FINITELY SUSLINIAN MODELS FOR PLANAR COMPACTA WITH APPLICATIONS TO JULIA SETS

ALEXANDER BLOKH, CLINTON CURRY, AND LEX OVERSTEEGEN

Abstract. A compactum $X \subset \mathbb{C}$ is **unshielded** if it coincides with the boundary of the unbounded component of $\mathbb{C}\setminus X$. Call a compactum X **finitely Suslinian** if every collection of pairwise disjoint subcontinua of X whose diameters are bounded away from zero is finite. We show that any unshielded planar compactum X admits a topologically unique monotone map $m_X : X \to X_{FS}$ onto a finitely Suslinian quotient such that any monotone map of X onto a finitely Suslinian quotient factors through m_X. We call the pair (X_{FS}, m_X) (or, more loosely, X_{FS}) the **finest finitely Suslinian model of** X.

If $f : \mathbb{C} \to \mathbb{C}$ is a branched covering map and $X \subset \mathbb{C}$ is a fully invariant compactum, then the appropriate extension M_X of m_X monotonically semiconjugates f to a branched covering map $g : \mathbb{C} \to \mathbb{C}$ which serves as a model for f. If f is a polynomial and J_f is its Julia set, we show that m_X (or M_X) can be defined on each component Z of J_f individually as the finest monotone map of Z onto a locally connected continuum.

1. Introduction

For us, a **compactum** is a non-empty compact metric space. A compactum is **degenerate** if all of its components are points. A **continuum** is a connected compactum. One way of describing the topology of a compactum X is by constructing a *model* for it, i.e. a compactum Y, simpler to describe than X, and a (monotone) onto map $m : X \to Y$ (a continuous onto map m is **monotone** if all m-preimages of continua are continua; we denote the family of all monotone maps by $\mathcal{M}$). If X carries an additional structure, it is nice if the map m preserves it (e.g., if there is a continuous map $f : X \to X$, the map m should be chosen so that f induces a continuous self-map on Y by $m(x) \mapsto m(f(x)))$. Then m is said to be a **monotone semiconjugacy** of the map $f : X \to X$ to the induced map $g : Y \to Y$ (if m is a homeomorphism, it is called a **conjugacy**).

Date: September 7, 2010.
2010 *Mathematics Subject Classification.* Primary 54F15; Secondary 37B45, 37F10, 37F20.
Key words and phrases. Continuum, finitely Suslinian, locally connected, monotone map, Julia set.
The first author was partially supported by NSF grant DMS-0901038.
The second author was partially supported by NSF grant DMS-0353825.
The third author was partially supported by NSF grant DMS-0906316.

Unless specified otherwise, *from now on all compacta we consider are planar.* A case of particular interest is when X is **unshielded**, i.e. $X \subset \mathbb{C}$ is the boundary of the unbounded component $U_\infty(X)$ of $\mathbb{C} \setminus X$. The following construction is due to Carathéodory. Recall that a space is **locally connected** if its topology has a basis of connected sets. If X is an unshielded continuum, then $U_\infty(X) \cup \{\infty\}$ is a simply connected open set in the sphere. Take the unique Riemann map $\varphi_X : \mathbb{D} \to U_\infty(X)$ with positive derivative at the origin. If X is locally connected, we may extend φ_X continuously to $\overline{\mathbb{D}}$, mapping $\mathbb{S}^1$ onto X. Declaring points $u, v \in \mathbb{S}^1$ equivalent if and only if $\varphi_X(u) = \varphi_X(v)$ and denoting this equivalence relation by $\approx$, we see that X is homeomorphic to the quotient space $\mathbb{S}^1/\approx$. Equivalence relations $\approx$ which arise in this way are called **laminations**. If J_P is the locally connected Julia set of a polynomial P, then φ_X semiconjugates $z^d|_{\mathbb{S}^1}$ to a map $\widetilde{P} : \mathbb{S}^1/\approx \to \mathbb{S}^1/\approx$.

A lamination can be defined in abstract circumstances as a closed equivalence relation $\approx$ on $\mathbb{S}^1$ such that convex hulls of $\approx$-classes are pairwise disjoint. Laminations therefore capture the external ray picture of unshielded continua. In order to model dynamical objects like the Julia set of a degree d polynomial, we may require that $\approx$ is **d-invariant**. This means that the image of a $\approx$-class under the angle d-tupling map is again a $\approx$-class, and classes map to each other in a consecutive-preserving way (loosely speaking, preserving the order of points on the circle).

There are even laminations for disconnected Julia sets; then $\approx$ is a closed equivalence relation defined on a Cantor subset $A \subset \mathbb{S}^1$, and the angle d-tupling map is replaced by a covering self-map of A. This models that, for a polynomial P with disconnected Julia set J_P, the neighborhood of ∞ on which P is conjugate to $z \mapsto z^d$ does not include the entire basin of infinity. In this case every external ray can be analytically continued until it runs into the Julia set unless it first runs into the preimage of an escaping critical point. In such a case, case one can take left- and right-sided limits of fully-defined external rays and define two external rays corresponding to the same angle. These angles are associated to (pre)critical points and to the gaps in the Cantor set A (see [GM93, Kiw04, LP96]).

By Kiwi [Kiw04] laminations correspond to a wider class of polynomials P, whose Julia sets may not be locally connected nor connected. More precisely, an n-periodic point a of P is called **irrationally neutral** if $(P^n)'(a) = e^{2\pi i\alpha}$ with α irrational. Also, given a lamination $\approx$ of $\mathbb{S}^1$, call a set $F \subset \mathbb{S}^1$ **$\approx$-saturated** if it is a union of a collection of $\approx$-classes. By [Kiw04], to every polynomial P without irrationally indifferent cycles we can associate a lamination $\approx$, a closed $\approx$-saturated set $F \subset \mathbb{S}^1$ and a monotone map $m : J_P \to F/\approx$ such that m is a semiconjugacy of $P|_{J_P}$ with an appropriately constructed map $f : F/\approx \to F/\approx$ (in the case that J_P is connected, then $F = \mathbb{S}^1$ and f is a map induced on $\mathbb{S}^1/\approx$ by z^d).

