A VOLUMETRIC PENROSE INEQUALITY FOR CONFORMALLY FLAT MANIFOLDS

FERNANDO SCHWARTZ

Abstract. We consider asymptotically flat Riemannian manifolds with nonnegative scalar curvature that are conformal to $\mathbb{R}^n \setminus \Omega$, $n \geq 3$, and so that their boundary is a minimal hypersurface. (Here, $\Omega \subset \mathbb{R}^n$ is open bounded with smooth mean-convex boundary.) We prove that the ADM mass of any such manifold is bounded below by $\frac{1}{2}\left(V/\beta_n\right)^{(n-2)/n}$, where V is the Euclidean volume of Ω and β_n is the volume of the Euclidean unit n-ball. Surprisingly, we do not require the boundary to be outermost.

1. Introduction

One of the major results in differential geometry is the positive mass inequality, which asserts that any asymptotically flat Riemannian manifold M with nonnegative scalar curvature has nonnegative ADM mass. Furthermore, the inequality is rigid, in that the ADM mass is strictly positive unless M is isometric to the Euclidean space $\mathbb{R}^n$. This inequality was proved in 1979 by Schoen and Yau [13] for manifolds of dimension $n \leq 7$ using minimal surface techniques. Witten [16] subsequently found a different argument based on spinors and the Dirac operator. (See also [2] and [11].) Witten's argument works for any spin manifold M, without any restrictions on the dimension.

A refinement to the positive mass inequality in the case when black holes are present is the Riemannian Penrose inequality. It asserts that any asymptotically flat manifold M with nonnegative scalar curvature containing an outermost minimal hypersurface of area A has ADM mass m that satisfies

$$m \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{RPI}$$

where ω_{n-1} is the area of the $(n-1)$-sphere $\mathbb{S}^{n-1}$. This inequality is also rigid, in that it is strict unless M is isometric to the Riemannian Schwarzschild manifold.[1] This inequality was first proved in three dimensions in 1997 by Huisken and Ilmanen [9] for the case of a single black hole. In 1999, Bray [3] extended this result, still in dimension three, to the general case of multiple black holes using a different technique. Later, Bray and Lee [5] generalized Bray's proof for dimensions $n \leq 7$, with the extra requirement that M be spin for the rigidity statement.

A special situation arises if we restrict ourselves to the case of *conformally flat* manifolds. There, the proof of the positive mass theorem follows from Green's formula. In view of this, Bray and Iga suggest in [4] that there should be a proof of the RPI for conformally flat manifolds that uses only properties of superharmonic

[1] See the beginning of §3 for the precise definition.

functions in $\mathbb{R}^n$. Moreover, in that work Bray and Iga proved the RPI with sub-optimal constant $c < 1$, on manifolds conformal to the flat metric on $\mathbb{R}^3$ minus the origin. Specifically, they proved that $m \geq c\sqrt{A/16\pi}$ where A is the infimum of the areas of all surfaces enclosing the origin.

In this paper we prove what we call a "volumetric" Penrose inequality for conformally flat manifolds using elementary techniques. An important point to note concerning our inequality is that it gives a weaker bound than the RPI in the cases where the latter is applicable. (See Remark 16 below.) On the other hand, our Theorem works in all dimensions $n \geq 3$. This is particularly interesting in dimensions 8 and above, where no such results (asides from spherically symmetric manifolds) were known to exist.[2]

The precise statement of our main theorem is the following.

Theorem 1. *Suppose that (M^n, g), $n \geq 3$, is an asymptotically flat n-dimensional manifold with nonnegative scalar curvature which is isometric to $(\mathbb{R}^n \backslash \Omega, u^{4/(n-2)}\delta_{ij})$, where $\Omega \subset \mathbb{R}^n$ is an open bounded set with smooth mean-convex boundary (i.e. having positive mean curvature), and u is normalized so that $u \to 1$ towards infinity. If the boundary of M is a minimal hypersurface, then*

$$m \geq \frac{1}{2}\left(\frac{V}{\beta_n}\right)^{\frac{n-2}{n}}, \tag{1}$$

where m is the ADM mass of (M, g), V is the volume of Ω with respect to the Euclidean metric, and β_n is the volume of the Euclidean unit n-ball.

