Constructing graphs over $\mathbb{R}^n$ with small prescribed mean-curvature

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Abstract

In this paper a convergent series expansion is constructed to solve the prescribed mean curvature equation $\pm\nabla\cdot(\nabla u/\sqrt{1\pm|\nabla u|^2}) = nH$ for n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ ($+$ sign) and $\mathbb{R}^{1,n}$ ($-$ sign) which are graphs $\{(x,u(x)) : x\in\mathbb{R}^n\}$ of a smooth function $u:\mathbb{R}^n\to\mathbb{R}$, and whose mean curvature function $H\in C^\alpha_0(\mathbb{R}^n,\mathbb{R})$ is not too large in norm and integrable. Our approach is inspired by the Maxwell-Born-Infeld theory of electromagnetism in $\mathbb{R}^{1,3}$, for which our method yields the first systematic way of explicitly computing the electrostatic potential u for regular charge densities $\rho\propto H$ and small Born parameter. Therefore, after the general n-dimensional problems have been treated with the help of nonlinear Hodge theory and Banach algebra estimates, our approach is reworked in more detail for $n=3$.

1 Introduction

In this paper we develop a constructive approach to the prescribed mean-curvature problem for hypersurfaces which are graphs of some scalar function u over the entire $\mathbb{R}^n$. While we use only standard tools from the theory of elliptic PDE, we believe that we combine these in a novel efficient way to find solutions to this so-called non-parametric prescribed mean-curvature problem. The scalar second-order problem for u is reformulated in terms of an equivalent first-order Hodge problem. It is then shown that for suitably small mean curvature functions the Hodge problem admits a convergent power series expansions which can be solved term-by-term through explicit quadratures. The convergence of this series in a Banach algebra can be proven with the help of complex analysis.

Our approach grew out of our attempts to solve the nonlinear Maxwell-Born-Infeld equations of the electromagnetic fields in $\mathbb{R}^{1,3}$, i.e. when $n = 3$. In the electrostatic limit, recalled in Appendix A, the electrostatic potential is proportional to the function u, and the charge density is proportional to the mean-curvature function. Thus our method also provides a systematic way of finding non-special electrostatic solutions to the Maxwell–Born–Infeld field equations when the charge density is suitably small. However, the smallness condition for H misses the important case of electrostatic fields generated by point charges [6, 7, 28, 23, 24, 22, 12], corresponding to maximal hypersurfaces with conical singularities [17].

In the next section we begin our quest by stating the scalar prescribed mean-curvature problem for graphs over $\mathbb{R}^n$ in both Euclidean and Minkowskian geometry. In section 3, we write down a formal power series solution to the scalar problem(s) for integrable prescribed curvature functions which are small in a C^α_0 norm; the rest of our paper is concerned with studying the convergence of the formal power series solution in all dimensions $n \in \mathbb{N}$. In section 4 we use nonlinear Hodge theory to first reformulate the scalar second-order PDE problem into a first-order system for two 1-forms, then state the corresponding formal power series solution for these nonlinear Hodge systems when the prescribed curvature function is small, and finally prove its convergence using Banach algebra arguments and a small amount of elementary complex analysis. Because of its special relevance to the mathematical theory of electromagnetism we also reformulate the three-dimensional problems in terms of equivalent vector problems, and also present some additional details; this is done in section 5. In section 6 we indicate some generalizations of our approach to related quasi-linear PDE problems in divergence form.

2 The scalar prescribed mean-curvature equation

The mean curvature function H of a hypersurface in $\mathbb{R}^{n+1}$ $(n = 1, 2, 3, ...)$ which is the graph $\{(x, u(x)) : x \in \mathbb{R}^n\}$ of a real-valued C^2 function u can be computed from u as

$$H = n^{-1}\nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \tag{1}$$

(see, e.g., [29], and the appendix to section 14 in [20]), while the analogous result for space-like hypersurfaces in Minkowski space $\mathbb{R}^{1,n}$ reads

$$H = -n^{-1}\nabla \cdot \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\,, \tag{2}$$

(see, e.g., [1, 3, 21, 25]). The inverse problem is to find a function u, satisfying certain asymptotic conditions, whose graph over $\mathbb{R}^n$ describes a hypersurface in $\mathbb{R}^{n+1}$, respectively $\mathbb{R}^{1,n}$, for which a prescribed function $H : \mathbb{R}^n \to \mathbb{R}$ is its mean curvature. In that case (1) and (2) become second-order quasi-linear elliptic PDEs for the unknown u, which we lump together as

$$\pm\nabla \cdot \frac{\nabla u}{\sqrt{1 \pm |\nabla u|^2}} = nH, \tag{3}$$

the upper sign for the Euclidean, the lower for the Minkowskian setting.

Regarding the problem of hypersurfaces which are graphs over $\mathbb{R}^n$, we note that the scalar prescribed mean-curvature equation can be solved in great generality when $n = 1$ (by simple quadratures, obviously). And for $n = 2$ powerful complex variable techniques can be put to work, see [15, 28, 21, 25, 2, 33, 26, 16], in particular when $H \equiv 0$. However, for $n \geq 3$ no general solution method exists. While one finds many special results and general (non-)existence and regularity results, there is not much in terms of explicitly constructing non-special solutions — except when H is small. For small H one can in principle compute solutions by running the iterations of the fixed point maps of the Schauder theory, or use the familiar variational methods.

In the remaining sections we develop an alternative, non-variational, more direct constructive PDE approach which produces entire graphs in all dimensions $n \in \mathbb{N}$ for quite general mean curvature functions H as long as they are small. Our approach treats the Euclidean and Minkowskian problems on an equal footing.

While we here are interested in entire solutions over $\mathbb{R}^n$ motivated by the electromagnetic Born–Infeld model, and also by some problems in the theory of spacetime structure (e.g. [1, 3, 18, 35]), with some mild modifications our approach should be adaptable to the prescribed mean-curvature equation in bounded domains with small Dirichlet data for u (e.g. [29, 5, 36, 34]); this then also includes the minimal / maximal hypersurface problem ($H \equiv 0$) with small Dirichlet data (e.g. [15, 26, 27, 16, 33]). The parametric prescribed mean-curvature problem (e.g. [37, 10, 11, 33, 39, 13]), which also captures embedded hypersurfaces which are not graphs over $\mathbb{R}^n$ and even non-embedded hypersurfaces, has a different structure, however.

3 Formal solution for small prescribed mean-curvature

Now let $H \in C_0^\alpha \cap L^1$ be small in the following sense: $H(x) = \epsilon H_0(x)$, where $H_0(x)$ is a fixed function, and $\epsilon \in \mathbb{R}_+$ can be chosen as small as we please. Then one can expand the scalar prescribed mean-curvature equation (3) into a hierarchy of (linear) Poisson equations all of which can be solved explicitly in the usual way. The problem is to prove the convergence of the so-obtained formal power series (in ϵ) solution.

Thus, for small but finite ϵ the analyticity properties of the nonlinearity of the scalar prescribed mean-curvature equation suggest to seek a solution u in form of the power series Ansatz

$$u(x) = \sum_{p=1}^{\infty} \epsilon^p u^{(p)}(x), \tag{4}$$

with each $u^{(p)}$ independent of ϵ. Inserting this Ansatz into the equation (3) and sorting according to powers of ϵ, we find a hierarchy of linear Poisson equations.

