POLYTOPALITY AND CARTESIAN PRODUCTS OF GRAPHS

JULIAN PFEIFLE, VINCENT PILAUD, AND FRANCISCO SANTOS

Abstract. We study the question of polytopality of graphs: when is a given graph the graph of a polytope? We first review the known necessary conditions for a graph to be polytopal, and we provide several families of graphs which satisfy all these conditions, but which nonetheless are not graphs of polytopes.

Our main contribution concerns the polytopality of Cartesian products of non-polytopal graphs. On the one hand, we show that products of simple polytopes are the only simple polytopes whose graph is a product. On the other hand, we provide a general method to construct (non-simple) polytopal products whose factors are not polytopal.

Even though graphs are perhaps the most prominent feature of polytopes, we are still far from being able to answer several basic questions regarding them. For applications, one of the most important ones is to bound the diameter of the graph in terms of the number of variables and inequalities defining the polytope [San10]. From a theoretical point of view, it is striking that we cannot even efficiently decide whether a given graph occurs as the graph of a polytope or not.

Steinitz' theorem from 1906 completely characterizes graphs of 3-dimensional polytopes as the 3-connected planar graphs [Ste22]. For higher dimensions, the situation is much more complicated: no general characterization of graphs of polytopes is known, even in dimension 4. In fact, Perles observed that absolutely every graph is an induced subgraph of the graph of some 4-dimensional polytope.

In this paper, we try to shed light upon these questions and study how polytopality behaves with respect to some common operations on graphs and polytopes. We start by reviewing in Section 1.1 some necessary conditions for a graph to be polytopal: Balinski's Theorem [Bal61], the d-Principal Subdivision Property [Bar67] and the Separation Property [Kle64]. One of our goals is to construct graphs satisfying these properties, but which nonetheless are not graphs of polytopes. We say that such graphs are non-polytopal for "non-trivial reasons". Moreover, since polytopes of different dimensions can have the same graph, it is also interesting to study the *polytopality range* of a graph, *i.e.* the set of possible dimensions of its realizations. For example, the polytopality range of the complete graph K_n on $n \geq 5$ vertices is $\{4, \dots, n-1\}$. Polytopes of dimension three are also special in this respect: the graph of a 3-polytope is never the graph of a d-polytope for any other d.

We then focus on graphs of simple polytopes. Apart from being regular, they are special in the sense that they leave no ambiguity: the whole face lattice of a simple polytope can be (efficiently) recovered from its graph [BML87, Kal88, Fri09]. In Section 1.2, we construct families of non-simply-polytopal graphs for non-trivial

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reasons. Our main tool is the remark that every induced cycle of length 3, 4 or 5 in the graph of a simple polytope defines a 2-dimensional face.

To close the first part of the paper, we study in Section 1.3 the behavior of polytopality with respect to the star-clique operation, which replaces a vertex of degree d by a d-clique. In dimension 3, this is the usual ΔY-operation, involved in one of the proofs of Steinitz' Theorem [Zie95].

The second part of this paper is dedicated to the study of the polytopality of Cartesian products of graphs. Cartesian products of polytopal graphs are automatically polytopal, and their polytopality range has been the subject of recent research [JZ00, Zie04, SZ10, MPP09]. The main contribution of this paper concerns the polytopality of Cartesian products of non-polytopal graphs. On the one hand, we show in Section 2.1 that products of simple polytopes are the only simple polytopes whose graph is a product. On the other hand, we provide in Section 2.2 a general method to construct (non-simple) polytopal products whose factors are not polytopal. To illustrate the possible behavior of polytopality under Cartesian product, we discuss various examples of products of a non-polytopal graph by a segment in Section 2.3.

1. Polytopality of graphs

Definition 1.1. *A graph G is polytopal if it is isomorphic to the graph of some polytope P. If P is d-dimensional, we say that G is d-polytopal.*

In small dimension, polytopality is easy to deal with. For example, 2-polytopal graphs are exactly cycles. The first interesting question is 3-polytopality, which is characterized by Steinitz' "Fundamental Theorem of convex types":

Theorem 1.2 (Steinitz [Ste22])**.** *A graph G is the graph of a 3-polytope P if and only if G is planar and 3-connected. Moreover, the combinatorial type of P is uniquely determined by G.* □

We refer to [Grü03, Zie95] for a discussion of three approaches for proving this fundamental theorem.

The first step to realizing a graph G is to understand the possible face lattice of a polytope whose graph is G. For example, it is often difficult to decide which cycles of G can define 2-faces of a d-polytope realizing G. In dimension 3, graphs of 2-faces are characterized by the following separation condition:

Theorem 1.3 (Whitney [Whi32])**.** *Let G be the graph of a 3-polytope P. The graphs of the 2-faces of P are precisely the induced cycles in G that do not separate G.* □

In contrast to the easy 2- and 3-dimensional worlds, d-polytopality becomes much more involved as soon as $d \geq 4$. As an illustration, the existence of neighborly polytopes (such as the well-known cyclic polytopes) proves that all possible edges can be present in the graph of a 4-polytope. Starting from a neighborly polytope, and stacking vertices on undesired edges, one can even learn the following:

Observation 1.4 (Perles)**.** *Every graph is an induced subgraph of the graph of a 4-polytope.*

1.1. **Necessary conditions for polytopality.** It is a long-standing question of polytope theory how to determine whether a graph is d-polytopal or not, without enumerating all d-polytopes with the same number of vertices. Here we recall some general necessary conditions and apply them to discuss polytopality of small examples.

Proposition 1.5. *A d-polytopal graph G satisfies the following properties:*

(1) Balinski's Theorem*: G is d-connected* [Bal61]*.*
(2) Principal Subdivision Property *(d-PSP): Every vertex of G is the principal vertex of a principal subdivision of K_{d+1}. Here, a* subdivision *of K_{d+1} is obtained by replacing edges by paths, and a* principal subdivision *of K_{d+1} is a subdivision in which all edges incident to a distinguished* principal vertex *are not subdivided* [Bar67]*.*
(3) Separation Property*: The maximal number of components into which G may be separated by removing $n > d$ vertices equals $f_{d-1}\big(C_d(n)\big)$, the maximum number of facets of a d-polytope with n vertices* [Kle64]*.* □

Remark 1.6. The principal subdivision property together with Steinitz' Theorem ensure that no graph of a 3-polytope is d-polytopal for $d \neq 3$. In other words, any 3-polytope is the unique polytopal realization of its graph. This property is also obviously true in dimension 0, 1 or 2. In contrast, it is strongly wrong in dimension 4 and higher as the complete graph shows: for every $n \geq 5$ and for every $d \in \{4, \dots, n-1\}$ there are polytopes whose graph is the complete graph K_n.

Before applying Proposition 1.5 on several examples, let us insist on the fact that these necessary conditions are not sufficient (see also Examples 1.17 and 1.25):

Example 1.7 (Non-polytopality of the complete bipartite graph [Bar67])**.** The complete bipartite graph $K_{m,n}$ is not polytopal, for any two integers $m, n \geq 3$, although $K_{n,n}$ satisfies all properties of Proposition 1.5 to be 4-polytopal as soon as $n \geq 7$.

Indeed, assume that $K_{n,m}$ is the graph of a d-polytope P. Then $d \geq 4$ because $K_{n,m}$ is non-planar. Consider the induced subgraph H of $K_{n,m}$ corresponding to some 3-face F of P. Because H is induced and has minimum degree at least 3, it contains a $K_{3,3}$ minor, so F was not a 3-face after all.

Example 1.8 (Circulant graphs)**.** Let n be an integer and S denote a subset of $\left\{1, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. The *circulant* graph $\Gamma_n(S)$ is the graph whose vertex set is $\mathbb{Z}_n$, and whose edges are the pairs of vertices with difference in $S \cup (-S)$. Observe that the degree of $\Gamma_n(S)$ is precisely $|S \cup (-S)|$ — in particular, the degree is odd only if n is even and S contains $n/2$ — and that $\Gamma_n(S)$ is connected if and only if $S \cup \{n\}$ is relatively prime. For example, Figure 1 represents all connected circulant graphs on at most 8 vertices.

Using Proposition 1.5 we can determine the polytopality of various circulant graphs:

Degree 2: A connected circulant graph of degree 2 is a cycle, and thus the graph of a polygon.

Degree 3: Up to isomorphism, the only connected circulant graphs of degree 3 are $\Gamma_{2m}(1, m)$ and $\Gamma_{4m+2}(2, 2m+1)$. When $m \geq 3$, the first one is not planar, and thus not polytopal. The second one is the graph of a prism over a $(2m+1)$-gon.