The present authors prove [BCO08] that every complex polynomial P with *connected* Julia set has a unique "best" lamination. This generalizes [Kiw04], albeit for connected Julia sets, by allowing P to have irrationally neutral cycles. The lamination $\approx$ comes with a monotone semiconjugacy $m : J_P \to \mathbb{S}^1/\approx$ which has the property of being the *finest monotone map of J_P onto a locally connected continuum* (defined in the next section). In [BCO08] we also provide a criterion for $\approx$ to have more than one equivalence class (equivalently, for J_P to have a non-degenerate locally connected monotone image).

A compactum X is called **finitely Suslinian** if, for every $\varepsilon > 0$, every collection of disjoint subcontinua of X with diameters at least ε is finite. By Lemma 2.9 [BO04], *unshielded planar locally connected continua are finitely Suslinian and vice versa*[1]. Thus, in the unshielded case the notion of finitely Suslinian generalizes the notion of local connectivity. There is another analogy to local connectivity too: by Theorem 1.4 [BMO07], *for an unshielded finitely Suslinian compactum $X \subset \mathbb{C}$ there exists a lamination $\approx$ of a closed set $F \subset \mathbb{S}^1$ such that X is homeomorphic to $F/\approx$*. This motivates us to extend onto finitely Suslinian compacta some results for locally connected continua and to look for good finitely Suslinian models of planar compacta. We need the following definition which applies to arbitrary maps (as customary in topology, by a map we always mean a *continuous* map).

Definition 1 (Finest models)**.** Let $X \subset \mathbb{C}$ be a compactum, P be a topological property (P could be the property of being locally connected, Hausdorff, etc) and $\mathcal{B}$ be a class of maps with domain X. The **finest $\mathcal{B}$-model of X with property P** is an onto map $\psi^P : X \to Y$, $\psi^P \in \mathcal{B}$ where Y is a topological space with property P such that any other map $\varphi^P : X \to Z$, $\varphi^P \in \mathcal{B}$ onto a space Z with property P can be written as the composition $g \circ \psi^P$ for some map $g : Y \to Z$.

Though we give the definition for any class $\mathcal{B}$, we are mostly interested in the class $\mathcal{M}$ of monotone maps because such maps do not change the structure of X too drastically; besides, we study planar compacta, and monotone maps of planar compacta with non-separating fibers keep them planar [Moo62]. In the monotone case we will use notation m^P instead of ψ^P (or just m if the property P is fixed). In fact, in the monotone case this concept of finest map has been studied before in the context of continua (cf [FS67]).

Lemma 2. *If the finest $\mathcal{B}$-model with property P exists, then it is unique up to a homeomorphism.*

[1]If X is not unshielded, this may fail as the closed unit disk $\overline{\mathbb{D}}$ is locally connected but not finitely Suslinian; Example 14 shows that there are nowhere dense locally connected planar continua which are not finitely Suslinian.

Proof. Suppose that $m_1 : X \to Y_1$ and $m_2 : X \to Y_2$ are finest $\mathcal{B}$-models. Then, by definition, we may factor m_1 as

$$m_1 : X \overset{m_2}{\to} Y_2 \overset{g_1}{\to} Y_1$$

and similarly factor the constituent map m_2 to obtain

$$m_1 : X \overset{m_1}{\to} Y_1 \overset{g_2}{\to} Y_2 \overset{g_1}{\to} Y_1.$$

However, since the composition is itself equal to m_1, we find that $g_1 : Y_2 \to Y_1$ and $g_2 : Y_1 \to Y_2$ are each other's inverse and hence homeomorphisms. Therefore, $Y_2 = g_2(Y_1)$ is homeomorphic to Y_1, $m_2 = g_2 \circ m_1$, and $m_1 = g_1 \circ m_2$. □

The following notion is a bit weaker than that defined in Definition 1.

Definition 3 (Top models)**.** Let $h : X \to Y$ be a map in $\mathcal{B}$ onto a compactum Y with property P such that there exists no map $h' : X \to Y'$ onto a compactum Y' with property P which refines h (i.e., if a map h'' is such that $h = h'' \circ h'$, then h'' must be a homeomorphism and Y' is homeomorphic to Y). Then (Y, h) is said to be a **top $\mathcal{B}$-model of X with property P**.

Observe that, while the finest model is finer than all others, a top model does not have another strictly finer model. This is the same as the greatest model and a maximal model in the sense of some partial order. Hence, if the finest $\mathcal{B}$-model of X with property P exists, it is the unique top $\mathcal{B}$-model of X with property P. So, if we have a top $\mathcal{B}$-model $h : X \to Y$ of X with property P and a $\mathcal{B}$-model $h' : X \to Y'$ of X with property P such that h is not finer than h', then the finest $\mathcal{B}$-model of X with property P does not exist. Example 14 provides a planar continuum X_1 with this situation in the case when P is the property of being finitely Suslinian and $\mathcal{B}$ is either the class of continuous maps or monotone maps; thus, X_1 has no finest continuous or monotone model with finitely Suslinian property.

Also, by the definitions if $\mathcal{B} \subset \mathcal{B}'$ are two classes of maps and the finest (top) $\mathcal{B}'$-model of X with property P is (Y, h) where h happens to belong to the smaller class $\mathcal{B}$, then (Y, h) is also the finest (top) $\mathcal{B}$-model of X with property P.