In particular, we find recursively that for each even p, i.e. $p = 2k$ for $k \in \mathbb{N}$, $u^{(p=2k)}$ satisfies

$$\Delta u^{(2k)} = 0, \tag{5}$$

so that $u^{(2k)}$ for general $k \in \mathbb{N}$ is an entire harmonic function. We also find that $u^{(1)}$ satisfies

$$\pm \Delta u^{(1)} = n H_0, \tag{6}$$

which is solved by $u^{(1)} \in C_0^{2,\alpha}$ given by

$$u^{(1)}(x) = \mp \tfrac{n}{|S^{n-1}|} \int \frac{H_0(y)}{|x-y|^{n-2}} \, \mathrm{d}^n y + h^{(1)}(x) \tag{7}$$

when $n \neq 2$, and by

$$u^{(1)}(x) = \mp\tfrac{2}{|S^1|} \int H_0(y) \ln \frac{1}{|x-y|}\, \mathrm{d}^2 y + h^{(1)}(x) \tag{8}$$

when $n = 2$. In (7) and (8), $h^{(1)}$ is an arbitrary entire harmonic function. Carrying on we also find recursively that for odd $p > 1$, i.e. $p = 2k+1$ for $k \in \mathbb{N}$, the functions $u^{(2k+1)}$ satisfy the linear Poisson PDE

$$\pm\Delta u^{(2k+1)} = \nabla \cdot U^{(2k+1)}, \tag{9}$$

where $U^{(2k+1)}$ is a polynomial in the $\nabla u^{(\ell)}$ with odd $\ell < 2k+1$, viz.

$$U^{(2k+1)} = -\sum_{h=1}^{k} \nabla u^{(2(k-h)+1)} \sum_{j=1}^{h} M_j^{\pm} \sum_{\substack{|\ell|_{2j}=\\ h-j}} \prod_{i=1}^{j} \nabla u^{(2\ell_{2i-1}+1)} \cdot \nabla u^{(2\ell_{2i}+1)}, \tag{10}$$

where $|\ell|_K = \sum\limits_{i=1}^{K} \ell_i$, the ℓ_i take any non-negative integer values, and where

$$M_j^{\mp} = (\pm 1)^j \frac{(2j-1)!!}{j!2^j} \tag{11}$$

is the j-th Maclaurin coefficient of $1/\sqrt{1 \mp z}$ (with $M_0^{\mp} := 1$). Assuming enough regularity of $U^{(2k+1)}$, the general regular solution to the linear Poisson PDE (9) is given by

$$u^{(2k+1)}(x) = \mp\tfrac{1}{|S^{n-1}|} \int \frac{\nabla \cdot U^{(2k+1)}(y)}{|x-y|^{n-2}}\, \mathrm{d}^n y + h^{(2k+1)}(x), \quad k \in \mathbb{N} \tag{12}$$

when $n \neq 2$, and by

$$u^{(2k+1)}(x) = \mp\tfrac{1}{|S^1|} \int \nabla \cdot U^{(2k+1)}(y) \ln \frac{1}{|x-y|}\, \mathrm{d}^2 y + h^{(2k+1)}(x), \quad k \in \mathbb{N} \tag{13}$$

when $n = 2$; here, $h^{(2k+1)}$ is an arbitrary entire harmonic function.

To remove the ambiguity in the choices of harmonic functions $h^{(p)}$, in this paper we impose the asymptotic condition on u that

$$u^{(1)}(x) \asymp \mp\tfrac{n}{|S^{n-1}|} \int H_0(y) \mathrm{d}^n y \frac{1}{|x|^{n-2}} \tag{14}$$

when $n \neq 2$, and by

$$u^{(1)}(x) \asymp \mp \tfrac{2}{|S^1|} \int H_0(y) \mathrm{d}^2 y \ln \frac{1}{|x|} \tag{15}$$

when $n = 2$. As a consequence, we find that $h^{(p)} \equiv 0$ for all $p \in \mathbb{N}$.

It follows that $u = \sum_{k=0}^{\infty} \epsilon^{2k+1} u^{(2k+1)}$, with $u^{(1)}$ given by (7) and $u^{(2k+1)}$ by (12), recursively for $k \in \mathbb{N}$ when $n \neq 2$, respectively given by their two-dimensional analogs when $n = 2$, and with vanishing harmonic functions $h^{(p)} \equiv 0$ for all $p \in \mathbb{N}$, is a *formal* series solution for our u problem (3), with $H = \epsilon H_0$. To prove the convergence of this formal power series solution we next resort to nonlinear Hodge theory.

4 Equivalent Hodge problems

For all dimensions $n \in \mathbb{N}$ the second-order prescribed mean-curvature equation (3) for either sign is equivalent to a nonlinear Hodge system of first order for some 1-form. There are two mutually dual formulations.

4.1 The first-order Hodge systems

Consider a 1-form ω satisfying

$$d\omega = 0; \tag{16}$$

$$\pm \delta \frac{\omega}{\sqrt{1 \pm |\omega|^2}} = nH \,. \tag{17}$$

Then this first-order system is equivalent to the second-order equation (3) by identifying $\omega \equiv \nabla u \cdot \mathrm{d}x$.

Dual to the above system is the following. Defining a 1-form

$$\tau = \pm \frac{\omega}{\sqrt{1 \pm |\omega|^2}} \,, \tag{18}$$

which can be inverted to yield

$$\omega = \pm \frac{\tau}{\sqrt{1 \mp |\tau|^2}} \,, \tag{19}$$

we see that the above first-order equations are equivalent to

$$\delta\tau = nH\,, \tag{20}$$

$$d\frac{\tau}{\sqrt{1 \mp |\tau|^2}} = 0\,. \tag{21}$$

Clearly, (21) implies that there is scalar σ such that

$$\frac{\tau}{\sqrt{1 \mp |\tau|^2}} = d\sigma. \tag{22}$$

Up to a sign and an additive constant, $\sigma = u$, of course.

We end this subsection with the observation that we can also carry out the differentiation in (21) and multiply with $(1 \mp |\tau|^2)^{3/2}$ to find the cubic version of (21),

$$0 = \left(1 \mp |\tau|^2\right) d\tau \pm d\left[\tfrac{1}{2}|\tau|^2\right] \wedge \tau. \tag{23}$$

We shall come back to this apparently simplest formulation in our next-to-last section.

4.2 The small curvature τ-hierarchy

For small $H = \epsilon H_0$, with $H_0 \in C_0^\alpha \cap L^1$, we can recycle our series solution Ansatz for either one of the two dual first-order Hodge problems, and solving one leads also to the solution of the other. Yet we found it more convenient to work with the τ problem in its square root version. Thus we set

$$\tau = \sum_{p=1}^{\infty} \epsilon^p \tau^{(p)}, \tag{24}$$

with each $\tau^{(p)}$ independent of ϵ. Inserting this Ansatz into the pair of equations (20), (21), and sorting according to powers of ϵ, we find a hierarchy of linear equations.

In particular, $\tau^{(1)}$ satisfies

$$\delta\tau^{(1)} = nH_0\,, \tag{25}$$

together with

$$d\tau^{(1)} = 0. \tag{26}$$

Consistency with our asymptotic conditions for u requires that for $H_0 \in C_0^\alpha \cap L^1$ we select the unique solution $\tau^{(1)} \in C_b^{1,\alpha}$ of the pair of linear equations (25), (26), given by the exact 1-form

$$\tau^{(1)}(x) := -d \int \frac{\frac{n}{|S^{n-1}|} H_0(y)}{|x-y|^{n-2}} \, \mathrm{d}^n y \tag{27}$$

for $n \neq 2$; when $n = 2$ the logarithmic kernel has to be used. Note that for $n \geq 2$ we even have $\tau^{(1)} \in C_0^{1,\alpha}$.

Having the exact 1-form (27), we next find that $\tau^{(2)}$ satisfies

$$\delta \tau^{(2)} = 0 \tag{28}$$

together with

$$d\tau^{(2)} = 0, \tag{29}$$

and $\tau^{(2)} \to 0$ at infinity. Clearly, the pair of linear equations (28), (29) has a unique solution $\tau^{(2)}$ which tends $\to 0$ at infinity, given by the trivial solution $\tau^{(2)} \equiv 0$.