The requirements that (M, g) be conformally flat and that the boundary of Ω have positive mean curvature appears to be quite stringent, but not so much from a topological point of view. For example, the manifolds we constructed in [14], which are the only known asymptotically flat manifolds with nonnegative scalar curvature having outermost minimal hypersurfaces which are not topological spheres,[3] are all conformally flat, and their respective Ω's have mean-convex boundary. Actually, since we do not require the boundary of M to be an *outermost* minimal hypersurface, there are many topologically-inequivalent examples of manifolds which satisfy the hypotheses of our theorem. Indeed, from the construction of [14] it follows fairly easily that one can find examples of scalar flat, asymptotically flat manifolds having minimal boundary which is, topologically, the boundary of any given handlebody in $\mathbb{R}^n$. Using appropriate scalings these can be made mean-convex as well.

Remark 2. For the special case of a Schwarzschild metric, the RHS of inequality (1) differs by a factor of 4 from being optimal.

Remark 3. Our theorem does not require that the boundary of M be outermost, in contrast with the standard RPI[4]. This may seem odd at first. Nevertheless, since a non-outermost minimal hypersurface bounds a domain that is contained in the

[2] At the time of submission the author found out that Lam, a student of Bray, has proved the positive mass inequality for graphs of asymptotically flat functions over $\mathbb{R}^n$, and the Riemannian Penrose inequality for graphs on $\mathbb{R}^n$ with convex boundaries, all this for $n \geq 3$.

[3] The outermost minimal hypersurfaces are, topologically, a product of spheres.

[4] This assumption is necessary in the RPI, for it is well known that counterexamples may be obtained by taking spherically symmetric metrics with fixed mass and arbitrarily large minimal, but not outermost, boundary, like in p. 358 of [9].

domain that the outermost minimal hypersurface bounds, inequality (1) only gives a weaker bound when applied to a non-outermost minimal boundary compared to it being applied to the external region of the outermost minimal hypersurface. Also, notice that we need to impose $u \to 1$ towards infinity to get rid of would-be counterexamples where the volume of Ω can be made arbitrarily large maintaining the mass bounded.

Outline of the proof. We first extend a theorem of Bray using Witten's positive mass theorem and obtain a lower bound for the ADM mass of (M, g) in terms of the capacity of its boundary. We then focus on finding an estimate for the capacity of the boundary. It turns out that, in the conformally flat case, this can be done using a spherical symmetrization trick, so long as we can find appropriate bounds for the conformal factor.

We should mention that Bray and Miao also exploit the relationship between mass and capacity in [6], but their estimates go in the opposite direction. In their beautiful work they find upper bunds for the capacity of surfaces in terms of the Hawking mass, all this inside asymptotically flat three dimensional manifolds with nonnegative scalar curvature. The proof of their main theorem relies on the monotonicity of the Hawking mass along inverse mean curvature flow; this is known to work only in dimension three. Their result was inspired by an earlier result of Bray and Neves [7], where similar techniques were used for computing Yamabe invariants.

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2. Preliminaries

We begin by recalling some classical facts about spherical symmetrization in $\mathbb{R}^n$.

Definition 4. Let u be a function in $W^{1,p}(\mathbb{R}^n)$. Its *spherical symmetrization*, $u^*(x) \equiv u^*(|x|)$, is the unique radially symmetric function on $\mathbb{R}^n$ which is decreasing on $|x|$, and so that the Lebesgue measure of the super-level sets of u^* equals the Lebesgue measure of the super-level sets of u. More precisely, u^* is defined as the unique decreasing spherically symmetric function on $\mathbb{R}^n$ so that $\mu\{u \geq K\} = \mu\{u^* \geq K\}$ for all $K \in \mathbb{R}$.

The following result is a classical theorem in analysis which can be traced back to a principle used by Pólya and Szegö [12]. (See also [15], [8].)

Symmetrization Theorem ([12]). *Spherical symmetrization preserves L^p norms and decreases $W^{1,p}$ norms.*

We need the above result for a calculation inside the proof of the main theorem. We now introduce the notion of ADM mass and capacity of asymptotically flat manifolds, and give some results of Bray and Witten concerning these quantities.