As Theorem 4 proves, the situation with *unshielded* planar continua and the finitely Suslinian property is better. From now on finest $\mathcal{M}$-models with finitely Suslinian property will be called *finest finitely Suslinian monotone models.*

Theorem 4. *Every unshielded compactum X has a finest finitely Suslinian monotone model $m_X : X \to X_{FS}$.*

This yields applications to the dynamics of branched covering maps of the plane, and in particular the study of Julia sets of polynomials, which are naturally occurring examples of unshielded compacta.

Theorem 5. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a branched covering map and X is an unshielded compactum which is fully invariant under f. Then X_{FS} can be embedded into the plane and the finest finitely Suslinian monotone model $m_X : X \to X_{FS}$ can be extended to the plane in such a way that the resulting map $M_X : \mathbb{C} \to \mathbb{C}$ semiconjugates f and a branched covering map $g : \mathbb{C} \to \mathbb{C}$.*

These results can be made stronger if f is a polynomial.

Theorem 6. *The finest finitely Suslinian monotone model $m_{J_P} : J_P \to J_{P_{FS}}$ of the Julia set of a polynomial P coincides on each component X of J_P with the finest monotone map m_X of X to a finitely Suslinian continuum. In particular:*

(1) *the finest finitely Suslinian monotone model of J_P is non-degenerate if and only if there exists a periodic component of J_P whose finest finitely Suslinian monotone model is non-degenerate;*
(2) *the set J_P is finitely Suslinian if and only if all periodic non-degenerate components of J_P are locally connected.*

By [BCO08], one can specify exactly the situations in which a non-degenerate finitely Suslinian model of a polynomial Julia set exists. This is because any periodic component of J_P is the Julia set of a polynomial-like map, which is hybrid equivalent (in particular, topologically conjugate) to a polynomial. Hence, summarizing the results of [BCO08], we conclude that a periodic component Y of J_P has a non-degenerate finitely Suslinian model if and only if one of the following is true:

(1) Y contains infinitely many periodic points, each of which separates Y,
(2) the topological hull of Y contains either a parabolic or attracting periodic point, or
(3) Y admits a *Siegel configuration*, which roughly means that are subcontinua of the Julia set, comprised of finitely many impressions and disjoint from all other impressions, which in essence correspond to the critical points on the boundaries of Siegel disks in locally connected Julia sets.

For all details, the reader is invited to read [BCO08], especially Section 5 thereof.

2. Topological Lemmas

First we introduce several useful notions. When speaking of limits of compacta, we always mean convergence in the Hausdorff sense.

Definition 7. A partition of a compactum X if said to be **upper semi-continuous** if for every pair of convergent sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ of points in X such that x_i, y_i belong to some element D_i of the partition, we have that the points $\lim_{i\to\infty} x_i$

and $\lim_{i\to\infty} y_i$ belong to some element D of the partition. In this case the equivalence relation $\sim$ induced by the partition is said to be **closed**. Equivalently, $\sim$ is said to be **closed** if its graph is closed in $X \times X$.

The following construction is less standard.

Definition 8. Let $\mathcal{A}$ be a family of subsets of a compactum X. An equivalence relation $\sim$ **respects** $\mathcal{A}$ if $\sim$ is closed and every member of $\mathcal{A}$ is contained in a $\sim$-class. If $\sim$ and $\approx$ are equivalence relations on a set X, we say that $\sim$ is **finer** than $\approx$ if $\sim$-classes are contained in $\approx$-classes. The **finest closed equivalence relation generated by** $\mathcal{A}$ is the finest equivalence relation $\sim_{\mathcal{A}}$ respecting $\mathcal{A}$.

Equivalently, one can define the **finest continuous map respecting** $\mathcal{A}$, i.e. a continuous map $\psi^{\mathcal{A}} : X \to Z$ such that for any map $f : X \to Q$ there exists a map $g : Z \to Q$ which can be composed with $\psi^{\mathcal{A}}$ to give $f = g \circ \psi^{\mathcal{A}}$.

Lemma 9 shows that the finest closed equivalence relation generated by $\mathcal{A}$ (and hence, the finest map respecting $\mathcal{A}$) exists and specifies its properties if elements of $\mathcal{A}$ are connected.

Lemma 9. *The finest closed equivalence relation generated by $\mathcal{A}$ exists and is therefore unique (thus, the finest map $\psi^{\mathcal{A}}$ respecting $\mathcal{A}$ exists and is well-defined). If $\mathcal{A}$ consists of connected subsets of a compactum X, then all $\sim_{\mathcal{A}}$-classes are continua and the finest continuous map respecting $\mathcal{A}$ is monotone.*

Proof. To see that $\sim_{\mathcal{A}}$ is well-defined, let $\Xi_{\mathcal{A}}$ be the set of all upper semi-continuous equivalence relations which respect $\mathcal{A}$. Then it is easy to see that the relation $\sim_{\mathcal{A}}$ defined by "$x \sim_{\mathcal{A}} y$ if and only if $x \sim y$ for all $\sim \in \Xi_{\mathcal{A}}$" is again a closed equivalence relation respecting $\mathcal{A}$, and that $\sim_{\mathcal{A}}$ is finer than all closed equivalence relations from $\Xi_{\mathcal{A}}$. It follows that the quotient map $X \to X/\sim_{\mathcal{A}}$ is in fact the finest continuous map which respects $\mathcal{A}$.

It suffices to show that all $\sim_{\mathcal{A}}$ classes are connected. According to [Nad92, Lemma 13.2], the equivalence relation $\sim$ whose classes are the components of $\sim_{\mathcal{A}}$-classes is also an upper semi-continuous equivalence relation, and $\sim$-classes are contained in $\sim_{\mathcal{A}}$-classes. Since elements of $\mathcal{A}$ are connected, it follows that $\sim$ still respects $\mathcal{A}$, so $\sim_{\mathcal{A}}$ classes are contained in $\sim$-classes. Therefore $\sim = \sim_{\mathcal{A}}$, and $\sim_{\mathcal{A}}$-classes are connected. □

It is quite easy to determine when a continuous function on X induces a continuous function on $X/\sim_{\mathcal{A}}$, as the following lemma shows.

