Elliptic operators with unbounded diffusion coefficients in L^p spaces

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Abstract

In this paper we prove that, under suitable assumptions on $\alpha > 0$, the operator $L = (1+|x|^\alpha)\Delta$ admits realizations generating contraction or analytic semigroups in $L^p(\mathbb{R}^N)$. For some values of α, we also explicitly characterize the domain of L. Finally, some informations about the location and composition of the spectrum are given.

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1 Introduction

In this paper we focus our attention on a class of elliptic operators with unbounded diffusion coefficients. We deal with operators of the form

$$Lu = (1+|x|^\alpha)\Delta u, \tag{1}$$

for positive values of α, on $L^p = L^p(\mathbb{R}^N, dx)$ with respect to the Lebesgue measure. The case $\alpha \le 2$ has been already investigated in literature and for this reason we shall assume $\alpha > 2$ throughout the paper, even when some argument easily extends to lower values of α. We refer to [2] where it is proved that the operator above generates a strongly continuous and analytic semigroup in L^p and in spaces of continuous functions. For $1 < p < \infty$ an explicit description follows from the a-priori estimates

$$\|(1+|x|^\alpha)D^2u\|_p \le C(\|u\|_p + \|(1+|x|^\alpha)\Delta u\|_p.$$

Similar estimates hold for a mor general class of operators, they can be deduced by some weigthed norm inequalities for Caldéron-Zygmund singular integrals. Muckenhoupt and Wheeden for example (see [12] or [14]) proved that estimates of the form

$$\|aD^2u\|_p \le C\|a\Delta u\|_p$$

are true for a weight a in some suitable Muckenhoupt classes. In particular the estimates above imply that

$$\||x|^\alpha D^2u\|_p \le C\||x|^\alpha \Delta u\|_p \tag{2}$$

and

$$\|(1+|x|^\alpha)D^2u\|_p \le C(\|u\|_p + \|(1+|x|^\alpha)\Delta u\|_p$$

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for $0 < \alpha < \frac{N}{p'}$ where p' is the conjugate exponent of p.
Similar estimates follow also by [6] where the author proved that certain singular integrals are convolution operators in weighted L^p spaces for the weight $1 + |x|^\alpha$, $-\frac{N}{p'} < \alpha < \frac{N}{p'}$.
We will prove that for certain values of $\alpha > 2$ the operator above admits realizations generating analytic semigroups in L^p for $1 < p < \infty$. Moreover for some values of α we will give also an explicit description of the domain by proving some a-priori estimates.
The starting point is a generation result of strongly continuous semigroups in spaces of continuous functions. It is known (see [11, Example 7.3]) that, if $N \geq 3$, $\alpha > 2$, the operator generates a strongly continuous semigroup in $C_0(\mathbb{R}^N)$, it has also been proved that both the semigroup and the resolvent are compact.

Notation. We use L^p for $L^p(\mathbb{R}^N, dx)$, where this latter is understood with respect to the Lebesgue measure. $C_b(\mathbb{R}^N)$ is the Banach space of all continuous and bounded functions in $\mathbb{R}^N$, endowed with the sup-norm, and $C_0(\mathbb{R}^N)$ its subspace consisting of all continuous functions vanishing at infinity. $C_c^\infty(\mathbb{R}^N)$ denotes the set of all C^∞ functions with compact support.

2 Solvability in spaces of continuos functions

The solvability of elliptic and parabolic problems associated to L in L^p depends on α, p, N. However, these restricions are not necessary in $C_b(\mathbb{R}^N)$ for a larger class of operators. Following [11], we recall the main results in spaces of continuos functions which will be useful for comparison throughout the paper

Let A be a second order elliptic partial differential operator of the form

$$Au(x) = \sum_{i,j=1}^N a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^N F_i(x) D_i u(x) \qquad x \in \mathbb{R}^N$$

under the following hypotheses on the coefficients: $a_{ij} = a_{ji}$, a_{ij}, F_i are real-valued locally Hölder continuous functions of exponent $0 < \alpha < 1$ and the matrix (a_{ij}) satisfies the ellipticity condition

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$$

for every x, $\xi \in \mathbb{R}^N$, with $\inf_K \lambda(x) > 0$ for every compact $K \subset \mathbb{R}^N$. The operator A is locally uniformly elliptic, that is uniformly elliptic on every compact subset of $\mathbb{R}^N$.
We endow A with its maximal domain in $C_b(\mathbb{R}^N)$ given by

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad p < \infty : \ Au \in C_b(\mathbb{R}^N)\}.$$

The main interest is in the existence of (spatial) bounded solutions of the parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N, \ t > 0, \\ u(0,x) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{3}$$

with initial datum $f \in C_b(\mathbb{R}^N)$. The unbounded interval $[0, \infty[$ can be changed to any bounded $[0,T]$ without affecting the results. Since the coefficients can be unbounded, the classical theory does not apply and existence and uniqueness for (3) are not clear. Quite surprisingly, existence is never a problem as stated in the following theorem.

Theorem 2.1 *There exists a positive semigroup $(T(t))_{t\geq 0}$ defined in $C_b(\mathbb{R}^N)$ such that, for any $f \in C_b(\mathbb{R}^N)$, $u(t,x) = T(t)f(x)$ belongs to the space $C_{loc}^{1+\frac{\alpha}{2},2+\alpha}((0,+\infty)\times\mathbb{R}^N)$, is a bounded solution of the following differential equation*

$$u_t(t,x) = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij}u(t,x) + \sum_{i=1}^{N} F_i(x)D_iu(t,x)$$

and satisfies

$$\lim_{t\to 0} u(t,x) = f(x)$$

pointwise.

When $f \in C_0(\mathbb{R}^N)$, then $u(t,\cdot) \to f$ uniformly as $t \to 0$. This, however, does not mean that $T(t)$ is strongly continuous on $C_0(\mathbb{R}^N)$ since this latter need not to be preserved by the semigroup

The idea of the proof is to take an increasing sequence of balls filling the whole space and, in each of them, to find a solution of the parabolic problem associated with the operator. Then the sequence of solutions so obtained is proved to converge to a solution of the problem in $\mathbb{R}^N$. More precisely, let us fix a ball $B_\rho = B_\rho(0)$ in $\mathbb{R}^N$ and consider the problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in B_\rho,\ \ t > 0, \\ u(t,x) = 0 & x \in \partial B_\rho,\ \ t > 0 \\ u(0,x) = f(x) & x \in \mathbb{R}^N. \end{cases} \tag{4}$$

Since the operator A is uniformly elliptic and the coefficients are bounded in B_ρ, there exists a unique solution u_ρ of problem (4). The next step consists in letting ρ to infinity in order to define the semigroup associated with A in $\mathbb{R}^N$. By using the parabolic maximum principle, it is possible to prove that the sequence u_ρ increases with ρ when $f \geq 0$ and is uniformly bounded by the sup-norm of f. In virtue of this monotonicity and since a general f can be written as $f = f^+ - f^-$, the limit

$$T(t)f(x) := \lim_{\rho\to\infty} u_\rho(t,x)$$

is well defined for $f \in C_b(\mathbb{R}^N)$ and one shows all relevant properties, using the interior Schauder estimates.

It is worth-mentioning that also the resolvent of A, namely $(\lambda - A)^{-1}$, is, for positive λ, the limit as $\rho \to \infty$ of the corresponding resolvents in the balls B_ρ. The construction then shows that, for positive $f \in C_b(\mathbb{R}^N)$ and $\lambda > 0$, both the semigroup $T(t)f$ (and the resolvent $(\lambda - A)^{-1}f$) select in a linear way the minimal solution among all bounded solutions of (3) (of $\lambda u - Au = f$). For this reason, from now on, the semigroup $T(t)$ will be called the minimal semigroup associated to A and will be denoted by $T_{\min}(t)$. Its generator (A, D), where $D \subset D_{\max}(A)$, will be denoted by $A_{\min}$

In contrast with the existence, the uniqueness is not guaranteed, in general, and relies on the existence of suitable Lyapunov functions. We do not deal here with such a topic and refer again to [11]. We only point out that uniqueness holds if and only if $D = D_{\max}(A)$, i.e. when $A_{\min}$ coincides with $(A, D_{\max}(A))$.

Let us specialize to our operator L.

Proposition 2.2 *Let $L = (1 + |x|^\alpha)\Delta$.*

(i) If $\alpha \leq 2$, the semigroup preserves $C_0(\mathbb{R}^N)$ and neither the semigroup nor the resolvent are compact.

(ii) If $\alpha > 2$ and $N = 1,\ 2$, the semigroup is generated by $(A, D_{max}(A))$, $C_0(\mathbb{R}^N)$ and L^p are not preserved by the semigroup and the resolvent and both the semigroup and the resolvent are compact.

(iii) If $\alpha > 2$, $N \geq 3$, then the semigroup is generated by $(A, D_{max}(A)) \cap C_0(\mathbb{R}^N)$, the resolvent and the semigroup map $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and are compact.

See ([11, Example 7.3]). In particular (ii) will imply that if $\alpha > 2$ and $N = 1, 2$, problem (3) cannot be solved in L^p. Observe also that (iii) and the discussion above show that $(T_{min}(t))_{t\geq 0}$ is strongly continuous on $C_0(\mathbb{R}^N)$.

3 Preliminary considerations in L^p

We consider the operator $\hat{L}_p = (L, \hat{D}_p)$ on any domain $\hat{D}_p$ contained in the maximal domain in $L^p(\mathbb{R}^N)$ defined by

$$D_{p,max}(L) = \{u \in L^p \cap W^{2,p}_{loc} :\ Lu \in L^p\}. \tag{5}$$

Note that $D_{p,max}(L)$ is the analogous of $D_{max}(L)$ for $p < \infty$. We are interested in solvability of elliptic and parabolic problems associated to L. We show that for certain values of p the equation

$$\lambda u - Lu = f$$

is not solvable in $L^p(\mathbb{R}^N)$ for positive λ.

In the following proposition we show that functions in $D_{p,max}(L)$ are globally in $W^{2,p}$.

Proposition 3.1

$$D_{p,max}(L) = \{u \in W^{2,p} : (1 + |x|^\alpha)\Delta u \in L^p\}.$$

Proof. It is clear that the right hand side is included in the left one. Conversely, if $u \in D_{p,max}(L)$, then $u,\ \Delta u \in L^p$ and we have to show that $u \in W^{2,p}$. Let $v \in W^{2,p}$ be such that $v - \Delta v = u - \Delta u$. Then $w = u - v \in L^p$ solves $w - \Delta w = 0$. Since w is a tempered distribution, by taking the Fourier transform it easily follows that $w = 0$, hence $u = v$. □

The next lemma shows that the resolvent operator in L^p, if it exists, is a positive operator.