Definition 5. Let $n \geq 3$. A Riemannian manifold (M^n, g) is said to be *asymptotically flat* if there is a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_1(0)$, and in this coordinate chart the metric g_{ij} satisfies

$$g_{ij} = \delta_{ij} + O(|x|^{-p}),\ g_{ij,k} = O(|x|^{-p-1}),\ g_{ij,kl} = O(|x|^{-p-2}),\ R_g = O(|x|^{-q}),$$

for some $p > (n-2)/2$ and some $q > n$, where the commas denote partial derivatives in the coordinate chart, and R_g is the scalar curvature of g.

For an asymptotically flat manifold (M, g), it is well known that the limit

$$m(g) = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu_j dA$$

exists, where ω_{n-1} is the area of the standard unit $(n-1)$-sphere, S_r is the coordinate sphere of radius r, ν is its outward unit normal, and dA is the Euclidean area element on S_r .

Definition 6. The quantity $m = m(g)$ from above is called the *ADM mass* of (M^n, g).

This notion of mass was first considered by Arnowitt, Deser, and Misner in [1]. Later, Bartnik showed that the ADM mass is a Riemannian invariant, independent of choice of asymptotically flat coordinates, cf. Section 4 of [2].

Definition 7. The *capacity* of the boundary Σ of a complete, asymptotically flat manifold (M^n, g) is

$$\mathcal{C}(\Sigma, g) = \inf\left\{\frac{1}{\omega_{n-1}}\int_M |\nabla\varphi|^2 dV\right\},$$

where the infimum is taken over all smooth $0 \leq \varphi(x) \leq 1$ which go to zero at infinity and equal to one on the boundary Σ.

Remark 8. The above definition of capacity differs slightly from the standard definition of capacity. We ask that the functions considered in the infimum satisfy the extra hypothesis $0 \leq \varphi(x) \leq 1$, which is required for the proof of Lemma 13. Nevertheless, this extra assumption does not affect the outcome of the infimum, since with or without it the infimum is attained by a positive harmonic function no greater than one. (Cf. equation (86) of [3].)

The following theorem of Bray is central to our purposes since it establishes a relationship between mass and capacity.

Bray's Theorem ([3])**.** *Let (M^n, g), $n \geq 3$ be an asymptotically flat manifold which is spin or has dimension less than 8, and so that it has nonnegative scalar curvature and minimal boundary Σ. Let m be its ADM mass. Then*

$$m \geq \frac{1}{2}\mathcal{C}(\Sigma, g),$$

with equality if and only if (M^n, g) is a Riemannian Schwarzschild manifold[5] *outside its outermost minimal hypersurface Σ.*

[5]For the precise definition of the Riemannian Schwarzschild manifold see the beginning of Section 3.

Remark 9. Bray's original version of the above theorem, which is Theorem 9 of [3], does not include the case of M being spin, but for our purposes this is a natural assumption. It is easy to see that a slight modification of Bray's proof, using Witten's positive mass theorem (included below for completeness) whenever necessary, gives a proof of the above statement.

Witten's Positive Mass Theorem ([16])**.** *Let (M^n, g), $n \geq 3$ be an asymptotically flat spin manifold with nonnegative scalar curvature which has multiple asymptotically flat ends and total mass m in a chosen end. Then*

$$m \geq 0,$$

with equality if and only if (M^n, g) is isometric to $(\mathbb{R}^n, \delta_{ij})$.

Finally, we cite a quick fact about spin geometry that we will use in the proof of the main theorem. This may be found e.g. in Michelsohn and Lawson's book [10].

Lemma 10 ([10])**.** *Let M be diffeomorphic to $\mathbb{R}^n \setminus \Omega$, where Ω is an open subset of $\mathbb{R}^n$ with smooth boundary. Then M is spin.*

3. Proof of Theorem 1

Throughout this section we will be using three different metrics:

(i) the Euclidean metric δ_{ij} on $\mathbb{R}^n$,
(ii) the conformally flay metric of (M, g) given by $g = u^{4/(n-2)}\delta_{ij}$, where $u > 0$ is a smooth function defined on $\mathbb{R}^n \setminus \Omega$,
(iii) the Riemannian Schwarzschild metric on $\mathbb{R}^n$ minus the origin:

$$s = \left(1 + \frac{m}{2}|x|^{2-n}\right)^{4/(n-2)} \delta_{ij},$$

where $|x|$ is the Euclidean norm of x, and m will be determined later.