Carrying on we now find recursively that each $\tau^{(p)}$ for even p, i.e. $p = 2k$ for $k \in \mathbb{N}$, satisfies

$$\begin{aligned} \delta \tau^{(2k)} &= 0 && (30) \\ d\tau^{(2k)} &= 0, && (31) \end{aligned}$$

with $\tau^{(2k)} \to 0$ at infinity, so that $\tau^{(2k)} \equiv 0$ for general $k \in \mathbb{N}$, while for odd $p = 2k+1$ with $k \in \mathbb{N}$, we find the pair of linear first-order PDE

$$\begin{aligned} \delta \tau^{(2k+1)} &= 0 && (32) \\ d\tau^{(2k+1)} &= dT^{(2k+1)}, && (33) \end{aligned}$$

where $T^{(2k+1)}$ is a polynomial in the $\tau^{(\ell)}$ with $\ell < 2k+1$, viz.

$$T^{(2k+1)} = -\sum_{h=1}^{k} \tau^{(2(k-h)+1)} \sum_{j=1}^{h} M_j^{\mp} \sum_{|\ell|_{2j}=h-j} \prod_{i=1}^{j} \tau^{(2\ell_{2i-1}+1)} \cdot \tau^{(2\ell_{2i}+1)}, \tag{34}$$

where $|\ell|_{2j}$ and $M_j^{\mp}$ have their previous meaning. The pair of linear first-order PDE (32), (33) has a unique solution $\tau^{(2k+1)} \in C_b^{1,\alpha}$ given by

$$\tau^{(2k+1)} = \mathbf{P} T^{(2k+1)}, \qquad k \in \mathbb{N} \tag{35}$$

where $\mathbf{P}: C_b^{1,\alpha} \to C_b^{1,\alpha}$ projects onto the co-exact subspace of $C_b^{1,\alpha}$. Since $\delta T^{(2k+1)} \in C_0^{0,\alpha} \cap L^1$, we have explicitly

$$\mathbf{P}T^{(2k+1)}(x) = T^{(2k+1)}(x) + d \int \frac{\frac{1}{|S^{n-1}|}\delta T^{(2k+1)}(y)}{|x-y|^{n-2}} \, \mathrm{d}^n y, \tag{36}$$

for $n \neq 2$; when $n = 2$ the logarithmic kernel has to be used. Note that $n > 1$, then we even have $\tau^{(2k+1)} \in C_0^{1,\alpha}$.

Thus we have *formally* solved our τ problem (20), (21) with $H = \epsilon H_0$ in terms of the formal series solution $\tau = \sum_{k=0}^{\infty} \epsilon^{2k+1}\tau^{(2k+1)}$, with $\tau^{(1)}$ given by (27) and $\tau^{(2k+1)}$ for $k \in \mathbb{N}$ recursively given by (36). We next investigate its convergence.

4.2.1 Absolute convergence of the formal τ series solution

To show that the formal series $\tau = \sum_{k=0}^{\infty} \epsilon^{2k+1}\tau^{(2k+1)}$ converges absolutely, in $C_b^{1,\alpha}$ when $n = 1$ and in $C_0^{1,\alpha}$ when $n \geq 2$, we first have to check that each $\tau^{(2k+1)}$ as given in (36) is in $C_b^{1,\alpha}$, respectively $C_0^{1,\alpha}$. But when $n = 1$ this follows inductively from the facts that $\tau^{(1)} \in C_b^{1,\alpha}$, that $C_b^{1,\alpha}$ is a Banach algebra, and that $\mathbf{P} : C_b^{1,\alpha} \to C_b^{1,\alpha}$; hence, each partial sum of $\tau = \sum_{k=0}^{\infty} \epsilon^{2k+1}\tau^{(2k+1)}$ is in $C_b^{1,\alpha}$. Similarly, when $n \geq 2$, then $\tau^{(1)} \in C_0^{1,\alpha}$ and so $\tau^{(2k+1)} \in C_0^{1,\alpha}$ and each partial sum of $\tau = \sum_{k=0}^{\infty} \epsilon^{2k+1}\tau^{(2k+1)}$ is in $C_0^{1,\alpha}$. It remains to show that the right hand side of the norm estimate $\|\tau\| \leq \sum_{k=0}^{\infty} \epsilon^{2k+1}\|\tau^{(2k+1)}\|$ converges.

We now estimate all norms $\|\tau^{(2k+1)}\|$ for $k \in \mathbb{N}$ in terms of the $2k+1$-th power of $\|\tau^{(1)}\|$. For notational convenience we extend the list of $T^{(2k+1)}$s defined for $k \in \mathbb{N}$ by (34) to the case $k = 0$ by defining

$$T^{(1)} := \tau^{(1)}. \tag{37}$$

Now, $\|\tau^{(2k+1)}\| = \|\mathbf{P}T^{(2k+1)}\|$ for $k \in \mathbb{N}$, and since $\mathbf{P} : C_b^{1,\alpha} \to C_b^{1,\alpha}$, respectively $\mathbf{P} : C_0^{1,\alpha} \to C_0^{1,\alpha}$, is a projector, we have the estimate: $\|\tau^{(2k+1)}\| \leq \|T^{(2k+1)}\|$. Substituting the RHS of (34) for $T^{(2k+1)}$, repeating the standard inequality $\|\sum_i \tau_i\| \leq \sum_i \|\tau_i\|$, then using the inequality $\|\tau_i\tau_j\| \leq \|\tau_i\|\|\tau_j\|$ valid in Banach algebras (here $C_b^{1,\alpha}$ or $C_0^{1,\alpha}$), and applying repeatedly the identity $\|\tau^{(2a+1)}\| = \|\mathbf{P}T^{(2a+1)}\|$ followed by the projector estimate $\|\mathbf{P}T^{(2a+1)}\| \leq \|T^{(2a+1)}\|$ for the various pertinent values of $a \geq 1$ (no estimate is necessary when $a = 0$), and using $|M_j^{\mp}| = M_j^- := M_j$, for $k \geq 1$ we obtain

$$\begin{aligned} \|T^{(2k+1)}\| &\leq \sum_{h=1}^{k} \|\tau^{(2(k-h)+1)}\| \sum_{j=1}^{h} |M_j^{\mp}| \sum_{|\ell|_{2j}=h-j} \prod_{i=1}^{2j} \|\tau^{(2\ell_i+1)}\| \\ &\leq \sum_{h=1}^{k} \|T^{(2(k-h)+1)}\| \sum_{j=1}^{h} M_j \sum_{|\ell|_{2j}=h-j} \prod_{i=1}^{2j} \|T^{(2\ell_i+1)}\|. \end{aligned} \tag{38}$$

Setting $k = 1$ and recalling the definition (37) we obtain the estimate

$$\|T^{(3)}\| \leq \tfrac{1}{2}\|\tau^{(1)}\|^3 \tag{39}$$

Now suppose that for all $k = 1, ..., k_*$ there exists some R_{2k+1} such that

$$\|T^{(2k+1)}\| \leq R_{2k+1}\|\tau^{(1)}\|^{2k+1}. \tag{40}$$

Then the estimate (38) guarantees that (40) is true also for $k = k_* + 1$, and since $k_* \geq 1$ is arbitrary in this induction step while (39) says that the estimate is true for $k_* = 1$, it follows that (38) is true for all $k \in \mathbb{N}$. Our inductive proof that (40) holds for all $k \in \mathbb{N}$ also yields that R_{2k+1} is recursively defined for $k \in \mathbb{N}$ by

$$R_{2k+1} = \sum_{h=1}^{k} R_{2(k-h)+1} \sum_{j=1}^{h} M_j \sum_{|\ell|_{2j}=h-j} \prod_{i=1}^{2j} R_{2\ell_i+1}, \tag{41}$$

with $R_1 := 1$.

