfunctions of the Hankel operator with kernel (1.4). M. Rosenblum observed that functions (1.5) are also eigenfunctions of operator (1.6) for $\gamma=0$ and $\alpha=1/4$. Since eigenfunctions of the self-adjoint differential operator L are orthogonal and complete, the same is true for eigenfunctions of the Hankel operator A with kernel (1.4). This yields the diagonalization of this operator.

Our approach is somewhat different. We prove the relation $LA=AL$ which shows that eigenfunctions of the operators L and A are the same. In particular, we obtain identity (1.8) without a recourse to the theory of special functions.

It is well-known that the integrability of differential equations of second order in terms of special functions has a deep group-theoretical interpretation (see, e.g., book [12] by N. Ya. Vilenkin). As far as Hankel operators are concerned, it is evident that the diagonalization of the Carleman operator can be explained by its invariance with respect to the group of dilations. The relation $LA=AL$ means that the operator A is invariant with respect to the group $\exp(-itL)$. In contrast to the Carleman operator, for other Hankel operators this invariance does not look obvious.

A commutator scheme is presented in Section 2 while specific examples of kernels singular at $x=\infty$ and $x=0$ are discussed in Sections 3 and 4, respectively. Hankel operators with regular kernels are considered in Section 5.

2. Commutator method

2.1. For a moment, we consider the operator L defined by formula (1.6) as a differential operator on the class $C^2(\mathbb{R}_+)$, but later it will be defined as a self-adjoint operator in the space $L^2(\mathbb{R}_+)$. Let the operator A be given by formula (1.1) where $a\in C^2(\mathbb{R}_+)$.

Let us commute the operators A and L. Suppose that $f\in C^2(\mathbb{R}_+)$ and that

$$\lim_{y\to 0}(y^2+\gamma y)f(y)=\lim_{y\to 0}(y^2+\gamma y)f'(y)=0 \tag{2.1}$$

as well as

$$\lim_{y\to\infty}a'(x+y)(y^2+\gamma y)f(y)=\lim_{y\to\infty}a(x+y)(y^2+\gamma y)f'(y)=0 \tag{2.2}$$

for all $x\geq 0$. Then integrating by parts, we find that

$$(ALf)(x)=\int_0^\infty\Big(-\frac{\partial}{\partial y}\big((y^2+\gamma y)a'(x+y)\big)+a(x+y)(\alpha y^2+\beta y)\Big)f(y)dy.$$

It follows that

$$((LA - AL)f)(x) = \int_0^\infty q(x,y)f(y)dy$$

where

$$\begin{aligned} q(x,y) = & -\frac{\partial}{\partial x}\big((x^2+\gamma x)a'(x+y)\big) + \frac{\partial}{\partial y}\big((y^2+\gamma y)a'(x+y)\big) \\ & + (\alpha x^2 - \alpha y^2 + \beta x - \beta y)a(x+y) \\ = & (x-y)\Big(-(z+\gamma)a''(z) - 2a'(z) + (\alpha z + \beta)a(z)\Big) \end{aligned}$$

and $z = x+y$. Thus, we arrive at the following general result.

Theorem 2.1. *Suppose that kernel a of a Hankel operator A satisfies the differential equation*

$$-(x+\gamma)a''(x) - 2a'(x) + (\alpha x + \beta)a(x) = 0. \tag{2.3}$$

Let $f \in C^2(\mathbb{R}_+)$ and let conditions (2.1) *and* (2.2) *hold. Then*

$$(LA - AL)f = 0. \tag{2.4}$$

Note that after a change of variables

$$a(x) = (x+\gamma)^{-1}b(x+\gamma) \tag{2.5}$$

in (2.3), we get the Schrödinger equation with the Coulomb potential

$$-b''(r) + (\alpha + \beta r^{-1})b(r) = 0. \tag{2.6}$$

2.2. In specific examples below, we are going to use Theorem 2.1 in the following way. If L is self-adjoint and has a simple spectrum, then the equality $LA = AL$ shows that A is a function F of L, i.e., the operators A and L have common eigenfunctions. For a calculation of the function F, we argue as follows. Suppose that a function ψ_μ satisfies conditions (2.1), (2.2) and the equation

$$-\big((x^2+\gamma x)\psi_\mu'(x)\big)' + (\alpha x^2 + \beta x)\psi_\mu(x) = \mu\psi_\mu(x). \tag{2.7}$$

Then according to equality (2.4) the same equation holds for the function $A\psi_\mu$ and hence, for some numbers $\lambda = \lambda_\mu$ and $\check\lambda = \check\lambda_\mu$,

$$(A\psi_\mu)(x) = \lambda\psi_\mu(x) + \check\lambda\check\psi_\mu(x) \tag{2.8}$$

where $\check\psi_\mu$ is a solution of the equation $L\check\psi_\mu = \mu\check\psi_\mu$ linearly independent of ψ_μ. Further, comparing asymptotics of the functions $\psi_\mu(x)$, $\check\psi_\mu(x)$ and $(A\psi_\mu)(x)$ as $x \to 0$ and $x \to \infty$, we see that $\check\lambda = 0$ and find $\lambda = \mathsf{F}(\mu)$ as a function of μ. Finally, if ψ_μ belong to the domain of some self-adjoint realization of the differential operator L, then, for a proper normalization of functions ψ_μ, the system of all ψ_μ is orthogonal and complete. In this case $A = \mathsf{F}(L)$. Note that this approach allows one to avoid precise definitions of commutators and references to the functional analysis.