Standard quantities depending on the metric, like covariant derivatives, volume forms, norms and so on, will be denoted by, respectively,

(i) $\nabla_0,\ dV_0,\ |\cdot|_0,$
(ii) $\nabla_g,\ dV_g,\ |\cdot|_g,$
(iii) $\nabla_s,\ dV_s,\ |\cdot|_s.$

We begin by proving an estimate for the conformal factor u, which is of independent interest. (Here is where we need that the boundary of Ω be mean-convex.)

Lemma 11. *Suppose that (M^n, g) is an asymptotically flat n-dimensional manifold with nonnegative scalar curvature which is isometric to $(\mathbb{R}^n \setminus \Omega, u^{4/(n-2)}\delta_{ij})$, where $\emptyset \neq \Omega \subset \mathbb{R}^n$ is an open bounded set with smooth mean-convex boundary. Assume that the boundary of M is minimal, and that u is normalized so that $u \to 1$ towards infinity. Then $u \geq 1$ on M.*

Proof. Recall that the transformation law for the scalar curvature under conformal changes of the metric is given by $R_g = \frac{4(n-1)}{n-2}u^{-(n+2)/(n-2)}(-\Delta_0 + \frac{n-2}{4(n-1)}R_0)u$, where Δ_0 is the Euclidean Laplacian and R_0 is the Euclidean scalar curvature,

namely $R_0 \equiv 0$. Since we assume that $R_g \geq 0$, it follows that u is superharmonic on M. Therefore, u achieves its minimum value at either infinity or at the boundary $\partial\Omega$. At infinity u goes to one. We now show that at the boundary it does not achieve its minimum, and so it must be everywhere greater or equal than one.
Claim. u does not achieve its minimum on the boundary of M.
From hypothesis, the boundary of M is a minimal hypersurface. This is, the mean curvature of the boundary of M is zero with respect to the metric $g = u^{4/(n-2)}\delta_{ij}$. Now, the transformation law for the mean curvature under the conformal change of the metric $g = u^{4/(n-2)}\delta_{ij}$ is given by $h_g = \frac{2}{n-2}u^{-n/(n-2)}(\partial_\nu + \frac{(n-2)}{2}h_0)u$, where h_0 is the Euclidean mean curvature and ν is the outward-pointing normal. Since we have assumed that the boundary of Ω is mean convex, i.e. that $h_0 > 0$, it follows that $\partial_\nu u < 0$ on the boundary of Ω. Thus, u may not achieve its minimum on the boundary of M. This proves the claim and the Lemma follows. □

We now bring spherical symmetrization into the picture. Suppose that $0 \leq \varphi \leq 1$ is a smooth function on $\mathbb{R}^n \setminus \Omega$ which is exactly 1 on the boundary $\partial\Omega$ and converges to 0 at infinity. We may extend this function to a function $\tilde{\varphi}$ defined on all of $\mathbb{R}^n$, given by

$$\tilde{\varphi} = \begin{cases} 1 & \text{in } \Omega, \\ \varphi & \text{outside } \Omega. \end{cases}$$

(Notice that $\tilde{\varphi}$ is Lipschitz.) Now consider $(\tilde{\varphi})^*$, the spherical symmetrization of $\tilde{\varphi}$, which is defined on all of $\mathbb{R}^n$. (See Definition 4.)

Definition 12. Let φ be as above, and let V be the Euclidean volume of Ω. We define φ^* to be the restriction of $(\tilde{\varphi})^*$ to $\mathbb{R}^n \setminus B_R(0)$, where $R = R(V)$ is the radius of the Euclidean ball of volume V, namely $R = (V/\beta_n)^{1/n}$.

Lemma 13. *Let φ be as above. Then*

$$\int_M |\nabla_0\varphi|_0^2 dV_0 \geq \int_{\mathbb{R}^n \setminus B_R(0)} |\nabla_0\varphi^*|_0^2 dV_0.$$

