LIMIT SETS OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. In this paper, we prove a limit set intersection theorem in relatively hyperbolic groups. We also show that a nonparabolic relatively quasiconvex subgroup cannot contain a proper conjugate of itself. Several well-known results on limit sets of geometrically finite Kleinian groups are derived in relatively hyperbolic groups. Lastly, we establish the dynamical quasiconvexity for undistorted subgroups of finitely generated groups with nontrivial Floyd boundary.

1. Introduction

The purpose of this paper is to establish a limit set intersection theorem in relatively hyperbolic groups. Following Anderson [3], a limit set intersection theorem for convergence groups describes the limit set $\Lambda(H \cap J)$ in terms of $\Lambda(H)$ and $\Lambda(J)$, where H, J are subgroups of a convergence group G. Ideally, we expect such a theorem has the following form

$$\Lambda(H) \cap \Lambda(J) = \Lambda(H \cap J) \cup E$$

where E is an exceptional set consisting of specific parabolic points of $\Lambda(H)$ and $\Lambda(J)$.

Such a limit set intersection theorem has been investigated in several different groups. In 1992, Susskind-Swarup [22] showed that the above decomposition of limit sets holds for pair of geometrically finite Kleinian subgroups. In [1], [2] and [3], using techniques specific to 3 manifolds, Anderson carried out a systematic study of the intersection of two finitely generated subgroups of 3 dimensional Kleinian groups and proved that the limit set intersection theorem holds in this context.

In 1987, Gromov introduced relatively hyperbolic groups as generalizations of many naturally occurred groups, for example, word hyperbolic groups and geometrically finite Kleinian groups, etc. In word hyperbolic groups, the above limit set intersection theorem has appeared in the work of Gitik-Mitra-Rips-Sageev [14], where the exceptional set E is proven to be empty. In this paper, we generalize these results in relatively hyperbolic groups as follows.

Theorem 1.1. *Let H, J be two relatively quasiconvex subgroups of a relatively hyperbolic group G. Then*

$$\Lambda(H) \cap \Lambda(J) = \Lambda(H \cap J) \bigsqcup E$$

2000 *Mathematics Subject Classification.* Primary 20F65, 20F67.

Key words and phrases. dynamical quasiconvexity, relative hyperbolicity, limit sets, Floyd boundary, undistorted subgroups.

The author is supported by the China-funded Postgraduates Studying Aboard Program for Building Top University.

where the exceptional set E consists of parabolic fixed points of $\Lambda(H)$ and $\Lambda(J)$, whose stabilizer subgroups in H and J have finite intersection. Equivalently, the set E comprises the limit points isolated in $\Lambda(H) \cap \Lambda(J)$.

Remark 1.2. In [7], Dahmani previously proved Theorem 1.1 for two fully quasiconvex subgroups, where the exceptional set E is empty.

One of its corollaries is the following well-known result, which is usually proved using geometrical methods, see for example, by Hruska [15] and independently by Martinez-Pedroza [19].

Corollary 1.3. *Let H, J be two relatively quasiconvex subgroups of a relatively hyperbolic group G. Then $H \cap J$ is relatively quasiconvex.*

We also study the conjugates of relatively quasiconvex subgroups and generalize a theorem of Mihalik-Towle [18] in the context of relatively hyperbolic groups.

Theorem 1.4. *Let H be relatively quasiconvex in a relatively hyperbolic group G and $|\Lambda(H)| \geq 2$. Then for any $g \in G \setminus H$, $gHg^{-1} \subseteq H$ implies that $gHg^{-1} = H$.*

In this paper, we take a dynamical approach to study relatively quasionvex subgroups, which is the main ingredient to establish the above results. This dynamical quasiconvexity was introduced by Bowditch [4] and recently proved by Gerasimov-Potyagailo [12] to be equivalent to relative quasiconvexity.

In the final section, we define dynamical quasiconvexity for subgroups of general convergence groups and in particular study this property in finitely generated groups with nontrivial Floyd boundary. Floyd boundary was introduced by Floyd [9] to compactify the Cayley graph of finitely generated groups, and later Karlsson [17] proved that G acts on its Floyd boundary as a convergence group when the Floyd boundary is nontrivial. Our last result is to establish the dynamical quasiconvexity of undistorted subgroups under this convergence action.