It turns out that in all our applications $\mathsf{F}(\mu) = \pi/\cosh\big(\pi\sqrt{\mu - 1/4}\big)$, and hence

$$A = \pi/\cosh\big(\pi\sqrt{L - 1/4}\big).$$

Actually, it is somewhat more convenient to parametrize eigenfunctions by the quasimomentum $k > 0$ related to μ and λ by the formulas

$$\mu = k^2 + 1/4 \in (1/4, \infty), \quad \lambda = \pi/\cosh \pi k \in (0,\pi). \tag{2.9}$$

Note that $\psi_k(x)$, $\psi_\mu(x)$ and $\psi_\lambda(x)$ denote the same function provided the parameters k, μ and λ are related by formulas (2.9).

The operator U defined by formula (1.3) is unitary and the operator UAU^* acts in $L^2(\mathbb{R}_+)$ as multiplication by the function $\lambda(k) = \pi/\cosh \pi k$. Indeed, according to the Fubini theorem it follows from the equation $A\psi_k = \lambda(k)\psi_k$ that for $g \in C_0^\infty(\mathbb{R}_+)$

$$\begin{aligned}(AU^*g)(x) &= \int_0^\infty dk g(k) \int_0^\infty dy a(x+y)\psi_k(y) \\ &= \int_0^\infty \lambda(k)\psi_k(x)g(k)dk = (U^*(\lambda g))(x),\end{aligned}$$

or equivalently

$$(UAf)(k) = \lambda(k)(Uf)(k), \quad \forall f \in L^2(\mathbb{R}_+). \tag{2.10}$$

Since $\lambda : \mathbb{R}_+ \to (0, \pi)$ is a smooth one-to-one mapping, the operator A has the simple absolutely continuous spectrum $[0, \pi]$.

To realize this scheme, it is convenient to study the cases of singularities at $x = \infty$ when $\gamma > 0$ and at $x = 0$ when $\gamma = 0$ separately.

3. Singularity at infinity

3.1. Set $\gamma = 2$. We first suppose that $\alpha = \beta = 0$. Then the function $a(x) = (x+2)^{-1}$ satisfies equation (2.3), and the corresponding operator

$$L = -\frac{d}{dx}p(x)\frac{d}{dx} \quad \text{where} \quad p(x) = x^2 + 2x.$$

Let $P_\nu(z)$ and $Q_\nu(z)$ be the Legendre functions (see, e.g., [3], Ch. 3) of the first and second kinds, respectively. They are defined as solutions of the equation

$$(1-z^2)u''(z) - 2zu'(z) + \nu(\nu+1)u(z) = 0, \quad z > 1,$$

satisfying the conditions $P_\nu(1) = 1$ and $Q_\nu(z) = -2^{-1}\ln(z-1) + c_\nu$ as $z \to 1+0$ (the value of the number c_ν is inessential). Then the functions $P_{-1/2+ik}(x+1)$ and $Q_{-1/2+ik}(x+1)$ satisfy the equation $Lu = (k^2 + 1/4)u$. We also note that (see formulas (2.10.2) and (2.10.5) of [3])

$$P_{-1/2+ik}(x+1) = m(k)x^{-1/2+ik} + \overline{m(k)}x^{-1/2-ik} + O(x^{-3/2}), \quad x \to \infty, \tag{3.1}$$

where

$$m(k) = \frac{\Gamma(ik)}{\sqrt{2\pi}\Gamma(1/2+ik)}2^{ik}. \tag{3.2}$$

The operator L is symmetric in the space $L^2(\mathbb{R}_+)$ on the domain $C_0^\infty(\mathbb{R}_+)$, but it is not essentially self-adjoint. Since both functions $P_{-1/2+ik}(x+1)$ and $Q_{-1/2+ik}(x+1)$ belong to L^2 in a neighborhood of the point $x = 0$, the defect indices of the operator L are $(1,1)$. One of self-adjoint extensions of L from $C_0^\infty(\mathbb{R}_+)$ (it will be also denoted by L) is defined on the domain $\mathcal{D}(L)$ consisting of functions $f(x)$ from the Sobolev class $\mathsf{H}^2_{loc}(\mathbb{R}_+)$ satisfying the boundary conditions

$$\exists \lim_{x\to 0} f(x), \quad f'(x) = o(x^{-1/2}),\ x \to 0, \tag{3.3}$$

(we call these boundary conditions regular); it is also required that $f \in L^2(\mathbb{R}_+)$ and $Lf \in L^2(\mathbb{R}_+)$. Actually, the direct integration by parts shows that the operator L is symmetric. Furthermore, using the appropriate Green function, we find that for all $h \in L^2(\mathbb{R}_+)$ the equation $(pf')' = h$ has a solution satisfying condition (3.3).

Thus the image of the operator L coincides with $L^2(\mathbb{R}_+)$, and hence L is self-adjoint (cf. §132, part II, of [1]).

3.2. For a study of the operator L, it is convenient to make a standard (see, e.g., book [11] by E. C. Titchmarsh) change of variables. Set

$$t = \omega(x) = \int_0^x p(y)^{-1/2} dy \quad \text{and} \quad f(x) = \omega'(x)^{1/2} \tilde{f}(\omega(x)) =: (F\tilde{f})(x). \tag{3.4}$$

The operator F is unitary in the space $L^2(\mathbb{R}_+)$, and the operator $\widetilde{L} = F^{-1}LF$ acts by the formula $\widetilde{L} = -d^2/dt^2 + q(\eta(t))$ where

$$q(x) = -16^{-1} p(x)^{-1} p'(x)^2 + 4^{-1} p''(x)$$

and $\eta = \omega^{-1}$ is the inverse function to ω (so that $x = \eta(t)$).

