

Classification of General Sequences by Frame-Related Operators

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Abstract

This note is a survey and collection of results, as well as presenting some original research. For Bessel sequences and frames, the analysis, synthesis and frame operators as well as the Gram matrix are well-known, bounded operators. We investigate these operators for arbitrary sequences, which in general lead to possibly unbounded operators. We characterize various classes of sequences in terms of these operators and vice-versa. Finally, we classify these sequences by operators applied on orthonormal bases.

Key words and phrases : analysis operator, synthesis operator, frame operator, Gram matrix, Bessel sequence, frame, Riesz basis, lower frame sequence, Riesz-Fischer sequence

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1 Introduction

Frames [12, 6, 10] have become an important topic in applied mathematics in the last decades. They are also used in many engineering applications, for

example in signal processing [5]. Clearly, they have some important advantages compared to orthonormal bases. However, even the frame condition cannot always be satisfied for the whole space, and so other classes of sequences have been investigated, for example, frame sequences, Bessel sequences, lower frame sequences, and Riesz-Fischer sequences [4, 7, 9, 10]. For such sequences, which need not be frames in general, the frame-related operators, i.e. the analysis, the synthesis and the frame operator can still be defined, see e.g. [7, 9]. In general, these operators can be unbounded.

In this note we want to give an overview of the connection between the properties of those operators and those of the sequences. We collect some existing results, extend them and add new, original results. Some known results are proved here for the sake of completeness, in particular, when there is no proof in the literature. Note that one result was obtained independently in [15]. While the paper [15] focuses on the investigation of sufficient conditions for the closability of the synthesis operator, the main aim of the present paper is to give a general overview of all the associated operators.

Our notation and some preliminary results are given in Section 2. In Section 3 we consider the operators associated to an arbitrary sequence, namely, the analysis, synthesis and ‘frame’ operators, as well as the operator based on the Gram matrix, and we investigate their properties. Section 4 is devoted to the characterization of sequences (Bessel and frame sequences, frames, Riesz bases, lower frame sequences, Riesz-Fischer sequences, complete sequences) via the associated operators. Finally, we consider operators that preserve the sequence type and present the classification via orthonormal bases.

2 Notation and Preliminaries

Throughout the paper we consider a sequence $\Psi = (\psi_k)_{k=1}^{\infty}$ with elements from a (infinite dimensional) Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The notation $\text{span}\{\Psi\}$ is used to denote the linear span of $(\psi_k)_{k=1}^{\infty}$ and $\overline{\text{span}}\{\Psi\}$ denotes the closed linear span of $(\psi_k)_{k=1}^{\infty}$. The sequence $(e_k)_{k=1}^{\infty}$ denotes an orthonormal basis of \mathcal{H} and $(\delta_k)_{k=1}^{\infty}$ denotes the canonical basis of ℓ^2 . The notion *operator* is used for a linear mapping. Given an operator F , we denote its domain by $\text{dom}(F)$, its range by $\text{ran}(F)$ and its kernel by $\text{ker}(F)$. First we recall the definitions of the basic concepts used in the paper.

Definition 2.1 *The sequence Ψ is called*

- *complete in \mathcal{H} if $\overline{\text{span}}\{\Psi\} = \mathcal{H}$;*
- *a Bessel sequence for \mathcal{H} with bound B if $B > 0$ and $\sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 \leq B \|f\|^2$ for every $f \in \mathcal{H}$;*
- *a lower frame sequence for \mathcal{H} with bound A if $A > 0$ and $A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2$ for every $f \in \mathcal{H}$;*

- a frame for \mathcal{H} with bounds A, B if it is a Bessel sequence for \mathcal{H} with bound B and a lower frame sequence for \mathcal{H} with bound A ;
- a frame-sequence if it is a frame for its closed linear span;
- a Riesz basis for \mathcal{H} with bounds A, B if Ψ is complete in \mathcal{H} , $A > 0$, $B < \infty$, and $A \sum |c_k|^2 \leq \|\sum c_k \psi_k\|^2 \leq B \sum |c_k|^2$ for all finite scalar sequences (c_k) (and hence, for all $(c_k)_{k=1}^\infty \in \ell^2$);
- a Riesz-Fischer sequence with bound A if $A > 0$ and $A \sum |c_k|^2 \leq \|\sum c_k \psi_k\|^2$ for all finite scalar sequences (c_k) (and hence, for all $(c_k)_{k=1}^\infty \in \ell^2$ such that $\sum_{k=1}^\infty c_k \psi_k$ converges in \mathcal{H}).

Lemma 2.2 Let $(d_k)_{k=1}^\infty$ be a sequence such that

$$\sum_{k=1}^{\infty} c_k d_k \text{ converges for all } (c_k)_{k=1}^\infty \in \ell^2. \quad (1)$$

Then $(d_k)_{k=1}^\infty \in \ell^2$.

Proof: If (1) is assumed to hold with absolute convergence of the series, the above statement is proved in [17, §30 1.(5)]. Similar proof (using $x_j = \frac{1}{i|M_i|} \overline{v_j}$ instead of $x_j = \frac{1}{iM_i} v_j$) leads to the validity of the above lemma. \square

We also need the following well-known results:

Proposition 2.3 Let an operator F be densely defined. Then

- [11, Prop. X.1.6] F^* is closed; in particular, every self-adjoint operator is closed;
- [11, Prop. X.1.6] F is closable if and only if F^* is densely defined;
- [21, Th. VIII.1] If F is closable, then its closure is $\overline{F} = (F^*)^*$ and $\overline{F}^* = F^*$;
- [11, Prop. X.1.13] $(\text{ran}(F))^\perp = \ker(F^*)$;
- [11, Prop. X.4.2] If F is closed, then F^*F is self-adjoint.

3 Associated Operators for Arbitrary Sequences

For any sequence Ψ , the associated analysis operator C , synthesis operator D , ‘frame’ operator S , and Gram operator G are (possibly unbounded) operators, defined as follows:

$$C : \text{dom}(C) \rightarrow \ell^2, \quad Cf = (\langle f, \psi_k \rangle)_{k=1}^\infty, \text{ where} \\ \text{dom}(C) = \{f \in \mathcal{H} : (\langle f, \psi_k \rangle)_{k=1}^\infty \in \ell^2\};$$

$$\begin{aligned}
D : \text{dom}(D) &\rightarrow \mathcal{H}, D(c_k)_{k=1}^\infty = \sum_{k=1}^\infty c_k \psi_k, \text{ where} \\
\text{dom}(D) &= \{(c_k)_{k=1}^\infty \in \ell^2 : \sum_{k=1}^\infty c_k \psi_k \text{ converges in } \mathcal{H}\}; \\
S : \text{dom}(S) &\rightarrow \mathcal{H}, Sf = \sum_{k=1}^\infty \langle f, \psi_k \rangle \psi_k, \text{ where} \\
\text{dom}(S) &= \{f \in \mathcal{H} : \sum_{k=1}^\infty \langle f, \psi_k \rangle \psi_k \text{ converges in } \mathcal{H}\}; \\
G : \text{dom}(G) &\rightarrow \ell^2, G(c_k)_{k=1}^\infty = (\sum_{l=1}^\infty G_{k,l} c_l)_{k=1}^\infty, \text{ where} \\
\text{dom}(G) &= \{(c_k)_{k=1}^\infty \in \ell^2 : \sum_{l=1}^\infty G_{k,l} c_l \text{ converges } \forall k \in \mathbb{N} \\
&\quad \text{and } (\sum_{l=1}^\infty G_{k,l} c_l)_{k=1}^\infty \in \ell^2\}, \\
&\text{and the Gram matrix } (G_{k,l})_{k,l} \text{ is defined by } G_{k,l} = \langle \psi_l, \psi_k \rangle, k, l \in \mathbb{N}.
\end{aligned}$$