Lemma 3.2 *Suppose that $\lambda \in \rho(\hat{L}_p)$ for some $\lambda \geq 0$. Then for every $0 \leq f \in L^p$,*

$$(\lambda - \hat{L}_p)^{-1} f \geq 0.$$

Proof. By density we may assume that $0 \leq f \in C_c^\infty(\mathbb{R}^N)$. Suppose first that $\lambda > 0$. We set $u = (\lambda - L_p)^{-1} f$. Suppose *supp* $f \subset B(R)$. Then u satisfies

$$\lambda u - Lu = f$$

in $B(R)$ and

$$\lambda u - Lu = 0$$

in $\mathbb{R}^N \setminus B(R)$. By local elliptic regularity ([7, Theorem 6.5.3]), $u \in C^{2,\beta}_{loc}(\mathbb{R}^N)$ for every $\beta < 1$. In $\mathbb{R}^N \setminus B(R)$, u satisfies

$$\Delta u = \frac{\lambda u}{1 + |x|^\alpha} \in L^p(\mathbb{R}^N).$$

By elliptic regularity, $u \in W^{2,p}(\mathbb{R}^N \setminus B(R))$. If $p > \dfrac{N}{2}$, we immediately deduce $u \in C_0(\mathbb{R}^N \setminus B(R))$. Otherwise $u \in L^{p_1}(\mathbb{R}^N \setminus B(R))$ where $\dfrac{1}{p_1} = \dfrac{1}{p} - \dfrac{2}{N}$ (with the usual modification when $p = N/2$). As before it follows $\Delta u \in L^{p_1}(\mathbb{R}^N \setminus B(R))$ and $u \in W^{2,p_1}(\mathbb{R}^N \setminus B(R))$. By iterating this procedure until $p_i > \frac{N}{2}$ we deduce $u \in C_0(\mathbb{R}^N)$. Therefore u attaints its minimum in a point $x_0 \in \mathbb{R}^N$. The equality

$$\lambda u(x_0) = (1 + |x_0|^\alpha)\Delta u(x_0) + f(x_0)$$

shows that $u(x_0) \geq 0$, since $\lambda > 0$, hence $u \geq 0$. If $\lambda = 0 \in \rho(\hat{L}_p)$, then $\lambda \in \rho(\hat{L}_p)$ for small positive values of λ and the thesis follows by approximation. $\square$

Lemma 3.3 *Suppose that $\lambda \in \rho(\hat{L}_p)$ for some $\lambda \geq 0$. Then for every $0 \leq f \in C_c(\mathbb{R}^N)$,*

$$(\lambda - \hat{L}_p)^{-1} f \geq (\lambda - L_{min})^{-1} f.$$

Proof. Let $0 \leq f \in C_c^\infty(\mathbb{R}^N)$ and set $u = (\lambda - \hat{L}_p)^{-1} f$. Proposition 3.2 and its proof show that that $0 \leq u \in D_{max}(L)$. Since $(\lambda - L_{min})^{-1} f$ is the minimal solution, we immediately have $u \geq (\lambda - L_{min})^{-1} f$. $\square$

Proposition 3.4 *Let $N \geq 3$, $\alpha > 2$, $p \leq \frac{N}{N-2}$. Then $\rho(\hat{L}_p) \cap [0, \infty[= \emptyset$.*

Proof. Let $\lambda > 0$ and $\chi_{B(0)} \leq f \leq \chi_{B(1)}$ be a smooth radial function. Denote by u the (minimal) solution of $\lambda u - Lu = f$ in $C_0(\mathbb{R}^N)$. Observe that from ([11, Example 7.3]) it follows that the above equation has a unique solution in $C_0(\mathbb{R}^N)$ (not in $C_b(\mathbb{R}^N)$). Hence, since the datum f is radial, the solution u is radial too, and solves

$$\lambda u(\rho) - (1 + \rho^\alpha) \left(u''(\rho) + \frac{N-1}{\rho} u'(\rho) \right) = f(\rho).$$

For $\rho \geq 1$, u solves the homogeneous equation

$$\lambda u(\rho) - (1 + \rho^\alpha) \left(u''(\rho) + \frac{N-1}{\rho} u'(\rho) \right) = 0.$$

Let us write u as $u(\rho) = \eta(\rho)\rho^{2-N}$ for a suitable function η. Elementary computations show that η satisfies

$$\lambda \eta(\rho) - (1 + \rho^\alpha) \left(\eta''(\rho) + \frac{3-N}{\rho} \eta' \right) = 0 \tag{6}$$

for $\rho \geq 1$. First observe that, since $f \neq 0$ is nonnegative, the strong maximum principle, see [5, Theorem 3.5]), implies that u (and so η) is strictly positive. We use Feller's theory to study the asymptotic behavior of the solutions of the previous equation (see [3, Section VI.4.c]). We introduce the Wronskian

$$W(\rho) = \exp\left\{ -\int_1^\rho \frac{3-N}{s} ds \right\} = \rho^{N-3}$$

and the functions

$$Q(\rho) = \frac{1}{(1 + \rho^\alpha) W(\rho)} \int_1^\rho W(s) ds = \frac{1}{N-2} \frac{1}{(1 + \rho^\alpha)\rho^{N-3}} (\rho^{N-2} - 1)$$

and

$$R(\rho) = W(\rho)\int_1^\rho \frac{1}{(1+s^\alpha)W(s)}ds = \rho^{N-3}\int_1^\rho \frac{1}{(1+s^\alpha)s^{N-3}}ds.$$

Since $\alpha > 2$ by assumption, we have $Q \in L^1(1,+\infty)$ and $R \notin L^1(1,\infty)$. This means that ∞ is an entrance endpoint. In this case there exists a positive decreasing solution η_1 of (6) satisfying $\lim_{\rho\to\infty}\eta_1(\rho) = 1$ and every solution of (6) independent of η_1 is unbounded at infinity. This shows that our solution u grows at infinity at least as ρ^{2-N} and therefore it does not belong to $L^p(\mathbb{R}^N)$. By Lemma 3.3 we deduce that $\lambda \notin \rho(\hat{L}_p)$. □

When $N = 1, 2$ and $\alpha > 2$, then (3) is never solvable in L^p.

Proposition 3.5 *Let $N = 1, 2$, $\alpha > 2$. Then $\rho(\hat{L}_p) \cap [0, \infty[= \emptyset$.*

This follows from Proposition 2.2 (ii), using Lemma 3.3. □

4 Solvability in L^p

In this section we investigate the solvability of the equation $\lambda u - Lu = f$ in L^p, for $\lambda \geq 0$. We start with $\lambda = 0$. Since the equation $-Lu = f$ is equivalent to $-\Delta u(x) = f(x)/(1+|x|^\alpha)$, we can express u and its gradient through an integral operator involving the Newtonian potential. For $f \in L^p$ we set

$$Tf(x) = u(x) = C_N \int_{\mathbb{R}^N} \frac{f(y)\,dy}{(1+|y|^\alpha)|x-y|^{N-2}} \tag{7}$$

and

$$Sf(x) = \nabla u(x) = C_N(N-2)\int_{\mathbb{R}^N} \frac{f(y)(y-x)\,dy}{(1+|y|^\alpha)|x-y|^N} \tag{8}$$

where $C_N = (N(2-N)\omega_N)^{-1}$ and ω_N is the Lebesgue measure of the unit ball in $\mathbb{R}^N$.

We prove a preliminary result which will be useful to prove estimates for the norm of the operator T in L^p.

Lemma 4.1 *Let $2 < \beta < N$. Then*

$$\frac{1}{N(2-N)\omega_N}\int_{\mathbb{R}^N}\frac{dy}{|x-y|^{N-2}|y|^\beta} = \frac{1}{(2-\beta)(N-\beta)}|x|^{2-\beta}.$$

Proof. Set

$$u(x) = \frac{1}{N(2-N)\omega_N}\int_{\mathbb{R}^N}\frac{dy}{|x-y|^{N-2}|y|^\beta}.$$

By writing $x,\ y$ in spherical coordinates, $x = s\eta$, $y = r\omega$, with $\eta,\ \omega \in S_{N-1}$, $s,\ r \in [0,+\infty)$, the expression of u becomes

$$\begin{aligned}
u(s\eta) &= \frac{1}{N(2-N)\omega_N}\int_{S_{N-1}}d\omega\int_0^\infty \frac{r^{N-1}dr}{|s\eta - r\omega|^{N-2}|r|^\beta} \\
&= \frac{1}{N(2-N)\omega_N}\int_{S_{N-1}}d\omega\int_0^\infty \frac{r^{N-1-\beta}dr}{s^{N-2}\left|\eta - \frac{r}{s}\omega\right|^{N-2}} \\
&= \frac{1}{N(2-N)\omega_N}s^{2-\beta}\int_{S_{N-1}}d\omega\int_0^\infty \frac{\xi^{N-1-\beta}d\xi}{|\eta - \xi\omega|^{N-2}}.
\end{aligned}$$

By the rotational invariance of the integral,

$$\begin{aligned} u(s\eta) &= \frac{1}{N(2-N)\omega_N} s^{2-\beta} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1-\beta} d\xi}{|e_1 - \xi\omega|^{N-2}} \\ &= \frac{1}{N(2-N)\omega_N} s^{2-\beta} \int_{\mathbb{R}^N} \frac{dy}{|e_1 - y|^{N-2}|y|^\beta}. \end{aligned}$$

where e_1 is the unitary vector in the canonical basis of S_{N-1}. Therefore $u(x) = C|x|^{2-\beta}$ with

$$C = \frac{1}{N(2-N)\omega_N} \int_{\mathbb{R}^N} \frac{dy}{|e_1 - y|^{N-2}|y|^\beta}.$$

To compute the constant C we note that u solves

$$\Delta u = \frac{1}{|x|^\beta}$$

or, in spherical coordinates,

$$u''(\rho) + \frac{N-1}{\rho} u'(\rho) = \frac{1}{\rho^\beta}.$$

Inserting $u(\rho) = C\rho^{2-\beta}$. we get $C = \dfrac{1}{(2-\beta)(N-\beta)}$. □

In the following lemma we investigate the boundedness of the operators T, S in weighted L^p-spaces. Even though we need here only the boundedness of T in the unweighted L^p-space, we prove the general result which will be of a central importance in the next sections.

Lemma 4.2 *Let $\alpha \geq 2$ and $N/(N-2) < p < \infty$. For every $0 \leq \beta, \gamma$ such that $\beta \leq \alpha - 2$, $\beta < \frac{N}{p'} - 2$ and $\gamma \leq \alpha - 1$, $\gamma < \frac{N}{p'} - 1$, there exists a positive constant C such that for any $f \in L^p$*

$$\begin{aligned} \||\cdot|^\beta u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}; \\ \||\cdot|^\gamma \nabla u\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}, \end{aligned}$$

where u is defined in (7).