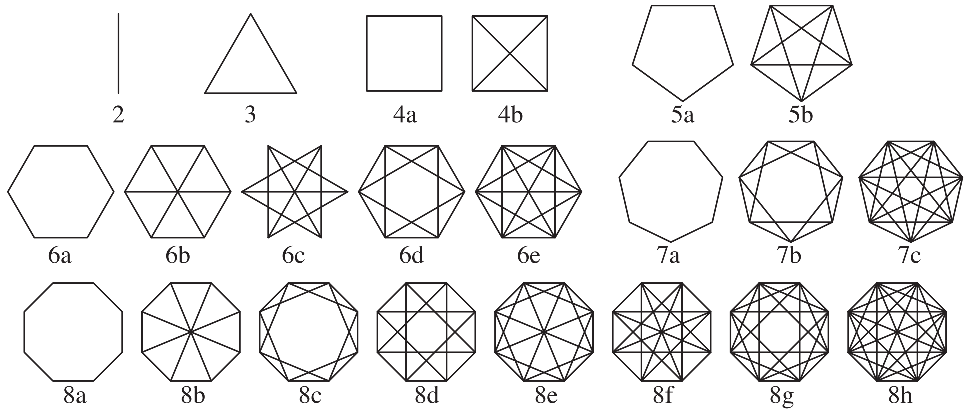

FIGURE 1. Connected circulant graphs with at most 8 vertices.

Degree 4: As soon as we reach degree 4, we cannot provide a complete description of polytopal circulant graphs, but we can discuss special cases, namely the circulant graphs $\Gamma_n(1, s)$ for $s \in \{2, 3, 4\}$:

(a) $s = 2$: For any $m \geq 2$, the graph $\Gamma_{2m}(1, 2)$ is the graph of an antiprism over an m-gon. In contrast, for any $m \geq 3$, the graph $\Gamma_{2m+1}(1, 2)$ is not polytopal: it is not planar and does not satisfy the principal subdivision property for dimension 4.

(b) $s \in \{3, 4\}$: For any $n \geq 7$, the graph $\Gamma_n(1, 3)$ is non-polytopal. Indeed, the 4-cycles induced by the vertices $\{1, 2, 3, 4\}$ and $\{2, 3, 4, 5\}$ should define 2-faces of any realization (because of Theorem 1.3 in dimension 3 and of Proposition 1.14, below, in dimension 4), but they intersect improperly. Similarly, for any $n \geq 9$, the graph $\Gamma_n(1, 4)$ is not polytopal.

Degree $n - 2$: The graph $\Gamma_{2m}(1, 2, \ldots, m - 1)$ is the only circulant graph with two vertices more than its degree. It is not planar when $m \geq 4$ and it is not $(2m - 2)$-polytopal since it does not satisfy the principal subdivision property in this dimension. However, it is always the graph of the m-dimensional cross-polytope, and when m is even, it is also the graph of the join of two $(m/2)$-dimensional cross-polytopes.

Degree $n - 1$: The complete graph on n vertices is the graph of any neighborly polytope, and its polytopality range is $\{4, \ldots, n-1\}$ (as soon as $n \geq 5$).

The sporadic cases developed above are sufficient to determine the polytopality range of all circulant graphs on at most 8 vertices, except the graphs 8e and 8f of Figure 1 that we treat separately now. None of them can be 3-polytopal since they are not planar. We prove that they are not 4-polytopal by discussing what could be the 3-faces of a possible realization:

- We start with the graph $\Gamma_8(1, 3, 4)$ represented in Figure 1(8f). Consider any subgraph of $\Gamma_8(1, 3, 4)$ induced by 6 vertices. If the distance between the two missing vertices is odd (resp. even), then the subgraph is not planar (resp. not 3-connected). Consequently, any subgraph of $\Gamma_8(1, 3, 4)$ induced by 7 vertices is not planar, while any subgraph of $\Gamma_8(1, 3, 4)$ induced by 5

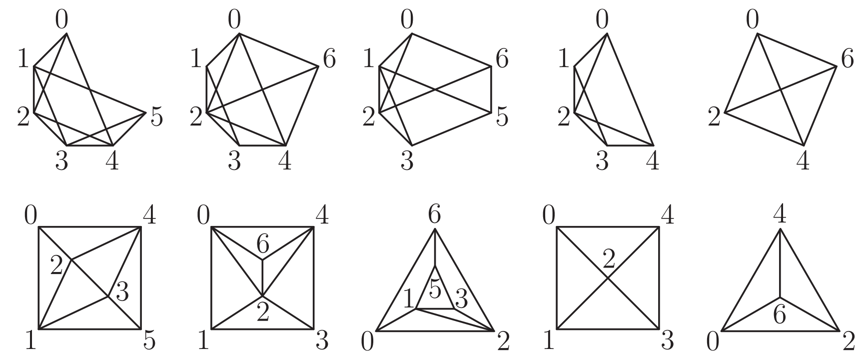

FIGURE 2. The 3-polytopal induced subgraphs of the circulant graph $\Gamma_8(1,2,4)$. The faces of the planar drawing below each of these subgraphs are the 2-faces of the corresponding 3-polytope.

vertices is not 3-connected. Thus, the only possible 3-faces are tetrahedra, but $\Gamma_8(1,3,4)$ contains only 4 induced K_4. Thus, $\Gamma_8(1,3,4)$ is not polytopal.

- The case of $\Gamma_8(1,2,4)$ is more involved. Up to rotation, its only 3-polytopal induced subgraphs are represented in Figure 2. Assume that the subgraph induced by $\{0,1,2,3,4,5\}$ defines a 3-face F in a realization P of $\Gamma_8(1,2,4)$. Then the triangle 123 is a 2-face of P and thus it should be contained in another 3-face of P. But any 3-face which contains 123 also contains either 0 or 4, and thus intersects improperly with F. Consequently, the subgraph induced by $\{0,1,2,3,4,5\}$ cannot define a 3-face of a realization of $\Gamma_8(1,2,4)$. For the same reason, the subgraphs induced by $\{0,1,2,3,4,6\}$ and $\{0,1,2,3,4\}$ cannot define 3-faces. Assume now that the subgraph induced by $\{0,1,2,3,5,6\}$ forms a 3-face in a realization P of $\Gamma_8(1,2,4)$. Then the triangle 123 is a 2-face of P and should be contained in another 3-face of P. The only possibility is the subgraph induced by $\{1,2,3,4,6,7\}$ which intersects improperly F. Finally, the only possible 3-faces are the two tetrahedra induced respectively by the odd and the even vertices, and thus, $\Gamma_8(1,2,4)$ is not polytopal.

We summarize these results in the following proposition:

Proposition 1.9. *The polytopality range of all the connected circulant graphs on at most* 8 *vertices, which are depicted on Figure 1, is given by the following table:*

2	*3*	*4a*	*4b*	*5a*	*5b*	*6a*	*6b*	*6c*	*6d*	*6e*
$\{1\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{2\}$	$\{4\}$	$\{2\}$	$\emptyset$	$\{3\}$	$\{3\}$	$\{4,5\}$

7a	*7b*	*7c*	*8a*	*8b*	*8c*	*8d*	*8e*	*8f*	*8g*	*8h*
$\{2\}$	$\emptyset$	$\{4,5,6\}$	$\{2\}$	$\emptyset$	$\{3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{4,5\}$	$\{4,5,6,7\}$

Example 1.10 (A graph whose polytopality range is $\{d\}$ [Kle64])**.** An interesting application of the separation property of Proposition 1.5 is the possibility to construct, for any integer d, a polytope whose polytopality range is exactly the singleton $\{d\}$. The construction, proposed by Klee [Kle64], consists of stacking a vertex on all facets of the cyclic polytope $C_d(n)$ (for example, on all facets of a

simplex). The graph of the resulting polytope can be separated into $f_{d-1}(C_d(n))$ isolated points by removing the n initial vertices, and thus is not d'-polytopal for $d' < d$, by the separation property. It can not be d'-polytopal for $d' > d$ either, since the stacked vertices have degree d (because the cyclic polytope is simplicial). Thus, the dimension of the resulting graph is not ambiguous.