Theorem 1.5. *If H is a undistorted subgroup of a finitely generated group G with nontrivial Floyd boundary, then H is dynamical quasiconvex.*

Remark 1.6. The class of groups with nontrivial Floyd boundary includes non-elementary finitely-generated relatively hyperbolic groups [11].

Lastly let us mention a connection of Theorem 1.5 with the following conjecture due to Olshanskii-Osin-Sapir [20].

Conjecture 1.7. *Suppose that a finitely generated group G has a nontrivial Floyd boundary. Is G hyperbolic relative to a collection of proper subgroups?*

We remark that Hruska proved that undistorted subgroups of relatively hyperbolic groups are relatively quasiconvex [15, Theorem 1.5] and thus dynamical quasiconvex by Gerasimov-Potyagailo [12]. Therefore one can thought of Theorem 1.5 as another positive evidence towards the above conjecture.

The paper is organized as follows. In Section 2, we introduce the definition of relatively hyperbolicity and dynamical quasiconvexity. In particular, using dynamical quasiconvexity, several well-known results on limit sets of geometrically finite Kleinian groups and word hyperbolic groups are derived in the context of relative hyperbolicity. In Section 3, we study the intersection of conical limit points and bounded parabolic points of relatively quasiconvex subgroups respectively, and then conclude with the proof of Theorem 1.1. In Section 4, we prove that a nonparabolic relatively quasiconvex subgroups cannot contain a proper conjugate of itself. In the final Section, we give a proof of Theorem 1.5 and restate several corollaries established in previous Sections in the setting of finitely generated groups with nontrivial Floyd boundary.

Acknowledgment. The author would like to sincerely thank Prof. Leonid Potyagailo for many helpful comments and inspired discussions during the course of this work. The author also thanks G. Hruska for pointing out several inaccuracies of references.

2. Preliminary results

Throughout the paper, G is a finitely generated group with a finite collection of subgroups $\mathbb{P} = \{P_1, P_2, ..., P_n\}$. Let M be a compact metrizable space containing at least three points.

A *convergence group action* is an action of a group G on M such that induced action of G on the space ΘM of distinct unordered triples of points of M is properly discontinuous.

Suppose G has a convergence group action on M. Then M is partitioned into a limit set $\Lambda(G)$ and discontinuous domain $M \setminus \Lambda(G)$. The *limit set* $\Lambda(G)$ of G is the set of limit points, where a *limit point* is an accumulation point of some $G-$orbit in M. An element $g \in G$ is *elliptic* if it has finite order. An element $g \in G$ is *parabolic* if it has infinite order and fixes exactly one point of M. An element $g \in G$ is *loxodromic* if it has infinite order and fixes exactly two points of M. An infinite subgroup $P \subset G$ is a *parabolic subgroup* if it contains no loxodromic element. A parabolic subgroup P has a unique fixed point in M, called a *parabolic point*. The stabilizer of a parabolic point is always a maximal parabolic group. A parabolic point p with stabilizer $G_p := Stab_G(p)$ is *bounded* if G_p acts cocompactly on $M \setminus \{p\}$. A point $z \in M$ is a *conical limit* point if there exists a sequence (g_i) in G and distinct points $a, b \in M$ such that $g_i(z) \to a$, while for all $q \in M \setminus \{z\}$ we have $g_i(q) \to b$.

Before discussing relative hyperbolicity, we recall the following well-known result on general convergence groups.

Lemma 2.1. [23, Theorem 3.A] *In a convergence group, a conical limit point can not be parabolic.*

In the literature, there are several definitions of relative hyperbolicty (see Farb [8], Bowditch [4] and Osin [21]). These different definitions are now proven to be equivalent(see Hruska [15] for a complete account) and provide convenient and complement viewpoints to study this class of groups. For the sake of the purpose of this paper, we use the following dynamical formulation of relatively hyperbolic groups.