In the case $p(x) = x^2 + 2x$ we have

$$\omega(x) = 2 \ln \big(x^{1/2} + (x+2)^{1/2} \big) - \ln 2 \tag{3.5}$$

and hence

$$\widetilde{L} = -\frac{d^2}{dt^2} + \tilde{q}(t) + 1/4, \tag{3.6}$$

where $\tilde{q}(t) = -4^{-1} \big(\eta^2(t) + 2\eta(t) \big)^{-1}$. Since $\omega(x) = (2x)^{1/2} + O(x)$ as $x \to 0$ and $\omega(x) = \ln(2x) + O(x^{-1})$ as $x \to \infty$, we see that $\eta(t) \sim t^2/2$ as $t \to 0$ and $\eta(t) \sim e^t/2$ as $t \to \infty$. It follows that $\tilde{q}(t) \sim -(4t^2)^{-1}$ as $t \to 0$ and $\tilde{q}(t) = O(e^{-t})$ as $t \to \infty$. Note that the operator $\widetilde{L}$ is self-adjoint on the domain $\mathcal{D}(\widetilde{L})$ consisting of functions $\tilde{f}(t)$ from the Sobolev class $\mathsf{H}^2_{loc}(\mathbb{R}_+)$ satisfying the boundary conditions

$$\exists \lim_{t \to 0} t^{-1/2} \tilde{f}(t), \quad \tilde{f}'(t) - (2t)^{-1} \tilde{f}(t) = o(t^{1/2}), \ t \to 0, \tag{3.7}$$

and such that $\tilde{f} \in L^2(\mathbb{R}_+)$, $\widetilde{L}\tilde{f} \in L^2(\mathbb{R}_+)$.

All usual results of spectral and scattering theories can be applied to the operator $\widetilde{L}$ and then used for the operator L. The operator $\widetilde{L}$ has a simple absolutely continuous spectrum coinciding with the interval $[1/4, \infty)$. It does not have eigenvalues because the equations $\widetilde{L}\tilde{u} = \mu\tilde{u}$, or equivalently $Lu = \mu u$, for $\mu \in \mathbb{R}$ do not have solutions from $L^2(\mathbb{R}_+)$ satisfying the regular boundary conditions at zero. The diagonalization of the operator $\widetilde{L}$ can be constructed (see, e.g., [11, 13]) in the following way. Let $\tilde{u}_k(t)$, $k > 0$, be a real-valued solution of the equation

$$\widetilde{L}\tilde{u}_k = (k^2 + 1/4)\tilde{u}_k \tag{3.8}$$

satisfying boundary conditions (3.7). It has the asymptotics

$$\tilde{u}_k(t) = m(k) e^{ikt} + \overline{m(k)} e^{-ikt} + o(1) \tag{3.9}$$

as $t \to \infty$. Then the operator $\widetilde{U}$ defined by the equation

$$(\widetilde{U}\tilde{f})(k) = (2\pi)^{-1/2} |m(k)|^{-1} \int_0^\infty \tilde{u}_k(t) \tilde{f}(t) dt, \tag{3.10}$$

is unitary in the space $L^2(\mathbb{R}_+)$ and $(\widetilde{U}\widetilde{L}\tilde{f})(k) = (k^2 + 1/4)(\widetilde{U}\tilde{f})(k)$.

Let us now make the change of variables (3.4) and set $U = F\widetilde{U}F^{-1}$. Note that

$$(2\pi)^{-1/2} |m(k)|^{-1} = \sqrt{k \tanh \pi k}$$

for the function $m(k)$ defined by equation (3.2). It follows that the operator U defined by the equation

$$(Uf)(k) = \sqrt{k \tanh \pi k} \int_0^\infty P_{-1/2+ik}(x+1) f(x) dx, \tag{3.11}$$

is unitary in the space $L^2(\mathbb{R}_+)$ and

$$(ULf)(k) = (k^2 + 1/4)(Uf)(k). \tag{3.12}$$

3.3. Now we return to the Hankel operator A. Observe that the function $P_{-1/2+ik}(x+1)$ satisfies[2] both boundary conditions (2.1) and (2.2). It follows from Theorem 2.1 that

$$\int_0^\infty (x+y+2)^{-1} P_{-1/2+ik}(y+1) dy = \lambda P_{-1/2+ik}(x+1) + \check{\lambda} Q_{-1/2+ik}(x+1). \tag{3.13}$$

Considering here the limit $x \to 0$, we see that $\check{\lambda} = 0$. Then we take the limit $x \to \infty$. It easily follows from (3.1) that the left-hand side of (3.13) equals

$$\begin{aligned} &2 \operatorname{Re} \Big(m(k) \int_0^\infty (x+y+2)^{-1} y^{-1/2+ik} dy \Big) + O(x^{-1}) \\ &\qquad = 2 \operatorname{Re} \Big(m(k) x^{-1/2+ik} \int_0^\infty (t+1)^{-1} t^{-1/2+ik} dt \Big) + O(x^{-1}), \end{aligned}$$

where we have set $y = xt$. Comparing this asymptotics with asymptotics (3.1) of the right-hand side of (3.13), we see that

$$\lambda = \int_0^\infty (t+1)^{-1} t^{-1/2+ik} dt = \pi (\cosh \pi k)^{-1} \tag{3.14}$$

and hence

$$\int_0^\infty (x+y+2)^{-1} P_{-1/2+ik}(y+1) dy = \pi (\cosh \pi k)^{-1} P_{-1/2+ik}(x+1). \tag{3.15}$$

It yields equation (2.10) with the operator U defined by formula (3.11). Since the operator U is unitary, we have recovered the result of F. Mehler [6].