Proof. Set $x = s\eta$, $y = \rho\omega$ with $s, \rho \in [0, +\infty)$, $\eta\omega \in S_{N-1}$, then

$$\begin{aligned} u(s\eta) &= \frac{1}{N(2-N)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{f(\rho\omega)\, \rho^{N-1}\, d\rho}{(1+\rho^\alpha)|s\eta - \rho\omega|^{N-2}} \\ &= \frac{1}{N(2-N)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{s^2 f(s\xi\omega)\, \xi^{N-1}\, d\xi}{(1+(s\xi)^\alpha)|\eta - \xi\omega|^{N-2}}. \end{aligned}$$

We compute the L^p norm of $|\cdot|^\beta u(\cdot)$. We start by integrating with respect to s the inequality above. We have, using Minkowski inequality for integrals,

$$\begin{aligned} &\left(\int_0^\infty |u(s\eta)|^p\, s^{\beta p + N - 1}\, ds \right)^{\frac{1}{p}} \leq \\ &\leq \frac{1}{N(N-2)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1}\, d\xi}{|\eta - \xi\omega|^{N-2}} \left(\int_0^\infty \frac{|f(s\xi\omega)|^p\, s^{N-1+2p+\beta p}\, ds}{(1+s^\alpha \xi^\alpha)^p} \right)^{\frac{1}{p}} \\ &= \frac{1}{N(N-2)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p}+\beta+2}} \left(\int_0^\infty \frac{|f(v\omega)|^p}{(1+v^\alpha)^p} v^{N-1+2p+\beta p}\, dv \right)^{\frac{1}{p}}. \end{aligned}$$

By recalling that $\beta \le \alpha - 2$ and since

$$\frac{v^{2+\beta}}{1+v^\alpha} \le \left(\frac{2+\beta}{\alpha-2+\beta}\right)^{\frac{2+\beta}{\alpha}} \frac{\alpha-2+\beta}{\alpha+2\beta},$$

we obtain

$$\left(\int_0^\infty |u(s\eta)|^p\, s^{\beta p+N-1}\, ds\right)^{\frac{1}{p}} \le \left(\frac{2+\beta}{\alpha-2+\beta}\right)^{\frac{2+\beta}{\alpha}} \frac{\alpha-2+\beta}{\alpha+2\beta} \frac{1}{N(N-2)\omega_N} \times$$
$$\times \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-2}\xi^{\frac{N}{p}+\beta+2}} \left(\int_0^\infty |f(v\omega)|^p v^{N-1}\, dv\right)^{\frac{1}{p}}.$$

Let us observe that, by Lemma 4.1 and the assumption $\beta < \frac{N}{p'} - 2$, we have

$$\frac{1}{N(N-2)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-2}\xi^{\frac{N}{p}+\beta+2}} = \frac{p^2}{(N+\beta p)(Np-N-\beta p-2p)}. \tag{9}$$

By applying Jensen's inequality with respect to probability measures

$$\frac{\xi^{N-1}}{c|\eta-\xi\omega|^{N-2}\xi^{\frac{N}{p}+\beta+2}}\, d\xi\, d\omega,$$

where c is the right-hand side in (9), we obtain

$$\int_0^\infty |u(s\eta)|^p\, s^{\beta p+N-1}\, ds \le \left(\frac{2+\beta}{\alpha-2+\beta}\right)^{\frac{(2+\beta)p}{\alpha}} \left(\frac{\alpha-2+\beta}{\alpha+2\beta}\right)^p \frac{c^{p-1}}{N(N-2)\omega_N} \times$$
$$\times \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-2}\xi^{\frac{N}{p}+\beta+2}} \int_0^\infty |f(v\omega)|^p v^{N-1}\, dv.$$

By integrating with respect to η on S_{N-1}, we obtain

$$\int_{\mathbb{R}^N} |u(x)|^p\, |x|^{\beta p}\, ds \le \left(\frac{2+\beta}{\alpha-2+\beta}\right)^{\frac{(2+\beta)p}{\alpha}} \left(\frac{\alpha-2+\beta}{\alpha+2\beta}\right)^p \frac{c^{p-1}}{N(N-2)\omega_N} \times$$
$$\times \int_{S_{N-1}} d\omega \int_{S_{N-1}} d\eta \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-2}\xi^{\frac{N}{p}+\beta+2}} \int_0^\infty |f(v\omega)|^p v^{N-1}\, dv.$$

A simple change of variables gives

$$\int_{S_{N-1}} d\eta \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-2}\xi^{\frac{N}{p}+\beta+2}} = \int_{S_{N-1}} d\eta \int_0^\infty \frac{t^{N-1} dt}{|\eta t-\omega|^{N-2} t^{\frac{N}{p'}-\beta}}$$
$$= \int_{\mathbb{R}^N} \frac{dy}{|y-\omega|^{N-2}|y|^{\frac{N}{p'}-\beta}}.$$

By applying Lemma 4.1 again it follows that that

$$\int_{\mathbb{R}^N} |u(x)|^p \, |x|^{\beta p} \, dx \le C^p \int_{\mathbb{R}^N} |f(x)|^p \, dx$$

with

$$C = \frac{p^2}{(N+\beta p)(Np-N-\beta p-2p)} \left(\frac{2+\beta}{\alpha-2+\beta}\right)^{\frac{2+\beta}{\alpha}} \frac{\alpha-2+\beta}{\alpha+2\beta}. \tag{10}$$

The L^p norm of $|\cdot|^\gamma \nabla u(\cdot)$ is estimated in a similar way but we shall not be as precise as before concerning the constants. By the representation formula,

$$|\nabla u(x)| \le C \int_{\mathbb{R}^N} \frac{|f(y)| \, dy}{(1+|y|^\alpha)|x-y|^{N-1}}.$$

and hence

$$\begin{aligned} |\nabla u(s\eta)| &\le C \int_{S_{N-1}} d\omega \int_0^\infty \frac{|f(\rho\omega)| \, \rho^{N-1} \, d\rho}{(1+\rho^\alpha)|s\eta-\rho\omega|^{N-1}} \\ &= C \int_{S_{N-1}} d\omega \int_0^\infty \frac{s|f(s\xi\omega)| \, \xi^{N-1} \, d\xi}{(1+(s\xi)^\alpha)|\eta-\xi\omega|^{N-1}}. \end{aligned}$$

By Minkowski inequality and since $\gamma \le \alpha - 1$,

$$\begin{aligned} &\left(\int_0^\infty |\nabla u(s\eta)|^p \, s^{\gamma p+N-1} \, ds\right)^{\frac{1}{p}} \\ &\qquad \le C \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} \, d\xi}{|\eta-\xi\omega|^{N-1}} \left(\int_0^\infty \frac{|f(s\xi\omega)|^p \, s^{N-1+p+\gamma p} \, ds}{(1+s^\alpha\xi^\alpha)^p}\right)^{\frac{1}{p}} \\ &\qquad = C \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-1}\xi^{\frac{N}{p}+\gamma+1}} \left(\int_0^\infty \frac{|f(v\omega)|^p}{(1+v^\alpha)^p} v^{N-1+p+\gamma p} \, dv\right)^{\frac{1}{p}} \\ &\qquad \le C \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-1}\xi^{\frac{N}{p}+\gamma+1}} \left(\int_0^\infty |f(v\omega)|^p v^{N-1} \, dv\right)^{\frac{1}{p}}. \end{aligned}$$

As before, the assumption $\gamma < \frac{N}{p'} - 1$ imples that the integral

$$\int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-1}\xi^{\frac{N}{p}+\gamma+1}} = \int_{\mathbb{R}^N} \frac{dy}{|\eta-y|^{N-1}|y|^{\frac{N}{p}+\gamma+1}}$$

is finite and independent of $\eta \in S^{N-1}$ (by the rotational invariance of the integrands). By applying Jensen's inequality we obtain

$$\begin{aligned} &\int_0^\infty |\nabla u(s\eta)|^p \, s^{\gamma p+N-1} \, ds \\ &\qquad \le C \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta-\xi\omega|^{N-1}\xi^{\frac{N}{p}+\gamma+1}} \int_0^\infty |f(v\omega)|^p v^{N-1} \, dv. \end{aligned}$$

Integration with respect to η on S_{N-1} yields

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p \, |x|^{\gamma p} \, ds$$
$$\leq C \int_{S_{N-1}} d\omega \int_{S_{N-1}} d\eta \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-1} \xi^{\frac{N}{p}+\gamma+1}} \int_0^\infty |f(v\omega)|^p v^{N-1} \, dv.$$

A simple change of variables gives

$$\int_{S_{N-1}} d\eta \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-1} \xi^{\frac{N}{p}+\gamma+1}} = \int_{S_{N-1}} d\eta \int_0^\infty \frac{t^{N-1} dt}{|\eta t - \omega|^{N-1} t^{N-\frac{N}{p}-\gamma}}$$
$$= \int_{\mathbb{R}^N} \frac{dy}{|y - \omega|^{N-1} |y|^{N-\frac{N}{p}-\gamma}}.$$

By the assumptions on γ, the last integral is convergent and independent of ω. It follows that

$$\int_{\mathbb{R}^N} |\nabla u(x)|^p \, |x|^{\gamma p} \, dx \leq C \int_{\mathbb{R}^N} |f(x)|^p \, dx$$

and the proof is complete. $\square$

By the previous lemma, the following estimate for the L^p-norm of the operator T immediately follows.

Corollary 4.3 *Let $\alpha \geq 2$ and $N/(N-2) < p < \infty$. Then*

$$\|T\|_p \leq \left(\frac{2}{\alpha - 2} \right)^{\frac{2}{\alpha}} \frac{\alpha - 2}{\alpha} \frac{p^2}{N(Np - N - 2p)}.$$

Proof. The estimate follows by setting $\beta = 0$ in (10). $\square$

Remark 4.4 The estimate with the constant C given by (10) is stable as $p \to \infty$ only if $\beta > 0$. On the other hand, the operator T is bounded also in L^∞ (and its norm will be computed later in Proposition 7.4). It is possible to prove that the operator T (with $\beta = 0$) is of weak-type $p - p$ with $p = N/(N-2)$ and then interpolate between $N/(N-2)$ and ∞ to obtain stable estimates for large p. The weak-type estimate is deduced as follows. Write Tf as the Riesz potential I_2 applied to the function $f(x)/(1+|x|^\alpha)$ to get, using the classical estimate of the Riesz potentials through the Hardy-Littlewood maximal function M,

$$|Tf(x)| \leq C \left(M \left(\frac{f(\cdot)}{1 + |\cdot|^\alpha} \right)(x) \right)^{1-2/N} \left\| \frac{f(\cdot)}{1 + |\cdot|^\alpha} \right\|_1^{2/N}$$

then Holder inequality to control the L^1-norms in terms of the $L^{N/(N-2)}$-norm of f and the weak $1-1$ estimate for M. Such a proof works only for $\beta = 0$ and gives constants depending on those of the Marzinkiewicz interpolation theorem and of the Hardy-Littlewood maximal function.

We can now prove the invertibility of L on $D_{p,max}(L)$, defined in (5).

Proposition 4.5 *Let $\alpha > 2$ and $N/(N-2) < p < \infty$. The operator L is closed and invertible on $D_{p,max}(L)$ and the inverse of $-L$ is the operator T defined in (7).*

PROOF. The closedness of L on $D_{p,max}(L)$ follows from local elliptic regularity. If $u \in D_{p,max}(L)$ satisfies $Lu = 0$, then $\Delta u = 0$ and then $u = 0$, since $u \in L^p$. This shows the injectivity of L. Finally, let $f \in L^p$ and $f_n \in C_c^\infty(\mathbb{R}^N)$ be such that $f_n \to f$ in L^p. Then $u_n = Tf_n \to u = Tf$ in L^p, since T is bounded (apply Lemma 4.2 with $\beta = 0$). By elementary potential theory

$$\Delta u_n(x) = \frac{f_n(x)}{1+|x|^\alpha}$$

hence $u_n \in D_{p,max}(L)$ and $Lu_n = f_n$. By the closedness of L, $u \in D_{p,max}(L)$ and $Lu = f$.