Recall that we want to show that $\sum_{k=0}^{\infty} \epsilon^{2k+1}\|\tau^{(2k+1)}\| < \infty$ for sufficiently small ϵ, so since we have shown that $\|\tau^{(2k+1)}\| \leq \|T^{(2k+1)}\| \leq R_{2k+1}\|\tau^{(1)}\|^{2k+1}$, it suffices to show that $\sum_{k=0}^{\infty} R_{2k+1}(\epsilon\|\tau^{(1)}\|)^{2k+1} < \infty$ for sufficiently small $\|\epsilon\tau^{(1)}\| =: \|\tau_g\|$. Setting now $\|\tau_g\| =: \xi$, we note that the formal power series $G(\xi) := \sum_{k=0}^{\infty} R_{2k+1}\xi^{2k+1}$ is nothing but the formal generating function of the R_{2k+1}, in the usual sense that, formally, $R_{2k+1} = G^{(2k+1)}(0)/(2k+1)!$. So our task is to show that the generating function is analytic about $\xi = 0$ with non-zero radius of convergence.

With the help of the recursion relation (41) we readily find that $G(\xi)$ is the positive inverse function of $g \mapsto \xi$ given by

$$\xi = 2g - \frac{g}{\sqrt{1-g^2}} \tag{42}$$

defined for positive ξ near $\xi = 0$, with $G(0) = 0$. Setting now $G(\xi) = \sin \Psi(\xi)$ we see that (42) becomes the function $\psi \mapsto \xi$ given by

$$\xi = 2\sin\psi - \tan\psi, \tag{43}$$

with $\xi = 0$ when $\psi = 0$. Since the function $\psi \mapsto \xi = 2\sin\psi - \tan\psi$ is analytic about $\psi = 0$ (with radius of convergence $= \pi/2$) and has unit derivative at $\psi = 0$, there now exists an open neighborhood of $\xi = 0$ on which there is defined a unique inverse function $\xi \mapsto \psi = \Psi(\xi)$ which vanishes for $\xi = 0$ and is analytic about $\xi = 0$, and also having unit derivative at $\xi = 0$. Since furthermore $\psi \mapsto \sin\psi$ is an entire function with unit derivative at $\psi = 0$, in total it now follows that there exists an open neighborhood of $\xi = 0$ on which there is defined an analytic function $\xi \mapsto g = G(\xi) = \sin\Psi(\xi)$ which vanishes at $\xi = 0$, has unit derivative at $\xi = 0$, and satisfies (42). Thus, in particular, the Maclaurin expansion of $G(\xi)$ converges to $G(\xi)$ and it generates the recursion coefficients R_{2k+1}.

We now determine the finite radius of convergence ξ_* of the power series for $G(\xi)$ about $\xi = 0$. Since $\psi \mapsto \sin\psi$ is an entire function, which vanishes for $\psi = 0$ and has unit derivative there, the radius of convergence of the Maclaurin series of $\xi \mapsto G(\xi) = \sin\Psi(\xi)$ coincides with the radius of convergence of the Maclaurin series of $\xi \mapsto \Psi(\xi)$. This radius of convergence in turn is determined by those ξ value(s) closest to $\xi = 0$ at which the derivative of $\psi \mapsto \xi = 2\sin\psi - \tan\psi$ vanishes (possibly asymptotically should $\xi \to \xi_\infty$ when $|\psi| \to \infty$ suitably). But this ψ derivative is $2\cos\psi - 1/\cos^2\psi$, and it vanishes iff $2\cos^3\psi = 1$. Clearly, $2^{1/3}\cos\psi \in \{1, e^{i2\pi/3}, e^{i4\pi/3}\}$, i.e. these three $\cos\psi$ points are situated on a circle in $\mathbb{C}$ of radius $2^{-1/3}$ centered at the origin. By the functional relation between cosine and sine it follows that the corresponding three $\sin\psi$ points are situated on the same circle in $\mathbb{C}$. A calculation now shows that also the three corresponding points $2\sin\psi - \tan\psi$ are situated on a circle in $\mathbb{C}$ centered at the origin. Its radius is the radius of convergence of $G(\xi)$ about $\xi = 0$, readily computed as

$$\xi_* = \left(2^{2/3} - 1\right)^{3/2}. \tag{44}$$

Thus we have shown that our formal series solution converges absolutely if $\|\tau_g\| < \left(2^{2/3} - 1\right)^{3/2}$. This completes our convergence proof.

We remark that because of our use of the projector estimates we cannot conclude that $\|\tau_g\| < \left(2^{2/3} - 1\right)^{3/2}$ is a necessary criterion for absolute convergence of our formal power series solution. Indeed, for any radially symmetric $H_0 \in C_0^\alpha$ the formal power series reduces to its first term, all other terms being identitcally zero, so convergence is a trivial issue and holds for any size of ϵ, then.

4.2.2 *Spin-off: series solution of the ω and u problems for small H*

The series solution for τ immediately yields the series solution for the ω-problem as a spin-off; namely,

$$\omega = \sum_{k\in\mathbb{N}} d \int \frac{\frac{1}{|S^{n-1}|}\delta T^{(2k+1)}(y)}{|x-y|^{n-2}} \, \mathrm{d}^n y. \tag{45}$$

Alternately, one can of course use the τ-series solution and then apply (19) to map τ to ω.

Lastly, the series solution for τ also yields the series solution for u as a spin-off; namely, defining

$$\rho(x) = \sum_{k=0}^{\infty} \delta T^{(2k+1)}(x) \tag{46}$$

we find that $\rho \in C_0^\alpha \cap L^1$, and so

$$u(x) = \tfrac{1}{|S^{n-1}|} \int \frac{\rho(y)}{|x-y|^{n-2}} \, \mathrm{d}^n y \tag{47}$$

for $n \neq 2$; for $n = 2$ we use the logarithmic kernel.

5 Prescribed mean-curvature graphs over $\mathbb{R}^3$

Because of its relevance to the electrostatic Maxwell–Born–Infeld equations, we now re-develop our solution theory in slightly more detail for the special dimension $n = 3$. The first step is to recast the prescribed mean-curvature problem for graphs over $\mathbb{R}^3$ in vector notation. There are two mutually dual first-order vector versions.

5.1 The equivalent first-order vector problems

Let u solve (3) for whichever sign and, for the pertinent sign, define the vector field w by

$$w = \pm \nabla u \tag{48}$$

Then, for the pertinent sign, w satisfies

$$\nabla \cdot \frac{w}{\sqrt{1 \pm |w|^2}} = 3H \tag{49}$$

together with

$$\nabla \times w = 0. \tag{50}$$

Given a solution w which vanishes at infinity, a solution u to (3) with monopole asymptotics at infinity is obtained by line integration of (48),

$$u(x) = \mp \int_x^\infty w(y) \cdot \mathrm{d}y\,. \tag{51}$$

The first-order system (49), (50) we call the "w problem," the second-order equation (3) is henceforth referred to as the "u problem."

