# Inversion of analytic characteristic functions and infinite convolutions of exponential and Laplace densities[1]

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**Abstract.**
We prove that certain quotients of entire functions are characteristic functions. Under some conditions, the probability measure corresponding to a characteristic function of that type has a density which can be expressed as a generalized Dirichlet series, which in turn is an infinite linear combination of exponential or Laplace densities. These results are applied to several examples.

**Keywords:** Infinite convolutions, characteristic function, entire functions.

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## 1 Introduction

There are many cases in the literature where a characteristic function $\varphi(t)$ of a probability measure can be written as $\varphi(t) = 1/g(it)$ for $t \in \mathbb{R}$, where $g(z)$ is an entire function of the complex variable $z$. Two important examples are the square of a Kolmogorov law and the Lévy area. Specifically, the characteristic function of the square of a Kolmogorov law is given by $\varphi_1(t) = \sqrt{2it}/\sin(\sqrt{2it})$, and thus we are setting $g_1(z) = \sin(\sqrt{2z})/\sqrt{2z}$. This example was studied by Dugué [6], who determined that its distribution function is the Jacobi theta function

$$F_1(x) = \vartheta_4(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-\pi^2 k^2 x/2}\,, \qquad x > 0\,.$$

The Lévy area was introduced by Lévy [14] (with a slightly different parametrization) as the random variable given by

$$\int_0^1 W_1(t)dW_2(t) - \int_0^1 W_2(t)dW_1(t)\,,$$

where $\{W_1(t),\ t \geq 0\}$ and $\{W_2(t),\ t \geq 0\}$ are two independent standard Brownian motions. Lévy [14] deduced that its characteristic function is $\varphi_2(t) = \operatorname{sech}(t)$. In this case we take $g_2(z) = \cos(z)$. The density was also computed by Lévy [14] and is

$$f_2(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi}{2}x\right) = \frac{e^{-\pi|x|/2}}{1+e^{-\pi|x|}} = \sum_{k=1}^{\infty}(-1)^{k+1}e^{-\frac{(2k-1)\pi}{2}|x|}\,, \qquad x \neq 0\,.$$

What these two examples share is that regardless of the behaviour of the entire functions $g_1(z)$ and $g_2(z)$, their inverses are characteristic functions. In fact, it is well known that for an entire function $g(z)$ of order $\rho < 2$ which has only real zeros and $g(0) = 1$, the inverse $\varphi(t) = 1/g(it)$ is a characteristic function (see

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Lukacs [15, pp. 88 and 212] for an equivalent formulation). This result follows from Hadamard factorization Theorem (see, for example, Levin [13, p. 26]) which states that an entire function of order $\rho < 2$ can be written as

$$g(z) = e^{cz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n} \ ,$$

where $\{a_n,\, n \geq 1\}$ are the zeros of $g(z)$ ($c$ must be real so that the characteristic function assertion is true). Note that such factorization of $g(z)$ induces a factorization of $\varphi(t)$, and each factor is of the type

$$\frac{1}{1 - it/a_n}\, e^{-it/a_n}$$

which is the characteristic function of a translated positive or negative exponential law. If $\rho < 1$, then we can leave aside the exponential part inside the infinite product for a canonical representation and $c = 0$. In any case $\varphi(t)$ is factorized as a convergent product of characteristic functions, and hence it is a characteristic function due to Lévy's continuity Theorem.

The two mentioned examples differ in terms of the density function. Notice that the density deduced from $F_1(x)$ can be considered an infinite linear combination of exponential densities, while $f_2(x)$ seems to be a mix of Laplace densities. The difference between both cases lies in the entire function $g(z)$. On the one hand, $g_1(z)$ has order $1/2$ and all its zeros are simple and positive; as opposed to the second example, where $g_2(z)$ has order $1$ and its zeros are simple but symmetric with respect to the origin.

The aim of this paper is to give some general results suggested by those and other similar examples. It will be shown that when $g(z)$ and $h(z)$ are both entire functions of order $\rho,\ \rho' \in (0, 2)$ respectively, satisfying $g(0) = h(0) = 1$ and with a further condition over their zeros then $\varphi(t) = h(it)/g(it)$ is a characteristic function. We prove that when $\rho,\ \rho' \in (0, 1)$, the zeros of $h(z)$ and $g(z)$ are simple and positive, and some other additional hypotheses, then the probability measure corresponding to $\varphi(t)$ has a density $f(x)$ that can be written as a sum of exponential type densities,

$$f(x) = -\sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x},\ x > 0.$$

This series is a generalized Dirichlet series (see Mandelbrojt [16]) which has very good properties; in particular, we prove that it is uniformly convergent on every compact subset of $(0, \infty)$, and the cumulative distribution function also turns out to be a generalized Dirichlet series.

When the zeros are simple and symmetric, positive and negative, the existence of a density can be also proved for $\rho = 1$ and $h(z) \equiv 1$. Then the density can be written as a series of Laplace type densities. This case is important because it covers some elements of the double Wiener chaos (see Janson [8]) as the double Itô-Wiener integrals where the kernel has symmetric zeros with multiplicity 2; in particular, the Lévy area belongs to this class.

We study some examples. In addition to the square of a Kolmogorov law and the Lévy area, we consider the law of the first hitting time of a Bessel process, whose characteristic function is expressed as a quotient of Bessel functions (Kent [12] and Borodin and Salminen [3]). We also show how this technique can be used to invert some Laplace transforms. In our last example we study a particular case of the Heston model used in mathematical finance and we prove that the general theory developed in the first part can be applied to it; such a study was the starting point of this paper.

To summarize, the paper is twofold. On the one hand, it shows a way to construct a rich family of characteristic functions. On the other, given a characteristic function that can be identified as belonging to that family, our results provide a procedure to invert that characteristic function.

## 2 Construction of characteristic functions

In this section we identify a rich family of characteristic functions which can be constructed by quotients of entire functions.

From now on we will consider that a *strictly increasing sequence of positive numbers*, $\{a_n\, n \geq 1\}$, is a sequence satisfying $0 < a_1 < a_2 < \cdots$ and such that $\lim_{n\to\infty} a_n = \infty$.

**Proposition 2.1.** *Consider two entire functions $g(z)$ and $h(z)$ of order lying in $(0,1)$ such that $g(0) = h(0) = 1$, with simple positive zeros given by the strictly increasing sequences of positive numbers $\{a_n,\, n \geq 1\}$ and $\{b_n,\, n \geq 1\}$ respectively. Assume that $a_n < b_n$ for all $n$. Then $\varphi(t) = h(it)/g(it)$ is a characteristic function of a probability measure on $[0,\infty)$.*

*Proof.* By Hadamard's factorization Theorem (see, for example, Levin [13, p. 26]),

$$g(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) \qquad \text{and} \qquad h(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{b_n}\right) .$$

We conclude from the convergence of both products that

$$\varphi(t) = \frac{h(it)}{g(it)} = \prod_{n=1}^{\infty}\frac{1 - it/b_n}{1 - it/a_n} = \prod_{n=1}^{\infty}\left(\frac{a_n}{b_n} + \left(1 - \frac{a_n}{b_n}\right)\left(1 - \frac{it}{a_n}\right)^{-1}\right) . \tag{1}$$

Since $0 < a_n/b_n < 1$ it follows that each factor of the above product is the characteristic function of the probability measure

$$\frac{a_n}{b_n}\,\delta_0 + \left(1 - \frac{a_n}{b_n}\right)\mathcal{E}\mathrm{xp}(a_n) ,$$

where $\delta_0$ is a Dirac measure at 0 and $\mathcal{E}\mathrm{xp}(a_n)$ is an exponential law with parameter $a_n$. The result is a consequence of Lévy's continuity Theorem. ■

**Remarks 2.2.**

**1)** The condition $g(0) = h(0) = 1$ is merely a way to ease the notation, in fact the same result would be true if we let $g(0) = h(0) \neq 0$.

**2)** By means of standard manipulations, Proposition 2.1 holds true for $h \equiv 1$. In this case each factor of $\varphi(t)$ is the characteristic function of the probability law $\mathcal{E}\mathrm{xp}(a_n)$.

**3)** A similar result to Proposition 2.1 is proved in Corollary 9.16 by Schilling *et al.* [19], where it is also deduced that the corresponding probability measure belongs to the Bondesson class, and if $h = 1$, then the probability measure is in the class of convolutions of exponential densities (for the definition of these classes see Schilling *et al.* [19, Pages 80 and 87]).

**4)** It is easy to show that the restriction of simple zeros in Proposition 2.1 can be relaxed if we let $a_n$ and $b_n$ have the same multiplicity for all $n$.