Definition 2.2. A convergence group action of G on M is *geometrically finite* if every point of M is either a conical limit point or a bounded parabolic point. In addition if $\mathbb{P}$ is a set of representatives of the conjugacy classes of maximal parabolic subgroups, then we say the pair (G, $\mathbb{P}$) is *relatively hyperbolic.*

Remark 2.3. In this paper, we consider a finitely generated relatively hyperbolic group. By the work of Drutu-Sapir [6], Osin [21] and Gerasimov [10], maximal parabolic subgroups are quasiconvex and finitely generated. So in the definition of relative hyperbolicity, we do not need impose the "finitely generated" condition on maximal parabolic subgroups as usually do in Bowditch [4].

From now on, unless explicitly stated, G is always assumed to be relatively hyperbolic untill the end of Section 4.

The following dynamical convexity was introduced by Bowditch and was proven to be equivalent to the geometric quasiconvexity in word hyperbolic groups [5].

Definition 2.4. A subgroup $H \subset G$ is *dynamically quasiconvex* if the following set

$$\{gH \in G/H : g\Lambda(H) \cap K \neq \emptyset \text{ and } g\Lambda(H) \cap L \neq \emptyset\}$$

is finite, whenever K and L are disjoint closed subsets of M.

Recently, Gerasimov-Potyagailo [12] proved that dynamical quasiconvexity coincides with relatively quasiconvexity in relatively hyperbolic groups, which answers a question of Osin in his book [21].

Theorem 2.5. [12] *Every subgroup H of G is dynamically quasiconvex if and only if it is relatively quasiconvex.*

In this paper, the dynamical quasiconvexity is pretty helpful when one deals with limit sets of relatively quasiconvex subgroups. Now let's first draw some direct consequences of dynamical quasiconvexity.

Lemma 2.6. *Let H be relatively quasiconvex in G and $|\Lambda(H)| \geq 2$. Then for any subgroup $H \subset J \subset G$ such that $\Lambda(H) = \Lambda(J)$, we have H is of finite index in J. In particular, J is relatively quasiconvex.*

Proof. Since $|\Lambda(H)| \geq 2$, we can pick two distinct points x and y from $\Lambda(H)$. Applying the dynamical quasiconvexity of H, we have each coset of H in J belongs to the following finite set

$$\{gH \in G/H : g\Lambda(H) \cap \{x\} \neq \emptyset \text{ and } g\Lambda(H) \cap \{y\} \neq \emptyset\},$$

using $\Lambda(H) = \Lambda(J)$. Thus H is of finite index in K.

Since H is relatively quasiconvex and of finite index in J, it is easy to see that the following wanted set in the definition of dynamical quasiconvexity

$$\{gJ \in G/J : g\Lambda(J) \cap \{x\} \neq \emptyset \text{ and } g\Lambda(J) \cap \{y\} \neq \emptyset\}$$

is finite and therefore it follows that J is also relatively quasiconvex. □

Corollary 2.7. *Let H, J be relatively quasiconvex in G and $\Lambda(H) = \Lambda(J)$. If $|\Lambda(H)| \geq 2$, then H and J are commensurable.*

Proof. Taking L as the stabilizer of common limit sets of H and J, we obtain that H and J are of finite index in L using Lemma 2.6. It thus follows that $H \cap J$ is of finite index in both H and J. □

Recall that the *commensurator* of H in G is defined as the subgroup of G, consisting of all $g \in G$ such that $H \cap gHg^{-1}$ has finite index in both H and gHg^{-1}.

Corollary 2.8. *Let H be relatively quasiconvex in G and $|\Lambda(H)| \geq 2$. Then H is of finite index in its commensurator. In particular, H is of finite index in its normalizer.*

Proof. Observe that the commensurater of H has the same limit set as $\Lambda(H)$. The conclusion now follows from Lemma 2.6. □

Remark 2.9. Corollary 2.8 has been proved using different method in Hruska-Wise [16]. We remark the hypothesis on the cardinality of $\Lambda(H)$ is necessary for the above lemma and corollaries, as it is easy to get counterexamples when we take H as parabolic subgroups.

3. Limit Sets of Intersections

In this section, we establish the limit set intersection theorem in the context of relatively hyperbolic groups, generalizing the results in geometrically finite Kleinian groups and word hyperbolic groups. For references on this topic, please see Anderson [1] [2] [3], Gitik-Mitra-Rips-Sageev [14] and Susskind-Swarup [22].