3.1 Domains

Lemma 3.1 *Given an arbitrary sequence Ψ , the following statements hold.*

- (i.1) $\text{dom}(S) \subseteq \text{dom}(C)$.
- (i.2) $\text{dom}(S) = \text{dom}(C)$ if and only if $\text{ran}(C) \subseteq \text{dom}(D)$.
- (ii) $\text{dom}(D) \subseteq \text{dom}(G)$ if and only if $\text{ran}(D) \subseteq \text{dom}(C)$.
- (iii) D is densely defined.
- (iv) [9, Prop. 4.5] *If $\sum_{l=1}^\infty |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every $k \in \mathbb{N}$, then C is densely defined.*
- (v) *If $\sum_{l=1}^\infty |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every $k \in \mathbb{N}$, then G is densely defined.*
- (vi) *If $\sum_{l=1}^\infty \sum_{k=1}^\infty |\langle \psi_l, \psi_k \rangle|^2 < \infty$, then $\text{dom}(G) = \text{dom}(D) = \ell^2$, G is a Hilbert-Schmidt operator and Ψ is a Bessel sequence for \mathcal{H} .*
- (vi) *If $\sum_{l=1}^\infty \sum_{k=1}^\infty |\langle \psi_l, \psi_k \rangle|^2 < \infty$, then $\text{dom}(G) = \text{dom}(D) = \ell^2$ and G is a Hilbert-Schmidt operator.*
- (vii) *If $\psi_k \in \text{dom}(S)$ for every $k \in \mathbb{N}$, then S is densely defined.*

Proof: (i.1) Assume that $f \in \text{dom}(S)$. Then $\sum_{k=1}^N \langle \langle f, \psi_k \rangle \psi_k, f \rangle \rightarrow \langle Sf, f \rangle$ as $N \rightarrow \infty$, which implies that $\sum_{k=1}^N |\langle f, \psi_k \rangle|^2$ converges as $N \rightarrow \infty$ and thus $f \in \text{dom}(C)$.

(i.2) If $\text{ran}(C) \subseteq \text{dom}(D)$, it is obvious that $\text{dom}(C) \subseteq \text{dom}(S)$ and the other inclusion is given in (i.1). Now assume that $\text{dom}(S) = \text{dom}(C)$ and take $Cf \in \text{ran}(C)$. Then $f \in \text{dom}(S)$, which implies that $Cf \in \text{dom}(D)$.

(ii) First observe that if $\sum_{l=1}^\infty c_l \psi_l$ converges in \mathcal{H} , then $\sum_{l=1}^\infty c_l \langle \psi_l, \psi_k \rangle$ converges for every $k \in \mathbb{N}$ and $\sum_{l=1}^\infty c_l \langle \psi_l, \psi_k \rangle = \langle Dc, \psi_k \rangle, \forall k \in \mathbb{N}$.

Assume that $\text{dom}(D) \subseteq \text{dom}(G)$ and take $Dc \in \text{ran}(D)$. Then $c \in \text{dom}(G)$, which implies that $(\langle Dc, \psi_k \rangle)_{k=1}^\infty \in \ell^2$ and thus, $Dc \in \text{dom}(C)$.

Now assume that $\text{ran}(D) \subseteq \text{dom}(C)$ and take $c \in \text{dom}(D)$. Then $Dc \in \text{dom}(C)$ and $\sum_{l=1}^\infty c_l \langle \psi_l, \psi_k \rangle = \langle Dc, \psi_k \rangle, \forall k \in \mathbb{N}$. Therefore, $c \in \text{dom}(G)$.

(iii) is clear, since the finite sequences are dense in ℓ^2 .

(iv) Since the statement is given in [9] without proof, for the sake of completeness we include a proof here. First observe that $(\overline{\text{span}}\{\Psi\})^\perp \subseteq \text{dom}(C)$. Assume that $\sum_{l=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every $k \in \mathbb{N}$. Then $\text{span}\{\Psi\} \subseteq \text{dom}(C)$. Therefore, $\mathcal{H} = \overline{\text{span}}\{\Psi\} \oplus (\overline{\text{span}}\{\Psi\})^\perp \subseteq \overline{\text{dom}(C)}$, which implies that $\text{dom}(C)$ is dense in \mathcal{H} .

(v) Let $\sum_{l=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every $k \in \mathbb{N}$. Then all the canonical vectors δ_k (and thus, all the finite sequences) belong to $\text{dom}(G)$. Then the conclusion follows as in (iii).

(vi) Assume that $\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 < \infty$. Then G is a Hilbert-Schmidt operator [20, Cor. 16.11]. Furthermore, Ψ is a Bessel sequence [10, Lemma 3.5.1] (see Prop. 4.4(a2)), which implies that $\text{dom}(D) = \ell^2$ [10, Theor. 3.2.3] (see Prop. 4.2).

(vii) Follows in a similar way as in (iv), having in mind that $(\overline{\text{span}}\{\Psi\})^\perp \subseteq \text{dom}(S)$. \square

Remarks 3.2

(1) Concerning Lemma 3.1(i):

If Ψ is a Bessel sequence for \mathcal{H} , then $\text{dom}(S) = \text{dom}(C) = \mathcal{H}$. Note that the equality $\text{dom}(S) = \text{dom}(C)$ might hold even in cases when Ψ is not a Bessel sequence. Consider for example the non-Bessel sequence

$$\Psi = (e_1, e_1, e_2, e_1, e_2, e_3, e_1, e_2, e_3, e_4, \dots).$$

Then $\text{dom}(C) = \{0\}$ and thus, Lemma 3.1(i.1) implies that $\text{dom}(S) = \{0\}$.

Note that the equality $\text{dom}(S) = \text{dom}(C)$ might fail. Consider for example the non-Bessel sequence $\Psi = (\frac{1}{2}e_1, 2e_2, \frac{1}{2^2}e_1, 2^2e_3, \frac{1}{2^3}e_1, 2^3e_4, \dots)$. As an example of an element which belongs to $\text{dom}(C)$ and does not belong to $\text{dom}(S)$ consider $h \in \mathcal{H}$ such that $\langle h, e_n \rangle = 2^{-2(n-1)}$, $\forall n \in \mathbb{N}$.

(2) Concerning Lemma 3.1(ii):

If Ψ is a Bessel sequence for \mathcal{H} , then $\text{dom}(D) = \text{dom}(G) = \ell^2$. For a non-Bessel sequence Ψ , the inclusion $\text{dom}(D) \subseteq \text{dom}(G)$ might fail. Take for example $\Psi = (e_1, e_2, e_1, e_3, e_1, e_4, \dots)$. Then $\delta_1 \in \text{dom}(D)$ and $\delta_1 \notin \text{dom}(G)$.

Note that a necessary condition for the validity of $\text{dom}(G) = \text{dom}(D)$ is that all the canonical vectors are in $\text{dom}(G)$, which holds if and only if $\sum_{l=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every k .