Dual to this first-order system of vector PDE is an alternate first-order system for a vector field v defined for a solution u of (3) by

$$v = \pm \frac{\nabla u}{\sqrt{1 \pm |\nabla u|^2}}. \tag{52}$$

The elementary inversion of (52) yields

$$\nabla u = \pm \frac{v}{\sqrt{1 \mp |v|^2}}. \tag{53}$$

For either sign, v satisfies

$$\nabla \cdot v = 3H\,. \tag{54}$$

On the other hand, since ∇u is curl-free, taking the curl of (53) gives

$$\nabla \times \frac{v}{\sqrt{1 \mp |v|^2}} = 0. \tag{55}$$

The two vector PDE (54), (55) also form a closed system of nonlinear first-order PDE for v which we call the "v problem." Having solved (54) plus (55) for v vanishing at infinity, a solution to the prescribed mean-curvature equation(s) (3) with monopole asymptotics is obtained in terms of a line integral of (53),

$$u(x) = \mp \int_x^\infty \frac{v(y)}{\sqrt{1 \mp |v|^2(y)}} \cdot \mathrm{d}y\,. \tag{56}$$

5.2 Hierarchical vector series solution for small H

For $H \in C_0^\alpha \cap L^1$ small in the sense that $H(x) = \epsilon H_0(x)$, where $H_0(x)$ is a fixed function, and $\epsilon \in \mathbb{R}_+$ as small as we please, our convergent power series solution method for the τ problem translates into the following power series solution method for the v problem in its original version, equivalently equation (78) with v_g given by (79). One can in fact work with either one of the two dual first-order vector problems, and solving one leads also to the solution of the other, but working with the v problem is more convenient.

5.2.1 Convergent series solution of the v problem

We begin by noting that in the limit of vanishing $\epsilon \to 0^+$ the v problem is solved by the trivial solution $v \equiv 0$. The analyticity properties of the nonlinearity then suggest to seek a solution of the nonlinear first-order vector problem for small but finite ϵ in form of the power series Ansatz

$$v(x) = \sum_{p=1}^{\infty} \epsilon^p v^{(p)}(x), \tag{57}$$

with each $v^{(p)}$ independent of ϵ. Inserting this Ansatz into the pair of equations (54), (55), and sorting according to powers of ϵ, we find a hierarchy of linear equations.

In particular, $v^{(1)}$ satisfies

$$\nabla \cdot v^{(1)} = 3H_0\,, \tag{58}$$

together with

$$\nabla \times v^{(1)} = 0. \tag{59}$$

For $H_0 \in C_0^\alpha \cap L^1$ the pair of linear equations (58), (59) has a unique solution $\in C_0^{1,\alpha}$ given by the gradient field

$$v^{(1)}(x) := -\nabla \int \frac{\frac{3}{4\pi} H_0(y)}{|x-y|}\, \mathrm{d}^3 y. \tag{60}$$

Having the gradient field (60), we next find that $v^{(2)}$ satisfies

$$\nabla \cdot v^{(2)} = 0 \tag{61}$$

together with

$$\nabla \times v^{(2)} = 0. \tag{62}$$

Clearly, the pair of linear equations (61), (62) is uniquely solved by the trivial solution $v^{(2)} \equiv 0$.

Carrying on we now find recursively that each $v^{(p)}$ for even p, i.e. $p = 2k$ for $k \in \mathbb{N}$, satisfies

$$\begin{aligned} \nabla \cdot v^{(2k)} &= 0 \qquad (63) \\ \nabla \times v^{(2k)} &= 0, \qquad (64) \end{aligned}$$

so that $v^{(2k)} \equiv 0$ for general $k \in \mathbb{N}$, while for odd $p > 1$, i.e. $p = 2k+1$ for $k \in \mathbb{N}$, we find the pair of linear first-order PDE

$$\begin{aligned} \nabla \cdot v^{(2k+1)} &= 0 \qquad (65) \\ \nabla \times v^{(2k+1)} &= \nabla \times V^{(2k+1)}, \qquad (66) \end{aligned}$$

where $V^{(2k+1)}$ is a polynomial in the $v^{(\ell)}$ with $\ell < 2k+1$, viz.

$$V^{(2k+1)} = -\sum_{h=1}^{k} v^{(2(k-h)+1)} \sum_{j=1}^{h} M_j^{\mp} \sum_{\substack{|\ell|_{2j}= \\ h-j}} \prod_{i=1}^{j} v^{(2\ell_{2i-1}+1)} \cdot v^{(2\ell_{2i}+1)}, \qquad (67)$$

where again $|\ell|_{2j}$ and $M_j^{\mp}$ have their earlier assigned meaning. The pair of linear first-order PDE (65), (66) has a unique solution given by

$$v^{(2k+1)} = \mathbf{P}V^{(2k+1)}, \qquad k \in \mathbb{N} \qquad (68)$$

where $\mathbf{P}\colon C_0^{1,\alpha} \to C_0^{1,\alpha}$ projects onto the solenoidal subspace of $C_0^{1,\alpha}$; for $\nabla \cdot V^{(2k+1)} \in C_0^{0,\alpha} \cap L^1$,

$$\mathbf{P}V^{(2k+1)}(x) = V^{(2k+1)}(x) + \nabla \int \frac{\frac{1}{4\pi} \nabla \cdot V^{(2k+1)}(y)}{|x-y|} \, \mathrm{d}^3 y. \qquad (69)$$

So far we have proved that $v = \sum_{k=0}^{\infty} \epsilon^{2k+1} v^{(2k+1)}$, with $v^{(1)}$ given by (60) and $v^{(2k+1)}$ for $k \in \mathbb{N}$ recursively given by (69), is a *formal* series solution for our v problem (54), (55) with $H = \epsilon H_0$. Almost verbatim to the proof of section 3 we can now prove its convergence in $C_0^{1,\alpha}$ for small enough ϵ. Thus the series solution is a classical solution.

5.2.2 *Spin-off: series solution of the w and u problems for small H*

By the duality between the v and w problems it follows right away that as a bonus of the solution to the v problem one also obtains the solution to the w problem. Specifically, by (48) and (53), if v solves the v problem then the solution to the w problem for the pertinent sign reads

$$w = \frac{v}{\sqrt{1 \mp |v|^2}}. \tag{70}$$

Inserting the series solution (57) for v into the RHS of (70), recalling that $v^{(p=2k)} \equiv 0$ while $v^{(1)}$ is given by (60) and the $v^{(p=2k+1)}$ by (68), (69), we find the series

$$w(x) = \sum_{p=1}^{\infty} \epsilon^p w^{(p)}(x), \tag{71}$$

with each $w^{(p)}$ independent of ϵ and given as follows:

$$w^{(1)}(x) = v^{(1)}(x), \tag{72}$$

and for odd $p = 2k+1$, with $k \in \mathbb{N}$,

$$w^{(2k+1)}(x) = \nabla \int \frac{\frac{1}{4\pi}\nabla \cdot V^{(2k+1)}(y)}{|x-y|}\, \mathrm{d}^3 y, \tag{73}$$

with $V^{(2k+1)}(x)$ given in terms of the $v^{(2k+1)}$ by (67); for all even $p = 2k$ with $k \in \mathbb{N}$, we have

$$w^{(2n)}(x) \equiv 0. \tag{74}$$

Each term in the w expansion is a gradient field, as it should, by (48).

Finally, having the solution to the w problem in form of a series of gradient fields converging in $C_0^{1,\alpha}$, we can invoke (48) one more time and directly read off the solution in $C_0^{2,\alpha}$ to the original u problem as the series

$$u(x) = \tfrac{1}{4\pi} \int \frac{\rho(y)}{|x-y|}\, \mathrm{d}^3 y, \tag{75}$$

where

$$\rho(x) = \pm \sum_{k=0}^{\infty} \epsilon^{2k+1} \nabla \cdot V^{(2k+1)}, \tag{76}$$

with $V^{(2k+1)}(x)$ given in terms of the $v^{(2k+1)}$ by (67) when $k \in \mathbb{N}$, and by (37) when $k = 0$.

We close this subsection with the remark that the solution to the w problem obtained here as spin-off of the solution to the v problem does not express $w^{(2k+1)}$ as an operator applied to some explicit expression involving all lower order $w^{(2(k-h)+1)}$, $h = 1, 2, ..., k$. We leave it to the interested reader to prove that the series solution Ansatz (71) inserted directly into the PDE of the w problem leads to expressions for the $w^{(p)}$ involving only all lower order $w^{(<p)}$, which can be converted into the expressions involving only all lower order $v^{(<p)}$ given here.