Theorem 4.6 *Let $\alpha > 2$, $N/(N-2) < p < \infty$ and $\lambda \geq 0$. The operator $\lambda - L$ is invertible on $D_{p,max}(L)$ and its inverse is a positive operator. Moreover, if $f \in L^p \cap C_0(\mathbb{R}^N)$, then $(\lambda - L)^{-1} f = (\lambda - L_{min})^{-1} f$.*

PROOF. Let ρ be the resolvent set of $(L, D_{p,max}(L))$ and observe that the proposition above shows that $0 \in \rho$. Lemma 3.2 with $\hat{D} = D_{p,max}(L)$ shows that if $0 \leq \lambda \in \rho$, than $(\lambda - L)^{-1} \geq 0$ and hence, by the resolvent equation, $(\lambda - L)^{-1} \leq (-L)^{-1} = T$ and therefore

$$\|(\lambda - L)^{-1}\| \leq \|T\| \tag{11}$$

where the norm above is the operator norm in L^p. Let $E = [0, \infty[\cap \rho$. Then E is non empty and open in $[0, \infty[$, since ρ is open, and closed since the operator norm of $(\lambda - L)^{-1}$ is bounded in E. Then $E = [0, \infty[$. To show the consistency of the resolvents we take $f \in C_c^\infty(\mathbb{R}^N)$ and let $u = (\lambda - L)^{-1} f$. As in Lemma 3.2 we see that $u \in C_0(\mathbb{R}^N) \cap C_{loc}^{2,\beta}(\mathbb{R}^N)$ for any $\beta < 1$. If $supp\ f \subset \mathbb{R}^N \setminus B(R)$, then the equation $Lu = \lambda u$ holds outside $B(R)$ and shows that u belongs to $D_{max}(L) \cap C_0(\mathbb{R}^N)$, which is the domain of L_{min} in $C_0(\mathbb{R}^N)$, see Proposition 2.2 (iii). Therefore $(\lambda - L_{min})^{-1} f = u = (\lambda - L)^{-1} f$. By density, this equality extends to all functions $f \in L^p \cap C_0(\mathbb{R}^N)$. □

It is worth mentioning that the resolvents of L in L^p and L^q are consistent, provided that $p, q > N/(N-2)$. This easily follows from above, together with a simple approximation argument, since both resolvents are consistent with $(\lambda - L_{min})^{-1}$. Observe also that estimate (11) shows only that the resolvent is bounded on $[0, \infty[$ and is not sufficient to apply the Hille-Yosida theorem and prove results for parabolic problems. This will be done in the next secion, under further restrictions on the admitted values for p.

5 Sectoriality in L^p

We prove that, for $2 < \alpha \leq (N-2)(p-1)$ and $N/(N-2) < p < \infty$, $(L, D_{p,max}(L))$ generates a strongly continuous semigroup of positive contractions, analytic for $\alpha < (N-2)(p-1)$, which coincides whith the minimal semigroup in $L^p \cap C_0(\mathbb{R}^N)$.

Theorem 5.1 *Let $N \geq 3$, $p > N/(N-2)$, $2 < \alpha \leq (p-1)(N-2)$. Then $(L, D_{p,max}(L))$ generates a positive semigroup of contractions in L^p. If $\alpha < (p-1)(N-2)$, the semigroup is also analytic.*

Proof. Take $f \in L^p(\mathbb{R}^N)$, $\rho > 0$, $\lambda \in \mathbb{C}$ and consider the Dirichlet problem in $L^p(B(\rho))$

$$\begin{cases} \lambda u - Lu = f & \text{in } B(\rho), \\ u = 0 & \text{on } \partial B(\rho). \end{cases} \tag{12}$$

According to Theorem 9.15 in [5], for $\lambda > 0$ there exists a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$. In order to show that the above problem is solvable for complex values of λ and to obtain estimates independent of ρ, we show that $e^{\pm i\theta}L$ is dissipative in $B(\rho)$ for $0 \le \theta \le \theta_0$ and a suitable $0 < \theta_0 \le \pi/2$. Set $u^\star = \overline{u}_\rho |u_\rho|^{p-2}$. Multiply Lu_ρ by $u^\star$ and integrate over $B(\rho)$. The integration by parts is straightforward when $p \ge 2$. For $1 < p < 2$, $|u_\rho|^{p-2}$ becomes singular near the zeros of u_ρ. It is possible to prove the the integration by parts is allowed also in this case (see [9]). Notice also that all boundary terms vanish since $u_\rho = 0$ at the boundary. So we get

$$\begin{aligned}\int_{B(\rho)} Lu_\rho\, u^\star dx = &- \int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4}|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx - \int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4}|Im(\overline{u}_\rho \nabla u_\rho)|^2 dx \\ &- \int_{B(\rho)} \overline{u}_\rho |u_\rho|^{p-2} \nabla(1+|x|^\alpha)\nabla u\, dx - (p-2)\int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4}\overline{u}_\rho \nabla u_\rho Re(\overline{u}_\rho \nabla u_\rho) dx.\end{aligned}$$

By taking the real and imaginary part of the left and the right hand side, we have

$$\begin{aligned}Re\left(\int_{B(\rho)} Lu_\rho\, u^\star dx\right) = &-(p-1)\int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4}|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx \\ &- \int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4}|Im(\overline{u}_\rho \nabla u_\rho)|^2 dx - \int_{B(\rho)} |u_\rho|^{p-2}\nabla(1+|x|^\alpha) Re(\overline{u}_\rho \nabla u_\rho)\, dx;\end{aligned}$$

$$\begin{aligned}Im\left(\int_{B(\rho)} Lu_\rho\, u^\star dx\right) = &-(p-2)\int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4} Im(\overline{u}_\rho \nabla u_\rho) Re(\overline{u}_\rho \nabla u_\rho) dx \\ &- \int_{B(\rho)} |u_\rho|^{p-2}\nabla(1+|x|^\alpha) Im(\overline{u}_\rho \nabla u_\rho)\, dx.\end{aligned}$$

By Hardy's inequality as stated in Proposition 8.7,

$$\begin{aligned}&\left|\int_{B(\rho)} |u_\rho|^{p-2}\nabla(1+|x|^\alpha) Re(\overline{u}_\rho \nabla u_\rho)\, dx\right| \le \alpha \int_{B(\rho)} |u_\rho|^{p-2}|x|^{\alpha-1}|Re(\overline{u}_\rho \nabla u_\rho)| dx \\ \le &\alpha \left(\int_{B(\rho)} |u_\rho|^{p-4}|x|^\alpha |Re(\overline{u}_\rho \nabla u_\rho)|^2 dx\right)^{\frac{1}{2}} \left(\int_{B(\rho)} |u_\rho|^p |x|^{\alpha-2} dx\right)^{\frac{1}{2}} \\ \le &\frac{p\alpha}{\alpha-2+N}\int_{B(\rho)} |u_\rho|^{p-4}(1+|x|^\alpha)|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx.\end{aligned}$$

It follows that

$$\begin{aligned}-Re\left(\int_{B(\rho)} Lu_\rho\, u^\star dx\right) \ge &\left(p-1-\frac{p\alpha}{\alpha-2+N}\right)\int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4}|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx \\ &+ \int_{B(\rho)} (1+|x|^\alpha)|u_\rho|^{p-4}|Im(\overline{u}_\rho \nabla u_\rho)|^2 dx\end{aligned}$$

and

$$\begin{aligned}
&\left|Im\Big(\int_{B(\rho)} Lu_\rho\, u^\star dx\Big)\right| \\
&\quad\le (p-2)\left(\int_{B(\rho)} |u_\rho|^{p-4}|x|^\alpha |Re(\overline{u}_\rho\nabla u_\rho)|^2dx\right)^{\frac12}\left(\int_{B(\rho)} |u_\rho|^{p-4}|x|^\alpha |Im(\overline{u}_\rho\nabla u_\rho)|^2dx\right)^{\frac12}\\
&\quad+\alpha\int_{B(\rho)} |u_\rho|^{p-2}|x|^{\alpha-1}|Im(\overline{u}_\rho\nabla u_\rho)|\,dx\le\\
&\quad\le\left(p-2+\frac{p\alpha}{\alpha-2+N}\right)\left(\int_{B(\rho)} |u_\rho|^{p-4}|x|^\alpha |Re(\overline{u}_\rho\nabla u_\rho)|^2dx\right)^{\frac12}\\
&\quad\times\left(\int_{B(\rho)} |u_\rho|^{p-4}|x|^\alpha |Im(\overline{u}_\rho\nabla u_\rho)|^2dx\right)^{\frac12}.
\end{aligned}$$

Setting

$$\begin{aligned}
B^2&=\int_{\mathbb{R}^N} |u_\rho|^{p-4}|x|^\alpha |Re(\overline{u}_\rho\nabla u_\rho)|^2dx,\\
C^2&=\int_{\mathbb{R}^N} |u_\rho|^{p-4}|x|^\alpha |Im(\overline{u}_\rho\nabla u_\rho)|^2dx,
\end{aligned}$$

we proved that

$$-Re\Big(\int_{\mathbb{R}^N} Lu_\rho\, u^\star dx\Big)\ge\left(p-1-\frac{p\alpha}{\alpha-2+N}\right)B^2+C^2$$

and

$$\left|Im\Big(\int_{B(\rho)} Lu_\rho\, u^\star dx\Big)\right|\le\left(p-2+\frac{p\alpha}{\alpha-2+N}\right)BC.$$

Observe that $p-1-\dfrac{p\alpha}{\alpha-2+N}$ is positive for $\alpha<(N-2)(p-1)$. In this case it is possible to determine a positive constant l_α, independent of ρ, such that

$$\left(p-1-\frac{p\alpha}{\alpha-2+N}\right)B^2+C^2\ge l_\alpha\left(p-2+\frac{p\alpha}{\alpha-2+N}\right)BC$$

and, consequently,

$$\left|Im\Big(\int_{B(\rho)} Lu_\rho\, u^\star dx\Big)\right|\le l_\alpha^{-1}\left\{-Re\Big(\int_{B(\rho)} Lu_\rho\, u^\star dx\Big)\right\}.$$

If $\tan\theta_\alpha=l_\alpha$, then $e^{\pm i\theta}L$ is dissipative in $B(\rho)$ for $0\le\theta\le\theta_\alpha$. The previous computations give also the dissipativity of L if $\alpha=(N-2)(p-1)$. Let us introduce the sector

$$\Sigma_\theta=\{\lambda\in\mathbb{C}\setminus\{0\}:|Arg\,\lambda|<\pi/2+\theta\}.$$

It follows from [13, Theorem I.3.9], that problem (12) has a unique solution for every $\lambda\in\Sigma_\theta$ and $0\le\theta<\theta_\alpha$ and that there exists a constant C_θ, independent of ρ, such that the solution u_ρ satisfies

$$\|u_\rho\|_{L^p(B(\rho))}\le\frac{C_\theta}{|\lambda|}\|f\|_{L^p}. \tag{13}$$

In the case $\alpha = (N-2)(p-1)$ the solutions u_ρ exist for $Re\,\lambda > 0$ and satisfy the estimate

$$\|u_\rho\|_{L^p(B(\rho))} \le \frac{1}{Re\,\lambda}\|f\|_{L^p}.$$

Moreover, if $\lambda > 0$ then $u_\rho \le 0$ if $f \le 0$ in $B(\rho)$. In fact, multiplying the equation

$$\lambda u_\rho - Lu_\rho = f$$

by $(u_\rho^+)^{p-1}$, integrating over $B(\rho)$ and proceeding as before we obtain

$$\lambda \int_{B(\rho)} (u_\rho^+)^p \; dx \le \int_{B(\rho)} f(u_\rho^+)^{p-1} \; dx \le 0.$$

Therefore $u_\rho^+ = 0$ and $u_\rho \le 0$.