(3) Concerning Lemma 3.1(iii):

If $\text{dom}(D) = \ell^2$, then $\sum_{k=1}^{\infty} c_k \psi_k$ converges unconditionally for every $(c_k) \in \ell^2$ and Ψ is a Bessel sequence for \mathcal{H} (see [10, Cor. 3.2.4 and 3.2.5]). Now one could wonder whether this is pointwise true for an arbitrary sequence, i.e. whether $c \in \text{dom}(D)$ implies that $\sum_{k=1}^{\infty} c_k \psi_k$ is already unconditionally convergent. This is not true in general, as shown by the following easy example. Consider the non-Bessel sequence $\Psi = (e_1, e_1, e_1, \dots)$ and $(c_k) = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots) \in \ell^2$. Then $\sum_{k=1}^{\infty} c_k \psi_k = (\ln 2)e_1$. However, if $(\epsilon_k) = (1, -1, 1, -1, \dots)$, then the series $\sum_{k=1}^{\infty} c_k \epsilon_k \psi_k = \sum_{k=1}^{\infty} \frac{1}{k} e_1$ does not converge, which by [18, Theor. 2.8] implies that $\sum_{k=1}^{\infty} c_k \psi_k$ is conditionally convergent.

(4) Concerning Lemma 3.1(iv),(v):

The assumption in (iv) and (v) is clearly satisfied when Ψ is a Bessel sequence for \mathcal{H} (in which case $\text{dom}(C) = \mathcal{H}$ and $\text{dom}(G) = \ell^2$). However, the validity of $\sum_{l=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every $k \in \mathbb{N}$ does not require Ψ to be a Bessel sequence for \mathcal{H} , consider for example $\Psi = (e_1, 2e_2, 3e_3, \dots)$.

(5) Concerning Lemma 3.1(vi):

For frames, the assumption in (vi) is equivalent to the Hilbert space being finite dimensional [2]. This is not necessarily valid any more for other sequences, consider for example $\Psi = (e_1, e_2/2, e_3/3, e_4/4, \dots)$, it satisfies (vi) and \mathcal{H} is infinite dimensional.

3.2 Connection between the associated operators

Having defined the operators associated to an arbitrary sequence, we will now investigate the relationships among them.

Proposition 3.3 *For an arbitrary sequence Ψ , the following statements hold.*

- (i) $C = D^*$ ¹.
- (ii) [9, Prop. 4.6] *If C is densely defined, then $D \subseteq C^*$.*
- (iii) $S = DC$.
- (iv) $CD \subseteq G$.
- (v) [7, Lemma 3.1] *C is closed.*
- (vi) *S is symmetric², but not necessarily densely defined; it is positive on $\text{dom}(S)$ and positive definite on $\overline{\text{span}}\{\Psi\} \cap \text{dom}(S)$.*
- (vii) *If C is densely defined, then S is closable.*
- (viii) *D is closable if and only if C is densely defined.*
- (ix) *D is closed if and only if C is densely defined and $D = C^*$.*
- (x) *If D is closed, then G is densely defined.*
- (xi) *If $\sum_{l=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every k , then G is closed.*

Proof: (i) First we show that $\text{dom}(D^*) \subseteq \text{dom}(C)$. Fix $f \in \text{dom}(D^*)$. This means that $\mathfrak{D} : \text{dom}(D) \rightarrow \mathbb{C}$, $\mathfrak{D}(c) := \langle Dc, f \rangle$, is a bounded functional. Since $\text{dom}(D)$ is dense in ℓ^2 , there is a unique bounded extension $\mathfrak{D}_0 : \ell^2 \rightarrow \mathbb{C}$. For $c = (c_k)_{k=1}^{\infty} \in \ell^2$, denote

$$c_N = (c_1, c_2, \dots, c_N, 0, 0, \dots) \in c_{00} \subseteq \text{dom}(D).$$

¹This result was obtained independently in [15].

²The terminology is not uniform in the literature. We use the definition of a symmetric operator given by [14, Def. XII.1.7], [22, Def. 13.3], namely, S is symmetric if $\langle Sf, g \rangle = \langle f, Sg \rangle, \forall f, g \in \text{dom}(S)$, without the assumption of a dense domain.

Clearly, $c_N \rightarrow c$ in ℓ^2 -norm as $N \rightarrow \infty$. Hence, $\mathfrak{D}_0 c_N \rightarrow \mathfrak{D}_0 c$ as $N \rightarrow \infty$. Therefore, $\mathfrak{D}_0 c_N = \sum_{k=1}^N c_k \langle \psi_k, f \rangle_{\mathcal{H}}$ converges as $N \rightarrow \infty$ for every $c \in \ell^2$. Now Lemma 2.2 implies that $(\langle \psi_k, f \rangle_{\mathcal{H}}) \in \ell^2$, which proves that f belongs to $\text{dom}(C)$.

Now we show that $\text{dom}(C) \subseteq \text{dom}(D^*)$. Let $f \in \text{dom}(C)$. For every $c = (c_k)_{k=1}^{\infty} \in \text{dom}(D)$,

$$\langle Dc, f \rangle_{\mathcal{H}} = \lim_{N \rightarrow \infty} \left\langle \sum_{k=1}^N c_k \psi_k, f \right\rangle_{\mathcal{H}} = \lim_{N \rightarrow \infty} \sum_{k=1}^N c_k \langle \psi_k, f \rangle_{\mathcal{H}} = \langle c, Cf \rangle_{\ell^2}. \quad (2)$$

Therefore, the functional $c \mapsto \langle Dc, f \rangle_{\mathcal{H}}$ is bounded on $\text{dom}(D)$ by $\|Cf\|$. Hence, $f \in \text{dom}(D^*)$.

Furthermore, (2) implies that $D^*f = Cf$ due to the uniqueness of D^* .

(iii) Assume that $f \in \text{dom}(S)$. Then $f \in \text{dom}(C)$ (see Lemma 3.1(i.1)). Furthermore, $Cf \in \text{dom}(D)$. Therefore, $\text{dom}(S) \subseteq \text{dom}(DC)$. The converse implication $\text{dom}(DC) \subseteq \text{dom}(S)$ is obvious. Now it is clear that $S = DC$.

(iv) Let $c = (c_k)_{k=1}^{\infty} \in \text{dom}(CD)$. Then $\sum_{l=1}^{\infty} c_l \langle \psi_l, \psi_k \rangle$ converges for every k and it is equal to $\langle Dc, \psi_k \rangle$. Furthermore, $Dc \in \text{dom}(C)$, which implies that $c \in \text{dom}(G)$ and $Gc = CDc$.

(v) Since the statement is given in [7] without proof, for the sake of completeness we refer to [23, Lemma 3.1] for a proof. Note that [23, Lemma 3.1] concerns sequences satisfying the lower p -frame condition, but the proof that C is closed does not use the validity of the lower p -frame condition.

(vi) Let $f, g \in \text{dom}(S)$. Then $\langle Sf, g \rangle = \sum_{k=1}^{\infty} \langle f, \psi_k \rangle \langle \psi_k, g \rangle = \langle f, Sg \rangle$, which means that S is symmetric. An example of a sequence Ψ with S being non-densely defined can be seen in Remark 3.2(1). Further, $\langle Sf, f \rangle = \sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 \geq 0$ for every $f \in \text{dom}(S)$. If $\sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 = 0$, then $f \in (\text{span}\{\Psi\})^{\perp}$, which implies that S is positive definite on $\overline{\text{span}\{\Psi\}} \cap \text{dom}(S)$.