**Proposition 2.3.** *Consider two even entire functions $g(z)$ and $h(z)$ of order lying in $(0,2)$ such that $g(0) = h(0) = 1$. Let $\{a_n,\, n \geq 1\}$ and $\{b_n,\, n \geq 1\}$ be two strictly increasing sequences of positive numbers; and let $\{\pm a_n,\, n \geq 1\}$ and $\{\pm b_n,\, n \geq 1\}$ be the simple zeros of $g(z)$ and $h(z)$ respectively; assume that $a_n < b_n$ for all $n$. Then $\varphi(t) = h(it)/g(it)$ is a characteristic function of a probability measure on $\mathbb{R}$.*

*Proof.* By Hadamard's factorization Theorem

$$g(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{a_n^2}\right) \qquad \text{and} \qquad h(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{b_n^2}\right)$$

due to the symmetry of the entire functions. The rest of the proof follows the same argument as the derivation of Proposition 2.1. In this setting each factor of $\varphi(t) = h(it)/g(it)$ is the characteristic function of the probability measure

$$\frac{a_n^2}{b_n^2}\,\delta_0 + \left(1 - \frac{a_n^2}{b_n^2}\right)\mathcal{L}\mathrm{aplace}(a_n) ,$$

where $\mathcal{L}\mathrm{aplace}(a_n)$ is a Laplace law with parameter $a_n$. ■

As pointed out in Remark 2.2 there is a straightforward generalization of Proposition 2.3 when $h \equiv 1$.

The statements of Propositions 2.1 and 2.3 are related to the main results of the following sections. The reader will notice that it is possible to state more general results using the same ideas and hence with similar derivations. We may state Proposition 2.3 without requesting the functions to be symmetric but with their zeros given by the sequences $\{a_n,\, n \geq 1\}$ and $\{b_n,\, n \geq 1\}$ where $a_n$ and $b_n$ have the same sign, and $|a_n| < |b_n|$. In such a case

$$\varphi(t) = \frac{h(it)}{g(it)} = e^{dit} \prod_{n=1}^{\infty} \left( \frac{a_n}{b_n} + \left(1 - \frac{a_n}{b_n}\right)\left(1 - \frac{it}{a_n}\right)^{-1} \right) e^{it\left(\frac{1}{b_n} - \frac{1}{a_n}\right)}$$

(assuming $d \in \mathbb{R}$) and each factor is the characteristic function of a translation of the probability measure

$$\left( \frac{a_n}{b_n}\, \delta_0 + \left(1 - \frac{a_n}{b_n}\right) \mathcal{E}\mathrm{xp}(a_n) \right) .$$

For example, we can set $\varphi(t) = \mathrm{Ai}(uit)/\mathrm{Ai}(vit)$, where $0 < u < v < \infty$ and $\mathrm{Ai}(z)$ is Airy's function. The Airy function is an entire function of order $3/2$ and its zeros are real and negative (see Katori and Tanemura [11]). In this case, although we can state that $\varphi(t)$ is a characteristic function we will not be able to write its distribution function explicitly.

### 2.1 Finite convolution of exponential and Laplace densities

The next two lemmas will prove useful for determining the density of the characteristic functions in Proposition 2.1 and 2.3; we will consider a finite product approximation of $\varphi(t)$ and determine its density so we can take the limit afterwards. This section and the one after will develop the first steps of the procedure.

Let us recall some standard notations. Given two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $P_1$ and $P_2$, we denote by $P_1 \star P_2$ its convolution

$$P_1 \star P_2(B) := \int_{\mathbb{R}} P_1(B - y) P_2(dy) ,$$

where $B \in \mathcal{B}(\mathbb{R})$ and $B - y = \{x - y,\, x \in B\}$. The characteristic function of $P_1 \star P_2$ is the product of the characteristic functions of $P_1$ and $P_2$. Moreover, if $P_1$ and $P_2$ are absolutely continuous with density $f_1$ and $f_2$ respectively, then the density of $P_1 \star P_2$ is given by the convolution of $f_1$ and $f_2$

$$f_1 \star f_2(x) := \int_{-\infty}^{+\infty} f_1(y) f_2(x - y)\, dy.$$

The convolution of $P_1 \star \cdots \star P_n$ (resp. $f_1 \star \cdots \star f_n$) is denoted by $\star_{j=1}^{n} P_j$ (resp. $\star_{j=1}^{n} f_j$).

Lemma 2.4 is well known in the literature (see, for example, problem 12 of chapter 1 of Feller [7]).

**Lemma 2.4.** *Fix $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$ and define the couple $A(n) := \prod_{i=1}^{n} \lambda_i$ and $B(k, n) := \prod_{\substack{i=1 \\ i \neq k}}^{n} (\lambda_k - \lambda_i)$. Then $\star_{j=1}^{n} \mathcal{E}\mathrm{xp}(\lambda_j)$ has density given by*

$$f_n(x) = (-1)^{n+1} A(n) \sum_{i=1}^{n} \frac{e^{-\lambda_i x}}{B(i, n)} \qquad x \geq 0 .$$

The proof of the following result uses an interesting property given by Bondesson (see [2]) in the context of generalized gamma convolutions that, in our setup, states that if $Y$ is a non–negative random variable with moment generating function $M_Y(u)$, for $u$ in a neighborhood of $0$, and $T$ is a centered normal random variable with variance 2, independent of $Y$, then the random variable $X := \sqrt{Y}\, T$ has moment generating function $M_X(u) = M_Y(u^2)$. The proof is completed in the line

$$M_X(u) = \mathbb{E}\big[e^{u\sqrt{Y}\,T}\big] = \mathbb{E}\Big[\mathbb{E}\big[e^{u\sqrt{Y}\,T}/Y\big]\Big] = \mathbb{E}\big[e^{u^2\,Y}\big] = M_Y(u^2) .$$

In fact, the next result is a consequence of the previous lemma and will be useful for deriving the density of the characteristic function of Proposition 2.3.

**Lemma 2.5.** *Fix* $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$ *and define* $E(k,n) = \prod_{\substack{i=1 \\ i\neq k}}^{n} (\lambda_k^2 - \lambda_i^2)$. *Then* $\star_{j=1}^{n} \mathcal{L}\text{aplace}(\lambda_j)$ *has density given by*

$$f_n(x) = \frac{(-1)^{n+1}}{2} A^2(n) \sum_{i=1}^{n} \frac{e^{-\lambda_i |x|}}{\lambda_i E(i,n)} \qquad x \in \mathbb{R} \, .$$

*Proof.* Let us consider the characteristic function

$$\tilde{\varphi}(t) = \prod_{j=1}^{n} \left(1 - \frac{it}{\lambda_j^2}\right)^{-1} ,$$

which corresponds to a random variable, $Y$, that is the sum of $n$ independent exponential random variables with parameters $\{\lambda_j^2,\, 1 \le j \le n\}$. By the previous lemma, the density of $Y$ is

$$f_Y(y) = (-1)^{n+1} A^2(n) \sum_{i=1}^{n} \frac{e^{-\lambda_i^2 y}}{E(i,n)} \qquad y \ge 0 \, .$$

Let $T$ be a centered normal random variable with variance 2, hence the random variable $X := \sqrt{Y}T$ has characteristic function

$$\varphi(t) = \prod_{j=1}^{n} \left(1 + \frac{t^2}{\lambda_j^2}\right)^{-1} ,$$

which corresponds to the sum of $n$ independent Laplace random variables with parameters $\{\lambda_j,\, 1 \le j \le n\}$. In order to find the density of $X$, consider the pair $(X,T)$ and compute its marginal density by means of a change of variables to the pair $(Y,T)$. The Jacobian determinant is $\frac{2|X|}{T^2}$, which means that the change of variables is an almost everywhere diffeomorphism of the plane. Fix $x \in \mathbb{R}$, then

$$\begin{aligned}
f_X(x) &= \mathbf{1}_{x\ge 0} \int_0^{\infty} f_{(X,T)}(x,t)dt + \mathbf{1}_{x<0} \int_{-\infty}^{0} f_{(X,T)}(x,t)dt \\
&= \mathbf{1}_{x\ge 0} \int_0^{\infty} f_{(Y,T)}\left(\frac{x^2}{t^2}, t\right) \frac{2|x|}{t^2} dt + \mathbf{1}_{x<0} \int_{-\infty}^{0} f_{(Y,T)}\left(\frac{x^2}{t^2}, t\right) \frac{2|x|}{t^2} dt \\
&= \int_0^{\infty} f_Y\left(\frac{x^2}{t^2}\right) f_T(t) \frac{2|x|}{t^2} dt \\
&= \int_0^{\infty} \sum_{i=1}^{n} \frac{(-1)^{n+1} A^2(n)}{E(i,n)} e^{-\lambda_i^2 x^2/t^2} e^{-t^2/4} \frac{1}{2\sqrt{\pi}} \frac{2|x|}{t^2} dt \\
&= \sum_{i=1}^{n} \frac{(-1)^{n+1} A^2(n) |x|}{\sqrt{\pi} E(i,n)} \int_0^{\infty} \frac{e^{-\lambda_i^2 x^2/t^2} e^{-t^2/4}}{t^2} dt \\
&= \frac{(-1)^{n+1}}{2} A^2(n) \sum_{i=1}^{n} \frac{e^{-\lambda_i |x|}}{\lambda_i E(i,n)}
\end{aligned}$$

as required. ■

## 2.2 Finite density approximation of the characteristic function

We derive here the density function for a finite approximation of the characteristic functions of Proposition 2.1 and 2.3.