Next we use weak compactness arguments to produce a function $u \in D_{p,max}(L)$ satisfying $\lambda u - Lu = f$. For definiteness, we consider the case $\alpha < (N-2)(p-1)$, the other one being simpler, and fix $\lambda \in \Sigma_\theta$, with $0 < \theta < \theta_\alpha$.

Let us fix a radius r and apply the interior L^p estimates ([5, Theorem 9.11]) together with (13) to the functions u_ρ with $\rho > r+1$

$$\|u_\rho\|_{W^{2,p}(B(r))} \le C_1[\|\lambda u_\rho - Lu_\rho\|_{L^p(B(r+1))} + \|u_\rho\|_{L^p(B(r+1))}] \le C_2\|f\|_{L^p}.$$

By weak compactness and a diagonal argument, we can find a sequence $\rho_n \to \infty$ such that the functions (u_{ρ_n}) converge weakly in $W^{2,p}_{loc}$ to a function u. Clearly u satisfies $\lambda u - Lu = f$ and, by (13),

$$\|u\|_{L^p} \le \frac{C_\theta}{|\lambda|}\|f\|_{L^p}. \tag{14}$$

In particular $u \in D_{p,max}(L)$ and, moreover, u is positive if $\lambda, f \ge 0$. To complete the proof we need only to show that $\lambda - L$ is injective on $D_{p,max}(L)$ for $\lambda \in \Sigma_\theta$. Let

$$E = \{r > 0 : \Sigma_\theta \cap B(r) \subset \rho(L, D_{p,max}(L))\}$$

and $R = \sup E$. Since $0 \in E$, by Proposition 4.5, R is positive. On the other hand the norm of the resolvent exists in $B(R) \cap \Sigma_\theta$ and is bounded by $C_\theta/|\lambda|$, by (14), hence cannot explode on the boundary of $B(R)$. This proves that $R = \infty$ and concludes the proof. □

Finally, let us show that on $L^p \cap C_0(\mathbb{R}^N)$ the semigroup coincide with T_{min} of Section 2. In particular, the semigroups are coherent in different L^p spaces (when they are defined).

Corollary 5.2 *Let $(T_p(t))$ be the semigroup generated by $(L, D_{p,max}(L))$ in L^p and $(T_{min}(t))$ be the minimal semigroup in $C_0(\mathbb{R}^N)$. Then for every $f \in C_0(\mathbb{R}^N) \cap L^p$, $T_p(t)f = T_{min}(t)f$. Moreover, if p, q are allowed in the above theorem and $f \in L^p \cap L^q$, then $T_p(t)f = T_q(t)f$.*

Proof. Since $(\lambda - L)^{-1}f = (\lambda - L_{min})^{-1}f$ for $f \in L^p \cap C_0(\mathbb{R}^N)$, see Theorem 4.6, the thesis follows by representing the semigroups as the limit of iterates of the corresponding resolvents. □

6 Domain Characterization

The main result of this section consists in showing that, for $N \geq 3$, $1 < p < \infty$, $2 < \alpha < N/p'$, the maximal domain $D_{p,max}(L)$ defined in (5) coincides with the weighted Sobolev space D_p defined by

$$D_p = \{u \in W^{2,p}_{loc}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) : \ (1+|x|^{\alpha-2})u, \ (1+|x|^{\alpha-1})\nabla u, \ (1+|x|^{\alpha})D^2u \in L^p(\mathbb{R}^N)\}$$

and endowed with its canonical norm.

Remark 6.1 *Observe that the assumption $2 < \alpha < N/p'$ forces p to be strictly greater than $\frac{N}{N-2}$, according with Proposition 3.4.*

The next lemma provides a core for D_p.

Lemma 6.2 *The space $C_c^\infty(\mathbb{R}^N)$ is dense in D_p with respect to the norm*

$$\|u\|_{D(A_p)} = \|u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha})D^2u\|_{L^p(\mathbb{R}^N)}.$$

Proof. Let us first observe that a function $u \in W^{2,p}(\mathbb{R}^N)$ with compact support can be approximated by a sequence of C^∞ functions with compact support, in the $D(A_p)$ norm. Indeed, if ρ_n are standard mollifiers, $u_n = \rho_n * u \in C_c^\infty(\mathbb{R}^N)$, $supp\, u_n \subset supp\, u + B(1)$ for any $n \in N$ and $u_n \to u$ in D_p since $(1+|x|^{\alpha-2})$, $(1+|x|^{\alpha-1})$, $(1+|x|^{\alpha})$ are bounded (uniformly with respect to n) on $supp\, u + B(1)$. Next we show that any function u in D_p can be approximated, with respect to the norm of $D(A_p)$, by a sequence of functions in $W^{2,p}(\mathbb{R}^N)$ each having a compact support. Let η be a smooth function such that $\eta = 1$ in $B(1)$, $\eta = 0$ in $\mathbb{R}^N \setminus B(2)$, $0 \leq \eta \leq 1$ and set $\eta_n(x) = \eta\left(\frac{x}{n}\right)$. If $u \in D_p$, then $u_n = \eta_n u$ are compactly supported functions in $W^{2,p}(\mathbb{R}^N)$, $u_n \to u$ in $L^p(\mathbb{R}^N)$, $(1+|x|^{\alpha-2})u_n \to (1+|x|^{\alpha-2})u$ in $L^p(\mathbb{R}^N)$ by dominated convergence. Concerning the convergence of the derivatives we have

$$(1+|x|^{\alpha-1})\nabla u_n = \frac{1}{n}(1+|x|^{\alpha-1})\nabla\eta\left(\frac{x}{n}\right)u + (1+|x|^{\alpha-1})\eta\left(\frac{x}{n}\right)\nabla u.$$

As before,

$$(1+|x|^{\alpha-1})\eta\left(\frac{x}{n}\right)\nabla u \to (1+|x|^{\alpha-1})\nabla u$$

in $L^p(\mathbb{R}^N)$. For the left term, since $\nabla\eta(x/n)$ can be different from zero only for $n \leq |x| \leq 2n$ we have

$$\frac{1}{n}(1+|x|^{\alpha-1})\left|\nabla\eta\left(\frac{x}{n}\right)\right||u| \leq C(1+|x|^{\alpha-2})|u|\chi_{\{n\leq|x|\leq 2n\}},$$

and the right hand side tends to 0 as $n \to \infty$. A similar argument shows the convergence of the second order derivatives in the weighted L^p norm. □

We can prove that L is closed on D_p.

Proposition 6.3 *assume that $2 < \alpha < N/p'$. Then there exists a positive constant C such that for any $u \in D_p$*

$$\begin{aligned} &\|u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha})D^2u\|_{L^p(\mathbb{R}^N)} \\ \leq\ & C\|Lu\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$ and set $f = -Lu$. Then $f \in C_c^2(\mathbb{R}^N)$, $(1+|x|^\alpha)\Delta u(x) = -f(x)$ or equivalently

$$-\Delta u(x) = \frac{f(x)}{1+|x|^\alpha}.$$

By elementary potential theory, u is given by (7). By setting $\beta = \alpha - 2$ and $\gamma = \alpha - 1$ in Lemma 4.2 and since $\alpha < \frac{N}{p'}$, we deduce that

$$\|u\|_{L^p(\mathbb{R}^N)} \le C\|Lu\|_{L^p(\mathbb{R}^N)},$$

$$\|(1+|x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)} \le C\|Lu\|_{L^p(\mathbb{R}^N)},$$

and

$$\|(1+|x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} \le C\|Lu\|_{L^p(\mathbb{R}^N)}$$

for every $u \in C_c^\infty(\mathbb{R}^N)$. In order to prove the estimates of the second order derivatives, we apply the classical Calderón- Zygmund inequality to $(1+|x|^\alpha)u$ and the estimates of the lower order derivates obtained above. We deduce

$$\begin{aligned}
&\|(1+|x|^\alpha)D^2u\|_{L^p(\mathbb{R}^N)} \\
&\qquad \le C(\alpha)[\|D^2((1+|x|^\alpha)u)\|_{L^p(\mathbb{R}^N)}) + \|(1+|x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)}] \\
&\qquad \le C(N,p,\alpha)[\|\Delta((1+|x|^\alpha)u)\|_{L^p(\mathbb{R}^N)} + \|Lu\|_{L^p(\mathbb{R}^N)}] \\
&\qquad \le C(N,p,\alpha)[\|(1+|x|^\alpha)\Delta u\|_{L^p(\mathbb{R}^N)} + \|Lu\|_{L^p(\mathbb{R}^N)}] = C(N,p,\alpha)\|Lu\|_{L^p(\mathbb{R}^N)}.
\end{aligned}$$

By Lemma 6.2, these estimates extend from $C_c^\infty(\mathbb{R}^N)$ to D_p. □

The following lemma is a tool to prove the equality $D_p = D_{p,max}(L)$. Once the latter equality has been proved, it is an obvious consequence of Proposition 4.5.

Lemma 6.4 *If $2 < \alpha < N/p'$, the operator $-(L, D_p)$ is invertible and its inverse is the operator T defined in (7).*

Proof. In fact, T is bounded in L^p, by Lemma 4.2 with $\beta = 0$, and the equality $u = -TLu$ holds for every $u \in C_c^\infty(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is a core for (L, D_p), see Lemma 6.2, then $u = -TLu$ for every $u \in D_p$. Since (L, D_p) is injective, the proof is complete. □

Theorem 6.5 *If $2 < \alpha < N/p'$, then D_p coincides with the maximal domain in L^p, that is*

$$D_p = \{u \in L^p \cap W^{2,p}_{loc} : \; Lu \in L^p\}.$$

Proof. The inclusion $D_p \subset D_{p,max}(L)$ is obvious. Let now $u \in D_{p,max}(L)$. By Corollary 6.4, there exists $v \in D_p$ such that $Lv = Lu$. Therefore $u - v$ belongs to the maximal domain of L and $L(u-v) = 0$, that is $\Delta(u-v) = 0$. Since $u, v \in L^p$, then $u = v$ and u belongs to D_p. □

Next we show that if $\alpha \ge N/p'$ then D_p is properly contained in $D_{p,max}(L)$

Proposition 6.6 *Let $N \ge 3$, $p > N/(N-2)$, $\alpha \ge N/p'$, $\alpha > 2$. Then D_p is a proper subset of $D_{p,max}(L)$.*

PROOF. Let $\chi_{B(1)} \le f \le \chi_{B(2)}$ be a smooth radial function. Denote by u the unique solution in $D_{p,max}(L)$ of $Lu(\rho) = (1+\rho^\alpha)f(\rho)$, see Proposition 4.5. Since the datum f is radial, by uniqueness, the solution u is radial too, hence it solves

$$u''(\rho) + \frac{N-1}{\rho}u'(\rho) = f(\rho).$$

For $\rho \ge 2$, u solves the homogeneous equation

$$u''(\rho) + \frac{N-1}{\rho}u'(\rho) = 0,$$

hence it is given by $u = c\rho^{2-N}$ for some positive c. Then

$$\int_{\mathbb{R}^N} |(1+|x|^{\alpha-2})u|^p dx \ge c_1 \int_2^\infty (1+\rho^{\alpha-2})^p \rho^{p(2-N)} \rho^{N-1}\ d\rho \ge C\int_2^\infty \rho^{p\alpha - Np+N-1} d\rho.$$

The last integral converges if and only if $\alpha < \dfrac{N}{p'}$. In a similar way one can show that $(1+|x|^{\alpha-1})\nabla u$ and $(1+|x|^\alpha)D^2 u$ are not in $L^p(\mathbb{R}^N)$. □

A partial characterization of $D_{p,max}(L)$ can be obtained from Lemma 4.2.