Remark 1.11 (Polytopality range)**.** What subsets of $\mathbb{N}$ can be polytopality ranges of graphs? We know that if a polytopality range contains 1, 2 or 3, then it is a singleton (Remark 1.6) and that every singleton is a polytopality range (Example 1.10), as well as any interval $\{4, \dots, n\}$ (complete graph). We suspect that any interval $\{m, \dots, n\}$ with $4 \le m \le n$ is a polytopality range. One way of getting non-singleton polytopality ranges is to project polytopes preserving their graph. For example, [MPP09] obtain that for any sequence of integers $n_1, \dots, n_r$ (with $n_i \ge 2$), the product $\triangle_{n_1} \times \cdots \times \triangle_{n_r}$ can be projected from dimension $\sum n_i$ until dimension $r + 3$ preserving its graph (a particular example of that is the projection of the simplex until dimension 4). This raises the question of whether there exist graphs whose polytopality range is not an interval of $\mathbb{N}$.

1.2. **Simple polytopes.** A d-polytope is *simple* if its vertex figures are simplices. In other words, its facet-defining hyperplanes are in general position, so that a vertex is contained in exactly d facets, and also in exactly d edges (and thus the graph of a simple d-polytope is d-regular). Surprisingly, a d-regular graph can be realized by at most one simple polytope:

Theorem 1.12 ([BML87, Kal88])**.** *Two simple polytopes are combinatorially equivalent if and only if they have the same graph.* □

This property, conjectured by Perles, was first proved by Blind and Mani [BML87]. Kalai [Kal88] then gave a very simple (but exponential) algorithm for reconstructing the face lattice from the graph, and Friedman [Fri09] showed that this can even be done in polynomial time.

As mentioned previously, the first step to find a polytopal realization of a graph is often to understand what the face lattice of this realization can look like. Theorem 1.12 ensures that if the realization is simple, there is only one choice. This motivates us to temporarily restrict the study of realization of regular graphs to simple polytopes:

Definition 1.13. *A graph is* simply d-polytopal *if it is the graph of a simple d-polytope.*

We can exploit properties of simple polytopes to obtain results on the simple polytopality of graphs. For us, the key property turns out to be that any k-tuple of edges incident to a vertex of a simple polytope is contained in a k-face. For example, this implies the following result:

Proposition 1.14. *All induced cycles of length 3, 4 and 5 in the graph of a simple d-polytope P are graphs of 2-faces of P.*

Proof. For 3-cycles, the result is immediate: any two adjacent edges of a 3-cycle induce a 2-face, which must be a triangle because the graph is induced.

Next, let $\{a, b, c, d\}$ be consecutive vertices of a 4-cycle in the graph of a simple polytope P. Any pair of edges emanating from a vertex lies in a 2-face of P. Let C_a be the 2-face of P that contains the edges $\operatorname{conv}\{a, b\}$ and $\operatorname{conv}\{a, d\}$. Similarly, let C_c be the 2-face of P that contains $\operatorname{conv}\{b, c\}$ and $\operatorname{conv}\{c, d\}$. If C_a and C_c were distinct, they would intersect improperly, at least in the two vertices b and d. Thus, $C_a = C_c = \operatorname{conv}\{a, b, c, d\}$ is a 2-face of P.

The case of 5-cycles is a little more involved. We first show it for 3-polytopes. If a 5-cycle C in the graph G of a simple 3-polytope does not define a 2-face, it separates G into two nonempty subgraphs A and B (Theorem 1.3). Since G is 3-connected, both A and B are connected to C by at least three edges. But the endpoints of these six edges must be distributed among the five vertices of C, so one vertex of C receives two additional edges, and this contradicts simplicity.

For the general case, we show that any 5-cycle C in a simple polytope is contained in some 3-face, and apply the previous argument (a face of a simple polytope is simple). First observe that any three consecutive edges in the graph of a simple polytope lie in a common 3-face. This is true because any two adjacent edges define a 2-face, and a 2-face together with another adjacent edge defines a 3-face. Thus, four of the vertices of C are already contained in a 3-face F. If the fifth vertex w of C lies outside F, then the 2-face defined by the two edges of C incident to w intersects improperly with F. □

Remark 1.15. Observe that there is an induced 6-cycle in the graph of the cube (resp. an induced p-cycle in the graph of a double pyramid over a p-cycle, for $p \geq 3$) which is not the graph of a 2-face. It is also interesting to notice that contrarily to dimension 3 (Theorem 1.3), the 2-faces of a 4-polytope are not characterized by a separation property: a pyramid over a cube has a non-separating induced 6-cycle which does not define a 2-face.

Corollary 1.16. *A simply polytopal graph cannot:*

(i) be separated by an induced cycle of length 3*,* 4 *or* 5*.*
(ii) contain two induced cycles of length 4 *or* 5 *which share* 3 *vertices.*
(iii) contain an induced $K_{2,3}$ *or an induced Petersen graph.* □

Proof. Parts (i) and (ii) are immediate consequences of Proposition 1.14 since the 2-faces of a polytope are non-separating cycles and pairwise intersect in at most one edge. Part (iii) arises from Part (ii) since $K_{2,3}$ (resp. the Petersen graph) contains two induced 4-cycles (resp. two 5-cycles) which share 3 vertices. □

Example 1.17 (An infinite family of non-polytopal graphs for non-trivial reasons [NdO09])**.** Consider the family of graphs suggested in Figure 3. The nth graph of this family is the graph G_n whose vertex set is $\mathbb{Z}_{2n+3} \times \mathbb{Z}_2$ and where the vertex (x, y) is related with the vertices $(x + y + 1, y)$, $(x + y, y + 1)$, $(x - y - 1, y)$ and $(x + y - 1, y + 1)$.

Observe first that the graphs of this family satisfy all necessary conditions of Proposition 1.5:

(1) They are 4-connected: when we remove 3 vertices, either the external cycle $\{i0 \,|\, i \in \mathbb{Z}_{2n+1}\}$ or the internal cycle $\{i1 \,|\, i \in \mathbb{Z}_{2n+1}\}$ remains a path, to which all the vertices are connected.
(2) They satisfy the principal subdivision property for dimension 4: the edges of a principal subdivision of K_5 with principal vertex 00 are colored in Figure 3.

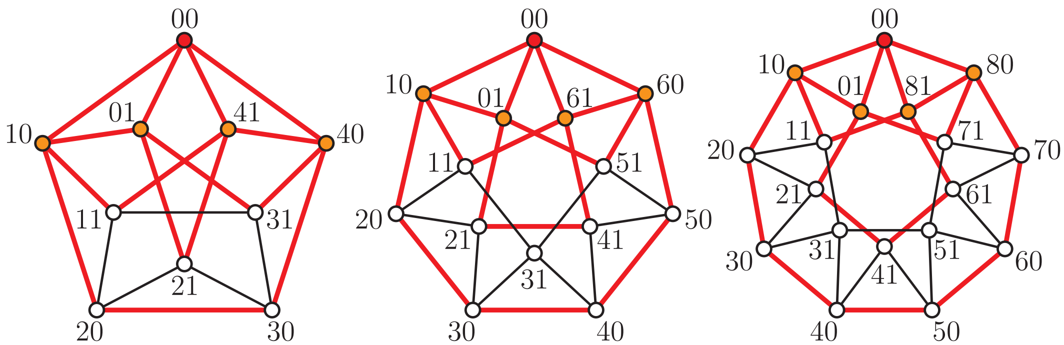

FIGURE 3. An infinite family of non-polytopal graphs (for non-trivial reasons). The vertex 00 is the principal vertex of a principal subdivision of K_5, whose edges are colored in red.

(3) They satisfy the separation property: the cyclic 4-polytope on m vertices has $\frac{m(m-3)}{2}$ facets, while removing m vertices from G_n cannot create more than m connected components.

Consider the first graph G_1 of this family (on the left in Figure 3). Since the 5-cycles induced by $\{00, 10, 20, 30, 40\}$ and $\{00, 10, 20, 21, 41\}$ share two edges, G_1 is not polytopal (because of Theorem 1.3 in dimension 3 and of Proposition 1.14 in dimension 4). In fact, Proposition 1.14 even excludes all graphs of the family:

Lemma 1.18. *None of the graphs of the infinite family suggested in Figure 3 is polytopal.*

Proof. Since they contain a subdivision of K_5, they are not 3-polytopal.

Denote by e_i the edge of the external cycle from vertex $i0$ to vertex $(i+1)0$. If the graph G_n were 4-polytopal, then all 3- and 4-cycles would define 2-faces. Now consider two consecutive angles e_1e_2 and e_2e_3 of the external cycle. Each of them defines a 2-face by simplicity. These two 2-faces must in fact coincide, since e_2 is already contained in a square and a triangular 2-face, none of which contain the angles e_1e_2 and e_2e_3. By iterating this argument, we obtain that the entire external cycle forms a 2-face.