5.3 The Helmholtz decomposition

As shown by Helmholtz, every vector field in $\mathbb{R}^3$ can be decomposed into a sum of a divergence-free and a curl-free field. The decomposition is not unique because a harmonic field can be transferred from one summand to the other. This non-uniqueness can be removed by imposing suitable boundary or asymptotic conditions on the individual summands; in our case asymptotic vanishing conditions. The Helmholtz decomposition sheds an interesting light on the w and v problems; or rather, on the v problem.

Indeed, for the w problem the Helmholtz decomposition just reverses the steps taken by us to arrive at it in the first place. Thus, we write $w = w_g + w_c$, where w_g is a gradient field (hence, curl-free) and w_c is a curl of some vector field (hence, divergence-free). Clearly, $\nabla \cdot w = \nabla \cdot w_g$ and $\nabla \times w = \nabla \times w_c$, so equation (50) together with the vanishing asymptotics for w_c implies right away that $w_c \equiv 0$, hence $w = w_g$, and (49) becomes a closed equation for $w_g = \pm\nabla u$. Of course, this is exactly the original second-order scalar equation (3).

As to the v-problem, writing $v = v_g + v_c$, where v_g is a gradient field (hence, curl-free) and v_c is a curl of some vector field (hence, divergence-free), and registering that $\nabla \cdot v = \nabla \cdot v_g$ and $\nabla \times v = \nabla \times v_c$, we get

$$\nabla \cdot v_g = 3H \tag{77}$$

(together with $\nabla \times v_g = 0$), and

$$\nabla \times \frac{(v_g + v_c)}{\sqrt{1 \mp |(v_g + v_c)|^2}} = 0 \tag{78}$$

(together with $\nabla \cdot v_c = 0$). We thereby have obtained a closed set of first-order vector equations for the vector field v_g separately from v_c, plus a conditionally closed set of first-order vector equations for the vector field v_c, conditioned on v_g being given. Both sets of PDE are supplemented by the asymptotic conditions that v_g and v_c vanish at spatial infinity; with this asymptotic condition the Helmholtz decomposition becomes unique.

As announced, the Helmholtz decomposition sheds an interesting light on the v problem. Since the curl-free component of the Helmholtz decomposition is determined completely and independently of the divergence-free Helmholtz component v_c of v, namely thusly: For $H \in C_0^\alpha \cap L^1$ the linear equation (77) has a unique gradient field solution $\in C_0^{1,\alpha}$ given by

$$v_g(x) := -\nabla \int \frac{\frac{3}{4\pi} H(y)}{|x-y|} \, \mathrm{d}^3 y, \tag{79}$$

we can reinterpret our power series approach as a method to solve the v_c-equation (78) given small v_g. Indeed, note that $v_g \equiv \epsilon v^{(1)}$ given earlier in (60) (note also that with $n = 3$ we have $\epsilon \tau^{(1)} \equiv v_g \cdot \mathrm{d}x$), so we can just separate the v_g part off from our power series solution for v. The remaining power series yields the solution v_c to (78) given small v_g.

5.4 The cubic nonlinearity of the v problem

We now come back to our earlier remark that the nonlinearity of the τ problem, i.e. here the first-order vector problem for v, is effectively cubic. As we will see, this implies some computational simplifications (though, interestingly enough, no simplifications for our solution theory, yet!). In addition, this "cubic version" of our v problem reveals some interesting a priori differential identities which are satisfied by any solution but which are obscured by the original formulation of the v problem.

Carrying out the curl operation in (55) and multiplying through with $(1 \mp |v|^2)^{3/2}$ we find that, away from singularities (which occur, for instance, when $1 - |v|^2 \to 0$), any solution v of (55) satisfies

$$0 = \left(1 \mp |v|^2\right) \nabla \times v \pm \left[\tfrac{1}{2} \nabla |v|^2\right] \times v. \tag{80}$$

This equation already exhibits a cubic nonlinearity, yet it can be further manipulated into a more concise alternate format.

Namely, using first the identity $\frac{1}{2}\nabla|v|^2 = (v \cdot \nabla)v + v \times (\nabla \times v)$ well-known from vector analysis, we find that (80) is equivalent to

$$0 = \left(1 \mp |v|^2\right) \nabla \times v \pm \left[(v \cdot \nabla)v + v \times (\nabla \times v)\right] \times v. \tag{81}$$

We next employ the identity $v \times (\nabla \times v) \times v = |v|^2 \nabla \times v - v(v \cdot \nabla \times v)$, well-known from vector algebra, which for any solution of (80) simplifies to $v \times (\nabla \times v) \times v = |v|^2 \nabla \times v$ because any solution of (80) satisfies (83). Inserting $v \times (\nabla \times v) \times v = |v|^2 \nabla \times v$ into (81) yields a cancellation and (81) becomes

$$\nabla \times v = \pm v \times (v \cdot \nabla) v. \tag{82}$$

This is perhaps the most concise cubic form of the curl equation for v.

The curl equation (82) paired with the divergence equation (54) forms a closed system of first-order vector PDE for v. By construction, this set of equations for v is equivalent to the scalar equations (3), for the appropriate choice of sign. To make them well posed the vector equations need to be supplemented by asymptotic conditions for v at spatial infinity, which as before we take to be vanishing in agreement with the monopole asymptotics of u at spatial infinity.

5.4.1 Spin-off: a-priori differential identities for solutions

The cubic version of the v problem reveals two a-priori differential identities which are satisfied by any solution of the v problem. Namely, dotting (80) with v yields

$$v \cdot (\nabla \times v) = 0 \tag{83}$$

wherever $\nabla \times v$ is defined; and dotting (80) with $\nabla |v|^2$ we see that

$$\nabla |v|^2 \cdot \nabla \times v = 0 \tag{84}$$

for any solution. Clearly, neither (83) nor (84) are generally true for arbitrary vector fields v.

5.4.2 The Helmholtz decomposition for the cubic version

The Helmholtz decomposition $v = v_g + v_c$ leads to analogous conclusions for the cubic version of the v problem. We find the previously obtained linear equation (77) for v_g (together with $\nabla \times v_g = 0$), while the nonlinear equation for v_c (given v_g) becomes

$$\nabla \times v_c = \pm (v_g + v_c) \times (v_g + v_c) \cdot \nabla (v_g + v_c) \tag{85}$$

(together with $\nabla \cdot v_c = 0$). This closed set of first-order vector equations for the vector fields v_g and v_c is supplemented by the asymptotic conditions that v_g and v_c vanish at spatial infinity.

As announced, given the gradient field (79), the remaining equation (85) now becomes a closed vector equation for the solenoidal field v_c. It remains to solve equation (85) with v_g given by (79).