Lemma 10. *If $f : X \to X$ is a continuous function which sends elements of $\mathcal{A}$ into $\sim_{\mathcal{A}}$-classes, then f induces a function $g : X/\sim_{\mathcal{A}} \to X/\sim_{\mathcal{A}}$ with $\psi^{\mathcal{A}} \circ f = g \circ \psi^{\mathcal{A}}$ (g maps the $\sim_{\mathcal{A}}$-class of x to the $\sim_{\mathcal{A}}$-class of $f(x)$).*

Proof. It is sufficient to show that the f-image of a $\sim_{\mathcal{A}}$-class is contained in a $\sim_{\mathcal{A}}$-class. Consider the fibers of $\psi^{\mathcal{A}} \circ f$. By assumption, f sends elements of $\mathcal{A}$ into $\sim_{\mathcal{A}}$-classes, so $\psi^{\mathcal{A}} \circ f$ is constant on the elements of $\mathcal{A}$. Therefore, the fibers of $\psi^{\mathcal{A}} \circ f$ form an upper semi-continuous partition of X which respects $\mathcal{A}$. Since $\psi^{\mathcal{A}}$ is the finest such map in the sense of Definition 8, there exists a map $g : X/\sim_{\mathcal{A}} \to X/\sim_{\mathcal{A}}$ with $\psi^{\mathcal{A}} \circ f = g \circ \psi^{\mathcal{A}}$ as desired. □

Remark 11. For later reference, we note that there is also a transfinite construction of the equivalence relation $\sim_{\mathcal{A}}$. To begin, let $\sim_0$ denote the equivalence relation such that $x \sim_0 y$ if and only if x and y are contained in a connected finite union of elements of $\mathcal{A}$. If an ordinal α has an immediate predecessor β for which $\sim_\beta$ is defined, we define $x \sim_\alpha y$ if there exist finitely many sequences of $\sim_\beta$ classes whose limits comprise a continuum containing x and y (here, the limit of non-closed sets is considered to be the same as the limit of their closures). In the case that α is a limit ordinal, we say $x \sim_\alpha y$ whenever there exists $\beta < \alpha$ such that $x \sim_\beta y$. Notice that the sequence of $\sim_\alpha$-classes of a point x (as α increases) is an increasing nest of connected sets, with the closure of each being a subcontinuum of its successor. It is also apparent that $\sim_\alpha$-classes are contained in $\sim_{\mathcal{A}}$-classes for all ordinals α.

Let us now show that $\sim_{\mathcal{A}} = \sim_\Omega$ where Ω is the smallest uncountable ordinal. To see this, we first note that $\sim_\Omega = \sim_{(\Omega+1)}$. This is because the sequence of closures of $\sim_\alpha$-classes containing a point x forms an increasing nest of subsets, no uncountable subchain of which can be strictly increasing in the plane. Therefore, all $\sim_\alpha$-classes have stabilized when $\alpha = \Omega$. This implies that $\sim_\Omega$ is a closed equivalence relation, since the limit of $\sim_\Omega$-classes is a $\sim_{(\Omega+1)}$-class, which we have shown is a $\sim_\Omega$-class again. Finally, $\sim_\Omega$ respects $\mathcal{A}$ and $\sim_\Omega$-classes are contained in $\sim_{\mathcal{A}}$-classes, so $\sim_{\mathcal{A}}$ and $\sim_\Omega$ coincide.

Let us become more specific and study finitely Suslinian compacta.

Definition 12 (Limit continuum and $\overset{\text{FS}}{\sim}$)**.** A subcontinuum C of X is said to be a **limit continuum** if there exists a sequence $(C_n)_{n=1}^\infty$ of pairwise disjoint subcontinua of X converging to C. We define $\overset{\text{FS}}{\sim}_X$ as the **finest equivalence relation respecting the family of limit continua** (if the context is clear, we may omit the subscript and refer simply to $\overset{\text{FS}}{\sim}$).

Note that this notion is slightly more general than the classical notion of *continuum of convergence* in continuum theory. Also, it is easy to see that a continuum is finitely Suslinian if and only if it contains no non-degenerate limit continua.

Lemma 13. *For any compactum X, the quotient $X/\overset{\text{FS}}{\sim}$ is finitely Suslinian.*

Proof. Let $(C_n)_{n=1}^\infty$ be a (without loss of generality convergent) sequence of pairwise disjoint subcontinua of $X/\overset{\text{FS}}{\sim}$. Let $m : X \to X/\overset{\text{FS}}{\sim}$ denote the quotient map. A

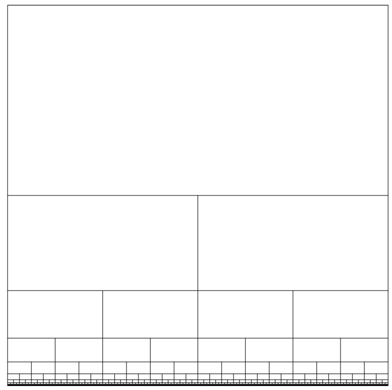

FIGURE 1. A continuum with no finest finitely Suslinian model.

subsequence of the preimages $(m^{-1}(C_n))_{n=1}^{\infty}$ converges to a continuum K. By definition of $\overset{\mathrm{FS}}{\sim}$ we have that $m(K)$ is a singleton, say, $\{a\}$, and continuity of m implies that $(C_n)_{n=1}^{\infty}$ converges to $\{a\}$. Since $(C_n)_{n=1}^{\infty}$ was arbitrary, we have that $X/\overset{\mathrm{FS}}{\sim}$ contains no non-degenerate limit continua and is therefore finitely Suslinian. □

Lemma 13, together with the characterization of finitely Suslinian compacta as those with no limit continua, suggests that $X/\overset{\mathrm{FS}}{\sim}$ could be the finest model of X. Such a fact would mean that any monotone map of X onto a finitely Suslinian compactum must collapse limit continua. However, in general this is not true.