**Lemma 2.6.** *Let $0 < a_1 < a_2 < \cdots < a_n$ and $0 < b_1 < b_2 < \cdots < b_n$ such that $a_i < b_i$ for $1 \le i \le n$. Write either*

*(a)* $g_n(z) = \prod_{i=1}^n (1 - z/a_i)$ *and* $h_n(z) = \prod_{i=1}^n (1 - z/b_i)$, *or*

*(b)* $g_n(z) = \prod_{i=1}^n \left(1 - z^2/a_i^2\right)$ *and* $h_n(z) = \prod_{i=1}^n \left(1 - z^2/b_i^2\right)$.

*Let $\varphi_n(t)$ be the characteristic function $\varphi_n(t) = h_n(it)/g_n(it)$. Then the corresponding law of $\varphi_n(t)$ is*

$$\left(\prod_{i=1}^n \frac{a_i}{b_i}\right) \delta_0 + \mu_n \ ,$$

*where $\mu_n$ is a finite measure on $(0, \infty)$ – case (a) – or on $\mathbb{R} \setminus \{0\}$ – case (b) –, with density given by*

$$\text{(a)} \quad \frac{d\mu_n}{dx} = -\sum_{i=1}^n \frac{h_n(a_i)}{g_n'(a_i)}\, e^{-a_j x} \quad x > 0 \qquad \text{or} \qquad \text{(b)} \quad \frac{d\mu_n}{dx} = -\sum_{i=1}^n \frac{h_n(a_i)}{g_n'(a_i)}\, e^{-a_j |x|} \quad x \neq 0 \, .$$

*Proof.* We will prove the result for the case (a); case (b) is similar. Since the convolution of two finite measures is absolutely continuous when one of them is, we abuse of the notation and write

$$\left(\frac{a_j}{b_j}\, \delta_0 + \left(1 - \frac{a_j}{b_j}\right) \mathcal{E}\text{xp}(a_j)\right)(x)$$

to denote the probability *density* function of the measure which is a mixture of a delta measure and an exponential distribution. Let us denote by $I_n^i$ the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinal $1 \le i \le n$. Then

$$\begin{aligned}
&\star_{j=1}^n \left(\frac{a_j}{b_j}\, \delta_0 + \left(1 - \frac{a_j}{b_j}\right) \mathcal{E}\text{xp}(a_j)\right)(x) = \\
&= \left(\prod_{i=1}^n \frac{a_i}{b_i}\right) \delta_0(x) + \sum_{i=1}^n \sum_{J \in I_n^{n-i}} \left[\prod_{k \in J} \frac{a_k}{b_k}\right] \left[\prod_{k \in J^c} \left(1 - \frac{a_k}{b_k}\right)\right] \left[\sum_{k \in J^c} \frac{a_k}{\prod_{r \in J^c \setminus \{k\}} \left(1 - \frac{a_k}{a_r}\right)} e^{-a_k x}\right] \\
&= \left(\prod_{i=1}^n \frac{a_i}{b_i}\right) \delta_0(x) + \sum_{k=1}^n \frac{e^{-a_k x} a_k}{\prod_{\substack{r=1 \\ r \neq k}}^n \left(1 - \frac{a_k}{a_r}\right)} \sum_{i=1}^n \sum_{\substack{J \in I_n^{n-i} \\ k \notin J}} \left[\prod_{r \in J} \frac{a_r}{b_r} \left(1 - \frac{a_k}{a_r}\right)\right] \left[\prod_{r \in J^c} \left(1 - \frac{a_r}{b_r}\right)\right] \\
&= \left(\prod_{i=1}^n \frac{a_i}{b_i}\right) \delta_0(x) + \sum_{k=1}^n e^{-a_k x} \frac{a_k}{\prod_{\substack{r=1 \\ r \neq k}}^n \left(1 - \frac{a_k}{a_r}\right)} \prod_{r=1}^n \left(\left(1 - \frac{a_r}{b_r}\right) + \frac{a_r}{b_r}\left(1 - \frac{a_k}{a_r}\right)\right) \\
&= \left(\prod_{i=1}^n \frac{a_i}{b_i}\right) \delta_0(x) + \sum_{k=1}^n e^{-a_k x} \frac{\prod_{r=1}^n \left(1 - \frac{a_k}{b_r}\right)}{\frac{1}{a_k} \prod_{\substack{r=1 \\ r \neq k}}^n \left(1 - \frac{a_k}{a_r}\right)} \ ,
\end{aligned}$$

where we have used Lemma 2.4. ■

**Remark 2.7.** If $h \equiv 1$ in Lemma 2.6 then the law of $\varphi_n(t)$ is just $\mu_n$.

Lemma 2.6 shows that the distribution has an atom at the origin. This causes a handicap towards applying the conventional limit theorems to the expressions of the densities therein. Section 4.2 will give further assumptions and criteria to overcome this difficulty. Finally, notice that the key point to this end is determining the behaviour of

$$\lim_{n \to \infty} \frac{d\mu_n}{dx} \ .$$

## 3 Fundamental lemmas

The following lemmas are essential for the purposes of the paper and they give the technical results to allow us to use the standard convergence results on the expressions of Lemma 2.6. Before that, let us remark on a key property of an entire function: the order of an entire function is greater or equal to the exponent of convergence of its zeros (see Titchmarsh [20, p. 251]). That means, let $f$ be an entire function of order $\rho$ and $\{a_n,\, n \geq 1\}$ its zeros, then

$$\rho \geq \inf\{\alpha > 0 \,:\, \sum_{n=1}^{\infty} |a_n|^{-\alpha} < \infty\}\,.$$

In particular, $\sum_{n=1}^{\infty} |a_n|^{-\beta} < \infty$ for $\beta > \rho$. The equality holds if $f$ has a representation as a canonical product, in fact Proposition 2.1 and 2.3 give such representation for $h$ and $g$.

**Lemma 3.1.** *Let $g(z)$ be an entire function of order $\rho \in (0,1)$ such that $g(0) = 1$, with simple positive zeros given by the strictly increasing sequence of positive numbers $\{a_n,\, n \geq 1\}$. Let $h(z)$ be another function that either*

*(a)* $h(z) \equiv 1$ *or*

*(b)* $h(z)$ *is an entire function of order $\rho' \in (0,1)$ such that $h(a_n) \neq 0$ for all $n$.*

*Then, for every $x > 0$, the series $\sum_{n\geq 1} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x}$ is convergent.*

*Proof.* We will prove the lemma for case (b); case (a) is similar and indeed easier. Consider the closed contour $D(R)$ in Figure 1, where $R > 0$, $\theta_0 \in (0, \pi/2)$ and $R \neq a_j$ for all $j$. Fix $x > 0$. The function $\frac{h(z)}{g(z)}\, e^{-xz}$ is

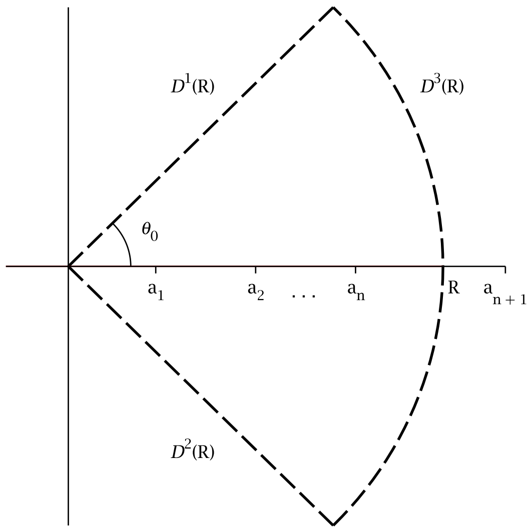


**Figure 1.** Dashed line is the contour $D(R)$ of integration.

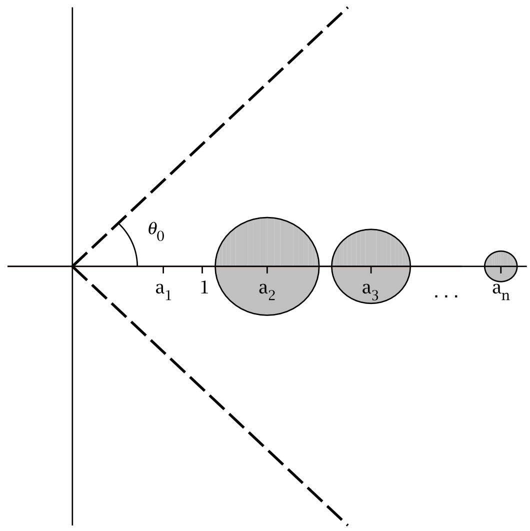


**Figure 2.** The function $|g(z)|$ is bounded below in the region excluded from the gray circles.

analytic in the region bounded by $D(R)$, except at the zeros $a_j$ that lie in the interior of $D(R)$. The Residue Theorem allows us to write

$$\frac{1}{2\pi i} \oint_{D(R)} \frac{h(z)}{g(z)}\, e^{-xz}\, dz = \sum_{a_j < R} \text{Res}(a_j) = \sum_{a_j < R} \frac{h(a_j)}{g'(a_j)}\, e^{-a_j x}\,.$$