Proof. Recall that $M = \mathbb{R}^n \setminus \Omega$. Since $\tilde{\varphi}$ is Lipschitz and is constant inside Ω, it follows that $\int_{\mathbb{R}^n\setminus\Omega} |\nabla_0\varphi|_0^2 dV_0 = \int_{\mathbb{R}^n} |\nabla_0\tilde{\varphi}|_0^2 dV_0$. From the Symmetrization Theorem applied to $\tilde{\varphi}$, we obtain that $\int_{\mathbb{R}^n} |\nabla_0\tilde{\varphi}|_0^2 dV_0 \geq \int_{\mathbb{R}^n} |\nabla_0(\tilde{\varphi})^*|_0^2 dV_0$. But since $0 \leq \tilde{\varphi} \leq 1$ is constant and equal to one on Ω, it follows that $(\tilde{\varphi})^*$ is also constant and equal to one on the ball $B_R(0)$, where $R = (V/\beta_n)^{1/n}$ and V is the Euclidean volume of Ω. This way, $\int_{\mathbb{R}^n} |\nabla_0(\tilde{\varphi})^*|_0^2 dV_0 = \int_{\mathbb{R}^n\setminus B_R(0)} |\nabla_0\varphi^*|_0^2 dV_0$. Putting this inequalities together gives a proof of the lemma. □

Lemma 14. *Let $g = u^{4/(n-2)}\delta_{ij}$. We have that*

(i) $|\nabla_g\varphi|_g^2 = u^{-4/(n-2)}|\nabla_0\varphi|_0^2$,
(ii) $dV_g = u^{2n/(n-2)}dV_0$.

Proof. Straightforward calculation. □

We now prove the main proposition in this section.

Proposition 15. *Let (M, g) be as in Theorem 1 and consider a smooth function $0 \leq \varphi \leq 1$ on M so that $\varphi = 1$ on ∂M and $\varphi \to 0$ towards infinity. Then*

$$\int_M |\nabla_g\varphi|_g^2 dV_g \geq \frac{1}{4}\int_{\mathbb{R}^n\setminus B_R(0)} |\nabla_s\varphi^*|_s^2 dV_s,$$

where $R = (V/\beta_n)^{1/n}$, V is the Euclidean volume of Ω, and s is the Schwarzschild metric of mass $m = 2R^{n-2}$.

Proof. Using Lemma 14 we obtain

$$\int_M |\nabla_g \varphi|_g^2 dV_g = \int_M u^{-4/(n-2)} |\nabla_0 \varphi|_0^2 u^{2n/(n-2)} dV_0 = \int_M u^2 |\nabla_0 \varphi|_0^2 dV_0$$
$$\geq (\inf_M u^2) \int_M |\nabla_0 \varphi|_0^2 dV_0,$$

but $u \geq 1$ by Lemma 11; this together with Lemma 13 gives

$$\geq \int_M |\nabla_0 \varphi|_0^2 dV_0 \geq \int_{\mathbb{R}^n \setminus B_R} |\nabla_0 \varphi^*|_0^2 dV_0.$$

Now, the Schwarzschild metric of mass $m = 2R^{n-2}$ has its unique minimal hypersurface at the coordinate sphere of radius R. Also, the Schwarzschild conformal factor $u_s = 1 + \frac{m}{2}|x|^{2-n}$ is bounded between 1 and 2 outside the minimal sphere. (It is exactly equal to 2 on the minimal sphere, and decreases to 1 towards infinity.) This, together with Lemmas 13 and 14 gives

$$\int_{\mathbb{R}^n \setminus B_R} |\nabla_0 \varphi^*|_0^2 dV_0 = \int_{\mathbb{R}^n \setminus B_R} u_s^{\frac{4}{n-2}} |\nabla_s \varphi^*|_s^2 u_s^{\frac{-2n}{n-2}} dV_s = \int_{\mathbb{R}^n \setminus B_R} u_s^{-2} |\nabla_s \varphi^*|_s^2 dV_s$$
$$\geq \frac{1}{4} \int_{\mathbb{R}^n \setminus B_R} |\nabla_s \varphi^*|_s^2 dV_s.$$

This ends the proof of the proposition. □

Proof of Theorem 1. M is spin from Lemma 10. Thus, we may apply Bray's Theorem and obtain that $m(g) \geq \frac{1}{2}\mathcal{C}(\Sigma, g)$. From Proposition 15 it follows that $\mathcal{C}(\Sigma, g) \geq \frac{1}{4}\mathcal{C}(S_R, s)$ where S_R is the Euclidean $(n-1)$-sphere of radius R inside $\mathbb{R}^n$, and s is the Schwarzschild metric like in the proof above. We deduce that $m(g) \geq \frac{1}{8}\mathcal{C}(S_R, s)$.