Proposition 3.1. *The Hankel operator with kernel $a(x) = (x+2)^{-1}$ has the simple absolutely continuous spectrum coinciding with the interval $[0, \pi]$. Its normalized eigenfunction corresponding to the spectral parameter $\lambda = \pi(\cosh \pi k)^{-1}$ is given by formula* (1.2).

We emphasize that equation (3.15) has been obtained as a direct consequence of the commutator method, without any use of the theory of special functions.

4. Singularity at zero

In the first three subsections we study the Hankel operator with kernel (1.4) and in subs. 4 – with kernel (1.7). In both cases $a(x) \sim x^{-1}$ as $x \to 0$ and $a(x)$ decays exponentially as $x \to \infty$. The corresponding operator L is defined by formula (1.6) where $\gamma = 0$.

4.1. Note that in the case $\gamma = 0$, after a change of variables

$$\psi(x) = x^{-1} \varphi(x)$$

[2] We are obliged to choose regular boundary conditions at zero since the function $Q_{-1/2+ik}(x+1)$ does not satisfy the second boundary condition (2.1)

in (2.7), we get again (cf. equation (2.6)) the Schrödinger equation

$$-\varphi''(x) + (\alpha + \beta x^{-1} - \mu x^{-2})\varphi(x) = 0 \tag{4.1}$$

with the Coulomb potential but with a non-zero orbital term. Below we set $\alpha = 1/4$.

Recall that the Whittaker function $W_{-\beta,p}(x)$ can be defined as the solution of equation (4.1) for $\mu = 1/4 - p^2$ such that

$$W_{-\beta,p}(x) = x^{-\beta} e^{-x/2}(1 + O(x^{-1})) \tag{4.2}$$

as $x \to \infty$. Of course, $W_{-\beta,-p}(x) = W_{-\beta,p}(x)$. In particular, the function $b(x) = W_{-\beta,1/2}(x)$ satisfies equation (2.6) (where $\alpha = 1/4$).

As far as asymptotics as $x \to 0$ are concerned (see §6.8 of [3]), we note that

$$W_{-\beta,ik}(x) = m(k)x^{1/2+ik} + \overline{m(k)}x^{1/2-ik} + O(x^{3/2}), \quad k > 0, \quad x \to 0, \tag{4.3}$$

where

$$m(k) = \Gamma(-2ik)\Gamma^{-1}(1/2 - ik + \beta). \tag{4.4}$$

If $p \geq 0$ and $-1/2 + p + \beta \neq -1, -2, \ldots$, we have as $x \to 0$

$$\begin{aligned} &W_{-\beta,p}(x) \sim \Gamma(2p)\Gamma(1/2 + p + \beta)^{-1} x^{1/2-p}, \quad p > 0, \\ &W_{-\beta,0}(x) \sim -\Gamma(1/2 + \beta)x^{1/2} \ln x. \end{aligned} \tag{4.5}$$

If $-1/2 + p + \beta = -n$ where $n = 1, 2, \ldots$, then taking into account formulas (6.9.4) and (6.9.36) of [3], we can express the Whittaker functions in terms of the Laguerre polynomials:

$$W_{-\beta,p}(x) = (-1)^{n-1}(n-1)! e^{-x/2} x^{p+1/2} L_{n-1}^{2p}(x). \tag{4.6}$$

If $\gamma = 0$ and $\alpha = 1/4$, then

$$L = -\frac{d}{dx} x^2 \frac{d}{dx} + x^2/4 + \beta x. \tag{4.7}$$

We emphasize that the coefficient β may be arbitrary. It turns out that the strong degeneracy of the function x^2 at $x = 0$ gives rise to a branch of the absolutely continuous spectrum of the operator L.

First, let us define L as a self-adjoint operator in the space $L^2(\mathbb{R}_+)$. We will check that the operator L is essentially self-adjoint on the domain $C_0^\infty(\mathbb{R}_+)$. Let[3] $(F\tilde{f})(x) = x^{-1/2}\tilde{f}(\ln x)$. Then the transformation $F : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$ is unitary and the operator $\widetilde{L} = F^{-1}LF$ acts by formula (3.6) where $\tilde{q}(t) = e^{2t}/4 + \beta e^t$. This is already a standard Sturm-Liouville operator in the space $\widetilde{\mathcal{H}} = L^2(\mathbb{R})$. The potential $\tilde{q}(t)$ tends to 0 as $t \to -\infty$ and to $+\infty$ as $t \to +\infty$. In particular, $\widetilde{L}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ which implies that L is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$ in the space $L^2(\mathbb{R}_+)$. Thus, a boundary condition at the point $x = 0$ is unnecessary. Since $\tilde{q}(t) \to \infty$ as $t \to \infty$, a quantum particle can evade to $-\infty$ only. This ensures that the spectrum of the operator $\widetilde{L}$ is simple.

4.2. The expansion over eigenfunctions of the operator $\widetilde{L}$ can be performed by the following standard procedure (see, e.g., [13], §5.4). As we will see later, in addition to the simple absolutely continuous spectrum which coincides with $[1/4, \infty)$, for $\beta < -1/2$ the operator $\widetilde{L}$ has a finite number of simple eigenvalues $\mu_1, \ldots, \mu_N$, $N = N(\beta)$, lying below the point 1/4. We denote by $\widetilde{\mathcal{H}}^{(p)}$ the subspace

[3] Note that the integral in (3.4) diverges for $p(x) = x^2$ and hence the definition of the operator F should be changed.

spanned by the corresponding eigenfunctions. Let $\tilde{u}_k(t)$, $k>0$, be a real-valued solution of equation (3.8) belonging to $L^2(\mathbb{R}_+)$. It has asymptotics (3.9) as $t\to-\infty$ with a function $m(k)$ which will be calculated later. Then the operator $\widetilde{U}:\widetilde{\mathcal{H}}\to L^2(\mathbb{R}_+)$ defined by the equation (cf. (3.10))

$$(\widetilde{U}\tilde{f})(k)=(2\pi)^{-1/2}|m(k)|^{-1}\int_{-\infty}^{\infty}\tilde{u}_k(t)\tilde{f}(t)dt, \tag{4.8}$$

is bounded, $\widetilde{U}\big|_{\widetilde{\mathcal{H}}^{(p)}}=0$, the mapping $\widetilde{U}:\widetilde{\mathcal{H}}\ominus\widetilde{\mathcal{H}}^{(p)}\to L^2(\mathbb{R}_+)$ is unitary and equation

$$(\widetilde{U}\widetilde{L}\tilde{f})(k)=(k^2+1/4)(\widetilde{U}\tilde{f})(k)$$

holds.