Example 14 (A continuum with no finest finitely Suslinian model)**.** Define a continuum X as follows, and as depicted in Figure 1 on page 8:

$$H_n = [0,1] \times \{1/2^n\},\ n \in \mathbb{N},$$
$$H = [0,1] \times \{0\},$$
$$V_{p/q} = \{p/q\} \times [0, 1/q],\ p/q \text{ a dyadic rational}$$
$$X_1 = \bigcup\{H_n \mid n \in \mathbb{N}\} \cup H \cup \bigcup\{V_{p/q} \mid 0 < p/q < 1 \text{ dyadic}\}$$

Observe that X_1 is a locally connected, not finitely Suslinian, nowhere dense and *not* unshielded in $\mathbb{C}$ continuum. There are two essentially different kinds of finitely Suslinian monotone quotients of X_1, depicted in Figure 2. One map, h, corresponds to identifying the unique maximal limit continuum $H = \lim_{n\to\infty} H_n$ to a point. Any finer (and not even necessarily monotone) map h' to a finitely Suslinian compactum would still keep images of H_n disjoint, implying that images of H_n must converge to a point which has to be the image of H. Thus, $h' = h$ and $(h(H), h)$ is a top finitely Suslinian model of X_1 which happens to be monotone.

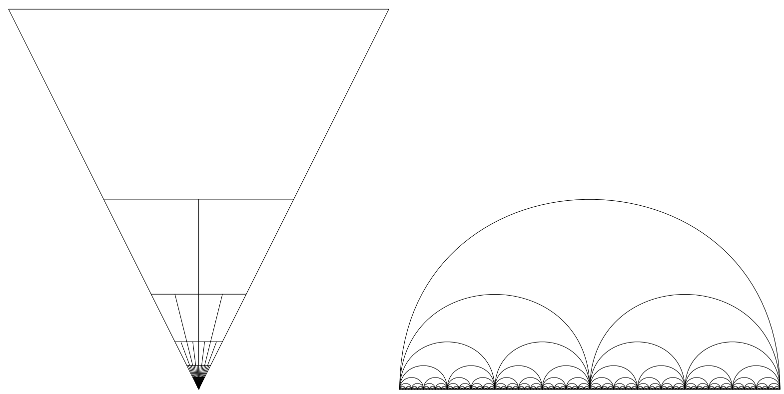

FIGURE 2. Essentially different finitely Suslinian quotients of the continuum depicted in Figure 1.

Other quotients of X_1 with finitely Suslinian images are maps φ_N which identify to points members of the collection $\{V_{p/q} \mid q > N\}$. This yields a sequence of maps $(\varphi_N)_{N=0}^{\infty}$, with φ_{N+1} finer than φ_N for all N. On the other hand, none of these maps can be compared with ψ in the sense that neither h is finer than φ_N nor φ_N is finer than h. As explained above, it follows from the definitions now that h is **not** the finest finitely Suslinian model of X_1 (neither is it the finest finitely Suslinian monotone model of X_1). It is worth noticing also that for any N the only maps finer than both φ_N and h are homeomorphisms (since the intersection of any fibers of h and φ_N is at most a point) and that the only maps finer than every map in $\{\varphi_N \mid N \in \mathbb{N}\}$ are homeomorphisms.

In the unshielded case the situation is better. First we need Definition 15.

Definition 15 (Irreducible continua)**.** Given two disjoint closed sets A, B, a continuum C is said to be **irreducible between** A **and** B if C intersects both A and B and does not contain a subcontinuum with the same property. Given a continuum D intersecting A and B, one can use Zorn's Lemma to find a subcontinuum $C \subset D$ irreducible between A and B.

We also need Lemma 16.

Lemma 16. *Let K be an irreducible continuum between ∂U and ∂V where U, V are open sets with disjoint closures. Then K is disjoint from both U and V.*

Proof. Set $K' = K \setminus \overline{V}$. Take a component Y of K' containing a point from ∂U. By the Boundary Bumping Theorem (Theorem 5.6 from [Nad92, Chapter V, p. 74]) $\overline{Y}$ intersects ∂V. Since K is irreducible, $\overline{Y} = K$ and hence K is disjoint from V. Similarly, K is disjoint from U. □

To prove our first theorem we need the following geometric lemma. It is a generalization of the fact that any homeomorphic copy of the letter θ embedded in the plane is not unshielded. For a planar continuum Y the set $\mathbb{C} \setminus U_\infty(Y)$ is called the **topological hull of** Y and is denoted by $T(Y)$.

Lemma 17. *Suppose a planar compactum X contains two disjoint continua X_1, $X_2 \subset X$ and three pairwise disjoint continua C^1, C^2, C^3 such that $X_i \cap C^j \neq \emptyset$ for all $i \in \{1,2\}$ and $j \in \{1,2,3\}$. Then X is not unshielded.*

Proof. By way of contradiction we assume that X is unshielded. Let us collapse the topological hulls $T(X_1)$ and $T(X_2)$ to points x_1 and x_2 and let $m : \mathbb{C} \to \mathbb{C}$ denote this monotone map (by Moore's Theorem [Moo62], the image is homeomorphic to the plane). Then $m(C^i) \cap m(C^j) = \{x_1, x_2\}$ for all $i \neq j$ and $m(X)$ is also unshielded. Put $Z^j = T(m(C^j))$. Then $Z^i \cap Z^j = \{x_1, x_2\}$ for all $i \neq j$. By Theorem 63.5 of [Mun00] for each $i \neq j$, $Z^i \cup Z^j$ separates $\mathbb{C}$ into precisely two components, one of which is bounded and denoted by $B_{i,j}$. It follows that for some choice of i, j, k the set Z_i intersects $B_{j,k}$, contradicting that $m(X)$ is unshielded. □

We use Lemma 17 to show that certain maps of unshielded compact sets collapse limit continua. Let $\mathcal{FM}$ be the class of all **finitely monotone** maps, i.e. such maps $h : X \to Y$ that for any continuum $Z \subset Y$ the set $h^{-1}(Z)$ consists of finitely many components.