Proposition 6.7 *Let $N \ge 3$, $p > N/(N-2)$, $\alpha \ge N/p'$. If $0 \le \beta < N/p'-2$ and $0 \le \gamma < N/p'-1$, then $|x|^\beta u$ and $|x|^\gamma \nabla u$ belong to L^p, for every $u \in D_{p,max}(L)$.*

PROOF. This follows immediately from Lemma 4.2, since the operator $-T$ defined in (7) is the inverse of $(L, D_{p,max}(L))$. □

Observe that for $\beta = \gamma = 0$ the above result has been already proved in Proposition 3.1.

If $\alpha > N/p'$, then $C_c^\infty(\mathbb{R}^N)$ is not a core for L. This fact also gives $D_p \ne D_{p,max}(L)$ in this case.

Proposition 6.8 *Let $\alpha > N/p'$. Then $L(C_c^\infty(\mathbb{R}^N))$ is not dense in $L^p(\mathbb{R}^N)$.*

PROOF. It is sufficient to observe that $0 \ne \dfrac{1}{1+|x|^\alpha} \in L^{p'}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (1+|x|^\alpha)\Delta u \frac{1}{1+|x|^\alpha}\ dx = \int_{\mathbb{R}^N} \Delta u\ dx = 0$$

for every $u \in C_c^\infty(\mathbb{R}^N)$. □

Proposition 6.9 *Let $\alpha = N/p'$. Then $L(C_c^\infty(\mathbb{R}^N))$ is dense in $L^p(\mathbb{R}^N)$.*

PROOF. Let $g \in L^{p'}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (1+|x|^\alpha)\Delta u \cdot g\, dx = 0$$

for every $u \in C_c^\infty(\mathbb{R}^N)$. It follows that

$$\Delta(g(1+|x|^\alpha)) = 0$$

in the distributional sense, hence in a classical sense. Set $h = g(1+|x|^\alpha)$. Since h is an harmonic function, it satisfies

$$|\nabla h(0)| \le \frac{C}{R^{N+1}} \int_{B(0,R)} |h| dx$$

for every $R > 0$. By assumption $g = \dfrac{h}{(1+|x|^\alpha)} \in L^{p'}(\mathbb{R}^N)$, therefore Hölder's inequality yields

$$|\nabla h(0)| \le \frac{C}{R^{N+1}} \int_{B(0,R)} \frac{|h|}{1+|x|^\alpha}(1+|x|^\alpha)dx \le CR^{-N-1+\frac{N}{p}+\alpha} = CR^{\alpha-\frac{N}{p'}-1} = CR^{-1}.$$

Letting R to infinity, we deduce that $|\nabla h(0)| = 0$. In a similar way one proves that $|\nabla h(x_0)| = 0$ for every $x_0 \in \mathbb{R}^N$. It means that $h = C$ for some constant C and $g = \dfrac{C}{1+|x|^\alpha} \in L^{p'}(\mathbb{R}^N)$. But $\dfrac{1}{1+|x|^\alpha} \in L^{p'}(\mathbb{R}^N)$ if and only if $\alpha > \frac{N}{p'}$, therefore $C = 0$ and, consequently, $g = 0$. This proves the density of $L(C_c^\infty(\mathbb{R}^N))$ in $L^p(\mathbb{R}^N)$. □

It can be proved that the a-priori estimates of Proposition 6.3 for $p = 2$ still hold in $C_c^\infty(\mathbb{R}^N)$ if $\alpha \neq N/2$. However, since $C_c^\infty(\mathbb{R}^N)$ is not a core for $(L, D_{2,max}(L)$, for $\alpha > N/2$ they do not extend to the domain of L. Next we show that the a-priori estimates fail even in $C_c^\infty(\mathbb{R}^N)$ if $\alpha = N/p'$, which is a core by the Proposition above.

Proposition 6.10 *Let $N \ge 3$, $\alpha = N/p'$. Then the estimates in Proposition 6.3 do not hold on $C_c^\infty(\mathbb{R}^N)$.*

PROOF. Let, $R \ge 2$, $\phi \in C_c^\infty(\mathbb{R}^N)$ be a radial function such that $\phi = 1$ in $B(R) \setminus B(2)$, $\phi = 0$ in $B(1) \cup \big(\mathbb{R}^N \setminus B(2R)\big)$, $\|\phi_R'\|_\infty \le \frac{C}{R}$, $\|\phi_R''\|_\infty \le \frac{C}{R^2}$ and set $u(\rho) = \phi_R \rho^{2-N}$, $N > 2$. Then $u \in C_c^\infty(\mathbb{R}^N)$ (we omit to indicate the dependence of u on R) and

$$\Delta u = u'' + \frac{N-1}{\rho}u' = \phi_R''(\rho)\rho^{2-N} + (3-N)\phi_R'(\rho)\rho^{1-N}.$$

A straightforward computation shows that, for $\alpha = \frac{N}{p'}$,

$$\int_{\mathbb{R}^N} |(1+|x|^\alpha)\Delta u|^p \, dx \le C$$

with C independent of R. On the other hand $u'(\rho) = \phi_R'(\rho)\rho^{2-N} + \phi_R(2-N)\rho^{1-N}$ and

$$\int_0^\infty (1+\rho^{\alpha-1})^p \rho^{N-1} |u'(\rho)|^p \, d\rho = \int_1^{2R} (1+\rho^{\alpha-1})^p \rho^{N-1} \left|\phi_R'(\rho)\rho^{2-N} + \phi_R(2-N)\rho^{1-N}\right|^p d\rho.$$

The last integral tends to ∞ as $R \to \infty$ since

$$\int_1^{2R} (1+\rho^{\alpha-1})^p \rho^{N-1} \left|\phi_R(2-N)\rho^{1-N}\right|^p \, d\rho \ge C \log R.$$

Therefore the L^p-norm of $(1+|x|^{\alpha-1})\nabla u$ cannot be controlled by the L^p-norm of Lu on $C_c^\infty(\mathbb{R}^N)$ Similarly one shows that the L^p-norm of $(1+|x|^\alpha)D^2u$ cannot be controlled by the L^p-norm of Lu.

□

7 The operator in $C_0(\mathbb{R}^N)$

Let

$$D(L) = D_{max}(L) \cap C_0(\mathbb{R}^N)$$

be the generator of $(T_{min}(t))_{t\geq 0}$ in C_0, see Proposition 2.2 (iii) and note that we need the only restriction $N, \alpha > 2$.

As in the L^p-case we give a description of the domain when $\alpha < N$ and a partial description when $\alpha \geq N$.

We need the analogous of Lemma 4.2 for $p = \infty$

Lemma 7.1 *Let $\gamma,\ \beta > 0$ such that $\gamma < N$ and $\gamma + \beta > N$. Set*

$$J(x) = \int_{\mathbb{R}^N} \frac{dy}{|x-y|^\gamma (1+|y|^\beta)}.$$

Then J is bounded in $\mathbb{R}^N$ and has the following behaviour as $|x|$ goes to infinity

$$J(x) \simeq \begin{cases} c_1 |x|^{N-(\gamma+\beta)} & \text{if} \quad \beta < N \\ c_2 |x|^{-\gamma} \log|x| & \text{if} \quad \beta = N \\ c_3 |x|^{-\gamma} & \text{if} \quad \beta > N \end{cases}$$

for suitable positive constants c_1, c_2, c_3.

Proof. Since $\dfrac{1}{|y|^\gamma}$ and $\dfrac{1}{1+|y|^\beta}$ are radial decreasing $J(x) \leq J(0) < \infty$. In order to prove the asymptotic behaviour, we write J in spherical coordinates. Set $x = s\eta$, $y = \rho\omega$ with $s, \rho \in [0, +\infty)$, $\eta\omega \in S_{N-1}$, then

$$\begin{aligned} J(s\eta) &= \int_{S_{N-1}} d\omega \int_0^\infty \frac{\rho^{N-1}\, d\rho}{(1+\rho^\alpha)|s\eta - \rho\omega|^{N-2}} \\ &= \int_{S_{N-1}} d\omega \int_0^\infty \frac{s^N \xi^{N-1}\, d\xi}{s^\gamma (1+(s\xi)^\beta)|\eta - \xi\omega|^\gamma} = \int_{S_{N-1}} d\omega \int_0^\infty \frac{s^{N-\gamma} \xi^{N-1}\, d\xi}{(1+s^\beta \xi^\beta)|e_1 - \xi\omega|^\gamma} \\ &= \int_{S_{N-1}} d\omega \int_0^{\frac{1}{2}} \frac{s^{N-\gamma} \xi^{N-1}\, d\xi}{(1+s^\beta \xi^\beta)|e_1 - \xi\omega|^\gamma} + \int_{S_{N-1}} d\omega \int_{\frac{1}{2}}^\infty \frac{s^{N-\gamma} \xi^{N-1}\, d\xi}{(1+s^\beta \xi^\beta)|e_1 - \xi\omega|^\gamma}. \end{aligned}$$

Set

$$J_1(s\eta) = \int_{S_{N-1}} d\omega \int_0^{\frac{1}{2}} \frac{s^{N-\gamma} \xi^{N-1}\, d\xi}{(1+s^\beta \xi^\beta)|e_1 - \xi\omega|^\gamma}$$

and

$$J_2(s\eta) = \int_{S_{N-1}} d\omega \int_{\frac{1}{2}}^\infty \frac{s^{N-\gamma} \xi^{N-1}\, d\xi}{(1+s^\beta \xi^\beta)|e_1 - \xi\omega|^\gamma}.$$