Consider a 3-face F containing the external cycle. The edge e_1 must also be contained in either the adjacent square or the adjacent triangle; without loss of generality, let it be the square. Then the triangle adjacent to the next edge e_2 must also be in F (because F already contains two of its edges). By the same reasoning, the square adjacent to e_3 is also contained in F, and iterating this argument (and using that n is odd) shows that in fact F contains all the squares and triangles of G_n, contradiction. □

1.3. **Truncation and star-clique operation.** We consider the polytope $\tau_v(P)$ obtained by cutting off a single vertex v in a polytope P. The set of inequalities defining $\tau_v(P)$ is that of P together with a new inequality satisfied strictly by all the vertices of P except v. The faces of $\tau_v(P)$ are:

(i) all the faces of P which do not contain v;
(ii) the truncations $\tau_v(F)$ of all faces F of P containing v; and
(iii) the vertex figure of v in P together with all its faces.

In particular, if v is a simple vertex in P, then the truncation of v in P replaces v by a simplex. On the graph of P, it translates into the following transformation:

Definition 1.19. *Let G be a graph and v be a vertex of degree d of G. The* star-clique operation *(at v) replaces vertex v by a d-clique K, and assigns one edge incident to v to each vertex of K. The resulting graph $\sigma_v(G)$ has $d-1$ more vertices and $\binom{d}{2}$ more edges.*

Remark 1.20. In degree 3, star-clique operations, usually called ΔY-transformations, are used to prove Steinitz' Theorem 1.2. The argument is that any 3-connected and planar graph can be reduced to the complete graph K_4 by a sequence of such transformations, which preserve polytopality (see [Zie95] for details).

Proposition 1.21. *Let v be a vertex of degree d in a graph G. Then $\sigma_v(G)$ is d-polytopal if and only if G is d-polytopal.*

Proof. If a d-polytope P realizes G, then the truncation $\tau_v(P)$ realizes $\sigma_v(G)$.

For the other direction, consider a d-polytope Q which realizes $\sigma_v(G)$. We first show that the d-clique replacing v forms a facet of Q. Let its vertices be denoted $v_1, \dots, v_d$. Observe that all these vertices have degree d in $\sigma_v(G)$. That is, Q is "simple at those vertices". This implies that for every subset S of neighbors of, say, v_1, there is a face of dimension $|S|$ containing S and v_1. In particular, there is a facet F of Q containing $v_1, \dots, v_d$. By simplicity of all these vertices, F cannot contain any other vertex.

Up to a projective transformation, we can assume that the d facets of Q adjacent to F intersect behind F. Then, removing the inequality defining F from the facet description of Q creates a polytope which realizes G. □

We can exploit Proposition 1.21 to construct several families of non-polytopal graphs. We need the following lemma:

Lemma 1.22. *Let v be a vertex of degree at least 4 in a 3-polytopal graph G. Then $\sigma_v(G)$ is not planar, and thus not 3-polytopal.*

Proof. Let A denote the cycle formed by the edges of G which are not incident to v, but belong to a 2-face incident to v in the 3-polytope realizing G (in other words, E denotes the link of v in the 3-polytope realizing G). Let B denote the set of all edges of G which are neither incident to v, nor contained in A. Then the minor of the graph $\sigma_v(G)$ obtained by contracting all edges of A and deleting all edges of B is the complete graph K_{n+1}, where $n = \deg_G(v)$. □

Corollary 1.23. *Any graph obtained from a 4-regular 3-polytopal graph by a finite nonempty sequence of star-clique operations is non-polytopal.*

Proof. No such graph can be 3-polytopal since it is not planar. If the resulting graph were 4-polytopal, Proposition 1.21 would assert that the original graph was also 4-polytopal, which would contradict Remark 1.6. □

Remark 1.24. This corollary fails in higher dimension: the graph obtained from a d-polytopal graph by a star-clique operation on a vertex of degree $\delta > d$ may still be polytopal. For example, the complete graph K_n is a $(n-1)$-regular 4-polytopal graph, and the graph $K_{n-1} \times K_2$ obtained by a star-clique operation on a vertex of K_n is still 4-polytopal [MPP09].

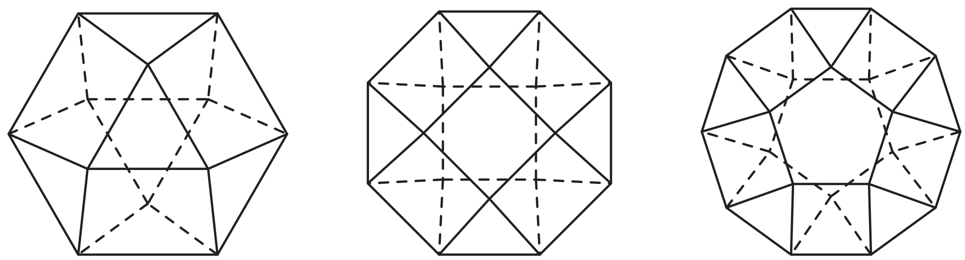

FIGURE 4. The graphs $✡_n$ for $n \in \{3, 4, 5\}$.

Example 1.25 (Another infinite family of non-polytopal graphs for non-trivial reasons)**.** For $n \geq 3$, consider the family of graphs suggested by Figure 4.

They are constructed as follows: place a regular $2n$-gon C_{2n} into the plane, centered at the origin. Draw a copy C'_{2n} of C_{2n} scaled by $\frac{1}{2}$ and rotated by $\frac{\pi}{2n}$, and lift the vertices of C'_{2n} alternately to heights 1 and -1 into the third dimension. The graph $✡_n$ is the graph of the convex hull of the result.

In other words, the graph $✡_n$ is the graph of the Minkowski sum of two pyramids over an n-gon (the first pyramid obtained as the convex hull of the even vertices of C'_{2n} together with the point $(0, 0, 1)$, and the second pyramid obtained as the convex hull of the odd vertices of C'_{2n} together with the point $(0, 0, -1)$).

Let $✡_n^\star$ be the result of successively applying the star-clique operation to all vertices on the intermediate cycle C_{2n}. Corollary 1.23 ensures that $✡_n^\star$ is not polytopal, although it satisfies all necessary conditions to be 4-polytopal (we skip this discussion which is similar to that in Example 1.17).

2. Polytopality of products of graphs

Define the *Cartesian product* $G \times H$ of two graphs G and H to be the graph whose vertex set is the product $V(G \times H) := V(G) \times V(H)$, and whose edge set is $E(G \times H) := \big(V(G) \times E(H)\big) \cup \big(E(G) \times V(H)\big)$. In other words, for $a, c \in V(G)$ and $b, d \in V(H)$, the vertices (a, b) and (c, d) of $G \times H$ are adjacent if either $a = c$ and $\{b, d\} \in E(H)$, or $b = d$ and $\{a, c\} \in E(G)$. Notice that this product is usually denoted by $G \square H$ in graph theory. We choose to use the notation $G \times H$ to be consistent with the Cartesian product of polytopes: if G and H are the graphs of the polytopes P and Q respectively, then the product $G \times H$ is the graph of the product $P \times Q$. In this section, we focus on the polytopality of products of non-polytopal graphs.

As already mentioned, the factors of a polytopal product are not necessarily polytopal: consider for example the product of a triangle by a path, or the product of a segment by two glued triangles (see Figure 5 and more generally Proposition 2.8). We neutralize these elementary examples by further requiring the product $G \times H$, or equivalently the factors G and H, to be regular (the degree of a vertex (v, w) of $G \times H$ is the sum of the degrees of the vertices v of G and w of H). In this case, it is natural to investigate when such regular products can be simply polytopal. The answer is given by Theorem 2.3.