Lemma 18. *Suppose that $\varphi : X \to Y$ is a finitely monotone map of an unshielded compact set X onto a finitely Suslinian compact set Y. If $C \subset X$ is a limit continuum, then $\varphi(C)$ is a point.*

Proof. Let $C \subset X$ be a limit continuum. Choose a sequence of continua $C_i \to C$. Consider two cases.

Case 1. *There are infinitely many distinct components of Y containing sets $f(C_i)$.*

Denote by T_i the component of Y which contains $f(C_i)$. We may refine the sequence $(C_i)_{i=1}^\infty$ so that all sets T_i are different. Since Y is finitely Suslinian, we may refine it further so that T_i converge to a point $t \in Y$. Hence $f(C) = \lim f(C_i) = \lim T_i = t$.

Case 2. *There are finitely many distinct components of Y containing all sets $f(C_i)$.*

Without loss of generality, we may assume that all C_n are contained in a single component T of X. Observe that then $\varphi(T) \subset Z$ where Z is a component of Y. By [BO04, Lemma 2.9], Z is locally connected. We suppose that $\varphi(C)$ is not a point and show that this contradicts the fact that X is unshielded. Let $z_1 = \varphi(x_1)$ and $z_2 = \varphi(x_2)$ be distinct points in $\varphi(C)$. Because Z is locally connected, there exist open, connected subsets $Z_1, Z_2 \subset Z$ with disjoint closures, containing $z_i \in Z_i$ for $i \in \{1,2\}$. Then $\varphi^{-1}(\overline{Z_1})$ and $\varphi^{-1}(\overline{Z_2})$ have finitely many components. After refining the sequence C_i we may assume that all sets C_i intersect a component A_1 of $\varphi^{-1}(\overline{Z_1})$ and a component A_2 of $\varphi^{-1}(\overline{Z_2})$. However, by Lemma 17 this is impossible.

□

3. The Existence of the Finest Map and Dynamical Applications in the Unshielded Case

3.1. **The existence of the finest map in the unshielded case.** We are ready to prove our first theorem which implies Theorem 4.

Theorem 19. *Let X be an unshielded compact set in the plane. Then the quotient map $m_X : X \to X/\overset{\mathrm{FS}}{\sim} = X_{FS}$ is the finest finitely Suslinian monotone model of X. Moreover, X_{FS} can be embedded into the plane and m_X can be extended to a monotone map $M_X : \mathbb{C} \to \mathbb{C}$ which collapses the topological hulls of $\overset{\mathrm{FS}}{\sim}$-classes and is one-to-one elsewhere.*

Proof. By Lemma 13, $X/\overset{\mathrm{FS}}{\sim}$ is a finitely Suslinian compactum. Now, suppose that $\varphi : X \to Z$ is monotone and Z is finitely Suslinian. Then φ collapses all limit continua by Corollary 18. Since $\overset{\mathrm{FS}}{\sim}$ is the finest equivalence relation respecting the collection of limit continua, we see that the quotient map $m_{FS} : X \to X/\overset{\mathrm{FS}}{\sim}$ is finer that φ, and is therefore the finest finitely Suslinian monotone model of X. The rest of the theorem follows from the Moore theorem [Moo62]. □

Observe that in fact Lemma 18 implies that the finest $\mathcal{FM}$-model of X with finitely Suslinian property is the same as the finest finitely Suslinian monotone model of X (despite the fact that the class $\mathcal{FM}$ of finitely monotone maps is much wider than the class $\mathcal{M}$ of monotone maps).

3.2. **Applications to Dynamical Systems.** First we show that sometimes the finest map is compatible with the dynamics. Recall that a set $A \subset X$ is **fully invariant** under a map $f : X \to X$ if $A = f^{-1}(A) = f(A)$.

Theorem 20. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a branched covering map and that X is a fully invariant unshielded compactum. Then there exists a branched covering map $g : \mathbb{C} \to \mathbb{C}$ such that $M_X \circ f = g \circ M_X$ and hence $X_{FS} = M_X(X)$ is fully invariant under g.*

Proof. By Lemma 18, m_X sends limit continua into $\overset{\mathrm{FS}}{\sim}$-classes. By Lemma 10 this and Theorem 19, in which the extension M_X of m_X onto $\mathbb{C}$ is described, implies that $g = M_X \circ f \circ M_X^{-1}$ is well-defined. If we now show that $\overset{\mathrm{FS}}{\sim}$-classes map *onto* image classes, then it will follow from the definition of a quotient map that g is open which implies the result by the Stoilow theorem [Sto56].

To see that the image of a $\overset{\text{FS}}{\sim}$-class is again a $\overset{\text{FS}}{\sim}$-class, we show that $\sim_\alpha$-classes map onto the union of $\sim_\alpha$-classes for every ordinal α, where $\sim_\alpha$ was defined in Remark 11 with $\mathcal{A}$ being the set of limit continua. Then, when $\alpha = \Omega$, we see that $\overset{\text{FS}}{\sim}$-classes map both into and over other $\overset{\text{FS}}{\sim}$-classes.