5.4.3 Cubic version of the hierarchical v series solution for small H

In this subsection we apply our solution method to the cubic v problem when H is small. We begin by deriving the cubic analog of the hierarchy of linear PDE. Since $H \in C_0^\alpha \cap L^1$ is small in the sense that $H(x) = \epsilon H_0(x)$, where $H_0(x)$ is a fixed function and $\epsilon \in \mathbb{R}_+$ is chosen so small that $|H(x)| \ll 1$ uniformly in x, as before we find $v_g = O(\epsilon)$, with v_g given in (79). Thus we write $v_g = \epsilon v^{(1)}$, and for the solenoidal part of v we make the power series Ansatz $v_c = \epsilon^{p_1} v^{(p_1)} + \epsilon^{p_2} v^{(p_2)} + \epsilon^{p_3} v^{(p_3)} + \dots$ with $1 < p_1 < p_2 < p_3 < \cdots$, and with each $v^{(p_k)}$ independent of ϵ. Inserting this Ansatz into the cubic vector PDE (85) and identifying p_1 with the smallest power on the RHS, p_2 with the next-to-smallest power, and so on, we find recursively that $p_1 = 3$, then $p_2 = 5$, and $p_k = 2k + 1$ for general $k \in \mathbb{N}$. Moreover, for $v = v_g + v_c$ with $v_g = \epsilon v^{(1)}$ and $v_c = \epsilon^3 v^{(3)} + \epsilon^5 v^{(5)} + \epsilon^7 v^{(7)} + \dots$ to be a solution of the pair of equations (54), (85), each $v^{(2k+1)}$ for $k \in \mathbb{N}$ has to satisfy

$$\nabla \times v^{(2k+1)} = \pm \sum_{\substack{h+i+j= \\ k-1}} v^{(2h+1)} \times (v^{(2i+1)} \cdot \nabla) v^{(2j+1)} \tag{86}$$

supplemented by the solenoidality condition

$$\nabla \cdot v^{(2k+1)} = 0. \tag{87}$$

Supposing that $v^{(2\ell+1)}$ is known for all $\ell \le k-1$, then (86), (87) is a pair of linear first-order PDE for $v^{(2k+1)}$, with vanishing conditions at spatial infinity for $v^{(2k+1)}$. Now $v^{(1)}$ is known, and so, by induction, it follows that (86), (87) successively determine $v^{(3)}$, then $v^{(5)}$, and so on — provided that each equation in this formal linear hierarchy of equations is solvable (in $C_0^{1,\alpha}$)! If we ignore for a moment that we already know that our formal series derived from the original version of the v problem converges absolutely to a classical solution, from which the solvability of the cubic hierarchy of linear equations follows as a corollary, then the solvability of the cubic hierarchy of linear equations is not at all obvious but needs to be verified. Since $\sum_{h+i+j=k-1} v^{(2h+1)} \times (v^{(2i+1)} \cdot \nabla) v^{(2j+1)} \in C_0^\alpha \cap L^1$ if $v^{(2\ell+1)} \in C_0^{1,\alpha} \cap L^2$ for all $\ell \le k-1$, it remains to be verified that the collected terms on the RHS of (86) have vanishing divergence at each order k. With some effort one can show term by term that (97) is identical to (68), (69). For instance, based on the identity $\nabla(a \cdot b) = (a \cdot \nabla) b + (b \cdot \nabla) a + a \times \nabla \times b + b \times \nabla \times a$ for any two vector fields a and b it is readily shown that

$$-v^{(1)} \times (v^{(1)} \cdot \nabla) v^{(1)} = \frac{1}{2} \nabla |v^{(1)}|^2 \times v^{(1)} = \nabla \times (\tfrac{1}{2} |v^{(1)}|^2 v^{(1)}), \tag{88}$$

proving the equality of the RHS of (97) and that of (68), (69) when $k = 1$. However, the procedure of proving equality term by term soon gets very complicated, and an inductive argument is needed, instead.

5.4.4 *Solvability of the cubic hierarchy*

Proceeding by induction, we will now show that, given $v_g = \epsilon v^{(1)}$, for each $k \in \mathbb{N}$ there is a $v^{(2k+1)}$ which vanishes at infinity and solves

$$\nabla \times v^{(2k+1)} = \pm \sum_{h+i+j=k-1} v^{(2h+1)} \times (v^{(2i+1)} \cdot \nabla) v^{(2j+1)}. \tag{89}$$

Since the formulas become rather long, we switch to the shorter notation $v^{[k]}$ for $v^{(2k+1)}$ for all $k = 0, 1, 2, 3, ...$; this mildly obscures the power of ϵ to which the terms belong, but shortens the length of the formulas considerably.

Step one is easily disposed of by checking explicitly that $v_g \times (v_g \cdot \nabla) v_g$ is divergence-free for any gradient field v_g. Indeed, since $\nabla \times v_g = 0$ for any gradient field v_g, first of all the well-known identity $(v \cdot \nabla) v = \frac{1}{2} \nabla |v|^2 - v \times (\nabla \times v)$ reduces to $(v_g \cdot \nabla) v_g = \frac{1}{2} \nabla |v_g|^2$, and second, $v_g \times \left(\frac{1}{2} \nabla |v_g|^2 \right) = -\nabla \times \left[\frac{1}{2} |v_g|^2 v_g \right]$, so that $v_g \times (v_g \cdot \nabla) v_g = -\nabla \times \left[\frac{1}{2} |v_g|^2 v_g \right]$ is a curl, i.e. divergence-free. Hence, given $\epsilon v^{[0]} = v_g$, for either sign of "$\pm$" there is a $v^{[1]}$ satisfying $\nabla \times v^{[1]} = \pm v^{[0]} \times (v^{[0]} \cdot \nabla) v^{[0]}$. Moreover, since v_g is given by (79) and $H \in C_0^\alpha \cap L^1$, we have that $v_g \times (v_g \cdot \nabla) v_g \in C_0^{1,\alpha} \cap L^1$, and so in particular there is a solution $v^{[1]}$ of $\nabla \times v^{[1]} = \pm v^{[0]} \times (v^{[0]} \cdot \nabla) v^{[0]}$ which vanishes at infinity.

As to the induction step, suppose that for all $k \le m$, and either sign of "$\pm$," there is a solution $v^{[k]}$ of (89) which vanishes at spatial infinity. We now show that then also $\sum_{h+i+j=m} v^{[h]} \times (v^{[i]} \cdot \nabla) v^{[j]}$ is divergence free and vanishes sufficiently rapidly at infinity.

We will need two equalities. Taking the dot product of both sides of (89) with $v^{[l]}$ and summing for $l + k = m'$ with $m' \le m$ gives

$$\begin{aligned} \sum_{k+l=m'} v^{[l]} \cdot \nabla \times v^{[k]} &= \pm \sum_{\substack{k+l \\ =m'}} \sum_{\substack{h+i+j \\ =k-1}} v^{[l]} \cdot \left[v^{[h]} \times (v^{[i]} \cdot \nabla) v^{[j]} \right] & (90) \\ &= \pm \sum_{\substack{h+i+j+l \\ =m'-1}} v^{[l]} \cdot \left[v^{[h]} \times (v^{[i]} \cdot \nabla) v^{[j]} \right] & (91) \\ &= 0, & (92) \end{aligned}$$

where the last equality follows by the anti-symmetry of the triple product $a \cdot (b \times c)$; (92) is the order m expansion analogue of (83).

Similarly, dotting both sides of (89) with $(v^{[l]}\cdot\nabla)v^{[p]}$ and summing over $k+l+p=m$ gives

$$\sum_{k+l+n=m} \nabla\times v^{[k]}\cdot((v^{[l]}\cdot\nabla)v^{[p]}) = \tag{93}$$

$$\sum_{\substack{k+l+n\\=m}}\sum_{\substack{h+i+j\\=k-1}} (v^{[h]}\times(v^{[i]}\cdot\nabla)v^{[j]})\cdot((v^{[l]}\cdot\nabla)v^{[p]}) = \tag{94}$$

$$\sum_{h+i+j+l+n=m-1} (v^{[h]}\times(v^{[i]}\cdot\nabla)v^{[j]})\cdot((v^{[l]}\cdot\nabla)v^{[p]}) = 0. \tag{95}$$