Proposition 3.1. *Let H be relatively quasiconvex in G. Suppose $J < G$ is infinite and let $z \in \Lambda(H) \cap \Lambda(J)$ be a conical limit point of J. Then $z \in \Lambda(H \cap J)$ and z is a conical limit point of $H \cap J$.*

Proof. Let $z \in \Lambda(K) \cap \Lambda(J)$ be a conical limit point of J. Then there exists a sequence $\{j_n\}$ in J and distinct points $a, b \in \Lambda(J)$ such that $j_n(z) \to a$, while for all $q \in \Lambda(J) \setminus \{z\}$ we have $j_n(q) \to b$. By the convergence property of $\{j_n\}$, we also have that $j_n(q) \to b$ for all $q \in M \setminus \{z\}$. In particular, we can choose q to be a limit point in $\Lambda(H) \setminus \{z\}$. Here we use the fact $|\Lambda(H)| \geq 2$, which follows from Lemma 2.1.

Take the closed neighborhoods U and V of a and b respectively, such that $U \cap V = \emptyset$. After passage to a subsequence of $\{j_n\}$, we can assume $j_n(z) \in U$ and $j_n(q) \in V$ for all n. This implies that $j_n H$ belongs to the following set for all n,

$$\{gH \in G/H : g\Lambda(H) \cap U \neq \emptyset \text{ and } g\Lambda(H) \cap V \neq \emptyset\}$$

By the dynamical quasiconvexity of H in G, the above set is finite. Thus $\{j_n H\}$ is a finite set of cosets. By taking further a subsequence of $\{j_n\}$, we suppose $j_n H = j_1 H$ for all n. We can write $j_n = j_1 h_n$ for each n, where $h_n \in H$. Then $j_1^{-1} j_n = h_n$ implies that $H \cap J$ is nontrivial and infinite.

It suffices now to prove that z is a conical limit point of $H \cap J$. By the convergence property of $\{j_n\}$, it follows that $h_n(z) = j_1^{-1} j_n(z) \to j_1^{-1}(a)$ and $h_n(q) = j_1^{-1} j_n(q) \to j_1^{-1}(b)$ for all $q \in M \setminus \{z\}$. Thus z is a conical limit point of $H \cap J$. □

We now study how bounded parabolic points intersect. Compared to conical points, the intersection of bounded parabolic points raises some complicated behaviour.

Proposition 3.2. *Let H, J be infinite subgroups of a countable convergence group G. If $z \in \Lambda(H) \cap \Lambda(J)$ be a bounded parabolic point of H and J, then z is either a bounded parabolic point of $H \cap J$, or an isolated point in $\Lambda(H) \cap \Lambda(J)$ and does not lie in $\Lambda(H \cap J)$.*

Proof. Since z is bounded parabolic of both H and J, there are compact subsets $K \subset \Lambda(H) \setminus z$ and $L \subset \Lambda(J) \setminus z$, such that $H_z K = \Lambda(H) \setminus z$ and $J_z L = \Lambda(J) \setminus z$. Here H_z and J_z are stabilizers of z in H and J, respectively. Let $P = H_z \cap J_z$.

We claim that there exists a compact subset $C \subset M \setminus z$ such that $\Lambda(H \cap J) \setminus z \subset PC$.

Note first that $\Lambda(H\cap J)\setminus z \subset (\Lambda(H)\cap\Lambda(J))\setminus z = H_zK\cap J_zL$. Therefore it suffices to show that there exists a compact subset $C \subset M\setminus z$ such that $H_zK\cap J_zL \subset PC$. Since G is countable, we formulate the following countable set

$$\mathcal{A} = \{h_nK\cap j_nL : (h_n, j_n) \in (H_z\setminus P)\times(J_z\setminus P); h_nK\cap j_nL \neq \emptyset\}. \tag{1}$$

We remark that it is possible that one set hK may have nontrivial intersections with two more sets j_1L and j_2L, but we count (h, j_1) and (h, j_2) differently in the above set $\mathcal{A}$. Note that $H_zK\cap J_zL \subset \cup\mathcal{A}$.