Let us first show that $\sim_0$-classes map over other $\sim_0$-classes. Indeed, let $f(x)$ and y belong to the same $\sim_0$-class. Then there exist finitely many limit continua $C_1 = \lim_{i\to\infty} C_i^1, \dots, C_n = \lim_{i\to\infty} C_i^n$ forming a chain joining $f(x)$ and y (i.e., so that $f(x) \in C_1$, $y \in C_n$, and $C_j \cap C_{j+1} \neq \emptyset$ for any $1 \le j < n$). Since f is an open map, there exists a convergent sequence $(D_i^1)_{i=1}^\infty \to D_1$ of continua such that $f(D_i^1) = C_i^1$ for each i and D_1 is a limit continuum which contains x. By continuity, $f(D_1) = C_1$, so D_1 contains the preimage of a point in C_2. We can now inductively find limit continua $D_2, \dots, D_n$ mapping onto $C_2, \dots, C_n$ and forming a chain from x to a preimage of y. Therefore, the $\sim_0$-class of $f(x)$ is contained in the image of the $\sim_0$-class of x.

Suppose now by induction that we have proven the claim for all ordinals less than α, and let $f(x) \sim_\alpha y$. If α has an immediate predecessor β (the other case is left as an easy exercise for the reader), there are finitely many sequences of $\sim_\beta$-classes $(K_i^1)_{i=1}^\infty, \dots, (K_i^n)_{i=1}^\infty$ which converge to a chain of continua joining $f(x)$ and y. By the inductive hypothesis, if $f(z) \in K_i^j$ then the $\sim_\beta$-class of z maps over K_i^j. One can therefore find, due to the openness of f, a convergent sequence $(L_i^1)_{i=1}^\infty \to L_1$ of $\sim_\beta$-classes such that $f(L_i^1) \supset K_i^1$ and $x \in L_1$. Note by continuity that $f(L_1) \supset K_1$. Proceeding as in the previous paragraph, we find similar limits $L_2, \dots, L_n$ forming a chain of continua which joins x to a preimage of y. We therefore see that the image of a $\sim_\alpha$-class is a union of $\sim_\alpha$-classes, and the proof is complete. □

Sometimes in the situation of Theorem 20 a naive but natural approach to the problem of constructing the finest finitely Suslinian model can be used.

Definition 21. By Theorem 19 for each component Y of X the finest equivalence relation on Y is $\overset{\text{FS}}{\sim}_Y$. Consider the equivalence relation $\sim_{n,X}$ defined as follows: $x \sim_{n,X} y$ if and only if x and y belong to the same component Y of X and $x \overset{\text{FS}}{\sim}_Y y$.

It is natural to find out if $\sim_{n,X}$ coincides with $\overset{\text{FS}}{\sim}_X$. Simple examples show that in general it is not true.

Example 22 (A map on a compactum with $\sim_n$ not coinciding with $\overset{\text{FS}}{\sim}$)**.** Define a map $f : \mathbb{C} \to \mathbb{C}$ as follows. Take a map from the real quadratic family $g_b(x) = bx(1-x)$ with $b > 4$. It is well known that then there exists a forward invariant Cantor set $A \subset [0, 1] \setminus \{0\}$ on which the map g_b acts as a full 2-shift. We define a map on the set $X = A \times [-1, 1]$ as $f(x, y) = (g_b(x), y)$. Evidently, f can be extended to a branched covering two-to-one map $f : \mathbb{C} \to \mathbb{C}$, however for brevity we will not give its full description here.

Observe that X is a fully invariant set. The equivalence relation $\overset{\text{FS}}{\sim}_X$ collapses X to a Cantor set, though all $\sim_{n,X}$-classes are points. Thus, in this case $\sim_{n,X} \neq \overset{\text{FS}}{\sim}_X$.

Example 22 shows that in some cases $\sim_n$ and $\overset{\text{FS}}{\sim}$ are distinct. Moreover, it also shows the mechanism of how this distinction occurs. However, the definition immediately implies that $\sim_{n,X}$ is finer than $\overset{\text{FS}}{\sim}_X$.

It turns out that the abberation $\overset{\text{FS}}{\sim}_X \neq \sim_{n,X}$ is impossible for polynomial maps. To show that we need some definitions. A point $x \in X$ of a planar compactum X is called **accessible (from $U_\infty(X)$)** if there is a curve $Q \subset U_\infty(X)$ with one endpoint at x (then one says that Q **lands** at x and that x is **accessible by** Q). We also need the definition of impression of an angle. For a continuum $X \subset \mathbb{C}$, let $\psi : \mathbb{C} \setminus \mathbb{D} \to U_\infty(K)$ denote the unique conformal isomorphism with real derivative at ∞. For an angle $\alpha \in \mathbb{S}^1$, we define the **impression** of the external ray at angle α as

$$\text{Imp}(\alpha) = \{\lim \psi(z_i) \,:\, z_i \to \alpha \text{ from within } \mathbb{D}\}.$$

Theorem 23. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. Then the equivalence relations $\sim_{n,J_P}$ and $\overset{\text{FS}}{\sim}_{J_P}$ coincide.*

Proof. A recent result by [KS06, QY06] claims that *all non-preperiodic components of a polynomial Julia set are points.* It is enough to show that if $x, y \in J_P$ and $x \overset{\text{FS}}{\sim} y$, then $x \sim_n y$. By Definition 12 it suffices to show that a limit continuum in J_P is contained in a $\sim_n$-class. Let $C \subset J_P$ be a limit continuum and $C_i \to C$ is a sequence of subcontinua of J_P which converge to C. Denote by T_i the component of J_P containing C_i and by T the component of J_P containing C.

If infinitely many C_i's are contained in T, then by Definition 12 C is contained in a $\sim_n$-class and we are done. Suppose that there are only finitely many C_i's in T. Then we may assume that a sequence of pairwise distinct components T_i converges to a limit continuum $C' \subset T$ where $C \subset C'$, and we need to show that C' is contained in one $\sim_n$-class. To do so, we consider two cases.