(vii) By (v), C is closed. Assume that C is densely defined. Then Proposition 2.3(e) implies that C^*C is self-adjoint and thus, closed. By (ii) and (iii), we have that $S = DC \subseteq C^*C$, which implies that S is closable.

(viii) and (ix) follow from (i), Lemma 3.1(iii) and Prop. 2.3(b),(c) applied with $F = D$.

(x) Let D be closed. By Lemma 3.1(iii) and Proposition 2.3(e), it follows that D^*D is self-adjoint, in particular, densely defined. Using (i) and (iv), it follows that $D^*D \subseteq G$, which implies that G is also densely defined.

(xi) Assume that $\sum_{l=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 < \infty$ for every $k \in \mathbb{N}$. Denote $c^n = (c_k^n)_{k=1}^{\infty}$, $c = (c_k)_{k=1}^{\infty}$, $d = (d_k)_{k=1}^{\infty}$. Let $c^n \in \text{dom}(G)$, $n \in \mathbb{N}$, $c^n \rightarrow c$ in ℓ^2 and $Gc^n \rightarrow d$ in ℓ^2 as $n \rightarrow \infty$. Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $\|c - c^N\|_{\ell^2} < \frac{\varepsilon}{2} (\sum_{l=1}^N |\langle \psi_l, \psi_k \rangle|^2)^{-1/2}$. Since $\sum_{l=1}^{\infty} c_l^N \langle \psi_l, \psi_k \rangle$ converges, there exists N_1 so that $|\sum_{l=i+1}^j c_l^N \langle \psi_l, \psi_k \rangle| < \frac{\varepsilon}{2}$, $\forall i, j > N_1$. Then, for every

$i, j > N_1$,

$$\begin{aligned} \left| \sum_{l=i+1}^j c_l \langle \psi_l, \psi_k \rangle \right| &\leq \left| \sum_{l=i+1}^j (c_l - c_l^N) \langle \psi_l, \psi_k \rangle \right| + \left| \sum_{l=i+1}^j c_l^N \langle \psi_l, \psi_k \rangle \right| \\ &\leq \left(\sum_{l=i+1}^j |c_l - c_l^N|^2 \right)^{1/2} \left(\sum_{l=i+1}^j |\langle \psi_l, \psi_k \rangle|^2 \right)^{1/2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Therefore, the series $\sum_{l=1}^{\infty} c_l \langle \psi_l, \psi_k \rangle$ converges. Furthermore,

$$\left| \sum_{l=1}^{\infty} (c_l - c_l^n) \langle \psi_l, \psi_k \rangle \right| \leq \|c - c^n\|_{\ell^2} \left(\sum_{l=1}^{\infty} |\langle \psi_l, \psi_k \rangle|^2 \right)^{1/2} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which implies that $\sum_{l=1}^{\infty} c_l^n \langle \psi_l, \psi_k \rangle \rightarrow \sum_{l=1}^{\infty} c_l \langle \psi_l, \psi_k \rangle$ as $n \rightarrow \infty$. Now, since $Gc^n \rightarrow d$ in ℓ^2 as $n \rightarrow \infty$ and since convergence in ℓ^2 implies convergence by coordinates, it follows that $\sum_{l=1}^{\infty} c_l^n \langle \psi_l, \psi_k \rangle \rightarrow d_k$ as $n \rightarrow \infty$. Therefore, $\sum_{l=1}^{\infty} c_l \langle \psi_l, \psi_k \rangle = d_k$ for every $k \in \mathbb{N}$, which implies that $c \in \text{dom}(G)$ and $Gc = d$. \square

Remark 3.4 (concerning Prop. 3.3(ii)): If C is densely defined, in general one should not expect the validity of $D = C^*$. A counterexample can be found in [9] after Corollary 4.7. When Ψ is a Bessel sequence for \mathcal{H} , then $D = C^*$ (see, e.g., [10, Theor. 3.2.3 and Lemma 3.2.1]).

Note that Proposition 3.3(ix) extends [9, Corollary 4.7].

3.3 Kernels

Lemma 3.5 *For an arbitrary sequence Ψ , the following statements hold.*

- (i) $\ker(S) = \ker(C) = (\overline{\text{span}\{\Psi\}})^\perp = (\text{ran}(D))^\perp$.
- (ii) $\ker(D) \subseteq \overline{\text{ran}(C)}^\perp$.

Proof: (i) Assume that $f \in \ker(S)$. Then $\sum_{k=1}^{\infty} |\langle f, \psi_k \rangle|^2 = \langle Sf, f \rangle = 0$, which implies that $f \in \ker(C)$. It is obvious that $\ker(C) \subseteq \ker(S)$. The equality $\ker(C) = \overline{\text{span}\{\Psi\}}^\perp$ is also obvious. The equality $\ker(C) = (\text{ran}(D))^\perp$ follows from Lemma 3.1(iii), Proposition 3.3(i) and Proposition 2.3(d) with $F = D$.

(ii) Follows easily using Prop. 3.3(i). \square

3.4 Ranges

Lemma 3.6 *For an arbitrary sequence Ψ , the following statements hold.*

- (i) $\text{ran}(S) \subseteq \text{ran}(D)$.
- (ii) *If $\text{ran}(S) = \text{ran}(D)$, then $\overline{\text{ran}(C)} \cap \text{dom}(D) = \text{ran}(C) \cap \text{dom}(D)$.*
- (iii.1) *Let Ψ be a Bessel sequence for \mathcal{H} . Then $\text{ran}(S) \subseteq \text{ran}(D) \subseteq \overline{\text{ran}(S)}$ (with dense inclusion).*
- (iii.2) *Let Ψ be a Bessel sequence for \mathcal{H} . Then $\text{ran}(S) = \text{ran}(D)$ if and only if Ψ is a frame sequence.*

Proof: (i) follows from Prop. 3.3(iii).

(ii) Assume that $\text{ran}(S) = \text{ran}(D)$. Let $c \in \overline{\text{ran}(C)} \cap \text{dom}(D)$. Since $Dc \in \text{ran}(D) = \text{ran}(S)$, there exists $f \in \text{dom}(S) \subseteq \text{dom}(C)$ so that $Dc = DCf$. Since $D_{\overline{\text{ran}(C)} \cap \text{dom}(D)}$ is injective (see Lemma 3.5(ii)), it follows that $c = Cf \in \text{ran}(C)$. Therefore, $\overline{\text{ran}(C)} \cap \text{dom}(D) \subseteq \text{ran}(C) \cap \text{dom}(D)$. The inverse inclusion is obvious.

(iii.1) Let $f \in \text{ran}(D)$. Since Ψ is a Bessel sequence for \mathcal{H} , it follows that $\ker D = \ker(C^*) = \overline{\text{ran}(C)}^\perp$, which implies that $f = Dc$ for some $c \in \overline{\text{ran}(C)}$. Therefore, $f = \lim_{n \rightarrow \infty} DCf_n = \lim_{n \rightarrow \infty} Sf_n$ for some $f_n \in \mathcal{H}, n \in \mathbb{N}$, such that $c = \lim_{n \rightarrow \infty} Cf_n$. This completes the proof.