Our study of polytopality of Cartesian products of graphs was inspired by Ziegler's prototype question:

Question 2.1 (Ziegler [CRM09])**.** *Is the product of two Petersen graphs polytopal?*

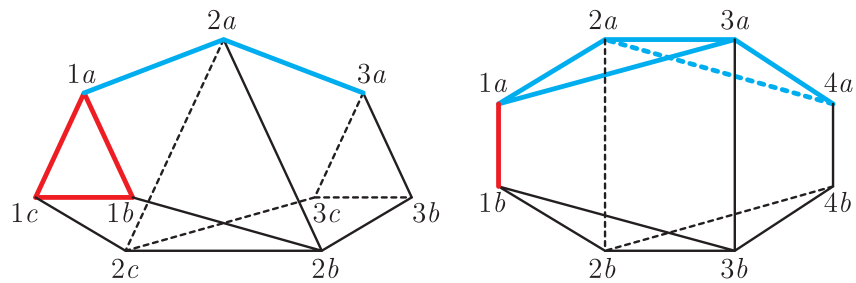

Figure 5. Polytopal products of non-polytopal graphs: the product of a triangle abc by a path 123 (left) and the product of a segment ab by two glued triangles 123 and 234 (right).

Incidentally, we already answered this question in the case of dimension 6 in Corollary 1.16: the product of two Petersen graphs cannot be simply polytopal since it contains an induced Petersen graph. However, we have no answer for dimensions 4 and 5.

Before starting, let us observe that the necessary conditions of Proposition 1.5 are preserved under Cartesian products in the following sense:

Proposition 2.2. *If two graphs G and H are respectively d- and e-connected, and respectively satisfy d- and e-PSP, then their product $G \times H$ is $(d+e)$-connected and satisfies $(d+e)$-PSP.*

Proof. The connectivity of a Cartesian product of graphs was studied in [CS99]. In fact, it is even proved in [Špa08] that

$$\kappa(G \times H) = \min(\kappa(G)|H|, \kappa(H)|G|, \delta(G) + \delta(H)) \geq \kappa(G) + \kappa(H),$$

where $\kappa(G)$ and $\delta(G)$ respectively denote the connectivity and the minimum degree of a graph G.

For the principal subdivision property, consider a vertex (v, w) of $G \times H$. Choose a principal subdivision of K_{d+1} in G with principal vertex v and neighbors N_v, and a principal subdivision of K_{e+1} in H with principal vertex w and neighbors N_w. This gives rise to a principal subdivision of K_{d+e+1} in $G \times H$ with principal vertex (v, w) and neighbors $(N_v \times \{w\}) \cup (\{v\} \times N_w)$. Indeed, for $x, x' \in N_v$, the vertices (x, w) and (x', w) are connected by a path in $G \times w$ by construction; similarly, for $y, y' \in N_w$, the vertices (v, y) and (v, y') are connected by a path in $v \times H$. Finally, for each $x \in N_v$ and $y \in N_w$, connect (x, w) to (v, y) via the path of length 2 that passes through (x, y). All these paths are disjoint by construction. □

2.1. **Simply polytopal products.** A product of simply polytopal graphs is automatically simply polytopal. We prove that the reciprocal statement is also true:

Theorem 2.3. *A product of graphs is simply polytopal if and only if its factors are.*

Applying Theorem 1.12, we obtain a strong characterization of the simply polytopal products:

Corollary 2.4. *The polytope realizing the above product of graphs is unique. Therefore, products of simple polytopes are the only simple polytopes whose graph is a product.* □

Let G and H be two connected regular graphs of degree d and e respectively, and assume that the graph $G \times H$ is the graph of a simple $(d+e)$-polytope P. By Proposition 1.14, for all edges a of G and b of H, the 4-cycle $a \times b$ is the graph of a 2-face of P.

Observation 2.5. *Let F be any facet of P, let v be a vertex of G, and let $\{x, y\}$ be an edge of H such that $(v, x) \in F$ and $(v, y) \notin F$. Then $G \times \{x\} \subset F$ and $G \times \{y\} \cap F = \emptyset$.*

Proof. Since the polytope is simple, all neighbors of (v, x) except (v, y) are connected to (v, x) by an edge of F. Let v' be a neighbor of v in G, and let C be the 2-face $\operatorname{conv}\{v, v'\} \times \operatorname{conv}\{x, y\}$ of P. If (v', y) were a vertex of F, the intersection $C \cap F$ would consist of exactly three vertices (because $(v, y) \notin F$), a contradiction. In summary, $(v', x) \in F$ and $(v', y) \notin F$, for all neighbors v' of v. Repeating this argument and using the fact that G is connected yields $G \times \{x\} \subset F$ and $G \times \{y\} \cap F = \emptyset$. □

Lemma 2.6. *The graph of any facet of P is either of the form $G' \times H$ for a $(d-1)$-regular induced subgraph G' of G, or of the form $G \times H'$ for an $(e-1)$-regular induced subgraph H' of H.*

Proof. Assume that the graph of a facet F is not of the form $G' \times H$. Then there exists a vertex v of G and an edge $\{x, y\}$ of H such that $(v, x) \in F$ and $(v, y) \notin F$. By Observation 2.5, the subgraph H' of H induced by the vertices $y \in H$ such that $G \times \{y\} \subset F$ is nonempty. We now prove that the graph $\operatorname{gr}(F)$ of F is exactly $G \times H'$.

The inclusion $G \times H' \subset \operatorname{gr}(F)$ is clear: by definition, $G \times \{y\}$ is a subgraph of $\operatorname{gr}(F)$ for any vertex $y \in H'$. For any edge $\{x, y\}$ of H' and any vertex $v \in G$, the two vertices (v, x) and (v, y) are contained in F, so the edge between them is an edge of F; if not, we would have an improper intersection between F and this edge. For the other inclusion, define $H'' := \{y \in H \mid G \times \{y\} \cap F = \emptyset\}$ and let $H''' := H \smallsetminus (H' \cup H'')$. If $H''' \neq \emptyset$, the fact that H is connected ensures that there is an edge between some vertex of H''' and either a vertex of H' or H''. This contradicts Observation 2.5.

We have proved that $G \times H' = \operatorname{gr}(F)$. The fact that F is a simple $(d+e-1)$-polytope and the d-regularity of G together ensure that H' is $(e-1)$-regular. □

Proof of Theorem 2.3. One direction is clear. For the other direction, proceed by induction on $d+e$, the cases $d = 0$ and $e = 0$ being trivial. Now assume that $d, e \geq 1$, that $G \times H = \operatorname{gr}(P)$, and that G is not the graph of a d-polytope. By Lemma 2.6, all facets of P are of the form $G' \times H$ or $G \times H'$, where G' (resp. H') is an induced $(d-1)$-regular (resp. $(e-1)$-regular) subgraph of G (resp. H). By induction, the second case does not arise. We fix a vertex w of H. Then induction tell us that $F_w := G' \times \{w\}$ is a face of P, and $G' \times H$ is the only facet of P that contains F_w by Lemma 2.6. This cannot occur unless F_w is a facet, but this only happens in the base case $H = \{w\}$. □

Example 2.7. Consider a graph G that is d-regular, d-connected, and satisfies d-PSP, but is not simply d-polytopal. Then, any product of G by a simply e-polytopal graph is $(d+e)$-regular, $(d+e)$-connected, satisfies $(d+e)$-PSP, but is not simply $(d+e)$-polytopal.

For example, the product of the circulant graph $C_8(1,4)$ by the graph of the d-dimensional cube is a non simply polytopal graph for non-trivial reasons. For any $m \geq 4$, the product of the circulant graph $C_{2m}(1,m)$ by a segment is non-polytopal for non-trivial reasons.

2.2. **Polytopal products of non-polytopal graphs.** In this section, we give a general construction to obtain polytopal products starting from a polytopal graph G and a non-polytopal one H. We need the graph H to be the graph of a *regular subdivision* of a polytope Q, that is, the graph of the upper[1] envelope (the set of all upper facets with respect to the last coordinate) of the convex hull of the point set $\{(q,\omega(q)) \,|\, q \in V(Q)\} \subset \mathbb{R}^{e+1}$ obtained by lifting the vertices of $Q \subset \mathbb{R}^e$ according to a *lifting function* $\omega : V(Q) \to \mathbb{R}$.

Proposition 2.8. *If G is the graph of a d-polytope P, and H is the graph of a regular subdivision of an e-polytope Q, then $G \times H$ is $(d+e)$-polytopal. In the case $d > 1$, the regular subdivision of Q can even have internal vertices.*

Proof. Let $\omega : V(Q) \to \mathbb{R}_{>0}$ be a lifting function that induces a regular subdivision of Q with graph H. Assume without loss of generality that the origin of $\mathbb{R}^d$ lies in the interior of P. For each $p \in V(P)$ and $q \in V(Q)$, we define the point $\rho(p,q) := (\omega(q)p, q) \in \mathbb{R}^{d+e}$. Consider

$$R := \operatorname{conv}\{\rho(p,q) \,|\, p \in V(P), q \in V(Q)\}.$$

Let g be a facet of Q defined by the linear inequality $\langle \psi \,|\, y\rangle \leq 1$. Then the inequality $\langle (0,\psi) \,|\, (x,y)\rangle \leq 1$ defines a facet of R, with vertex set $\{\rho(p,q) \,|\, p \in P, q \in g\}$, and isomorphic to $P \times g$.