First, assume that T is periodic of period m. Then it is well-known that $P^m|_T$ is a so-called **polynomial-like** map (see [DH85]) for which T plays the role of its filled-in Julia set. That is, there exist two simply connected neighborhoods $U \subset V$ of T such that $f : U \to V$ is a branched covering map and there exist a polynomial f with connected Julia set J_f and two neighborhoods $U' \subset V'$ of J_f such that $P|_U$ is (quasi-conformally) conjugate to $f|_{U'}$ by a homeomorphism φ and $\varphi(T) = K_f$ where K_f is the **filled-in Julia set of** f (i.e., the topological hull of J_f). We will use **Böttcher coordinates** for f and consider a conformal map $\psi : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \overline{\mathbb{D}}$ which conjugates $f|_{\mathbb{C} \setminus K_f}$ and $z^d|_{\mathbb{C} \setminus \overline{\mathbb{D}}}$.

We claim that there exists an angle α such that $C' \subset \varphi^{-1}(\mathrm{Imp}(\alpha))$. Consider continua $\psi \circ \varphi(T_i) = T_i' \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ and show that they converge to a unique point in $\mathbb{S}^1$. Indeed, otherwise we may assume that they converge to an non-degenerate arc $I \subset \mathbb{S}^1$. For any $t \in \mathbb{S}^1$ let $R_t \subset \mathbb{C} \setminus \mathbb{D}$ be the half-line from t to infinity, orthogonal to $\mathbb{S}^1$ at t. Choose $\beta \in I$ such that the $R' = \psi^{-1}(R_\beta)$ is a curve in $\mathbb{C} \setminus K_f$ landing at a point $b \in T$. We may assume that β is not an endpoint of I.

We need Theorem 2 of [LP96] which claims that if $x \in T$ is an accessible point from $\mathbb{C} \setminus T$ by a curve l, then x is accessible from $\mathbb{C} \setminus J_P$ by a curve R which is homotopic to l among all curves in $\mathbb{C} \setminus T$ landing at x. By this result we can find a curve $L \subset \mathbb{C} \setminus K_f$ which lands at b and is disjoint from $\varphi(J_P)$. Then the curve $\psi(L) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ lands at β while being disjoint from all sets T_i' which clearly contradicts the assumption that these sets converge to the arc I.

Thus, we may assume that $T_i' \to \alpha \in \mathbb{S}^1$ which, by the definition of impression, implies that T_i converge into the set $\varphi^{-1}(\mathrm{Imp}(\alpha))$ and so $C' \subset \varphi^{-1}(\mathrm{Imp}(\alpha))$. Now we need Lemma 16 of [BCO08] by which the set $\mathrm{Imp}(\alpha)$ is contained in one $\sim_{n,J_f}$-class which implies (after we apply the homeomorphism φ^{-1} to this) that the set $\varphi^{-1}(\mathrm{Imp}(\alpha))$ is contained in one $\sim_{n,T}$-class as desired. This completes the consideration of the case of a periodic T.

Now, suppose that T is not periodic. Then by [KS06, QY06] T is preperiodic and we can choose $n > 0$ such that $P^n(T)$ is a periodic component of J_P. Since by the above all limit continua in $P^n(T)$ are contained in $\sim_n$-classes, it is easy to use pullbacks to see that all limit continua in T are contained in $\sim_n$-classes too. This completes the proof. □

The following two corollaries easily follow.

Corollary 24. *The finest finitely Suslinian model of J_P has at least one non-degenerate component if and only if there exists a periodic component of J_P which has a non-degenerate finitely Suslinian model.*

Observe that a dynamical criterion for a connected Julia set to have a non-degenerate finitely Suslinian model is obtained in [BCO08].

Corollary 25. *The set J_P is finitely Suslinian if and only if all periodic components of J_P are locally connected.*

References

[BCO08] A. Blokh, C. Curry, L. Oversteegen, *Locally connected models for Julia sets*, to appear in Adv. Math. (2010), doi:10.1016/j.aim.2010.08.011

[BMO07] Alexander Blokh, Michał Misiurewicz, and Lex Oversteegen, *Planar finitely Suslinian compacta*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3755–3764.

[BO04] Alexander Blokh and Lex Oversteegen, *Backward stability for polynomial maps with locally connected Julia sets*, Trans. Amer. Math. Soc. **356** (2004), no. 1, 119–133.

[DH85] A. Douady, J. H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4) **18** (1985), 287–343.

[FS67] R. W. FitzGerald and P. M. Swingle, *Core decomposition of continua*, Fund. Math. **61** (1967), 33–50.

[GM93] L. Goldberg, J. Milnor, *Fixed points of polynomial maps. II. Fixed point portraits*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 1, 51–98.

[Kiw04] Jan Kiwi, $\mathbb{R}$*eal laminations and the topological dynamics of complex polynomials*, Adv. Math. **184** (2004), no. 2, 207–267.

[KS06] O. Kozlovski, S. van Strien, *Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials*, Proc. Lond. Math. Soc. (3), **99** (2009), 275–296.

[LP96] G. Levin, F. Przytycki, *External rays to periodic points,* Israel J. Math. **94** (1996), 29–57.

[Moo62] R. L. Moore, *Foundations of point set theory. Revised edition*, AMS Colloquium Publications **13** (1962), AMS, Providence, R.I.

[Mun00] J. Munkres, *Topology*, 2nd edition, Prentice Hall (2000).

[Nad92] Sam B. Nadler, Jr., *Continuum theory*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker Inc., New York (1992).

[QY06] W. Qiu, Y. Yin, *Proof of the Branner-Hubbard conjecture on Cantor Julia sets,* Sci. China Ser. A, **52** (2009), no. 1, 45–65

[Sto56] S. Stoilow, *Leçons sur les principes topologique de la theorie des fonctions analytique*, 2nd edition, Paris (1956).

(A. Blokh) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170

E-mail address, A. Blokh: `ablokh@math.uab.edu`

(C. Curry) Institute for Mathematical Sciences, Stony Brook University, Stony Brook, NY 11794

E-mail address, C. Curry: `clintonc@math.sunysb.edu`

(L. Oversteegen) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170

E-mail address, L. Oversteegen: `overstee@math.uab.edu`