(iii.2) Assume that $\text{ran}(S) = \text{ran}(D)$. Since Ψ is a Bessel sequence for \mathcal{H} , it follows that $\text{dom}(D) = \mathcal{H}$ and (ii) implies that $\text{ran}(C)$ is closed. Now [10, Cor. 5.5.3] (see Prop. 4.1(b)) implies that Ψ is a frame sequence. Conversely, if Ψ is a frame sequence, then $\text{ran}(S) = \text{ran}(D) = \overline{\text{span}}\{\Psi\}$ (see, e.g., [10, Lemma 5.1.5 and Theor. 5.5.1]). \square

4 Classification of General Sequences

In this section, we turn to the central topic of the paper, namely, the classification of arbitrary sequences. We consider two different methods of classification, first in terms of the associated operators, then via orthonormal bases.

Note that throughout this section, we use a rather unconventional numbering technique to increase the comparability for the convenience of the reader. The items (a), (b), ... refer always to the same class of sequences, Bessel sequences, frame sequences, ... Therefore, if no result is available for a given class, we simply omit the corresponding item.

4.1 Classification by the associated operators

In the next four propositions, we proceed with the classification of arbitrary sequences in terms of the operators C, D, S , and G .

Proposition 4.1 *Given a sequence Ψ , the following statements hold.*

- (a1) [18, Lemma 11.8] Ψ is a Bessel sequence for \mathcal{H} if and only if $\text{dom}(C) = \mathcal{H}$.
- (a2) Ψ is a Bessel sequence for \mathcal{H} with bound B if and only if $\text{dom}(C) = \mathcal{H}$ and C is bounded with $\|C\| \leq \sqrt{B}$.
- (b) [10, Cor. 5.5.3] Ψ is a frame sequence if and only if $\text{dom}(C) = \mathcal{H}$ and $\text{ran}(C)$ is closed.
- (c) [10, Cor. 5.5.3] Ψ is a frame for \mathcal{H} if and only if $\text{dom}(C) = \mathcal{H}$, $\text{ran}(C)$ is closed and C is injective.
- (d) Ψ is a Riesz basis for \mathcal{H} if and only if $\text{dom}(C) = \mathcal{H}$ and C is bijective.
- (e) [7, Lemma 3.1] Ψ is a lower frame sequence for \mathcal{H} if and only if C is injective and $\text{ran}(C)$ is closed.
- (f) [25, Ch.4 Sec.2] Ψ is a Riesz-Fischer sequence if and only if C is surjective.
- (g) Ψ is complete in \mathcal{H} if and only if C is injective.

Proof: (a2) Having (a1) proved, then (a2) is obvious.

(d) By [10, Theor. 5.4.1] and [16, Prop. 5.1.5], Ψ is a Riesz basis for \mathcal{H} if and only if Ψ is a frame for \mathcal{H} and C is surjective. The rest follows from (c).

(g) follows from Lemma 3.5(i). \square

Proposition 4.2 *Given a sequence Ψ , the following statements hold.*

- (a1) [10, Cor. 3.2.4 and Theor. 3.2.3] Ψ is a Bessel sequence if and only if $\text{dom}(D) = \ell^2$.
- (a2) [10, Theor. 3.2.3] Ψ is a Bessel sequence with bound B if and only if $\text{dom}(D) = \ell^2$ and D is bounded with $\|D\| \leq \sqrt{B}$.
- (b1) [10, Cor. 5.5.2] Ψ is a frame sequence if and only if $\text{dom}(D) = \ell^2$ and $\text{ran}(D)$ is closed.
- (b2) Ψ is a frame sequence if and only if $\text{ran}(D)$ is closed and $\text{ran}(D) \subseteq \text{dom}(C)$.
- (b3) Ψ is a frame sequence if and only if $\text{dom}(D) = \ell^2$ and $\text{ran}(D) = \text{ran}(S)$.
- (c) [10, Theor. 5.5.1] Ψ is a frame if and only if $\text{dom}(D) = \ell^2$ and D is surjective.
- (d) Ψ is a Riesz basis for \mathcal{H} if and only if $\text{dom}(D) = \ell^2$ and D is bijective.
- (e) Ψ is a lower frame sequence for \mathcal{H} if and only if $\text{ran}(D)$ is dense in \mathcal{H} and $\text{ran}(D^*)$ is closed.

- (f) Ψ is a Riesz-Fischer sequence if and only if D is injective and D^{-1} is bounded on $\text{ran}(D)$.
- (g) Ψ is complete in \mathcal{H} if and only if $\text{ran}(D)$ is dense in \mathcal{H} .

Proof: (b2) Assume that $\text{ran}(D)$ is closed and $\text{ran}(D) \subseteq \text{dom}(C)$. Then $\mathcal{H} = \text{ran}(D) \oplus (\text{ran}(D))^\perp \subseteq \text{dom}(C)$, which by Proposition 4.1(a1) implies that Ψ is a Bessel sequence for \mathcal{H} . Now apply (a1) and (b1).

Conversely, assume that Ψ is a frame sequence. By (b1), $\text{ran}(D)$ is closed, and clearly, $\text{dom}(C) = \mathcal{H} \supseteq \text{ran}(D)$.

(b3) follows from Lemma 3.6(iii.2) and (a1).

(d) By [10, Theor. 5.4.1 and 6.1.1], Ψ is a Riesz basis for \mathcal{H} if and only if Ψ is a frame for \mathcal{H} and D is injective. The rest follows from (c).

(e) By Propositions 4.1(e) and 3.3(i), Ψ is a lower frame sequence for \mathcal{H} if and only if D^* is injective and $\text{ran}(D^*)$ is closed. Since D is densely defined, Proposition 2.3(d) completes the proof.

(f) is clear (see, e.g., [19, V.4.3 Theorem 2 and the Remark after that]).

(g) follows from Lemma 3.5(i). \square

Note that (f) (a result about the Riesz-Fischer sequences) is missing in Proposition 4.3. The operator S does not distinguish between a frame which is a Riesz-Fischer sequence and a frame which is not a Riesz-Fischer sequence. For example, an orthonormal basis and a Parseval frame have the same frame operator, the identity. This can also be seen in Proposition 4.3(d), where the properties of $(S^{-1}\psi_k)$ have to be taken into account.

Proposition 4.3 *Given a sequence Ψ , the following statements hold.*

- (a1) Ψ is a Bessel sequence for \mathcal{H} if and only if $\text{dom}(S) = \mathcal{H}$.
- (a2) Ψ is a Bessel sequence for \mathcal{H} with bound B if and only if $\text{dom}(S) = \mathcal{H}$ and S is bounded with $\|S\| \leq B$.
- (b1) Ψ is a frame sequence if and only if $\text{dom}(S) = \mathcal{H}$ and $\text{ran}(S)$ is closed.
- (b2) Ψ is a frame sequence if and only if $\text{dom}(S) = \mathcal{H}$ and $\text{ran}(S) = \overline{\text{span}}\{\Psi\}$.
- (c1) [8, Theorem 2.1] Ψ is a frame for \mathcal{H} if and only if $\text{dom}(S) = \mathcal{H}$ and S is surjective.
- (c2) Ψ is a frame for \mathcal{H} if and only if $\text{dom}(S) = \mathcal{H}$ and S is bijective.
- (d) Ψ is a Riesz basis for \mathcal{H} if and only if $\text{dom}(S) = \mathcal{H}$, S is bijective and $(S^{-1}\psi_k)_{k=1}^\infty$ is biorthogonal to Ψ .
- (e) Ψ is a lower frame sequence for \mathcal{H} if and only if S is injective and $\text{ran}(C)$ is closed.
- (g) Ψ is complete in \mathcal{H} if and only if S is injective.