Our objective is to prove that there is an increasing sequence, $\{R_n,\, n \geq 1\}$, towards infinity such that the limit

$$\lim_n \oint_{D(R_n)} \frac{h(z)}{g(z)}\, e^{-xz}\, dz \tag{2}$$

exists and is finite. The key point of the proof is that an entire function of order $\rho < 1$ can be bounded below in the region excluded from small circles around its zeros. Since the zeros of $g(z)$ are located on the positive real axis, convenient bounds of $1/|g(z)|$ can be obtained. Specifically, the proof is based on:

**1.** For every $\varepsilon > 0$, $g(z) = \mathcal{O}(e^{\epsilon|z|})$ and $h(z) = \mathcal{O}(e^{\epsilon|z|})$. This is due to $\sum_n |a_n|^{-1} < \infty$, and hence we can apply Titchmarsh's issue 15 [20, p. 286].

**2.** As a consequence of the previous statement, for every $\varepsilon > 0$ there is an increasing sequence $\{R_n, n \geq 1\}$ such that $\lim_n R_n = \infty$ and $|g(R_n e^{i\theta})| > \exp\{-\varepsilon R_n\}$ uniformly on $\theta \in [0, 2\pi]$. See Titchmarsh [20, p. 276, it. 8.75].

**3.** For every $\varepsilon > 0$, there is $\theta_0 \in (0, \pi/2)$ and $r_0 > 0$ such that

$$|g(re^{\pm i\theta_0})| > \exp\{-\varepsilon r\}\ ,\ \text{for } r \geq r_0.$$

This is deduced from Titchmarsh [20, p. 273, it. 8.71]. Take straight lines through the origin with angles $\theta_0$ and $-\theta_0$ that do not intersect with any of the circles with centers $a_j$ and radius $1/a_j$ (for $a_j > 1$), see Figure 2. Titchmarsh proves that for all $\varepsilon' > 0$ exists $r_0'$ such that if $r \geq r_0'$ we have $|g(z)| > \exp\{-r^{\rho+\varepsilon'}\}$ in the region excluded from these discs. Take $\varepsilon'$ satisfying $\rho + \varepsilon' < 1$ and thus for sufficiently large $r$ we have $r^{\rho+\varepsilon'} < \varepsilon r$. Choose $r_0$ a slightly larger than $r_0'$ in order to obtain the claimed inequality.

Now we are ready to complete the proof. Fix $\varepsilon > 0$ such that $2\varepsilon < x\cos\theta_0$, where $\theta_0$ is the angle depending on $\varepsilon$ such that property **3** is fulfilled. Denote by $D^1(R_n)$ and $D^2(R_n)$ the lower and upper straight segment of $D(R_n)$ while $D^3(R_n)$ stands for the arch. All paths are considered with the corresponding orientation. Then we divide the integral into three parts :

$$\oint_{D(R_n)} \frac{h(z)}{g(z)}\, e^{-xz}\, dz = \int_{D^1(R_n)} + \int_{D^2(R_n)} + \int_{D^3(R_n)} .$$

We first consider the integral over $D^3(R_n)$. Due to points **1** and **2**, we can bound the module of this integral in the following way:

$$\begin{aligned}\left|\int_{D^3(R_n)} \frac{h(z)}{g(z)}\, e^{-xz}\, dz\right| &\leq R_n \int_{-\theta_0}^{\theta_0} \left|\frac{h(R_n e^{i\theta})}{g(R_n e^{i\theta})}\right| e^{-xR_n\cos\theta}\, d\theta \\ &\leq R_n e^{-xR_n\cos\theta_0} \int_{-\theta_0}^{\theta_0} \left|\frac{h(R_n e^{i\theta})}{g(R_n e^{i\theta})}\right| d\theta \leq K\theta_0 R_n e^{-R_n(x\cos\theta_0 - 2\varepsilon)},\end{aligned}$$

for $R_n$ large enough. This goes to zero as $n \to \infty$.

The integral over $D^1(R_n)$ can be parametrized as

$$\int_{D^1(R_n)} \frac{h(z)}{g(z)}\, e^{-xz}\, dz = e^{-i\theta_0} \int_0^{R_n} \frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}}\, dr\ .$$

We claim that

$$\int_0^\infty \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}}\right| dr < \infty. \tag{3}$$

According to observations **1** and **3**, there is $r_0 > 0$ depending on $\varepsilon$ such that

$$\int_{r_0}^\infty \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}}\right| dr = \int_{r_0}^\infty \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\right| e^{-xr\cos\theta_0}\, dr \leq K \int_{r_0}^\infty e^{-r(x\cos\theta_0 - 2\varepsilon)}\, dr < \infty.$$

Since $g(re^{-i\theta_0}) \neq 0$ for $r \geq 0$, it turns out that the function $\left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\right| e^{-xr\cos\theta_0}$ is continuous on $[0, r_0]$ and hence

$$\int_0^{r_0} \left|\frac{h(re^{-i\theta_0})}{g(re^{-i\theta_0})}\, e^{-xre^{-i\theta_0}}\right| dr < \infty.$$

Adding up the two upper bounds we obtain (3). For $D^2(R_n)$ the computations are equivalent and the limit (2) exists and is finite. ■

**Lemma 3.2.** *Under assumptions of Lemma 3.1, the series*

$$\sum_{n\geq 1} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x} \tag{4}$$

*converges absolutely for* $x > 0$*, and the convergence is uniform on any compact subset of* $(0, \infty)$.

*Proof.* The above expression is a generalized Dirichlet series. As a consequence, if (4) is convergent for some $x_0$, then it is so for all $x > x_0$ and the convergence is uniform on every compact subset of the half-line (see Mandelbrojt [16, p. 9]). Therefore the second part of the lemma follows from Lemma 3.1. Each general Dirichlet series has associated an abscissa $\sigma_c$ of convergence and an abscissa $\sigma_a$ of absolute convergence. In general these two values are not equal but their distance is bounded (see Mandelbrojt [16, p. 11]) by

$$0 \leq \sigma_a - \sigma_c \leq \limsup_{n\to\infty} \frac{\ln n}{a_n} \ .$$

From the previous observation we know that $\sigma_c \leq 0$; we claim that the above limit is zero to derive the first part of the result. The claim is proved through Jensen's formula which gives the relationship $n(r) = \mathcal{O}(r^{\rho+\varepsilon})$ for the entire function $g(z)$, where $\varepsilon > 0$ and $n(r)$ stands for the number of zeros with norm less than or equal to $r$ (see Titchamrsh [20, p. 249]). Choose $\varepsilon$ such that $\rho + \varepsilon < 1$, then $n \leq K a_n^{\rho+\varepsilon}$ for some positive constant $K$. Finally

$$\ln n \leq n - 1 \leq K a_n^{\rho+\varepsilon} - 1 \ ,$$

and hence $\sigma_c = \sigma_a$. ■

**Remark 3.3.** Straightforward manipulations lead to generalizing Lemmas 3.1 and 3.2 for the case where the simple zeros of $g(z)$ are $\{\pm a_n,\, n \geq 1\}$, but still of an order lying in $(0, 1)$.

The next result somehow extends case (a) of Lemma 3.1 but with a penalization of extra hypotheses. As will be shown in the examples, this particular case is also of great interest.