Now note that

$$\begin{aligned}
&\sum_{h+i+j=m} v^{[h]}\times(v^{[i]}\cdot\nabla)v^{[j]} = \\
&\sum_{h+i+j=m} v^{[h]}\times\Big(\tfrac{1}{2}\nabla(v^{[i]}\cdot v^{[j]}) - v^{[i]}\times(\nabla\times v^{[j]})\Big) = \\
&\sum_{h+i+j=m} \Big(-\nabla\times\big(\tfrac{1}{2}(v^{[i]}\cdot v^{[j]})v^{[h]}\big) + \tfrac{1}{2}(v^{[i]}\cdot v^{[j]})\nabla\times v^{[h]} \\
&\qquad -(v^{[h]}\cdot\nabla\times v^{[j]})v^{[i]} + (v^{[h]}\cdot v^{[i]})\nabla\times v^{[j]}\Big) = \\
&\sum_{h+i+j=m} \Big(-\nabla\times(\tfrac{1}{2}(v^{[i]}\cdot v^{[j]})v^{[h]}) + \tfrac{3}{2}(v^{[h]}\cdot v^{[i]})\nabla\times v^{[j]}\Big),
\end{aligned}$$

where we used that $\sum_{h+i+j=m}(v^{[h]}\cdot\nabla\times v^{[j]})v^{[i]}=0$, by (92). And so,

$$\begin{aligned}
\nabla\cdot\sum_{h+i+j=m} v^{[h]}\times(v^{[i]}\cdot\nabla)v^{[j]} &= \tfrac{3}{2}\sum_{h+i+j=m}\nabla(v^{[h]}\cdot v^{[i]})\cdot\nabla\times v^{[j]} = \\
\sum_{h+i+j=m}\Big[\Big((v^{[h]}\cdot\nabla)v^{[i]} &+ (v^{[i]}\cdot\nabla)v^{[h]}\Big)\cdot\nabla\times v^{[j]} \\
&+\Big((v^{[h]}\times\nabla\times v^{[i]} + v^{[i]}\times\nabla\times v^{[h]})\Big)\cdot\nabla\times v^{[j]}\Big].
\end{aligned}$$

By (95), $\sum_{h+i+j=m}\big((v^{[h]}\cdot\nabla)v^{[i]} + (v^{[i]}\cdot\nabla)v^{[h]}\big)\cdot\nabla\times v^{[j]} = 0$. Moreover, by the anti-symmetry of the triple product $a\cdot(b\times c)$ we have

$$\sum_{h+i+j=m}(v^{[h]}\times(\nabla\times v^{[i]}) + v^{[i]}\times(\nabla\times v^{[h]}))\cdot\nabla\times v^{[j]} = 0. \tag{96}$$

Thus $\sum_{h+i+j=m} v^{[h]} \times (v^{[i]} \cdot \nabla) v^{[j]}$ is divergence free, as claimed.

The upshot is: for all $k \in \mathbb{N}$ the pair of equations (86), (87) has a unique $C_0^{1,\alpha} \cap L^2$ solution given by

$$v^{(2k+1)}(x) = \pm \tfrac{1}{4\pi} \sum_{\substack{h+i+j\\=k-1}} \int (v^{(2h+1)} \times (v^{(2i+1)} \cdot \nabla) v^{(2j+1)})(y) \times \frac{x-y}{|x-y|^3} \mathrm{d}^3 y. \tag{97}$$

In addition we have $\epsilon v^{(1)} = v_g$ given by (79).

5.4.5 Convergence of the cubic v series

Multiplying (97) by ϵ^{2k+1} and summing over k and adding $\epsilon v^{(1)} = v_g$ given by (79) yields a formal series solution to the cubic v problem, viz.

$$v(x) = \pm \tfrac{1}{4\pi} \sum_{k=0}^{\infty} \epsilon^{2k+1} \sum_{\substack{h+i+j=\\k-1}} \nabla \times \int \frac{(v^{(2h+1)} \times (v^{(2i+1)} \cdot \nabla) v^{(2j+1)})(y)}{|x-y|} \mathrm{d}^3 y. \tag{98}$$

We now have to address the convergence of this formal series solution.

Since the combinatorial structure of the RHS of (97) is considerably simpler than that of (68), (69), it would seem that a convergence proof is more readily forthcoming than our previous proof. Curiously, the obviously simpler series expansion for the cubic version of the v problem has not at all yielded to our attempts of proving its convergence *directly*, i.e. without recourse to the original version of the first-order vector problem with its more complicated nonlinearity. So we are finally forced to recall the origin of the cubic version of the v problem to conclude that (98) converges absolutely to a classical solution for small enough ϵ.

Summary: Except for the first few low-order terms, the series solution to our first-order v vector problem and its spin-off, the solutions to the first-order w vector problem and the original second-order scalar u problem, soon involves terms which look more and more unwieldy. The cubic version of the v problem offers relief, by having to evaluate fewer integrals. But there is also a caveat: it would be good to have a simple convergence proof directly for the cubic hierarchy.

6 Related quasi-linear problems in divergence form

Our power series technique of solving the prescribed mean-curvature problem for graphs over $\mathbb{R}^n$ can handle more general quasilinear problems in divergence form. In particular, let $f(s) = 1 + as + bs^2 + \cdots$ be given and analytic about $s = 0$, then the equation

$$\nabla \cdot \Big(f(|\nabla u|)\nabla u \Big) = \rho, \tag{99}$$

with $\rho \in C_0^\alpha \cap L^1$ small enough and asymptotic monopole condition for u, can be solved with the same solution techniques developed here for the scalar prescribed mean-curvature equation. In particular, such type of problems occur in the theory of stationary compressible fluid flows, see [4, 30, 31, 32], and other versions of nonlinear electrostatics [14].

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A The electrostatic analog

In Maxwell's classical electromagnetic theory, Coulomb's law states that an electrostatic charge density ρ in $\mathbb{R}^3$ is the source for the electric displacement density field D,

$$\nabla \cdot D = 4\pi\rho \qquad \text{Coulomb's law}\,, \tag{100}$$

while Faraday's law says that the electric field strength E, when stationary, is curl-free:

$$\nabla \times E = 0 \qquad \text{Faraday's law (stationary)}\,. \tag{101}$$

The two fields E and D need to be linked by an "aether law," with the help of which one can eliminate either D or E and obtain a closed nonlinear set of first-order PDE for the remaining field. Note that "aether" is used here merely as a shorthand for "electromagnetic vacuum;" historically Maxwell of course thought of "*the* classical aether."

Of particular relevance for the prescribed mean-curvature problem is the aether law proposed in the 1930s by Max Born [6], which reads

$$D = \frac{E}{\sqrt{1-\beta^4|E|^2}} \qquad \text{Born's law} \tag{102}$$

which can be inverted to yield

$$E = \frac{D}{\sqrt{1+\beta^4|D|^2}} \qquad \text{Born's law (reverse)}. \tag{103}$$

Here, $\beta \in (0,\infty)$ is a hypothetical new constant of nature (in the dimensionless notation of [22]). The limit $\beta \to 0$ of Born's law is Maxwell's law of the "pure aether", $D = E$.

The system of electrostatic Maxwell–Born(–Infeld) equations is clearly equivalent to the presribed mean-curvature problem in Minkowski spacetime $\mathbb{R}^{1,3}$. Replacing $\beta^4 \to -\beta^4$ in (102) yields a "pseudo Born's law," in which case the electrostatic Maxwell–Born(–Infeld) equations are equivalent to the presribed mean-curvature problem in Euclidean space $\mathbb{R}^4$.

Remarks: Born's aether law (102) is only the electrostatic special case of his electromagnetic aether law (replace $|E|^2$ by $|E|^2 - |B|^2$, and add a counterpart for the magnetic induction field B and field strength H), see [6, 7], and also App. VI in [8]. Born's electromagnetic aether law was subsequently generalized by Born and Infeld [9] to a much more interesting law, yet in the electrostatic limit all these aether laws coincide. Also higher-dimensional generalizations of Maxwell–Born–Infeld theory with geometric significance exist; see, for instance, [19, 38].

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