Let f be a facet of P defined by the linear inequality $\langle \phi \,|\, x\rangle \leq 1$. Let c be a cell of the subdivision of Q, and let $\psi_0 h + \langle \psi \,|\, y\rangle \leq 1$ be the linear inequality that defines the upper facet corresponding to c in the lifting. Then we claim that the linear inequality

$$\chi(x,y) = \psi_0\langle \phi \,|\, x\rangle + \langle \psi \,|\, y\rangle \leq 1$$

selects a facet of R with vertex set $\{\rho(p,q) \,|\, p \in f, q \in c\}$ that is isomorphic to $f \times c$. Indeed,

$$\chi\big(\rho(p,q)\big) = \chi(\omega(q)p, q) = \psi_0\omega(q)\langle \phi \,|\, p\rangle + \langle \psi \,|\, q\rangle \leq 1,$$

where equality holds if and only if $\langle \phi \,|\, p\rangle = 1$ and $\psi_0\omega(q) + \langle \psi \,|\, q\rangle = 1$, so that $p \in f$ and $q \in c$.

The above set $\mathcal{F}$ of facets of R contains all facets: indeed, any $(d+e-2)$-face of a facet in $\mathcal{F}$ is contained in precisely two facets in $\mathcal{F}$. Since the union of the edge sets of the facets in $\mathcal{F}$ is precisely $G \times H$, it follows that the graph of R equals $G \times H$.

A similar argument proves the same statement in the case when $d > 1$ and H is a regular subdivision of Q with internal vertices (meaning that not only the vertices of Q are lifted, but also a finite number of interior points). □

We already mentioned two examples obtained by such a construction in the beginning of this section (see Figure 5): the product of a polytopal graph by a path and the product of a segment by a subdivision of an n-gon with no internal vertex. Proposition 2.8 even produces examples of regular polytopal products which are not simply polytopal:

[1]The unusual convention we adopt here of defining a subdivision as the projection of the upper facets of the lifting simplifies the presentation of the construction.

Example 2.9. Let H be the graph obtained by a star-clique operation from the graph of an octahedron. It is non-polytopal (Corollary 1.23), but it is the graph of a regular subdivision of a 3-polytope (see Figure 6). Consequently, the product of H by any regular polytopal graph is polytopal. Thus, there exist regular polytopal products which are not simply polytopal.

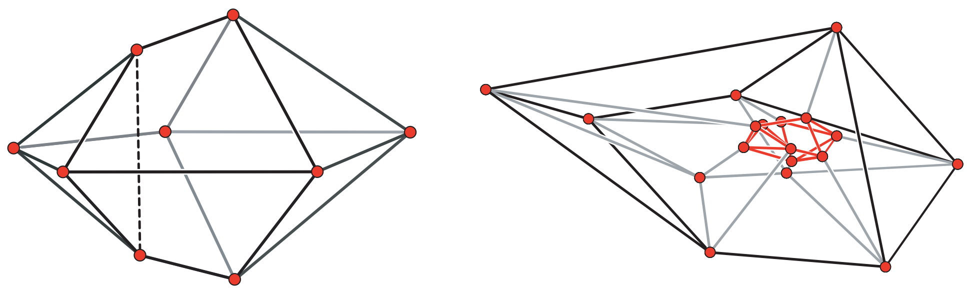

FIGURE 6. A non-polytopal 4-regular graph H which is the graph of a regular subdivision of a 3-polytope (left) and the Schlegel diagram of a 4-polytope whose graph is the product of H by a segment (right).

Finally, Proposition 2.8 also produces polytopal products of two non-polytopal graphs:

Example 2.10 (Product of dominos)**.** Define the *p-domino graph* D_p to be the product of a path P_p of length p by a segment. Let $p, q \geq 2$. Observe that D_p and D_q are not polytopal and that $D_p \times P_q$ is a regular subdivision of a 3-polytope. Consequently, the product of dominos $D_p \times D_q$ is a 4-polytopal product of two non-polytopal graphs (see Figure 7).

Finally, let us observe that the product $D_p \times D_q = P_p \times P_q \times (K_2)^2$ can be decomposed in different ways into a product of two graphs. However, in any such decomposition, at least one of the factors is non-polytopal.

2.3. **Product with a segment.** In this section, we complete our list of examples of products of a segment by a regular graph H. The goal is to illustrate all possible behaviors of such a product regarding polytopality:

(1) If H is polytopal, then $K_2 \times H$ is polytopal. However, some ambiguities can appear:
   (a) The dimension can be ambiguous. For example, $K_2 \times K_n$ is realized by the product of a segment by any neighborly polytope. See [MPP09] for a discussion on dimensional ambiguity of products of complete graphs.
   (b) The dimension can be unambiguous, but the combinatorics of the polytope can be ambiguous. In this case, H is not simply polytopal (Theorem 1.12). In Proposition 2.14, we determine all possible realizations of the graph of a prism over an octahedron.
   (c) There can be no ambiguity at all. This happens for example if H is simply 3-polytopal.

(2) If H is not polytopal, then $K_2 \times H$ is not simply polytopal (Theorem 2.3). However:

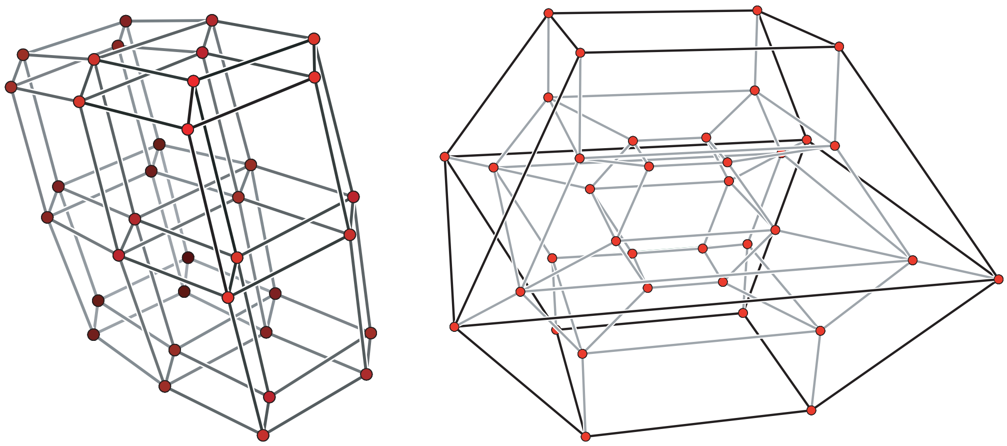

FIGURE 7. The graph of the product of two 2-dominos (left) and the Schlegel diagram of a realizing 4-polytope (right).

(a) $K_2 \times H$ can be polytopal in smaller dimension (Example 2.9).
(b) $K_2 \times H$ can be non-polytopal. This happens for example when H is the complete graph $K_{n,n}$ (Proposition 2.11) or when H is non-polytopal and 3-regular (Proposition 2.13).

Proposition 2.11. *For $n \geq 3$, the graph $K_2 \times K_{n,n}$ is not polytopal.*

To prove this proposition, we will need the following well known lemma:

Lemma 2.12. *A 3-polytope with no triangular facet has at least 8 vertices.*

Proof. Let P be a 3-polytope. For $k \geq 3$, denote by v_k the number of vertices of degree k and by p_k the number of 2-faces with k vertices. By double counting and Euler's Formula (see [Grü03, Chapter 13] for details),

$$v_3 + p_3 = 8 + \sum_{k \geq 5} (k-4)(v_k + p_k).$$

The lemma immediately follows. □

Proof of Proposition 2.11. Observe that $K_2 \times K_{n,n}$ is not d-polytopal for $d \leq 3$ because it contains a $K_{3,3}$-minor, and for $d = n + 1$ by Theorem 2.3.

The proof proceeds by contradiction. Suppose that $K_2 \times K_{n,n}$ is the graph of a d-polytope P, for some d with $3 \leq d \leq n$, and consider a 3-face F of P. Since $K_2 \times K_{n,n}$ contains no triangle, Lemma 2.12 says that F has at least 8 vertices. Denote by A and B the two maximal independent sets in $K_{n,n}$, and by A_0, B_0, A_1, B_1 their corresponding copies in the Cartesian product $K_2 \times K_{n,n}$. We discuss the possible repartition of the vertices of F in these sets.