**Lemma 3.4.** *Consider an even entire function* $g(z)$ *of order* $1$ *such that* $g(0) = 1$ *and* $g(z) = \mathcal{O}(e^{A|z|})$ *for some positive constant* $A$*. Let* $\{a_n,\, n \geq 1\}$ *be a strictly increasing sequence of positive numbers; let* $\{\pm a_n,\, n \geq 1\}$ *be the simple zeros of* $g(z)$*; assume that the following limit exists:*

$$\lim_{n\to\infty} \frac{n}{a_n} = \delta > 0 \ . \tag{5}$$

*Then, for every* $x > 0$*, the series* $\sum_{n\geq 1} \frac{1}{g'(a_n)}\, e^{-a_n x}$ *is absolutely convergent and the convergence is uniform on any compact subset of* $(0, \infty)$.

*Proof.* It is clear from (5) that $\sigma_a = \sigma_c$, thus if we prove the plain convergence of the series in $(0, \infty)$ the result will follow. The proof is very similar to that of Lemma 3.1 but the fact that the order of $g(z)$ is 1 demands some modifications. We will consider as before the closed contour $D(R)$ in Figure 1, where $R > 0$, $\theta_0 \in (0, \pi/2)$ and $R \neq a_j$ for all $j$. Our objective is to prove that there is an increasing sequence, $\{R_n,\, n \geq 1\}$, towards infinity such that

$$\lim_n \oint_{D(R_n)} \frac{1}{g(z)}\, e^{-xz} \tag{6}$$

exists and is finite for a fixed $x > 0$. Split the above integral as in Lemma 3.1 into $D(R_n) = D^1(R_n) \cup D^2(R_n) \cup D^3(R_n)$.

The function $g(z)$ admits an expression as a canonical product given by

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right) \ .$$

Define the function

$$\widetilde{g}(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n^2}\right)$$

of order $1/2$ and notice $\widetilde{g}(z^2) = g(z)$. Fortunately, the function $g(z)$ inherits the good properties of $\widetilde{g}(z)$. To start with, since $\widetilde{g}(z)$ has order $1/2$, for every $\varepsilon > 0$ there is a sequence $\widetilde{R}_n \nearrow \infty$ such that for all $\theta \in [0, 2\pi]$

$$|\widetilde{g}(\widetilde{R}_n e^{i\theta})| > M_{\widetilde{g}}(\widetilde{R}_n)^{-\epsilon},$$

where $M_{\widetilde{g}}(r) = \max_{\{|z|=r\}} |\widetilde{g}(z)|$ (see Titchmarsh [20, p. 275, it. 8.74]). Since $g(z)$ is even, we deduce that there is a sequence $R_n := \widetilde{R}_n^{1/2} \nearrow \infty$ such that for all $\theta \in [0, 2\pi]$

$$|g(R_n e^{i\theta})| > M_g(R_n)^{-\epsilon} .$$

Given that $g(z) = \mathcal{O}(e^{A\,|z|})$, for $R_n$ large enough,

$$\left|\frac{1}{g(R_n e^{i\theta})}\right| \leq C e^{\varepsilon A R_n} \quad \text{uniformly on } \theta \in [0, 2\pi].$$

Now consider $\varepsilon$ such that $A\varepsilon < x\cos\theta_0$, and the bound

$$\begin{aligned} \left|\int_{D^3(R_n)} \frac{1}{g(z)}\, e^{-xz}\, dz\right| &\leq R_n \int_{-\theta_0}^{\theta_0} \left|\frac{1}{g(R_n e^{i\theta})}\right| e^{-xR_n\cos\theta}\, d\theta \\ &\leq R_n e^{-xR_n\cos\theta_0} \int_{-\theta_0}^{\theta_0} \left|\frac{1}{g(R_n e^{i\theta})}\right| d\theta \leq K\theta_0 R_n e^{-R_n(x\cos\theta_0 - A\varepsilon)}, \end{aligned}$$

for $R_n$ large enough. This goes to zero as $n \to \infty$.

In order to bound the integral over $D^1(R_n)$ consider the following Remark in Levin [13, p. 82, eqn. $(2')$]:

$$\ln|\widetilde{g}(\widetilde{r}e^{i\widetilde{\theta}})| = \widetilde{r}^{1/2}\pi\delta\sin(\widetilde{\theta}/2) + \frac{\mathcal{O}(r^{1/2})}{\sin(\widetilde{\theta}/2)}$$

for $\widetilde{\theta} \in (0, 2\pi)$ and as $\widetilde{r} \to \infty$; that asymptotic result is based on the fact that $\widetilde{g}$ is a canonical product of order 1/2. Notice that $\widetilde{g}(\widetilde{r}e^{i\widetilde{\theta}}) = g(\widetilde{r}^{1/2}e^{i\widetilde{\theta}/2}) := g(re^{i\theta}) = g(re^{i(\theta+\pi)})$, hence the above asymptotic equation is translated into $g(z)$ as

$$\ln|g(re^{i\theta})| = r\pi\delta|\sin(\theta)| + \frac{\mathcal{O}(r)}{|\sin(\theta)|}$$

for $\theta \in (0, 2\pi) \setminus \{\pi\}$. Therefore

$$\left|\int_{D^1(R_n)} \frac{1}{g(z)}\, e^{-xz}\, dz\right| \leq \int_0^{\infty} \left|\frac{1}{g(re^{-i\theta_0})}\right| e^{-xr\cos\theta_0}\, dr \sim \int_0^{\infty} e^{-r\left(x\cos\theta_0 + \pi\delta|\sin(\theta_0)| + \frac{\mathcal{O}(r)}{r|\sin(\theta_0)|}\right)}\, dr\ ,$$

where $\sim$ means that both converge or diverge together. Clearly, the last integral is finite. The same derivation is valid for the integral over $D^2(R_n)$ and (6) holds. ■

# 4 Construction of density functions

This section will derive the density function associated to the characteristic functions of Proposition 2.1 and Proposition 2.3 under the restrictions of Lemma 3.4.

Lemma 2.6 shows the density function for a finite product approximation of the characteristic function. From Lévy's continuity theorem, this means a convergence in distribution of the associated probability measures. We first show a pointwise convergence of the distribution function to finally obtain the convergence of densities. In order to avoid Dirac's delta measure at 0, instead of using the distribution function $F(x)$ we work with $\overline{F}(x) := 1 - F(x)$.

**Theorem 4.1.** *Under assumptions of Proposition 2.1, the probability measure on $[0,\infty)$ corresponding to the characteristic function $\varphi(t) = h(it)/g(it)$ is absolutely continuous on $(0,\infty)$ with (perhaps defective) density given by*

$$f(x) = -\sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)}\, e^{-a_n x}, \qquad x > 0\ .$$

*Proof.* Recall $h_n(z)$, $g_n(z)$ and $\varphi_n(t)$ from Lemma 2.6 case (a). Denote by $F_n(x)$ the distribution function corresponding to the characteristic function $\varphi_n(t)$. It is clear that $\varphi_n(t) \to \varphi(t)$ pointwise, where $\varphi(t)$ is defined in (1). Since $a_n \neq 0$ for all $n$, it follows that $\varphi(t)$ is continuous at $0$. Therefore there exists a distribution function $F(x)$, such that

$$\lim_{n\to\infty} \overline{F}_n(x) = \overline{F}(x)$$

for all $x$ where $F(x)$ is continuous. From the expression of the density of $F_n(x)$ we deduce that

$$\overline{F}_n(x) = -\sum_{i=1}^{n} \frac{h_n(a_i)}{a_i g_n'(a_i)}\, e^{-a_i x}, \qquad x > 0\ .$$

Our objective is to prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{h_n(a_i)}{a_i g_n'(a_i)}\, e^{-a_i x} = \sum_{i=1}^{\infty} \frac{h(a_i)}{a_i g'(a_i)}\, e^{-a_i x}, \qquad x > 0\ . \tag{7}$$

The proof of (7) is done using the dominated convergence theorem. First notice that for all $i \geq 1$,

$$\lim_{n\to\infty} g_n'(a_i) = g'(a_i). \tag{8}$$

This is proved in the following way: On the one hand,

$$\lim_{n\to\infty} g_n'(a_i) = -\frac{1}{a_i} \lim_{n\to\infty} \prod_{\substack{j=1\\ j\neq i}}^{n} \Big(1 - \frac{a_i}{a_j}\Big) = -\frac{1}{a_i} \prod_{\substack{j=1\\ j\neq i}}^{\infty} \Big(1 - \frac{a_i}{a_j}\Big). \tag{9}$$

On the other hand, for $z \neq a_i$,

$$\frac{g(z)}{1 - z/a_i} = \prod_{\substack{j=1\\ j\neq i}}^{\infty} \Big(1 - \frac{z}{a_j}\Big).$$

The function on the right hand side is entire, so by analytic continuation,

$$\lim_{z\to a_i} \frac{g(z)}{1 - z/a_i} = \prod_{\substack{j=1\\ j\neq i}}^{\infty} \Big(1 - \frac{a_i}{a_j}\Big). \tag{10}$$

Further, the function $1/g(z)$ has a simple pole at $a_i$. Hence,

$$\lim_{z\to a_i} (z - a_i)\frac{1}{g(z)} = \mathrm{Res}(1/g, a_i) = \frac{1}{g'(a_i)}. \tag{11}$$

Combining (9)–(11) we obtain (8).