Proof: (a1) Let $\text{dom}(S) = \mathcal{H}$. Validity of the upper frame inequality follows from the first part of the proof in [8, Theorem 2.1]. For the sake of completeness, we sketch a proof. If $\text{dom}(S) = \mathcal{H}$, then $\sum |\langle f, \psi_k \rangle|^2 = \langle Sf, f \rangle < \infty$ for every $f \in \mathcal{H}$, which implies that $\text{dom}(C) = \mathcal{H}$ and the rest follows from Proposition 4.1(a1). For the other direction, see the text after Example 5.1.4 in [10, Sec. 5.1].

(a2) See (a1); the boundedness of S can be found, e.g., in the proof of [10, Lemma 5.1.5].

(b1) Assume that $\text{dom}(S) = \mathcal{H}$ and $\text{ran}(S)$ is closed. By (a1), Ψ is a Bessel sequence for \mathcal{H} . Then Lemma 3.6(iii.1) implies that $\text{ran}(D)$ is closed, which by Proposition 4.2(b1) implies that Ψ is a frame sequence.

Conversely, if Ψ is a frame sequence, the conclusion follows from [8, Theorem 2.1] (see (c1)).

(b2) follows from [8, Theorem 2.1] (see (c1)).

(c2) follows from (c1) and [10, Lemma 5.1.5].

(d) Use (c2) and the fact that Ψ is a Riesz basis for \mathcal{H} if and only if Ψ is a frame for \mathcal{H} and $(S^{-1}\psi_k)_{k=1}^{\infty}$ is biorthogonal to Ψ [10, Theor. 5.4.1 and 6.1.1].

(e) By Lemma 3.5(i), S is injective if and only if C is injective. The rest follows from Proposition 4.1(e).

(g) follows from Lemma 3.5(i). \square

Proposition 4.4 *For a sequence Ψ , the following statements hold.*

(a1) Ψ is a Bessel sequence for \mathcal{H} if and only if $\text{dom}(G) = \ell^2$.

(a2) [10, Lemma 3.5.1] Ψ is a Bessel sequence for \mathcal{H} with bound B if and only if $\text{dom}(G) = \ell^2$ and G is bounded with $\|G\| \leq B$.

(a3) If Ψ is a Bessel sequence for \mathcal{H} , then $G|_{\text{ran}(C)}$ is an injective operator from $\text{ran}(C)$ into $\text{ran}(C)$ and $\text{ran}(G|_{\text{ran}(C)})$ is dense in $\text{ran}(C)$.

(b) Ψ is a frame sequence if and only if $\text{dom}(G) = \ell^2$ and $G|_{\text{ran}(C)}$ is a bounded operator from $\text{ran}(C)$ onto $\text{ran}(C)$ with bounded inverse.

(c) Ψ is a frame for \mathcal{H} if and only if Ψ is complete in \mathcal{H} , $\text{dom}(G) = \ell^2$ and $G|_{\text{ran}(C)}$ is a bounded operator from $\text{ran}(C)$ onto $\text{ran}(C)$ with bounded inverse.

(d) [10, Theor. 3.6.6] Ψ is a Riesz basis for \mathcal{H} if and only if Ψ is complete in \mathcal{H} and G is a bounded invertible operator on \mathcal{H} .

(f) [25, Ch.4 Sec.2] Ψ is a Riesz-Fischer sequence with bound A if and only if the “sections” G_n of G satisfy the inequality $A\|c\| \leq \|G_n c\|$ for all finite sequences and all n , where the “section” G_n is the matrix $(\langle \psi_l, \psi_k \rangle)_{l,k=1}^n$.

Proof: (a1) Let $\text{dom}(G) = \ell^2$. Then G admits a matrix representation via the orthonormal basis $(\delta_k)_{k=1}^{\infty}$ of ℓ^2 and thus, G is bounded [1, Sec. 29

Theor. 1].³ Since [1] is in Russian, for convenience of the reader we add a sketch of a proof. Let $n \in \mathbb{N}$. Consider the operator G_n defined by $G_n(c)_{c=1}^\infty = (\sum_{l=1}^\infty c_l \langle \psi_l, \psi_k \rangle)_{k=1}^n$. Then $\text{dom}(G_n) = \ell^2$ and G_n is bounded. Therefore, by the Banach-Steinhaus theorem, G is bounded. Now the rest follows from [10, Lemma 3.5.1] (see (a2)).

(a3) follows from [10, Lemma 3.5.2] and Proposition 3.3(i).

(b) If Ψ is a frame sequence, the statement follows from [10, Prop.5.2.2] and Proposition 3.3(i). Conversely, assume that $\text{dom}(G) = \ell^2$ and $G|_{\text{ran}(C)}$ is a bounded operator from $\text{ran}(C)$ onto $\text{ran}(C)$ with bounded inverse. Then Ψ is a Bessel sequence for \mathcal{H} and thus $G = CD$. Consider the subspace $\overline{\text{span}}\{\Psi\}$ of \mathcal{H} and the operator $C_1 = C|_{\overline{\text{span}}\{\Psi\}}$. By Lemma 3.5(i), C_1 is a bijection from $\overline{\text{span}}\{\Psi\}$ onto $\text{ran}(C)$. Since $\text{ran}(D) \subseteq \overline{\text{span}}\{\Psi\}$, for every $c \in \text{ran}(C)$ one has $G|_{\text{ran}(C)}c = CDc = C_1D|_{\text{ran}(C)}c$ and thus, $C_1D|_{\text{ran}(C)}(G|_{\text{ran}(C)})^{-1}d = d$ for every $d \in \text{ran}(C)$. Therefore,

$$C_1^{-1} = D|_{\text{ran}(C)}(G|_{\text{ran}(C)})^{-1},$$

which implies that C_1^{-1} is bounded. Thus, the bounded injective operator C_1 has a bounded inverse on $\text{ran}(C_1)$, which implies that C_1 has closed range [24, Cor. IV.3.6] (for a reference in English, see [13, Ex. VI.9.15(i)]). Therefore, by Proposition 4.1(c), Ψ is a frame for $\overline{\text{span}}\{\Psi\}$.

(c) follows from (b). \square

Note that again some items are missing, as in Proposition 4.3. It is impossible to have some connection of properties of the Gram matrix to completeness. For example, the standard orthonormal basis (e_1, e_2, e_3, \dots) as well as (e_2, e_3, e_4, \dots) have the same Gram matrix. The problem concerning sequences that satisfy the lower frame condition is still open.