Assume first that F has at least three vertices in A_0; let x, y, z be three of them. Then it cannot have more than two vertices in B_0, because otherwise its graph would contain a copy of $K_{3,3}$. In fact, there must be exactly two vertices u, v in B_0: since any vertex of F has degree at least 3, and each vertex in A_0 can only be connected to vertices in B_0 or to its corresponding neighbor in A_1, each vertex of F in A_0 must have at least, and thus exactly, two neighbors in B_0 and one in A_1. Thus, F also has at least three vertices in A_1, and by the same reasoning, there must be exactly two vertices in B_1; call one of them w. But now $\{x, y, z\}$ and $\{u, v, w\}$ are the two maximal independent sets of a subdivision of $K_{3,3}$ included in F.

By symmetry and Lemma 2.12, F has exactly two vertices in each of the sets A_0, B_0, A_1, B_1. Since all these vertices have degree 3, we have proved that P's only 3-faces are combinatorial cubes whose graphs are Cartesian products of K_2 with 4-cycle in $A_0 \cup B_0 = K_{n,n}$. However, this 4-cycle is not contained in any other 3-face, which is an obstruction to the existence of P. □

Proposition 2.13. *If H is a non-polytopal and 3-regular graph, then $K_2 \times H$ is non-polytopal.*

Proof. We distinguish two cases:

(i) If H contains a K_4-minor, then $K_2 \times H$ is not 3-polytopal because it contains a K_5-minor, and it is not 4-polytopal by Theorem 2.3. Since $K_2 \times H$ is 4-regular, these are the only possibilities.

(ii) Otherwise, H is a series-parallel graph. Thus, it can be obtained from K_2 by a sequence of *series* and *parallel* extensions, *i.e.* subdividing or duplicating an edge. Since duplicating an edge creates a double edge, and subdividing an edge yields a vertex of degree two, H is either not simple or not 3-regular; since our graphs are simple by assumption, this case cannot occur. □

To complete our collection of examples of products with a segment, we examine the possible realizations of the graph of the prism over the octahedron:

Proposition 2.14. *The graph of the prism over the octahedron is realized by exactly four combinatorially different polytopes.*

In order to exhibit four different realizations, we recall the situation and the proof of Proposition 2.8. Given the graph G of a d-polytope P and the graph H of a regular subdivision of an e-polytope Q defined by a lifting function $\omega : V(Q) \to \mathbb{R}$, we construct a $(d+e)$-polytope with graph $G \times H$ as follows: we start from the product $P \times Q$ and we lift each face $\{p\} \times Q$ using ω. This subdivides $\{p\} \times Q$, creating the subgraph $\{p\} \times H$ of the product $G \times H$. Observe now that the deformation can be different at each vertex of P: we can use a different lifting function at each vertex of P, and produce combinatorially different polytopes.

To come back to our example, denote by H the graph of the octahedron. Observe that the octahedron has four regular subdivisions with no additional edges: the octahedron itself (for a constant lifting function), and the three subdivisions into two Egyptian pyramids glued along their square face (for a lifting function that vanishes in the common square face and is negative at the other two vertices). This leads to four combinatorially different realizations of $K_2 \times H$: in our previous construction, we can choose either the octahedron at both ends of the segment (thus obtaining the prism over the octahedron), or the octahedron at one end and the glued Egyptian pyramids at the other, or the glued Egyptian pyramids at both ends of the segment (and this leads to two possibilities according to whether we choose the same square or two orthogonal squares to subdivide the two octahedra).

In fact, by the same argument, we can even slightly improve Proposition 2.8:

Observation 2.15. *Let G be the graph of a d-polytope P and H be the graph of an e-polytope Q. For each $v \in G$, choose a lifting function $\omega_v : V(Q) \to \mathbb{R}$, and denote by H_v the graph of the corresponding regular subdivision of Q. Then the graph obtained by replacing in $G \times H$ the subgraph $\{v\} \times H$ by $\{v\} \times H_v$ is polytopal.*

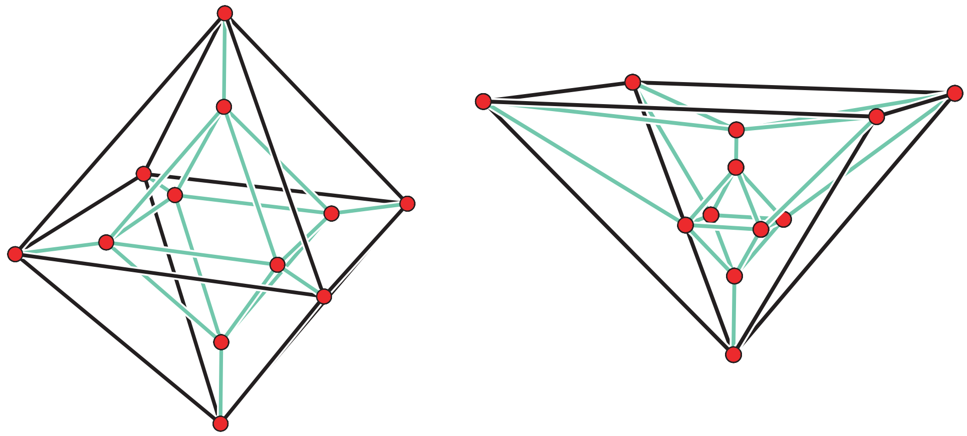

FIGURE 8. The prism over the octahedron (left) and a combinatorially different polytope with the same graph (right).

The result remains true if we allow the use of extra, perhaps interior, points of Q as vertices of the subdivisions, as long as $d > 1$ and all the subdivisions of Q have the same ones.

It remains to prove that any realization of $K_2 \times H$ is combinatorially equivalent to one of the four described above. First, the dimension is unambiguous: $K_2 \times H$ can only be 4-polytopal (by Remark 1.6 and Theorem 2.3). In particular, any realization is almost simple, in the following sense:

Definition 2.16. *A d-polytope is* almost simple *if its graph is $(d+1)$-regular.*

The vertex figures of a simple polytope are all simplices, which implies that any two incident edges in a simple polytope lie in a common 2-face. For almost simple polytopes, the vertex figures are almost as restricted: they are *$(d-1)$-circuits*, that is, $(d-1)$-polytopes with $d+1$ vertices. This implies the following property:

Proposition 2.17. *Let $\{v, w\}$ be an edge of an almost simple d-polytope P. Then:*

(a) either $\{v, w\}$ together with any other edge incident to v forms a 2-face;

(b) or there exists exactly one more edge incident to v which does not form a 2-face with $\{v, w\}$. In this case any two 2-faces both incident to $\{v, w\}$ lie in a 3-face.

Proof. Consider the vertex figure F_v of v. It is a $(d-1)$-polytope with $d+1$ vertices, one of which, say $\overline{w}$, corresponds to the edge $\{v, w\}$. This vertex $\overline{w}$ can be adjacent to either d or $d-1$ vertices of F_v. The first case corresponds to statement (a). In the second case, $\overline{w}$ has exactly one missing edge in F_v (corresponding to a missing 2-face in P), but the edge figure of $\{v, w\}$ is a $(d-2)$-simplex. This implies statement (b). □

With this in mind, we can finally prove Proposition 2.14:

Proof of Proposition 2.14. We introduce some notations: let $V := \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}\}$ denote the 6 vertices of the octahedron such that $\{1, \bar{1}\}$, $\{2, \bar{2}\}$ and $\{3, \bar{3}\}$ are the three missing edges, and let a and b denote the two endpoints of the segment factor. We denote the vertices of $K_2 \times H$ by $\{1a, 1b, \bar{1}a, \ldots, \bar{3}b\}$. We call *horizontal* *edges* the edges of the form $\{ia, ib\}$, for $i \in V$, and *vertical edges* the edges of the form $\{ix, jx\}$, for $i \in V$, $j \in V \smallsetminus \{i, \bar{i}\}$ and $x \in \{a, b\}$.