Fix $x > 0$, set $\Lambda = \{a_i,\ i \geq 1\}$ and denote the counting measure by $\Gamma$. By (8),

$$\lim_{n\to\infty} \frac{h_n(y)}{y g_n'(y)}\, e^{-yx} = \frac{h(y)}{y g'(y)}\, e^{-yx}, \qquad \forall\, y \in \Lambda\ .$$

Now observe that for $n > j$

$$\frac{h_{n+1}(a_j)}{g'_{n+1}(a_j)} = \frac{h_n(a_j)}{g'_n(a_j)} \, \frac{1 - a_j/b_{n+1}}{1 - a_j/a_{n+1}} \, ,$$

and since $a_{n+1} < b_{n+1}$ we conclude that

$$\left| \frac{h_n(a_j)}{g'_n(a_j)} \right| \le \left| \frac{h_{n+1}(a_j)}{g'_{n+1}(a_j)} \right| \le \left| \frac{h(a_j)}{g'(a_j)} \right| \, . \tag{12}$$

Finally, let $\Phi(y)$ be the function

$$\Phi(y) := \left| \frac{h(y)}{y g'(y)} \right| \mathbf{1}_{[0,1)}(y) e^{-yx} + \left| \frac{h(y)}{g'(y)} \right| \mathbf{1}_{[1,\infty)}(y) e^{-yx}, \qquad y > 0 \, .$$

Then

$$\left| \frac{h_n(y)}{y g'_n(y)} e^{-yx} \right| \le \left| \frac{h_n(y)}{y g'_n(y)} \right| \mathbf{1}_{[0,1)}(y) e^{-yx} + \left| \frac{h_n(y)}{g'_n(y)} \right| \mathbf{1}_{[1,\infty)}(y) e^{-yx} \le \Phi(y), \qquad y > 0 \, .$$

By Lemma 3.2,

$$\int_\Lambda |\Phi| d\Gamma = \sum_{i:\, a_i < 1} \left| \frac{h(a_i)}{a_i g'(a_i)} \right| e^{-a_i x} + \sum_{i:\, a_i \ge 1} \left| \frac{h(a_i)}{g'(a_i)} \right| e^{-a_i x} < \infty \, .$$

Hence we can apply the dominated convergence theorem to prove (7). It follows

$$\overline{F}(x) = - \sum_{i=1}^{\infty} \frac{h(a_i)}{a_i g'(a_i)} \, e^{-a_i x}, \qquad x > 0 \, .$$

Also from Lemma 3.2, we check that the set of continuity of $F(x)$ is $(0, \infty)$. Moreover, from the uniform convergence on compact sets of (4),

$$F'(x) = f(x) = - \sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)} \, e^{-a_n x}, \quad x > 0$$

as required. ∎

**Remark 4.2.** Standard manipulations show that Theorem 4.1 is also true for $h \equiv 1$.

**Remark 4.3.** Notice that inequality (12) also holds true for Lemma 2.6 case (b). Thus, Theorem 4.1 for such $g(z)$ and $h(z)$ with orders lying in $(0, 1)$ still holds and concludes that the limiting density is

$$f(x) = - \sum_{n=1}^{\infty} \frac{h(a_n)}{g'(a_n)} \, e^{-a_n |x|}, \quad x \ne 0 \, .$$

Moreover, the same proof of Theorem 4.1 will be valid for functions $g(z)$ and $h(z)$ of order lying in $[1, 2)$, provided that $\sigma_a \le 0$ for the series (4).

The next result is an extension of the preceding theorem on the existence of a density function. One would like to generalize such a result for characteristic functions $\varphi(t) = h(it)/g(it)$ where $g(z)$ and $h(z)$ have zeros $\{\pm a_n,\, n \ge 1\}$ and $\{\pm b_n,\, n \ge 1\}$ and orders lying in $[1, 2)$. Entire functions of order greater than or equal to one are much more difficult to treat than those that have order less than one. Therefore we will restrict the study to the setup of Lemma 3.4. As pointed out in the above remark, we only need the absolute convergence of the general Dirichlet series to follow the same proof of Theorem 4.1 and obtain the density

$$f(x) = - \sum_{n=1}^{\infty} \frac{1}{g'(a_n)} \, e^{-a_n |x|}, \quad x \ne 0 \, . \tag{13}$$

However, we would like to present a different proof, Theorem 4.4, which shows that both problems, the exponential and Laplace convolutions, are the head and tail of the same coin. It is worth remarking that the random variables corresponding to that case are in the homogeneous second Wiener chaos (see Janson [8, chap. 6]), so our result is a step forward in the study of the densities of such an interesting space of random variables; in particular, this case includes the Lévy area.

**Theorem 4.4.** *Under assumptions of Lemma 3.4, the probability measure on $\mathbb{R}$ corresponding to the characteristic function $\varphi(t) = 1/g(it)$ is absolutely continuous on $\mathbb{R} \setminus \{0\}$ with (perhaps defective) density given by (13).*

*Proof.* Recall the expressions of $g(z)$ and $\widetilde{g}(z)$ defined in the proof of Lemma 3.4. Notice that

$$2a_j \widetilde{g}'(a_j^2) = g'(a_j) \ . \tag{14}$$

Due to Proposition 2.1 and Theorem 4.1, $\varphi(t) = 1/\widetilde{g}(it)$ is a characteristic function of a non–negative random variable, denoted by $Y$, with density

$$f_Y(x) = \sum_{j=1}^{\infty} \frac{1}{\widetilde{g}'(a_j^2)} e^{-a_j^2 x}, \quad x > 0 \ .$$

Fix $x \neq 0$ and proceed heuristically as in Lemma 2.5 using the afore mentioned Bondesson argument and (14) to obtain the required expression (13). To make the argument accurate we need to apply Fubini's theorem since $f_Y(x)$ is an infinite series. It turns out that the absolute convergence of (4) allows us to use Fubini's result. ■

## 4.1 Infinite convolution of exponential and Laplace densities

For the sake of completeness we will rewrite Remark 4.2 and Theorem 4.4 in a way that will extend Lemmas 2.4 and 2.5. Moreover, we prove that in this setup the resulting density is continuous in $[0, \infty)$ or $\mathbb{R}$.

Following Wintner, for a sequence of densities, $\{f_n,\ n \geq 1\}$, we will say that $\star_{n=1}^{\infty} f_n$ is a convergent infinite convolution if the product

$$\prod_{n=1}^{\infty} \psi_n(t) \ , \qquad \text{where} \qquad \psi_n(t) = \int_{-\infty}^{\infty} e^{itx} f_n(x) dx \ ,$$

is uniformly convergent in every fixed finite $t$–interval.

**Proposition 4.5.** *Let $\{\lambda_n\, n \geq 1\}$ be a strictly increasing sequence of positive numbers such that for some $\rho \in (0,1)$, $\sum_{n\geq 1} \lambda_n^{-\rho} < \infty$. Then $\star_{n=1}^{\infty} \mathcal{E}\mathrm{xp}(\lambda_n)$ converges to a continuous density on $[0,\infty)$ which can be written as*

$$\star_{n=1}^{\infty}\, \mathcal{E}\mathrm{xp}(\lambda_n)(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(-1)^{n+1} A(n)}{B(i,n)}\, e^{-\lambda_i x} = \sum_{i=1}^{\infty} \lambda_i \left[ \prod_{\substack{k=1\\ k\neq i}}^{\infty} \left(1 - \frac{\lambda_i}{\lambda_k}\right) \right]^{-1} e^{-\lambda_i x} \ , \quad x > 0, \tag{15}$$

*and*

$$\star_{n=1}^{\infty} \mathcal{E}\mathrm{xp}(\lambda_n)(0) = 0.$$

*Proof.* We will first point out that the infinite convolution is convergent. To this end we will use a very useful result from Wintner [21] that ensures the convergence of the infinite convolution if

$$\sum_{n=1}^{\infty} M_n < \infty \qquad \text{where} \qquad M_n = \mathbb{E}[|\mathcal{E}\mathrm{xp}(\lambda_n)|] = \int_{-\infty}^{\infty} |x| \lambda_n\, e^{-\lambda_n x} \mathbf{1}_{[0,\infty)} dx \ .$$

Clearly $M_n = \lambda_n^{-1}$ and the condition to guarantee the convergence of the infinite convolution is fulfilled. Moreover, if one term of the infinite convolution has a continuous density of bounded variation, then so has the infinite convolution; and the continuous function $\star_{n=1}^{m} \mathcal{E}\mathrm{xp}(\lambda_n)$ tends, as $m \to \infty$, to the infinite convolution uniformly in every bounded range (see Wintner [22]). It suffices to notice that $\mathcal{E}\text{xp}(\lambda_1) \star \mathcal{E}\text{xp}(\lambda_2)$ is continuous and of bounded variation. Hence, for all $x \geq 0$,

$$\star_{n=1}^{\infty} \mathcal{E}\text{xp}(\lambda_n)(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(-1)^{n+1} A(n)}{B(i,n)} \, e^{-\lambda_i x}$$

To prove the second equality in (15), let

$$g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) .$$

Then $g(z)$ is an entire function of order less than 1 and apply Theorem 4.1 to $x > 0$, and the computations done in the first part of the proof of that theorem. The value of the density at zero is guarantied by Wintner [22] result about the continuity on $\mathbb{R}$ of the density. ∎

Note that the series in the right hand side of (15) may be divergent at $x = 0$.