Concerning J_2, we have

$$\lim_{s\to+\infty} s^{\gamma+\beta-N} J_2 = \int_{S_{N-1}} d\omega \int_{\frac{1}{2}}^\infty \frac{\xi^{N-1}\, d\xi}{\xi^\beta |e_1 - \xi\omega|^\gamma} = \int_{\mathbb{R}^N \setminus B(0,\frac{1}{2})} \frac{dy}{|y|^\beta |e_1 - y|^\gamma} = C > 0$$

for some positive contant C. Therefore

$$J_2(x) \simeq C|x|^{N-(\gamma+\beta)} \tag{15}$$

as $|x| \to \infty$. Let us estimate the remaining term. We have

$$J_1(s\eta) = \int_{S_{N-1}} d\omega \int_0^{\frac{1}{2}} \frac{s^{N-\gamma}\xi^{N-1}\, d\xi}{(1+s^\beta\xi^\beta)|e_1-\xi\omega|^\gamma} \quad = \quad s^{-\gamma}\int_{S_{N-1}} d\omega \int_0^{\frac{s}{2}} \frac{t^{N-1}\, dt}{(1+t^\beta)\left|e_1-\frac{t}{s}\omega\right|^\gamma}.$$

Since $\frac{1}{2} \le \left|e_1 - \frac{t}{s}\omega\right| \le \frac{3}{2}$,

$$c_1 J_1 \le s^{-\gamma}\int_0^{\frac{s}{2}} \frac{t^{N-1}\, dt}{(1+t^\beta)} \le c_2 J_1$$

for some positive $c_1,\ c_2$. Evidently

$$s^{-\gamma}\int_0^{\frac{s}{2}} \frac{t^{N-1}\, dt}{(1+t^\beta)} \simeq \begin{cases} |s|^{N-(\gamma+\beta)} & \text{if} \quad \beta < N, \\ |s|^{-\gamma}\log|x| & \text{if} \quad \beta = N, \\ |s|^{-\gamma} & \text{if} \quad \beta > N \end{cases}$$

as s goes to infinity. From (15) and the last estimate the aymptotic behaviour of J follows. □

The following two results are deduced from the lemma above as Theorem 6.5 and Proposition 6.7 are deduced from Lemma 4.2.

Theorem 7.2 *Let $2 < \alpha < N$. Then*

$$D(L) = \{u \in C_0 :\ (1+|x|^{\alpha-2})u,\ (1+|x|^{\alpha-1})\nabla u,\ (1+|x|^\alpha)\Delta u \in C_0\}$$

Proposition 7.3 *Let $N \ge 3$, $\alpha \ge N$. If $0 \le \beta < N-2$ and $0 \le \gamma < N-1$, then for every $u \in D(L)$, $|x|^\beta u$ and $|x|^\gamma \nabla u$ belong to C_0.*

Finally, we compute the operator norm in C_0 of the operator $T = (-L)^{-1}$ defined in (7)

Proposition 7.4 *If $N \ge 3$ and $\alpha > 2$ then*

$$\|T\|_\infty = \frac{\pi}{(N-2)\alpha\sin\left(\frac{2}{\alpha}\pi\right)}.$$

Proof. We have

$$\|T\| = \frac{1}{N(N-2)\omega_N}\sup_{x\in\mathbb{R}^N} J(x) = \frac{1}{N(N-2)\omega_N}J(0) = \frac{1}{N(N-2)\omega_N}\int_{\mathbb{R}^N}\frac{dy}{|y|^{N-2}(1+|y|^\alpha)}.$$

Since

$$\frac{1}{N\omega_N}\int_{\mathbb{R}^N}\frac{dy}{|y|^{N-2}(1+|y|^\alpha)} = \int_0^\infty \frac{s^{N-1}}{s^{N-2}(1+s^\alpha)}ds = \frac{1}{\alpha}\int_0^\infty \frac{t^{\frac{2}{\alpha}-1}}{1+t}dt = \frac{\pi}{\alpha\sin\left(\frac{2}{\alpha}\pi\right)}$$

the proof is complete. □

8 Discreteness and location of the spectrum

Throughout this section, to unify the notation, when $p = \infty$, L^p stands for C_0 and $D_{p,max}(L)$ for $D(L)$.

Proposition 8.1 *If $N/(N-2) < p < \infty$, $2 < \alpha \le \infty$, then the resolvent of $(L, D_{p,max}(L))$ is compact in L^p.*

Proof. Let us prove that $D_{p,max(L)}$ is compactly embedded into L^p for $p < \infty$. Let $\mathcal{U}$ be a bounded subset of $D_{p,max}(L)$. Fixing $0 < \beta < \alpha - 2$, $N/p' - 2$ in Lemma 4.2 we obtain $\int_{\mathbb{R}^N} |(1+|x|^\beta)u|^p \le M$ for some positive M and for every $u \in \mathcal{U}$. Then, given $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|>R} |u|^p < \varepsilon^p$$

for every $u \in \mathcal{U}$. Let $\mathcal{U}'$ be the set of the restrictions of the functions in $\mathcal{U}$ to $B(R)$. Since the embedding of $W^{2,p}(B(R))$ into $L^p(B(R))$ is compact, the set $\mathcal{U}'$ which is bounded in $W^{2,p}(B(R))$ is totally bounded in $L^p(B(R))$. Therefore there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in L^p(B(R))$ such that

$$\mathcal{U}' \subseteq \bigcup_{i=1}^{n} \{f \in L^p(B(R)) : \ \|f - f_i\|_{L^p(B(R))} < \varepsilon\}.$$

Set $\overline{f}_i = f_i$ in $B(R)$ and $\overline{f}_i = 0$ in $\mathbb{R}^N \setminus B(R)$. Then $\overline{f}_i \in L^p(\mathbb{R}^N)$ and

$$\mathcal{U} \subseteq \bigcup_{i=1}^{n} \{f \in L^p(\mathbb{R}^N) : \ \|f - \overline{f}_i\|_{L^p(\mathbb{R})} < 2\varepsilon\}.$$

It follows that $\mathcal{U}$ is relatively compact in $L^p(\mathbb{R}^N)$. The compactness of the resolvent of $(L, D(L))$ in C_0 follows similarly from the results of the previous section or from ([11, Example 7.3]). $\square$

Clearly, the spectrum of L consists of eigenvalues. Let us show that it is independent of p.

Corollary 8.2 *If $N/(N-2) < p \le \infty$, $2 < \alpha < \infty$, then the spectrum of $(L, D_{p,max}(L))$ is independent of p.*

Proof. Let ρ_p, ρ_q be the resolvent sets in L^p, L^q, respectively. Then $0 \in \rho_p \cap \rho_q$ and the inverse of L in L^p and in L^q is given by the operator $-T$ defined in (7), see Proposition 4.5. This shows the consistency of the resolvents at 0 and, since $\rho_p \cap \rho_q$ is connected, the consistency of the resolvents at any point of $\rho_p \cap \rho_q$, see [1, Proposition 2.2]. An application of [1, Proposition 2.6] concludes the proof. $\square$

In order to have more information on the spectrum of L, we introduce the Hilbert space L^2_μ, where $d\mu(x) = (1+|x|^\alpha)^{-1}dx$, endowed with its canonical inner product. Note that the measure μ is finite if and only if $\alpha > N$. We consider also the Sobolev space

$$H = \{u \in L^2_\mu : \nabla u \in L^2\}$$

endowed with the inner product

$$(u, v)_H = \int_{\mathbb{R}^N} (u\bar{v}\, d\mu + \nabla u \cdot \nabla \bar{v}\, dx)$$

and let $\mathcal{V}$ be the closure of C_c^∞ in H, with respect to the norm of H. Observe that Sobolev inequality

$$\|u\|_{2^*}^2 \le C_2^2 \|\nabla u\|_2^2 \tag{16}$$

holds in $\mathcal{V}$ but not in H (consider for example the case where $\alpha > N$ and $u = 1$). Here $2^* = 2N/(N-2)$ and C_2 is the best constant for which the equality above holds.

Lemma 8.3 *If $\alpha > N$, the embedding of $\mathcal{V}$ in L^2_μ is compact.*

Proof. The proof is very similar to that of Proposition 8.1 once one notes that on any ball $B(R)$ the measure μ is bounded above and below from zero. Therefore, it suffices to show that given $\mathcal{U}$ a bounded subset of $\mathcal{V}$ and $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|>R} |u|^2 \, d\mu < \varepsilon^2$$

for every $u \in \mathcal{U}$. This easily follows from (16) since

$$\int_{|x|>R} |u|^2 \, d\mu \le \left(\int_{|x|>R} |u|^{\frac{2N}{N-2}} \, dx \right)^{1-\frac{2}{N}} \left(\int_{|x|>R} \frac{1}{(1+|x|^\alpha)^{\frac{N}{2}}} \, dx \right)^{\frac{2}{N}}.$$

$\square$

Next we introduce the continuous and weakly coercive symmetric form

$$a(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} \, dx \tag{17}$$

for $u, v \in \mathcal{V}$ and the self-adjoint operator $\mathcal{L}$ defined by

$$D(\mathcal{L}) = \{u \in L^2_\mu : \text{there exists } f \in L^2_\mu : a(u,v) = -\int_{\mathbb{R}^N} f \bar{v} \, d\mu \text{ for every } v \in \mathcal{V}\} \qquad \mathcal{L}u = f.$$

Since $a(u,u) \ge 0$, the operator $\mathcal{L}$ generates an analytic semigroup of contractions in $e^{t\mathcal{L}}$ in L^2_μ. An application of the Beurling-Deny criteria shows that the generated semigroup is positive and L^∞-contractive. For our purposes we need to show that the resolvent of $\mathcal{L}$ and of $(L, D_{p,max}(L))$ are coherent. This is done in the following proposition.

Proposition 8.4

$$D(\mathcal{L}) \subset \{u \in \mathcal{V} \cap W^{2,2}_{loc} : (1+|x|^\alpha)\Delta u \in L^2_\mu\}$$

and $\mathcal{L}u = (1+|x|^\alpha)u$ for $u \in D(\mathcal{L})$. If $\lambda > 0$ and $f \in L^p \cap L^2_\mu$, then

$$(\lambda - \mathcal{L})^{-1} f = (\lambda - L)^{-1} f.$$

Proof. The first part of the proposition easily follows from local elliptic regularity, testing with any $v \in C_c^\infty$ in (17). To show the coherence of the resolvents we consider $f \in C_c^\infty$, *supp* $f \subset B(R)$ and $u = (\lambda - L_{min})^{-1} f$. Then $u \in D(L)$ solves

$$\Delta u = \frac{\lambda u}{1+|x|^\alpha}$$

outside $B(R)$ and is a C^2-function. Theorem 4.6 implies that $u \in D_{p,max}(L)$ for every $p > N/(N-2)$. If $N > 4$, then $u \in D_{2,max}(L)$ hence $\nabla u \in L^2$, see Proposition 3.1, and clearly $u \in L^2_\mu$. This yields $u \in H$ but not yet $u \in \mathcal{V}$. To show that u can be approximated with a sequence of C_c^∞-functions, in the norm of H, we fix a smooth C^∞ function η such that $\eta \equiv 1$ in $B(1)$ and $\eta \equiv 0$ outside $B(2)$ and set $\eta_n(x) = \eta(x/n)$. Clearly $\eta_n u \to u$ in L^2_μ. Concerning the gradients we have $\nabla(\eta_n u) = \eta_n \nabla u + u \nabla \eta_n$. The term $\eta_n \nabla u$ converges to ∇u in L^2, since $\nabla u \in L^2$ and we have to show that $u \nabla \eta_n \to 0$ in L^2. Since $u \in L^{2^*}$ we can use Hölder's inequality to deduce

$$\int_{\mathbb{R}^N} |u|^2 |\nabla \eta_n|^2 \, dx \le \frac{C}{n^2} \int_{n \le |x| \le 2n} |u|^2 \, dx \le C_1 \left(\int_{n \le |x| \le 2n} |u|^{2^*} \right)^{1-\frac{2}{N}} \tag{18}$$

which tends to zero as $n \to \infty$. This shows u can be approximated with a sequence of $W^{1,2}$ compactly supported functions and to produce a sequence of smooth approximants it is now sufficient to use convolutions. Then $u \in \mathcal{V}$ and, by integration by parts,

$$a(u,v) = -(\lambda u - f, v)_{L^2_\mu},$$

that is $u \in D(\mathcal{L})$ and $\lambda u - \mathcal{L} u = f$. By density, this shows the coeherence of the resolvents of L_{min} and $\mathcal{L}$ for $\lambda > 0$, hence of $\mathcal{L}$ and $(L, D_{p,max}(L))$, see Theorem 4.6. The cases $N = 3, 4$ require some variants, since $p = 2$ does not safisfy the inequality $p > N/(N-2)$. To show that u belongs to L^2_μ we use the fact that $u \in L^p$ with $p = 2N/(N-2)$ and therefore

$$\int_{\mathbb{R}^N} |u|^2 \, d\mu \le \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \, dx \right)^{1-\frac{2}{N}} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^\alpha)^{\frac{N}{2}}} \, dx \right)^{\frac{2}{N}}$$