The functions $u_k(x)=x^{-1/2}\tilde{u}_k(\ln x)$, $k>0$, satisfy the equation

$$-(x^2u_k'(x))'+4^{-1}x^2u_k(x)+\beta xu_k(x)=(k^2+1/4)u_k(x) \tag{4.9}$$

and can be expressed in terms of Whittaker functions:

$$u_k(x)=x^{-1}W_{-\beta,ik}(x). \tag{4.10}$$

It follows from (4.3) that the function $\tilde{u}_k(t)=e^{t/2}u_k(e^t)$ has as $t\to-\infty$ asymptotics (3.9) with the function $m(k)$ defined by (4.4). Calculating $|m(k)|$ and making in (4.4) the change of variables $t=\ln x$, we find that the operator $U=F\widetilde{U}F^{-1}$ is given by the equation

$$(Uf)(k)=\pi^{-1}\sqrt{k\sinh 2\pi k}|\Gamma(1/2-ik+\beta)|\int_0^{\infty}x^{-1}W_{-\beta,ik}(x)f(x)dx.$$

It is bounded, $U\big|_{\mathcal{H}^{(p)}}=0$, the mapping $U:\mathcal{H}\ominus\mathcal{H}^{(p)}\to L^2(\mathbb{R}_+)$ is unitary and equation (3.12) holds. Here $\mathcal{H}^{(p)}$ is the subspace spanned by the eigenfunctions $\psi_1,\dots,\psi_N$ of the operator L.

Let us calculate these functions. The function $u_p(x)=x^{-1}W_{-\beta,p}(x)$ for $p\geq 0$ satisfies equation (4.9) where the role of k^2 is played by $-p^2$. In view of (4.2) it belongs to L^2 at infinity. However, it follows from asymptotics (4.5) that it does not belong to L^2 in a neighborhood of the point $x=0$ unless $-1/2+p+\beta=-n$ where $n=1,2,\dots$. Moreover, in view of (4.6) for all $\beta=-1/2$ the function $u_0\notin L^2$. Thus, if $\beta\geq-1/2$, the operator L is purely absolutely continuous. If $\beta<-1/2$, it also has the eigenvalues $\mu_n=1/4-(|\beta|+1/2-n)^2$ where $n=1,2,\dots$ and $n<|\beta|+1/2$ (then $p>0$). According to formula (4.6) the corresponding eigenfunctions equal

$$\psi_n(x)=e^{-x/2}x^{p-1/2}L_{n-1}^{2p}(x),\quad p=|\beta|+1/2-n. \tag{4.11}$$

4.3. Now we return to the Hankel operator A with kernel (1.4). It follows from (4.2) that $a(x)$ exponentially decays as $x\to\infty$, and it follows from the first formula (4.5) for $p=1/2$ that $a(x)\sim x^{-1}$ as $x\to 0$. Observe that in view of asymptotics (4.2) and (4.3), function (4.10) satisfies both boundary conditions (2.1) and (2.2). Hence it follows from Theorem 2.1 that

$$\int_0^{\infty}a(x+y)y^{-1}W_{-\beta,ik}(y)dy=\lambda(k)x^{-1}W_{-\beta,ik}(x)+\check{\lambda}(k)x^{-1}M_{-\beta,ik}(x) \tag{4.12}$$

where the Whittaker function $M_{-\beta,ik}$ is the solution of equation (2.6) exponentially growing as $x\to\infty$. Therefore considering the limit $x\to\infty$ in (4.12), we see that necessarily $\check{\lambda}(k) = 0$. Then we take the limit $x \to 0$ and use asymptotics (4.3). Since $a(x) \sim x^{-1}$ as $x \to 0$, we have

$$\begin{aligned}\int_0^\infty a(x+y)y^{-1}W_{-\beta,ik}(y)dy =& 2\operatorname{Re}\Big(m(k)\int_0^\infty (x+y)^{-1}y^{-1/2+ik}dy\Big) + O(x^{1/2})\\ =& 2\lambda(k)\operatorname{Re}\big(m(k)x^{-1/2+ik}\big) + O(x^{1/2})\end{aligned}$$

where $\lambda(k)$ is again given by formula (3.14). This yields equation (1.8).