**Proposition 4.6.** *Let $\{\lambda_n \, n \geq 1\}$ be a strictly increasing sequence of positive numbers such that $\sum_{n\geq 1} \lambda_n^{-2} < \infty$. Then $\star_{n=1}^{\infty} \mathcal{L}\text{aplace}(\lambda_n)$ converges to a continuous density on $\mathbb{R}$ which can be written as*

$$\star_{n=1}^{\infty} \mathcal{L}\text{aplace}(\lambda_n)(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(-1)^{n+1} A^2(n)}{E(i,n)} e^{-\lambda_i |x|} \, , \qquad x \in \mathbb{R} \, .$$

*Proof.* As in Proposition 4.5, we start by showing that the infinite convolution is convergent. Jessen and Wintner show in [9] that if

$$\sum_{n=1}^{\infty} M_n^1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} M_n^2$$

are convergent, where $M_n^1$ and $M_n^2$ are the first and second moment of $\mathcal{L}\text{aplace}(\lambda_n)$ respectively, then so is the infinite convolution. One can check that $M_n^1 = 0$ and $M_n^2 = \lambda_n^{-2}$ to obtain the convergence of the infinite convolution. Therefore we can apply Wintner [22] to obtain the proposition since a Laplace density is continuous and of finite variation. ∎

## 4.2 Existence of densities

In this section we give three different conditions that guarantee that the probability measure corresponding to $h(it)/g(it)$ is absolutely continuous.

**Proposition 4.7.** *Under the hypotheses of Theorem 4.1, if $h(z) \equiv 1$, then the probability measure corresponding to $1/g(it)$ is absolutely continuous and its density is continuous.*

This result follows from Proposition 4.5.

In some cases it is known that the probability measure corresponding to $h(it)/g(it)$ has no atom at zero. Since that probability measure is concentrated on $[0, \infty)$, this suffices for the existence of a density.

**Proposition 4.8.** *Assume the hypothesis of Theorem 4.1 and denote by $\mu$ the probability measure corresponding to $h(it)/g(it)$. If $\mu(\{0\}) = 0$, then $\mu$ is absolutely continuous.*

The following lemma gives a sufficient condition in order to apply a classical criterion for the existence of a continuous density.

**Lemma 4.9.** *Let $g(z)$ and $h(z)$ be two entire functions of order $\rho \in (0,1)$, both with non-zero positive simple real zeros, and the zeros of $g(z)$ different from the zeros of $h(z)$. Denote by $n_g(r)$ (respectively $n_h(r)$) the number of the zeros of $g(z)$ with module less than $r$. Assume the existence of the limits*

$$\delta = \lim_{r\to\infty} \frac{n_g(r)}{r^\rho} > 0 \qquad \textit{and} \qquad \delta' = \lim_{r\to\infty} \frac{n_h(r)}{r^\rho} > 0,$$

*with $\delta' < \delta$. Then*

$$\int_{-\infty}^{\infty} \left| \frac{h(it)}{g(it)} \right| dt < \infty \,. \tag{16}$$

*Proof.* Levin ([13, p. 82, eqn. (2')]) proves that if $\theta \in (0, 2\pi)$ and $r \to \infty$ then

$$\log |g(re^{i\theta})| = \frac{\pi\delta r^\rho \cos\big(\rho(\theta - \pi)\big)}{\sin(\pi\rho)} + \frac{\mathcal{O}(r^\rho)}{\sin(\theta/2)} \,. \tag{17}$$

The analogous holds true for $h(re^{i\theta})$ with $\delta'$ instead of $\delta$.

Split integral (16) into two parts as

$$\int_{-\infty}^{\infty} \Big|\frac{h(it)}{g(it)}\Big| dt = \int_0^\infty \Big|\frac{h(te^{i\pi/2})}{g(te^{i\pi/2})}\Big| dt + \int_0^\infty \Big|\frac{h(te^{i3\pi/2})}{g(te^{i3\pi/2})}\Big| dt. \tag{18}$$

Due to (17) there is $r_0 > 0$ and $C > 0$ such that for $r > r_0$ we get

$$\begin{aligned}\left|\frac{h(te^{i\pi/2})}{g(te^{i\pi/2})}\right| &= \exp\{-\cos(\rho\pi/2)\pi(\delta - \delta')\csc(\pi\rho) r^\rho + \mathcal{O}(r^\rho)\} \\ &= \exp\{-r^\rho\big(\cos(\rho\pi/2)\pi(\delta-\delta')\csc(\pi\rho) + \mathcal{O}(r^\rho)/r^\rho\big)\} \le e^{-Cr^\rho}.\end{aligned}$$

Hence

$$\int_{r_0}^{\infty} \Big|\frac{h(te^{i\pi/2})}{g(te^{i\pi/2})}\Big| dt \le \int_{r_0}^\infty e^{-Cr^\rho}\, dr < \infty$$

since the last expression can be reduced to a convergent gamma integral. For the integral over $[0, r_0]$ there is a straightforward bound using the same sort of derivations used in Lemma 3.1. The other integral on the right hand side of (18) is bounded in a similar way. ■

With the notations of this lemma,

**Proposition 4.10.** *Under the hypothesis of Theorem 4.1, and assume the existence of the limits*

$$\delta = \lim_{r\to\infty} \frac{n_g(r)}{r^\rho} > 0 \qquad \textit{and} \qquad \delta' = \lim_{r\to\infty} \frac{n_h(r)}{r^\rho} > 0,$$

*with $\delta' < \delta$. Then the probability measure corresponding to $h(it)/g(it)$ has a continuous density.*

# 5 Examples

We will now see different situations where we can apply the results obtained in the previous sections. Most results are known, but here we get them all using the same technique.

## 5.1 Lévy area

Let $\varphi(t) = \operatorname{sech}(tT)$ be the characteristic function for the Lévy area (see Lévy [14]), where the time component of the process varies in $[0, T]$. Here we set $h \equiv 1$ and $g(z) = \cos(zT)$, where $g(z)$ has order 1. It is clear that $\{\pm(2k-1)\pi/2T,\ k \ge 1\}$ and $\{\pm(-1)^k/T,\ k \ge 1\}$ are the poles and the residues of $1/g(z)$ respectively, thus $\varphi(t)$ fulfills Proposition 2.3. Moreover, $g(z)$ is even and of exponential type, thus we apply Theorem 4.4 to obtain

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{T}\, e^{-\frac{(2k-1)\pi}{2T}|x|} \qquad x \neq 0 \,.$$

Proposition 4.6 ensures that the density function is continuous in $\mathbb{R}$.

### 5.2 The first hitting time of a Bessel process

The second example is a characteristic function obtained from

$$k(z) = z^{-\nu/2} 2^{\nu/2} \Gamma(\nu+1) J_\nu(\sqrt{2z}) ,$$

where $J_\nu(z)$ is the Bessel function of first kind and order $\nu > -1$. Consider $0 < u < v < \infty$ and let $h(z) = k(u^2 z)$ and $g(z) = k(v^2 z)$. The probability measure of the corresponding characteristic function $\varphi(t) = h(it)/g(it)$ describes the first hitting time of the point $v$ by a Bessel process of order $\nu$ that starts at $u$, see Kent [12], and can be expressed as

$$\varphi(t) = \left(\frac{v}{u}\right)^\nu \frac{J_\nu(u\sqrt{2it})}{J_\nu(v\sqrt{2it})} .$$

From the Taylor expansion of

$$\left(\frac{2}{z}\right)^\nu J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)4^n} z^{2n} ,$$

we can deduce the order of $k(z)$, since the order of an entire function is

$$\rho = \limsup_{n\to\infty} \frac{n\ln(n)}{\ln(1/|c_n|)} ,$$

where $c_n$ are the coefficients of the Taylor expansion, see Levin [13, p. 6]. Due to the Stirling formula the above limit for $h(z)$ and $g(z)$ is $1/2$. Denote by $\{j_{\nu,k},\ k \geq 1\}$the positive zeros in order of magnitude of the Bessel function $J_\nu(z)$ and by $a_k = j_{\nu,k}^2/(2v^2)$ the zeros of $g(z)$. Finally we can apply Theorem 4.1 to obtain the density

$$f(x) = \sum_{k=1}^{\infty} \frac{j_{\nu,k} v^{\nu-2} J_v(j_{\nu,k} u/v)}{u^\nu J_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2}{2v^2}x} \qquad x > 0 , \tag{19}$$

where we have used

$$\frac{d}{dz} J_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z) .$$

Equation (19) is also derived by Borodin and Salminen [3, p. 387]. Notice that the distribution function is absolutely continuous in $[0,\infty)$ since the probability distribution gives no mass to $\{0\}$ due to the continuous paths of the Bessel process.