4.2 Operators that preserve the sequence properties

Proposition 4.5 *Given a sequence Ψ , the following statements hold.*

- (a) *Let Ψ be a Bessel sequence for \mathcal{H} with bound B and let $F : \text{span}\{\Psi\} \rightarrow \mathcal{H}$ be a bounded operator. Then $(F\psi_k)_{k=1}^\infty$ is a Bessel sequence for \mathcal{H} with bound $B\|F\|^2$.*
- (b) *Let Ψ be a frame sequence with bounds A, B and let $F : \overline{\text{span}}\{\Psi\} \rightarrow \overline{\text{span}}\{\Psi\}$ be a bounded surjective operator. Then $(F\psi_k)_{k=1}^\infty$ is a frame sequence with frame bounds $A\|F^\dagger\|^{-2}$ and $B\|F\|^2$, where F^\dagger ⁴ denotes the pseudo-inverse of F .*
- (c) [10, Cor. 5.3.2] *Let Ψ be a frame for \mathcal{H} with bounds A, B and let $F : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded surjective operator. Then $(F\psi_k)_{k=1}^\infty$ is a frame for \mathcal{H} with bounds $A\|F^\dagger\|^{-2}$ and $B\|F\|^2$, where F^\dagger denotes the pseudo-inverse of F .*

³For a matrix representation using frames refer to [3].

⁴For the pseudo-inverse of a bounded operator with closed range see, e.g., [10, App. A.7].

- (d) Let Ψ be a Riesz basis for \mathcal{H} with bounds A, B and let $F : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded bijective operator. Then $(F\psi_k)_{k=1}^\infty$ is a Riesz basis for \mathcal{H} with bounds $A\|F^{-1}\|^{-2}$, $B\|F\|^2$.
- (e) Let Ψ be a lower frame sequence for \mathcal{H} with bound A and let the operator $F : \overline{\text{span}\{\Psi\}} \rightarrow \mathcal{H}$ be bounded and such that F^* has a bounded inverse on $\text{ran}(F^*)$ (equivalently, F be bounded and surjective). Then $(F\psi_k)_{k=1}^\infty$ is a lower frame sequence for \mathcal{H} with bound $A\|(F^*)^{-1}\|^{-2}$.
- (f) Let Ψ be a Riesz-Fischer sequence with bound A and let the operator $F : \text{span}\{\Psi\} \rightarrow \mathcal{H}$ have a bounded inverse $F^{-1} : \text{ran}(F) \rightarrow \mathcal{H}$ with bound K . Then $(F\psi_k)_{k=1}^\infty$ is a Riesz-Fischer sequence with bound AK^{-2} .
- (g) Let Ψ be complete in \mathcal{H} and let $F : \text{span}\{\Psi\} \rightarrow \mathcal{H}$ be bounded with $\text{ran}(F)$ being dense in \mathcal{H} . Then $(F\psi_k)_{k=1}^\infty$ is complete in \mathcal{H} .

Proof: (a) Let \overline{F} denote the bounded extension of F on $\overline{\text{span}\{\Psi\}}$ without increasing the norm. For every $f \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} |\langle f, F\psi_k \rangle|^2 = \sum_{k=1}^{\infty} \left| \langle \overline{F}^* f, \psi_k \rangle \right|^2 \leq B \|\overline{F}^* f\|^2 \leq B \|F\|^2 \cdot \|f\|^2.$$

(b) follows from [10, Cor. 5.3.2] (see (c)).

(d) By (c), $(F\psi_k)_{k=1}^\infty$ is a frame for \mathcal{H} and thus, complete in \mathcal{H} (see Proposition 4.2(c)). Furthermore, for every finite sequence (c_k) ,

$$\left\| \sum c_k F\psi_k \right\|^2 = \left\| F \left(\sum c_k \psi_k \right) \right\|^2 \geq \frac{1}{\|F^{-1}\|^2} A \sum |c_k|^2$$

and similar, $\left\| \sum c_k F\psi_k \right\|^2 \leq \|F\|^2 B \sum |c_k|^2$.

(e) A bounded operator from a Banach space into a Banach space is surjective if and only if its adjoint has a bounded inverse on the range of the adjoint (see, e.g., [22, Theorem 4.15] and [19, V.4.3 Theorem 2 and the Remark after that]). For every $f \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} |\langle f, F\psi_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle F^* f, \psi_k \rangle|^2 \geq A \|F^* f\|^2 \geq \frac{A}{\|(F^*)^{-1}\|^2} \|f\|^2.$$

(f) As in (d), $\left\| \sum c_k F\psi_k \right\|^2 \geq K^{-2} A \sum |c_k|^2$ for every finite sequence (c_k) .

(g) Let Ψ be complete in \mathcal{H} and let \overline{F} denote the bounded extension of F on $\overline{\text{span}\{\Psi\}} = \mathcal{H}$. Assume that $f \in \mathcal{H}$ and $\langle f, F\psi_k \rangle = 0, \forall k \in \mathbb{N}$. Then $\overline{F}^* f = 0$. Since $\text{ran}(\overline{F})$ is dense in \mathcal{H} , it follows that \overline{F}^* is injective (see Proposition 2.3(d)). Therefore, $f = 0$. \square

4.3 Classification with Orthonormal Bases

Another way of classifying the sequences is to examine how an orthonormal basis behaves under application of a given class of operators. This we do in the present subsection. From now on, D_Ψ denotes the synthesis operator for Ψ and $C_{(e_k)}$ denotes the analysis operator for $(e_k)_{k=1}^\infty$.

Proposition 4.6 *Let $(e_k)_{k=1}^\infty$ be an orthonormal basis for \mathcal{H} .*

- (a) *The Bessel sequences for \mathcal{H} are precisely the families $(Ve_k)_{k=1}^\infty$, where $V : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator.*
- (b) *The frame sequences for \mathcal{H} are precisely the families $(Ve_k)_{k=1}^\infty$, where $V : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator with closed range.*
- (c) [10, Theor. 5.5.5] *The frames for \mathcal{H} are precisely the families $(Ve_k)_{k=1}^\infty$, where $V : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded and surjective operator.*
- (d) [10, Def. 3.6.1 and Theor. 3.6.6] *The Riesz bases for \mathcal{H} are precisely the sequences $(Ve_k)_{k=1}^\infty$, where $V : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded bijective operator.*
- (e) *The lower frame sequences for \mathcal{H} are precisely the families $(Ve_k)_{k=1}^\infty$, where $V : \text{dom}(V) \rightarrow \mathcal{H}$ is a densely defined operator such that $e_k \in \text{dom}(V)$, $\forall k \in \mathbb{N}$, V^* is injective with bounded inverse on $\text{ran}(V^*)$, and $V(\sum_{k=1}^n c_k e_k) \rightarrow V(\sum_{k=1}^\infty c_k e_k)$ as $n \rightarrow \infty$ for every $\sum_{k=1}^\infty c_k e_k \in \text{dom}(V)$.*
- (f) [7, Prop. 2.3] *The Riesz-Fischer sequences are precisely the families $(Ve_k)_{k=1}^\infty$, where V is an operator having all e_k in the domain and which has a bounded inverse $V^{-1} : \text{ran}(V) \rightarrow \mathcal{H}$.*
- (g) *The complete sequences are precisely the families $(Ve_k)_{k=1}^\infty$, where $V : \text{dom}(V) \rightarrow \mathcal{H}$ is a densely defined operator such that $e_k \in \text{dom}(V)$, $\forall k \in \mathbb{N}$, $\text{ran}(V)$ is dense in \mathcal{H} (equivalently, the adjoint V^* is injective) and $V(\sum_{k=1}^n c_k e_k) \rightarrow V(\sum_{k=1}^\infty c_k e_k)$ as $n \rightarrow \infty$ for every $\sum_{k=1}^\infty c_k e_k \in \text{dom}(V)$.*

Proof: (a) If $V : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator, the conclusion follows from Proposition 4.5(a).