On the other hand, the rigidity statement of Bray's Theorem gives the explicit value of $\mathcal{C}(S_R, s)$, namely that $m(s) = \frac{1}{2}\mathcal{C}(S_R, s) = 2R^{n-2} = 2(V/\beta_n)^{(n-2)/n}$. From this it follows that

$$m(g) \geq \frac{1}{2}\left(\frac{V}{\beta_n}\right)^{\frac{n-2}{n}},$$

as desired. □

Remark 16. We now check that the volumetric Penrose inequality is weaker than the RPI. Let A_0, A_g denote the area of the boundary of M with respect to the Euclidean metric δ_{ij} and the metric g, respectively. Also let V denote the volume of Ω with respect to the Euclidean metric as before. From Lemma 11, it follows that $A_g > A_0$, since $u > 1$ on the boundary. This way,

$$\frac{1}{2}\left(\frac{V}{\beta_n}\right)^{\frac{n-2}{n}} = \left(\frac{V}{A_0^{\frac{n}{n-1}}}\right)^{\frac{n-2}{n}} \left(\frac{\omega_{n-1}^{\frac{n}{n-1}}}{\beta_n}\right)^{\frac{n-2}{n}} \frac{1}{2}\left(\frac{A_0}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}$$
$$< \left[I(\Omega)^{\frac{n-2}{n}}\right] \frac{1}{2}\left(\frac{A_g}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

where $I(\Omega)$ is the isoperimetric ratio of Ω given by $I(\Omega) = \left(\frac{V}{A_0^{\frac{n}{n-1}}}\right)\left(\frac{\beta_n}{\omega_{n-1}^{\frac{n}{n-1}}}\right)^{-1}$.
Clearly $I(\Omega) \leq 1$, so it follows that the volumetric Penrose inequality gives a strictly weaker lower bound for the mass compared to the RPI, provided the latter holds.

References

[1] R. Arnowitt, S. Deser, and C. W. Misner, *Coordinate invariance and energy expressions in general relativity.*, Phys. Rev. (2) **122** (1961), 997–1006. MR0127946 (23 #B991)

[2] Robert Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693, DOI 10.1002/cpa.3160390505. MR849427 (88b:58144)

[3] Hubert L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267.

[4] Hubert L. Bray and Kevin Iga, *Superharmonic functions in* $\mathbf{R}^n$ *and the Penrose inequality in general relativity*, Comm. Anal. Geom. **10** (2002), no. 5, 999–1016. MR1957660 (2003k:83066)

[5] Hubert L. Bray and Dan A. Lee, *On the Riemannian Penrose inequality in dimensions less than eight*, Duke Math. J. **148** (2009), no. 1, 81–106. MR2515101

[6] Hubert Bray and Pengzi Miao, *On the capacity of surfaces in manifolds with nonnegative scalar curvature*, Invent. Math. **172** (2008), no. 3, 459–475, DOI 10.1007/s00222-007-0102-x. MR2393076 (2009h:53071)

[7] Hubert L. Bray and André Neves, *Classification of prime 3-manifolds with Yamabe invariant greater than* $\mathbb{RP}^3$, Ann. of Math. (2) **159** (2004), no. 2, 407–424. MR2052359

[8] Keijo Hildén, *Symmetrization of functions in Sobolev spaces and the isoperimetric inequality*, Manuscripta Math. **18** (1976), no. 3, 215–235. MR0409773 (53 #13525)

[9] Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437. MR1916951 (2003h:53091)

[10] H. Blaine Lawson Jr. and Marie-Louise Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR1031992 (91g:53001)

[11] Thomas Parker and Clifford Henry Taubes, *On Witten's proof of the positive energy theorem*, Comm. Math. Phys. **84** (1982), no. 2, 223–238. MR661134 (83m:83020)

[12] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J., 1951. MR0043486 (13,270d)

[13] Richard Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR526976 (80j:83024)

[14] Fernando Schwartz, *Existence of outermost apparent horizons with product of spheres topology*, Comm. Anal. Geom. **16** (2008), no. 4, 799–817. MR2471370 (2009m:53100)

[15] Giorgio Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372. MR0463908 (57 #3846)

[16] Edward Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402. MR626707 (83e:83035)

Department of Mathematics, University of Tennessee, Knoxville, USA
E-mail address: fernando@math.utk.edu