#### 5.2.1 Exit time from a $n$–dimensional sphere by a Brownian motion

Let $T_n$ denote the random variable of the total time spent by an $n$–dimensional Brownian motion starting at $0$ inside the sphere $S^{n-1}(r)$ of radius $r > 0$ and $n \geq 3$. Let $P_n$ denote the first exit time for a $n$–dimensional Brownian motion starting at $0$ from the sphere $S^{n-1}(r)$ for $n \geq 1$. Ciesielski and Taylor [4] show the remarkable equality of the distribution functions of $T_n$ and $P_{n-2}$ for $n \geq 3$, and they derive the distribution function of $T_n$ using methods developed by Kac [10]. They first compute the solution for $n = 3$ and then make a guess for the general framework. Finally they compare the result with the distribution of $P_n$ which was computed by Lévy.

We can use Theorem 4.1 to derive the density function of $T_n$ since Ciesielski and Taylor give its characteristic function. In fact they establish the following result

$$\mathbb{E}[e^{zT_n}] = \frac{(r\sqrt{2z})^{\nu-1}}{2^{\nu-1}\Gamma(\nu) J_{\nu-1}(r\sqrt{2z})} = \prod_{i=1}^{\infty} \left(1 - \frac{2r^2 z}{j_{\nu-1,i}^2}\right)^{-1} ,$$

where $\nu = (n-2)/2$. This is a particular case of the previous example, where we let $h(z) = 1$ and $g(z) = k(r^2 z)$. Use the same arguments as before to derive the density function

$$f(x) = \frac{1}{2^{\nu-1}\Gamma(\nu)r^2} \sum_{k=1}^{\infty} \frac{j_{\nu-1,k}^{\nu}}{J_\nu(j_{\nu-1,k})} e^{-\frac{j_{\nu-1,k}^2}{2r^2}x} \qquad x > 0 \; .$$

Notice that $T_n$ is absolutely continuous due to Proposition 4.7, and we can set $f(0) = 0$ to obtain a continuous density.

### 5.2.2 The area under a squared Bessel bridge

The characteristic function of $T_n$ is easily representated when $\nu = n + 1/2$ for $n \in \mathbb{N}$. Let $\nu = 3/2$ and $r = 1$ to obtain the expression

$$\varphi(t) = \frac{\sqrt{-2it}}{\sinh(\sqrt{-2it})} \; .$$

Revuz and Yor [18, p. 465] give the Laplace transform of the area under a squared Bessel process starting at any point and arriving at zero. It turns out that the above characteristic function corresponds to a Bessel process of order 2 starting and arriving at zero. Such functions appear recursively in the literature and many authors have studied them, for instance [1] and [17].

The factorization of the characteristic function is particularly easy, and one can check the identity

$$\frac{1}{g(z)} = \prod_{k=1}^{\infty} \left(1 - \frac{2z}{\pi^2 k^2}\right)^{-1} \; ,$$

where the residues of $1/g(z)$ are $\{(-1)^k \pi^2 k^2, \; k \geq 1\}$. Finally, the density function can be written as

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} (-1)^{k+1} \pi^2 k^2 e^{-\pi^2 k^2 x/2} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases} \; ,$$

as stated in [1]. This characteristic function also corresponds (with a change of parameters) to the square of a Kolmogorov law. The corresponding distribution function is

$$\vartheta_4 = 1 + 2\sum_{k=1}^{\infty} (-1)^k e^{-\pi^2 k^2 x/2} \qquad \text{for } x > 0 \; ,$$

which was obtained by Dugué [6].

## 5.3 Inverse Laplace transform

Theta functions and related expressions have proved useful for manipulations of functionals of Brownian motion. For instance, Borodin and Salminen use the inverse Laplace transform of

$$\varphi(z) = \frac{v \sinh(u\sqrt{2z})}{u \sinh(v\sqrt{2z})} \quad 0 < u < v \; ,$$

which turns out to be

$$\mathcal{L}^{-1}(\varphi)(y) = \frac{v}{u} \sum_{k=-\infty}^{\infty} \frac{v - u + 2kv}{\sqrt{2\pi} y^{3/2}} e^{-\frac{(v-u+2kv)^2}{2y}} \qquad y > 0 \; . \tag{20}$$

Let us consider $\varphi(t) = h(it)/g(it)$, where $g(z) = \frac{\sinh(v\sqrt{2z})}{v\sqrt{2z}}$ and similar for $h(z)$. Standard manipulations lead to the computation of the sequences $\{-\frac{k^2\pi^2}{2v^2}, \; k \geq 1\}$ and $\{(-1)^{k+1} \frac{k\pi}{u} \sin\left(\frac{u}{v} k\pi\right), \; k \geq 1\}$, which are the poles and the residues of $h(z)/g(z)$ respectively. As pointed out in the previous example, both functions $h(z)$ and $g(z)$ are entire functions of order $1/2$ and $\varphi(t)$ satisfies the hypothesis of Proposition 2.1. Notice that the poles are negative and hence we have to apply a generalization of Theorem 4.1 for the negative case, this is straightforward and we obtain the expression

$$f(y) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k\pi}{u} \sin\left(\frac{u}{v} k\pi\right) e^{\frac{k^2\pi^2}{2v^2} y} \quad y < 0 \,.$$

Since we consider the Laplace transform we need to change the sign of $y$ in the above expression and consider it in the range $(0, \infty)$. After doing that, it turns out that the above series is equal to (20) due to Poisson summation formulae. Moreover, we are able to say that the density function $f(y)$ is continuous in $[0, \infty)$ due to Proposition 4.10 since $n_g(r) = \left[\frac{v\sqrt{2r}}{\pi}\right]$, where $[x]$ stands for the integer part of $x$ and similar for $n_h(r)$.

### 5.4 Heston density function

The authors proved in [5] that the density function of the Heston model is $\mathcal{C}^\infty$ and can be expressed as an infinite convolution of Bessel type densities. We give here another expression in a particular case. In fact the search for such an expression for the general case was the starting point for the present paper, thus we believe it is worth recording it, even though it is only a partial result.

The Heston model for the log-spot is driven by the following system of stochastic differential equations

$$\left.\begin{aligned} dX_t &= -\frac{1}{2} V_t \, dt + \sqrt{V_t} \, dZ_t \\ dV_t &= a(b - V_t) \, dt + c\sqrt{V_t} \, dW(t) \end{aligned}\right\}$$

where $a, b$ and $c$ are real positive constants. The processes $W$ and $Z$ are two standard correlated Brownian motions such that $\langle Z, W\rangle_t = \rho t$ for some $\rho \in [-1, 1]$. For the particular case of interest we set $2ab = c^2$ and consider the volatility process $V$ to start at $0$. Then the complex moment generating function of log–spot is

$$\mathbb{E}[e^{zX_t}] = \frac{e^{z(x_0 - \rho ct/2)} e^{a}}{\cosh(P(z)) + \frac{a - \rho cz}{P(z)} \sinh(P(z))} = \frac{e^{z(x_0 - \rho ct/2)}}{g(z)} \,,$$

where $x_0$ is the initial point for the process $X$ and $P(z) = \sqrt{(a - c\rho z)^2 + c^2(z - z^2)}$. The term $e^{z(x_0 - \rho ct/2)}$ is merely a decentralisation term, while the function $g(z)$ is an entire function of order $1/2$ with all zeros being real (see [5]). The function $g(z)$ has negative zeros for $\rho = -1$, while for $\rho = 1$ and $a \geq c$ all zeros are positive. In any case we can apply Theorem 4.1 to obtain an expression of the density function as a series of exponential type densities. Obviously, the zeros of $g(z)$ must be computed numerically.

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