It remains to calculate eigenvalues $\lambda_1, \ldots, \lambda_N$ of the operator A. The corresponding eigenfunctions are given by formula (4.11). We proceed again from equation (4.12) where the role of ik is played by $p = |\beta| + 1/2 - n$. As before considering the limit $x \to \infty$, we see that $\check{\lambda}_n = 0$ and hence

$$\int_0^\infty a(x+y)e^{-y/2}y^{p-1/2}L_{n-1}^{2p}(y)dy = \lambda_n e^{-x/2}x^{p-1/2}L_{n-1}^{2p}(x). \tag{4.13}$$

It follows from (1.4) and (4.2) that the left-hand side here equals

$$\Gamma(1+\beta)x^{-\beta-1}e^{-x/2}\int_0^\infty e^{-y}y^{p-1/2}L_{n-1}^{2p}(y)dy\,\big(1+O(x^{-1})\big), \quad x \to \infty.$$

Putting together formulas (2.8.46) and (10.12.33) of [3], we see that

$$(n-1)!\int_0^\infty e^{-y}y^{p-1/2}L_{n-1}^{2p}(y)dy = \Gamma(p+n-1/2). \tag{4.14}$$

Recall also that $L_{n-1}^{2p}(x)$ is a polynomial of degree $n-1$ with the coefficient $(-1)^{n-1}/(n-1)!$ at x^{n-1}. Hence it follows from relation (4.13) that

$$\lambda_n = (-1)^n\pi/\sin\pi\beta, \quad n = 1, 2, \ldots, \quad n < |\beta| + 1/2. \tag{4.15}$$

Since eigenfunctions of the operator L are orthogonal and complete, we have recovered the result of M. Rosenblum [9].

Proposition 4.1. *The Hankel operator A with kernel* (1.4) *has the simple absolutely continuous spectrum coinciding with the interval* $[0, \pi]$. *Its normalized eigenfunction corresponding to a point $\lambda = \pi(\cosh \pi k)^{-1}$ from the continuous spectrum is given by the formula*

$$\psi_k(x) = \pi^{-1}\sqrt{k\sinh 2\pi k}|\Gamma(1/2 - ik + \beta)|\, x^{-1}W_{-\beta,ik}(x), \quad k > 0.$$

Moreover, if $\beta < -1/2$, then the operator A has eigenvalues (4.15) *with the corresponding eigenfunctions defined by* (4.11).

4.4. Next, we turn to the Hankel operator with singular kernel (1.7) which probably was not considered in the literature. Recall that the MacDonald function is defined by the relation $K_p(z) = 2^{-1}ie^{\pi ip/2}H_p^{(1)}(iz)$ where $H_p^{(1)}$ is the Hankel function. Now the function $b(x) = xa(x)$ satisfies the Schrödinger equation (2.3) for the zero energy $\alpha = 0$ and the coupling constant $\beta = 2$. Of course, we could have taken arbitrary $\beta > 0$, but we have to exclude negative β since in this case the function $b(x)$ grows as $x \to \infty$.

It follows from the well-known properties of $H_p^{(1)}$ that function (1.7) has asymptotics

$$a(x) = 4\pi^{1/2}x^{-3/4}e^{-\sqrt{8x}}(1 + O(x^{-1/2})) \tag{4.16}$$

as $x \to \infty$ and $a(x) \sim x^{-1}$ as $x \to 0$.

The corresponding operator

$$L = -\frac{d}{dx}x^2\frac{d}{dx} + 2x$$

can be studied quite similarly to operator (4.7). For example, the operator $\widetilde{L} = F^{-1}LF$ acts by formula (3.6) where $\tilde{q}(t) = 2e^t$. A solution of equation (2.7) where $\mu = k^2 + 1/4$ belonging to L^2 at infinity can be expressed again in terms of the MacDonald function

$$u_k(x) = x^{-1/2}K_{2ik}(\sqrt{8x}).$$

According to formulas (7.2.12) and (7.2.13) of [3] we have

$$u_k(x) = m(k)x^{-1/2+ik} + \overline{m(k)}x^{-1/2-ik} + O(x^{1/2}), \quad x \to 0,$$

where

$$m(k) = i\pi 2^{-1+ik}\big(\Gamma(1+2ik)\sinh 2\pi k\big)^{-1}.$$

Calculating $|m(k)|$ and using (4.8), we see that formula (1.3) now looks as

$$(Uf)(k) = 2\pi^{-1}\sqrt{k\sinh 2\pi k}\int_0^\infty x^{-1/2}K_{2ik}(\sqrt{8x})f(x)dx. \tag{4.17}$$

The operator L does not have eigenvalues because the functions $x^{-1/2}K_{2p}(\sqrt{8x})$ for $p \geq 0$ do not belong to L^2 in a neighborhood of the point $x = 0$. Thus, similarly to subs. 4.2, we see that the operator U defined by formula (4.17) is unitary in the space $L^2(\mathbb{R}_+)$ and the operator L has the simple absolutely continuous spectrum $[1/4, \infty)$.

Theorem 2.1 implies that

$$\int_0^\infty a(x+y)y^{-1/2}K_{2ik}(\sqrt{8y})dy = \lambda(k)x^{-1/2}K_{2ik}(\sqrt{8x}) + \check{\lambda}(k)x^{-1/2}H^{(2)}_{2ik}(i\sqrt{8x}) \tag{4.18}$$

(the Hankel function $H^{(2)}_{2ik}(iz)$ exponentially increases as $z \to \infty$) for some constants $\lambda(k)$ and $\check{\lambda}(k)$. Since the integral in (4.18) (exponentially) decays as $x \to \infty$, necessarily $\check{\lambda}(k) = 0$. Comparing the asymptotics of the left- and right-hand sides of (4.18) as $x \to 0$ and using that $a(x) \sim x^{-1}$ as $x \to 0$, we find that the constant $\lambda(k)$ is again given by formula (3.14). Thus, similarly to the previous subsection, we obtain

Proposition 4.2. *The Hankel operator A with kernel* (1.7) *has the simple absolutely continuous spectrum coinciding with the interval* $[0, \pi]$. *Its normalized eigenfunction corresponding to a spectral point $\lambda = \pi(\cosh \pi k)^{-1}$ is given by the formula*

$$\psi_k(x) = 2\pi^{-1}\sqrt{k\sinh 2\pi k}\,x^{-1/2}K_{2ik}(\sqrt{8x}), \quad k > 0.$$