Conversely, assume that Ψ is a Bessel sequence for \mathcal{H} . Then the operator $V = D_\Psi C_{(e_k)}$ is bounded and $\Psi = (Ve_k)_{k=1}^\infty$.

(b) If $V : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator with closed range, then [10, Prop. 5.3.1] implies that $(Ve_k)_{k=1}^\infty$ is a frame sequence.

Conversely, assume that Ψ is a frame sequence. Consider the operator $V = D_\Psi C_{(e_k)}$. Since $C_{(e_k)}$ is a bounded bijection (see Proposition 4.1(d)) and D_Ψ is a bounded operator with closed range (see Proposition 4.2(b1)), it follows that V is bounded and $\text{ran}(V)$ is closed. Clearly, $\Psi = (Ve_k)_{k=1}^\infty$.

(e) Let Ψ be a lower frame sequence for \mathcal{H} . Consider the operator $V := D_\Psi C_{(e_k)}$. It is clear that $\text{dom}(V)$ contains all e_k , $k \in \mathbb{N}$, (and thus, V is densely defined) and $(Ve_k)_{k=1}^\infty = \Psi$. Using the fact that $C_{(e_k)}$ is a bounded

bijection of \mathcal{H} onto ℓ^2 , and $C_{(e_k)}$ maps bijectively $\text{dom}(V)$ onto $\text{dom}(D)$, it is not difficult to see that $\text{dom}(V^*) = \text{dom}((D_\Psi)^*) = \text{dom}((C_{(e_k)})^*(D_\Psi)^*)$ and $V^* = (C_{(e_k)})^*(D_\Psi)^*$. Using Proposition 3.3(i), it follows that $V^* = D_{(e_k)}C_\Psi$. Now Propositions 4.1(e) and 4.2(d) imply that V^* is injective. Furthermore, $(V^*)^{-1} = (C_\Psi)^{-1}(D_{(e_k)})^{-1}|_{\text{ran}(D_{(e_k)}C_\Psi)}$ and $(V^*)^{-1}$ is bounded on $\text{ran}(V^*)$. Now assume that $\sum_{i=1}^{\infty} c_i e_i \in \text{dom}(V)$. Then $(c_k)_{k=1}^{\infty} = C_{(e_k)}(\sum_{i=1}^{\infty} c_i e_i) \in \text{dom}(D_\Psi)$, which implies that $\sum_{k=1}^{\infty} c_k \psi_k$ converges in \mathcal{H} and $V(\sum_{i=1}^{\infty} c_i e_i) = \sum_{k=1}^{\infty} c_k \psi_k$. For every $n \in \mathbb{N}$, $\sum_{k=1}^n c_k e_k \in \text{dom}(V)$ and $V(\sum_{k=1}^n c_k e_k) = \sum_{k=1}^n c_k \psi_k$. Therefore, $V(\sum_{k=1}^n c_k e_k) \rightarrow V(\sum_{i=1}^{\infty} c_i e_i)$ as $n \rightarrow \infty$.

Conversely, assume that $V : \text{dom}(V) \rightarrow \mathcal{H}$ satisfies the conditions in the statement of (e). Let $f \in \mathcal{H}$ be such that $\sum_{k=1}^{\infty} |\langle f, V e_k \rangle|^2 < \infty$ and denote $N := \sum_{k=1}^{\infty} |\langle f, V e_k \rangle|^2$. First we prove that f belongs to $\text{dom}(V^*)$. Let $\sum_{k=1}^n c_k e_k$ be an arbitrary element of $\text{dom}(V)$. For every $n \in \mathbb{N}$, $\sum_{k=1}^n c_k e_k$ belongs to $\text{dom}(V)$ and

$$\begin{aligned} \left| \left\langle V \left(\sum_{k=1}^n c_k e_k \right), f \right\rangle \right| &\leq \left(\sum_{k=1}^n |c_k|^2 \right)^{1/2} \left(\sum_{k=1}^n |\langle V e_k, f \rangle|^2 \right)^{1/2} \\ &\leq \sqrt{N} \left\| \sum_{k=1}^n c_k e_k \right\|. \end{aligned}$$

By the assumptions, $V(\sum_{k=1}^n c_k e_k) \rightarrow V(\sum_{k=1}^{\infty} c_k e_k)$ as $n \rightarrow \infty$. Therefore, taking the limit as $n \rightarrow \infty$ in the above inequalities, it follows that

$$\left| \left\langle V \left(\sum_{k=1}^{\infty} c_k e_k \right), f \right\rangle \right| \leq \sqrt{N} \left\| \sum_{k=1}^{\infty} c_k e_k \right\|.$$

Hence, $f \in \text{dom}(V^*)$ and

$$\sum_{k=1}^{\infty} |\langle f, V e_k \rangle|^2 = \|V^* f\|^2 \geq \frac{1}{\|(F^*)^{-1}\|} \|f\|^2.$$

(f) Since the statement in [7, Prop. 2.3] is given without proof, for the sake of completeness we add a proof here. One of the directions follows from Proposition 4.5(f). For the other direction, assume that Ψ is a Riesz-Fischer sequence with bound A . Consider then the operator $V = D_\Psi C_{(e_k)}$. Then the domain of V contains all e_k , $(V e_k)_{k=1}^{\infty} = \Psi$, and $\|V f\| = \|D_\Psi C_{(e_k)} f\| \geq \sqrt{A} \|C_{(e_k)} f\| = \sqrt{A} \|f\|$ for every $f \in \text{dom}(V)$. Therefore, V is injective with bounded inverse on $\text{ran}(V)$.

(g) Let Ψ be complete in \mathcal{H} . Consider the operator $V := D_\Psi C_{(e_k)}$. In a similar way as in (e), it follows that V has the desired properties.

Conversely, assume that $V : \text{dom}(V) \rightarrow \mathcal{H}$ satisfies the conditions in the statement of (g). Let $f \in \mathcal{H}$ be such that $\langle f, V e_k \rangle = 0$, $\forall k \in \mathbb{N}$. First we prove that f belongs to $\text{dom}(V^*)$. Every finite sum $\sum c_k e_k$ belongs to $\text{dom}(V)$ and $\langle V(\sum c_k e_k), f \rangle = 0$. Now, let $\sum_{k=1}^{\infty} c_k e_k \in \text{dom}(V)$. By the above, it

follows that $\langle V(\sum_{k=1}^n c_k e_k), f \rangle = 0, \forall n \in \mathbb{N}$. By assumption, $V(\sum_{k=1}^n c_k e_k) \rightarrow V(\sum_{k=1}^{\infty} c_k e_k)$ as $n \rightarrow \infty$. Therefore, $\langle V(\sum_{k=1}^{\infty} c_k e_k), f \rangle = 0$. Hence, $\langle Vh, f \rangle = 0, \forall h \in \text{dom}(V)$, which implies that $f \in \text{dom}(V^*)$. Therefore, the assumptions $\langle f, V e_k \rangle = 0, \forall k \in \mathbb{N}$, imply that $\langle V^* f, e_k \rangle = 0, \forall k \in \mathbb{N}$, which implies that $V^* f = 0$ and the injectivity of V^* implies that $f = 0$. This ends the proof that $(V e_k)_{k=1}^{\infty}$ is complete in \mathcal{H} . \square

Note that, in each of the statements in Proposition 4.6, the orthonormal basis can be replaced by a Riesz basis, in view of Proposition 4.6(d).

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