Next we show that ∇u belongs to L^2. Since $u \in C^2$ and

$$\int_{\mathbb{R}^N} \nabla u \nabla v \, dx = \int_{\mathbb{R}^N} (\lambda u - f) v \, d\mu$$

for every $v \in C_c^\infty$, the same equality holds for every $v \in W^{1,2}$ having compact support. Taking $v_n = \eta_n u$ we get

$$\int_{\mathbb{R}^N} \eta_n |\nabla u|^2 dx = \int_{\mathbb{R}^N} (\lambda u - f) \eta_n u \, d\mu - \int_{\mathbb{R}^N} u \nabla u \cdot \nabla \eta_n \, dx.$$

Since

$$\int_{\mathbb{R}^N} u \nabla u \cdot \nabla \eta_n \, dx = -\frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \Delta \eta_n \, dx$$

we can proceed as in (18) to show that this term tends to zero and hence $\nabla u \in L^2$. From now one, the proof proceeds as in the case $N > 4$. □

We can now strengthen Corollary 8.2.

Proposition 8.5 *If $N/(N-2) < p \le \infty$, $2 < \alpha < \infty$, then the spectrum of $(L, D_{p,max}(L))$ lies in $]-\infty, 0[$ and consists of a sequence λ_n of eigenvalues, which are simple poles of the resolvent and tend to $-\infty$. Each eigenspace is finite dimensional and independent of p.*

Proof. Since the resolvents of $(L, D_{p,max}(L))$ and $(\mathcal{L}, (D(\mathcal{L}))$ are coherent and compact in L^p, L^2_μ, respectively all the assertions except the density of the eigenfuctions follow from [1, Proposition 2.2] (see also [10, Proposition 5.2] for more details). □

Observe that 0 is in the resolvent set of L, since it is injective. This is clear in L^p or C_0 because $\Delta u \in L^p$ implies $u = 0$. However, constant functions are in H if $\alpha > N$ and this explains why we work with $\mathcal{V}$ (constant functions are never in $\mathcal{V}$ since $\mathcal{V}$ embeds into L^{2^*}).

Next we show some methods to estimate the first eigenvalue λ_1.

Proposition 8.6 *The following estimates hold*

$$\lambda_1 \leq -\left(\frac{\alpha-2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha-2} \frac{(N-2)^2}{4} \tag{19}$$

and

$$\lambda_1 \leq -(N-2)\frac{\alpha \sin\left(\frac{2}{\alpha}\pi\right)}{\pi}. \tag{20}$$

Proof. By Corollary 4.3, we obtain

$$\|L^{-1}\| \leq \left(\frac{2}{\alpha-2}\right)^{\frac{2}{\alpha}} \frac{\alpha-2}{\alpha} \frac{p^2}{N(Np-N-2p)}$$

By classical spectral theory then

$$|\lambda_1| \geq \left(\frac{\alpha-2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha-2} \frac{N(Np-N-2p)}{p^2}.$$

The function appearing on the right hand side attaints its maximum for $p = \dfrac{2N}{N-2}$ where it reaches the value

$$\left(\frac{\alpha-2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha-2} \frac{(N-2)^2}{4}.$$

Since the spectrum of L is independent of p we obtain (19). (20) is obtained in a similar way from Proposition 7.4. $\square$

Observe that the coefficient

$$\left(\frac{\alpha-2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha-2}$$

is always greater than or equal to 1, and it is 1 for $\alpha = 2, \infty$. Then (19) improves the estimate $\lambda_1 \leq -(N-2)^2/4$ which can be obtained using the classical Hardy inequality. On the other hand (20) is better than (19) for large α and small N, but worse for α close to 2 or large N.

Since $\mathcal{L}$ is self-adjoint in L^2_μ, its eigenvalues can be computed through the Raleigh quotients and, in particular,

$$-\lambda_1 = \min\left\{\int_{\mathbb{R}^N} |\nabla u|^2\, dx : \int_{\mathbb{R}^N} |u|^2\, d\mu = 1\right\}.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^2\, d\mu &\leq \left(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\right)^{1-2\frac{2}{N}} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^\alpha)^{N/2}}\, dx\right)^{2/N} \\ &\leq C_2^2 \int_{\mathbb{R}^N} |\nabla u|^2\, dx \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^\alpha)^{N/2}}\, dx\right)^{2/N},\end{aligned}$$

it follows that $-\lambda_1 \geq \left(C_2^2 L(\alpha)\right)^{-1}$ where C_2 is given by [15] and

$$L(\alpha) = \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^\alpha)^{N/2}}\, dx\right)^{2/N}.$$

Obesrve also that, when $\alpha \to \infty$, then (formally) λ_1 tends to the first eigenvalue of the Dirichlet Laplacian in the unit ball.

Appendix

Here we prove a Hardy-type inequality used throughout the paper.

Proposition 8.7 *Let $u \in W^{1,p}(\mathbb{R}^N)$ with compact support, $1 < p < \infty$, $\gamma \geq 0$. Then*

$$\int_{\mathbb{R}^N} |u|^p |x|^\gamma \leq \left(\frac{p}{\gamma+N}\right)^2 \int_{\mathbb{R}^N} |u|^{p-4} |Re(\overline{u}\nabla u)|^2 |x|^{\gamma+2}.$$

Proof. Let first $u \in C_c^\infty(\mathbb{R}^N)$ and set $g(t) = u(tx)$. Then

$$\begin{aligned} |u(x)|^p &= |g(1)|^p = -p \int_1^\infty |g|^{p-2} Re\left(\overline{g}\frac{\partial g}{\partial t}\right) dt \\ &= -p \int_1^\infty |u(tx)|^{p-2} Re(\overline{u}\nabla u(tx)) x dt. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} |u(x)|^p |x|^\gamma dx \leq p \int_1^\infty dt \int_{\mathbb{R}^N} |u(tx)|^{p-2} |Re(\overline{u}(tx)\nabla u(tx))| |x|^{\gamma+1} dx \\ &= p \int_1^\infty dt \int_{S_{N-1}} d\omega \int_0^\infty |u(tr\omega)|^{p-2} |Re(\overline{u}(tr\omega)\nabla u(tr\omega)| r^{\gamma+N} dr \\ &= p \int_1^\infty \frac{1}{t^{\gamma+N+1}} dt \int_{S_{N-1}} d\omega \int_0^\infty |u(s\omega)|^{p-2} |Re(\overline{u}(s\omega)\nabla u(s\omega)| s^{\gamma+N} ds \\ &= \frac{p}{\gamma+N} \int_{\mathbb{R}^N} |u(x)|^{p-2} |Re(\overline{u}(x)\nabla u(x)| |x|^{\gamma+1} dx. \end{aligned}$$

By density this inequality holds for every $u \in W^{1,p}(\mathbb{R}^N)$ having compact support. At this point Hölder's inequality yields

$$\int_{\mathbb{R}^N} |u(x)|^p |x|^\gamma dx \leq \frac{p}{\gamma+N} \left(\int_{\mathbb{R}^N} |u(x)|^p |x|^\gamma dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u(x)|^{p-4} |Re(\overline{u}(x)\nabla u(x)| |x|^{\gamma+2} dx\right)^{\frac{1}{2}}.$$

$\square$

References

[1] W. Arendt: Gaussian estimates and interpolation of the spectrum in L^p, *Diff. Int. Eq.,* **Vol. 7,n. 5** (1994), 1153-1168.

[2] S. Fornaro, L. Lorenzi: Generation results for elliptic operators with unbounded diffusion coefficients in L^p and C_b-spaces, *Discrete and continuous dynamical sistems,* **Vol. 18, N.4** (2007), 747-772.

[3] K.J. Engel, R. Nagel: *One parameter semigroups for linear evolutions equations,* Springer-Verlag, Berlin, (2000).

[4] A. Friedman: *Partial Differential Equations of parabolic Type,* Prentice Hall, New Jersey, (1964).

[5] D. Gilbarg, N. Trudinger: *Elliptic Partial Differential Equations of Second Order,* Second edition, Springer, Berlin, (1983).

[6] Paul Krée: Sur les multiplicateurs dans $\mathcal{F}L^p$ avec poids, *Ann. Inst. Fourier, Grenoble,* **Vol. 16, N. 2** (1966), 91-121.

[7] N.V. Krylov: *Lectures on elliptic and parabolic equations in Holder spaces,* Graduate studies in mathematics, American Mathematical Society, (1996)

[8] A. Lunardi: *Analytic semigroups and optimal regularity in parabolic problems,* Progress in Nonlinear Differential Equations and Their Applications **16**, Birkäuser, (1995).

[9] G. Metafune, C. Spina: An integration by parts formula in Sobolev spaces, *Mediterranean Journal of Mathematics,* **Vol. 5, N. 3,** (2008), 359-371.

[10] G. Metafune, D. Pallara: Trace formulas for some singular differential operators and applications, *Math. Nachr.,* **Vol. 211** (2000), 127-157.

[11] G. Metafune, D. Pallara, M. Wacker: Feller Semigroups on $\mathbb{R}^N$, *Semigroup Forum,* **Vol. 65** (2002), 159-205.

[12] B. Muckenhoupt, R. Wheeden: Weighted Norm Inequalities for Singular and Fractional Integrals, *Transactions of the America Mathematical Society,* **Vol. 161,** (1971), 249-258.

[13] A. Pazy: *Semigroups of linear operators and applications to partial differential equations,* Applied mathematical sciences **44**, New York : Springer-Verlag, (1983).

[14] E. M. Stein, G. Weiss: *Introduction to Fourier analysis on euclidean spaces,* Princeton University Press , (1971).

[15] G. Talenti: Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.,* **Vol. 4, N. 110** (1976), 353-372.