As a by-product of our considerations, we obtain the equation

$$x^{1/2}\int_0^\infty (x+y)^{-1/2}K_1(\sqrt{x+y})y^{-1/2}K_{2ik}(\sqrt{y})dy = \pi(\cosh\pi k)^{-1}K_{2ik}(\sqrt{x}).$$

We have not found this equation in the literature on special functions. Note, however, that it can formally be deduced from the Shanker equation (1.8) if one uses the relation (formula (6.9.19) of [3])

$$\lim_{\beta\to\infty}\Gamma(\beta+1)W_{-\beta,m}(x/\beta) = 2x^{1/2}K_{2m}(2x^{1/2}).$$

4.5. The Carleman operator A trivially fits into the scheme exposed above. Now the operator A commutes with operator (1.6) for $\alpha = \beta = \gamma = 0$. This operator has the absolutely continuous spectrum of multiplicity 2 coinciding with $[1/4, \infty)$. It has eigenfunctions $x^{-1/2+ik}$ for all $k \in \mathbb{R}$ which are also eigenfunctions of the operator A. The relation between the spectral parameters λ and k is again given by formula (3.14) so that the operator A has the absolutely continuous spectrum of multiplicity 2 coinciding with $[0, \pi]$.

5. Regular kernels

5.1. Let us here consider kernels $a(x)$ which decay rapidly as $x \to \infty$ and have finite limits as $x \to 0$. We set $\gamma = 2$ and distinguish the cases $\alpha > 0$, β is arbitrary and $\alpha = 0$, $\beta > 0$. Let $\alpha = 1/4$ and $\beta = 2$ in the first and second cases, respectively. If $\alpha = 1/4$, then the solution of equation (2.6) is given (see subs. 4.1) by the formula $b(r) = W_{-\beta,1/2}(r)$ where $W_{-\beta,1/2}$ is the Whittaker function. If $\alpha = 0$ and $\beta = 2$, then the solution of (2.6) equals $b(r) = r^{1/2} K_1(\sqrt{8r})$ where K_1 is the MacDonald function (see subs. 4.4). The corresponding functions (2.5) decay exponentially at infinity and have finite limits as $x \to 0$. It follows that the operators A are compact.

Let the function $\omega(x)$ be defined by formula (3.5), $\eta = \omega^{-1}$ and

$$\tilde{q}(t) = -4^{-1}\big(\eta^2(t) + 2\eta(t)\big)^{-1} + \alpha\eta^2(t) + \beta\eta(t). \tag{5.1}$$

Since $\eta(t) \sim e^t/2$, the potential $\tilde{q}(t) \to +\infty$ as $t \to \infty$. It follows that the operators $\widetilde{L}$ and hence L have now discrete spectra. We point out that these operators are again defined by formulas (3.6) and (1.6) on functions satisfying boundary conditions (3.7) and (3.3), respectively.

Theorem 2.1 implies that the operators A and L have common eigenfunctions. Apparently, eigenfunctions of the operator L cannot be expressed in terms of standard special functions. However, in their terms we can calculate eigenvalues $\lambda_1, \lambda_2, \dots$ of the operator A. Indeed, suppose that $\psi_\mu \in \mathcal{D}(L)$ and $L\psi_\mu = \mu\psi_\mu$. Then the function $\tilde{\psi}_\mu(t) = (\mathsf{F}^{-1}\psi_\mu)(t)$ satisfies the equation

$$\tilde{\psi}_\mu''(t) + \tilde{q}(t)\tilde{\psi}_\mu(t) = (\mu - 1/4)\tilde{\psi}_\mu(t)$$

where $\tilde{q}(t)$ is function (5.1). For a suitable normalization, asymptotics of $\tilde{\psi}_\mu(t)$ as $t \to \infty$ is given (see, e.g., book [7]) by the semiclassical formula

$$\tilde{\psi}_\mu(t) \sim \tilde{q}(t)^{-1/4} \exp\Big(- \int_0^t \tilde{q}(s)^{1/2} ds\Big).$$

It follows that $\tilde{\psi}_\mu(t) \sim e^{-t/2} \exp(-4^{-1}e^t - \beta t)$ in the first case and $\tilde{\psi}_\mu(t) \sim e^{-t/4} \exp(-2^{3/2} e^{t/2})$ in the second case. Returning to the eigenfuctions $\psi_\mu(x)$, we find that

$$\psi_\mu(x) \sim x^{-1-\beta} e^{-x/2} \quad \text{and} \quad \psi_\mu(x) \sim x^{-3/4} e^{-\sqrt{8x}}, \quad x \to \infty, \tag{5.2}$$

in the first and second cases, respectively.

On the other hand, using asymptotics (4.2) and (4.16) for function (1.7), we see that

$$(A\psi_\mu)(x) \sim x^{-1-\beta} e^{-x/2} \int_0^\infty e^{-y/2} \psi_\mu(y) dy$$

and

$$(A\psi_\mu)(x) \sim \sqrt{2\pi} x^{-3/4} e^{-\sqrt{8x}} \int_0^\infty \psi_\mu(y) dy$$

as $x \to \infty$ in the first and second cases, respectively. Comparing these relations with relations (5.2) and using the equation $A\psi_\mu = \lambda_\mu \psi_\mu$, we get expressions for eigenvalues of the operators A:

$$\lambda_\mu = \int_0^\infty e^{-y/2}\psi_\mu(y)dy \quad \text{and} \quad \lambda_\mu = \sqrt{2\pi}\int_0^\infty \psi_\mu(y)dy \tag{5.3}$$

in the first and second cases, respectively.