We first use "almost simplicity" to study the possible 2-faces of a realization P of $K_2 \times H$. Assume that there exists a 2-face F which is neither a triangle nor a square. It has to contain an angle between a horizontal edge and a vertical edge, say without loss of generality $\{1a, 1b\}$ and $\{1a, 2a\}$. By inducedness, the next edges of F are necessarily $\{2a, \bar{1}a\}$ and $\{\bar{1}a, \bar{1}b\}$. Since the edges $\{1a, 1b\}$ and $\{1a, 2a\}$ form an angle of F, the two edges $\{2a, 2b\}$ and $\{1b, 2b\}$ cannot form an angle: otherwise the 4-cycle $(1a, 1b, 2b, 2a)$ would form a square face which improperly intersects F. Similarly, since the edges $\{2a, \bar{1}a\}$ and $\{\bar{1}a, \bar{1}b\}$ form an angle, the edges $\{2a, 2b\}$ and $\{2b, \bar{1}b\}$ cannot form an angle. Thus, $\{2a, 2b\}$ is adjacent to two missing angles, which is impossible by Proposition 2.17. We conclude that the 2-faces of any realization of $K_2 \times H$ can only be squares and triangles.

We now use this information on the 2-faces to understand the possible 3-faces of P. Assume that none of the angles of the 4-cycles $(1a, 2a, \bar{1}a, \bar{2}a)$, $(1a, 3a, \bar{1}a, \bar{3}a)$, and $(2a, 3a, \bar{2}a, \bar{3}a)$ forms a 2-face. Then for each $i \in V$, the vertex ia has already two missing angles. Consequently, the remaining angles necessarily form a 2-face of P by Proposition 2.17. By inducedness, we obtain all the triangles of the a-copy of H, and any two adjacent of these triangles are contained in a common 3-face. This 3-face is necessarily an octahedron.

Assume now that one of the angles of the 4-cycles $(1a, 2a, \bar{1}a, \bar{2}a)$, $(1a, 3a, \bar{1}a, \bar{3}a)$, and $(2a, 3a, \bar{2}a, \bar{3}a)$ forms a 2-face. By symmetry, we can suppose that it is the angle defined by the edges $\{1a, 2a\}$ and $\{2a, \bar{1}a\}$. Let F denote the corresponding 2-face of P. By inducedness, the last vertex of F cannot be either $3a$ or $\bar{3}a$, and F is necessarily the square $(1a, 2a, \bar{1}a, \bar{2}a)$. It is now easy to see that none of the angles of the 4-cycle $(1a, 3a, \bar{1}a, \bar{3}a)$ (resp. $(2a, 3a, \bar{2}a, \bar{3}a)$) can be an angle of a 2-face of P: otherwise, this 4-cycle would be a 2-face of P (by a symmetric argument), which would intersect improperly with F. All together, this implies that the vertices $3a$ and $\bar{3}a$ both have already two missing angles, and thus, that all the other angles form 2-faces by Proposition 2.17. Furthermore, any two 2-faces adjacent to an edge $\{3a, ia\}$, with $i \in \{1, \bar{1}, 2, \bar{2}\}$, form a 3-face. This implies that all angles adjacent to a vertex ia, except the angles of the 4-cycles $(1a, 3a, \bar{1}a, \bar{3}a)$ and $(2a, 3a, \bar{2}a, \bar{3}a)$ form a 2-face.

Since the two above cases can occur independently at both ends of the segment (a, b), we obtain the claimed result. □

2.4. **Topological products.** To finish, we come back to Ziegler's motivating question 2.1: "is the product of two Petersen graphs polytopal?" We proved in Theorem 2.3 that it is not 6 polytopal, but the question remains open in dimension 4 and 5.

Proposition 2.18. *The product of two Petersen graphs is the graph of a cellular decomposition of* $\mathbb{RP}^2 \times \mathbb{RP}^2$.

Proof. The Petersen graph is the graph of a cellular decomposition of the projective plane $\mathbb{RP}^2$ with 6 pentagons (see Figure 9). Consequently, the product of two Petersen graphs is the graph of a cellular decomposition of $\mathbb{RP}^2 \times \mathbb{RP}^2$. The maximal cells of this decomposition are 36 products of two pentagons. □

This proposition tells that understanding the possible 4-faces of a realization, and their possible incidence relations (as we did for example in Proposition 2.11) is

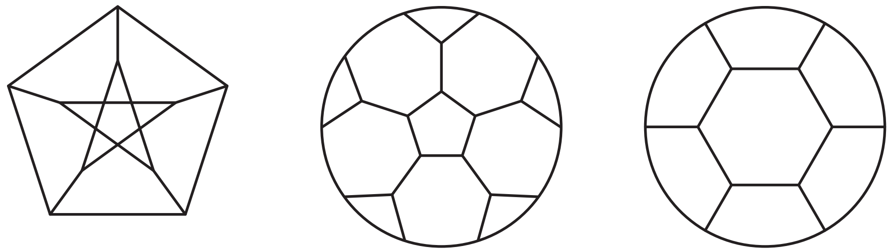

FIGURE 9. The Petersen graph (left), its embedding on the projective plane (middle) and the embedding of $K_{3,3}$ on the projective plane (right). Antipodal points on the circle are identified.

not enough to decide polytopality of the product of two Petersen graphs. Observe that the same remark holds for the product of any graphs of cellular decompositions of manifolds: for example, the product of a triangle by the Petersen graph is the graph of a cellular decomposition of $\mathcal{S}^1 \times \mathbb{RP}^2$.

Another interesting example is the product of a triangle by $K_{3,3}$. Indeed, in contrast with $K_2 \times K_{3,3}$, the graph $K_3 \times K_{3,3}$ is the graph of a cellular decomposition of the manifold $\mathcal{S}^1 \times \mathbb{RP}^2$. To see this, embed $K_{3,3}$ in the projective plane $\mathbb{RP}^2$ as in Figure 9, and multiply this embedding by a triangle. This cell decomposition is, however, not *strongly regular*. Here, following [EKZ03], we say that a cell decomposition is strongly regular if every closed cell is embedded and the intersection of every two of them is a (perhaps empty) closed cell. Our decomposition fails to have the second property because the embedding of $K_{3,3}$ in the projective plane already fails to have it: the central hexagon in the embedding of Figure 9 improperly intersects the three squares. Consequently, in the product with the triangle, each of the three hexagonal prisms improperly intersects three cubes. This can be solved by a "Dehn surgery", replacing the chain of three hexagonal prisms by a chain of six triangular prisms with the same boundary — see Figure 10. In fact, it turns out that the cellular decomposition of $\mathcal{S}^1 \times \mathbb{RP}^2$ obtained in this way is the unique (strongly regular) combinatorial manifold whose graph is $K_{3,3} \times K_3$. This example and other results will be discussed in a future publication.

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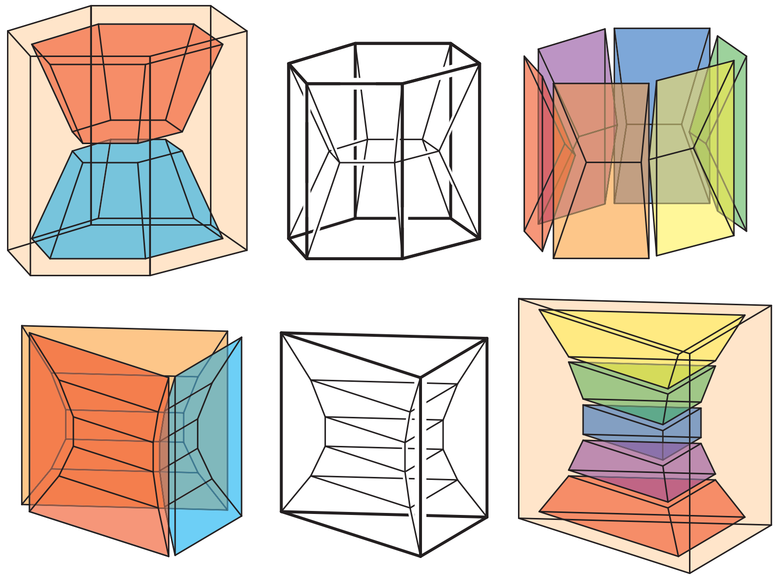

FIGURE 10. A decomposition of the 3-dimensional sphere (middle) using a cycle of three hexagonal prisms (left) and a cycle of six triangular prisms (right).

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Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain
E-mail address: julian.pfeifle@upc.edu

Équipe Combinatoire et Optimisation, Université Pierre et Marie Curie, Paris, France
E-mail address: vpilaud@math.jussieu.fr

Departamento de Matemáticas Estadística y Computación, Universidad de Cantabria, Santander, Spain
E-mail address: francisco.santos@unican.es