Thus we have obtained the following results.

Proposition 5.1. *The Hankel operator A with kernel*

$$a(x) = (x+2)^{-1}W_{-\beta,1/2}(x+2)$$

and the differential operator (1.6) *for $\gamma = 2$ and $\alpha = 1/4$ have common eigenfunctions. If $\psi_\mu \in \mathcal{D}(L)$, $L\psi_\mu = \mu\psi_\mu$ and $\psi_\mu(x)$ has the first asymptotics* (5.2) *as $x \to \infty$, then $A\psi_\mu = \lambda_\mu\psi_\mu$ where λ_μ is determined by the first formula* (5.3)*.*

Proposition 5.2. *The Hankel operator A with kernel*

$$a(x) = (x+2)^{-1/2}K_1(\sqrt{8(x+2)})$$

and the differential operator (1.6) *for $\gamma = 2$, $\alpha = 0$ and $\beta = 2$ have common eigenfunctions. If $\psi_\mu \in \mathcal{D}(L)$, $L\psi_\mu = \mu\psi_\mu$ and $\psi_\mu(x)$ has the second asymptotics* (5.2) *as $x \to \infty$, then $A\psi_\mu = \lambda_\mu\psi_\mu$ where λ_μ is determined by the first second formula* (5.3)*.*

5.2. Finally, we consider kernel (1.4) for exceptional values $\beta = -l$ where $l = 1, 2, \ldots$. To be more precise, we now set

$$a(x) = (-1)^{l-1}(l-1)!^{-1}x^{-1}W_{l,1/2}(x) = e^{-x/2}L^1_{l-1}(x), \quad l = 1, 2, \ldots, \tag{5.4}$$

(here we have taken formula (4.6) into account). The Hankel operator A with this kernel has rank l. Here we show how this simple example fits into the scheme exposed above.

The spectral analysis of the corresponding operator (4.7) remains the same as in subs. 4.2. In addition to the absolutely continuous spectrum $[1/4, \infty)$, the operator L has eigenvalues $\mu_n = 1/4 - (l + 1/2 - n)^2$ where $n = 1, \ldots, l$.

However, instead of the absolutely continuous spectrum, the operator A has the zero eigenvalue of infinite multiplicity. Indeed, as in subs. 4.3, Theorem 2.1 yields equation (4.12) where again $\check\lambda(k) = 0$. Observe that $a(x)$ and hence in view of (4.3) the integral in the left-hand side of (4.12) have finite limits as $x \to 0$. Therefore it follows from (4.3) that necessarily $\lambda(k) = 0$ for all $k > 0$. Hence the kernel of the operator A is spanned by the functions $x^{-1}W_{l,ik}(x)$, $k > 0$.

Eigenfunctions $\psi_n(x)$ corresponding to non-zero eigenvalues λ_n of the operator A are defined by formula (4.11) where $p = l + 1/2 - n$, $n < l + 1/2$, and λ_n can be found from equation (4.13):

$$\int_0^\infty L^1_{l-1}(x+y)e^{-y}y^{p-1/2}L^{2p}_{n-1}(y)dy = \lambda_n x^{p-1/2}L^{2p}_{n-1}(x). \tag{5.5}$$

Recall that $L^\alpha_p(x)$ is a polynomial of degree p with the coefficient $(-1)^p/p!$ at x^p. Comparing coefficients at the highest power x^{l-1} in the left- and right-hand sides of (5.5) and taking into account formula (4.14), we find that

$$\lambda_n = (-1)^{n-l}\frac{(n-1)!}{(l-1)!}\int_0^\infty e^{-y}y^{p-1/2}L^{2p}_{n-1}(y)dy = (-1)^{n-l}.$$

Thus, we have obtained the following result.

Proposition 5.3. *The Hankel operator A with kernel* (5.4) *has rank l. Its non-zero eigenvalues are given by the formula $\lambda_n = (-1)^{n-l}$ where $n = 1, \dots, l$, and the corresponding eigenfunctions $\psi_n(x)$ are defined by equality* (4.11) *where $p = l + 1/2 - n$.*

References

[1] N. I. Akhieser and I. M. Glasman, *The theory of linear operators in Hilbert space*, vols. I, II, Ungar, New York, 1961.
[2] T. Carleman, *Sur les équations intégrales singulières à noyau réel et symetrique*, Almqvist and Wiksell, 1923.
[3] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions.* Vol. 1, 2, McGraw-Hill, New York-Toronto-London, 1953.
[4] J. S. Howland, *Spectral theory of operators of Hankel type.* I, II, Indiana Univ. Math. J. **41** (1992), no. 2, 409–426 and 427–434.
[5] W. Magnus, *On the spectrum of Hilbert's matrix*, Amer. J. Math. **72** (1950), 405-412.
[6] F. G. Mehler, Math. Ann. **18** (1881), 161-194.
[7] F. W. J. Olver, *Asymptotics and special functions*, Academic Press, 1974.
[8] V. V. Peller, *Hankel operators and their applications*, Springer Verlag, 2002.
[9] M. Rosenblum, *On the Hilbert matrix*, I, II, Proc. Amer. Math. Soc. **9** (1958), 137-140, 581-585.
[10] H. Shanker, *An integral equation for Whittaker's confluent hypergeometric function*, Proc. Cambridge Philos. Soc. **45** (1949), 482-483.
[11] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Vol. 1, Oxford, 1946.
[12] N. Ya. Vilenkin, *Special functions and the theory of representations of groups*, Nauka, Moscow, 1965. (Russian)
[13] D. R. Yafaev, *Mathematical scattering theory. Analytic theory*, American Mathematical Society, Providence, RI, 2010.

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