Define the set $\mathcal{B} = \{j_n^{-1}h_n : j_n^{-1}h_nK\cap L \neq \emptyset\}$ and we will show that $\mathcal{B}$ is finite. Suppose not. By the convergence property, the infinite set $\{j_n^{-1}h_n\}$ contains an infinite subsequence $\{j_{n_i}^{-1}h_{n_i}\}$, which converges locally compactly to b on $M\setminus a$, for $a, b \in M$. Since $j_n^{-1}h_n$ are parabolic elements in G, we have $a = b$ by using Lemma 2.5 in Bowditch [5]. Furthermore, it follows that $a = b = z$, since z is the unique fixed point of $j_n^{-1}h_n$. Observing that $K \subset M\setminus z$ and $L \subset M\setminus z$, and using $j_{n_i}^{-1}h_{n_i}K\cap L \neq \emptyset$, we can conclude that the subsequence $\{j_{n_i}^{-1}h_{n_i}\}$ is a finite set. This is impossible since we assumed $\mathcal{B}$ is an infinite set.

Since $\mathcal{B}$ is a finite set, say for example, $\{j_1^{-1}h_1, ..., j_r^{-1}h_r\}$. Without loss of generality, we first consider the elements in $\{j_n^{-1}h_n\}$ of the form $j_n^{-1}h_n = j_1^{-1}h_1$. Then $j_nj_1^{-1} = h_nh_1^{-1} \in H_z\cap J_z = P$, and we can write $j_n = p_nj_1$ and $h_n = p_nh_1$ for some $p_n \in P$. Thus it follows that $h_nK\cap j_nL = p_n(h_1K\cap j_1L)$ for each $j_n^{-1}h_n = j_1^{-1}h_1$.

We can do the rewrite process similarly for other elements in $\{j_n^{-1}h_n\}$, and finally we obtain $H_zK\cap J_zL \subset \cup\mathcal{A} \subset PC$, where C is defined as $\bigcup_{i=1}^{r}(h_iK\cap j_iL)$. The claim is proved.

Since there exists a compact subset $C \subset M$ such that the following holds

$$\Lambda(H\cap J)\setminus z \subset (\Lambda(H)\cap\Lambda(J))\setminus z \subset PC, \tag{2}$$

the following two cases are examined to complete the proof of proposition,

- P **is finite:** Since the righthand of (2) is a compact set, there exists an open neighborhood of z disjoint with $\Lambda(H)\cap\Lambda(J)$. Thus it follows that z is an isolated point of $\Lambda(H)\cap\Lambda(J)$ and does not lie in $\Lambda(H\cap J)$.
- P **is infinite:** P acts cocompactly on $\Lambda(H\cap J)\setminus z$ and thus z is a bounded parabolic point of $H\cap J$.

□

Summarizing the above results, we can now conclude with the proof of Theorem 1.1

Proof of Theorem 1.1. The limit set of relatively quasiconvex subgroup consists of conical limit points and bounded parabolic points (See (QC-1) definition of relatively quasiconvexity in Hruska [15]). Thus the above decomposition of $\Lambda(H)\cap\Lambda(J)$ follows from Propositions 3.1 and 3.2. □

Remark 3.3. In the word hyperbolic group(i.e.: without nontrivial peripheral subgroups), the exceptional set E is empty. Thus limit sets of two relatively quasiconvex subgroups intersect at least in two points once they intersects. But in the relative case, it is possible that their limit sets are intersecting at only one (necessarily parabolic) point. Here we just remind you of parabolic subgroups $< z+1 >$

and $< z+i >$ in 3 dimensional Kleinian group, which have a common parabolic point, but have trivial group intersection.

4. Proper Conjugates of Relatively Quasiconvex Subgroups

The aim of this section is to show that nonparabolic relatively quasiconvex subgroups cannot contain a proper conjugate of itself, which generalizes a theorem of Mihalik-Towle [18] in relatively hyperbolic groups.

According to [14], a relatively quasiconvex subgroup H is said to be *maximal* in its limit set if $H = stab(\Lambda(H))$. Note that Lemma 2.6 shows any nonparabolic relatively quasiconvex subgroup is of finite index in the stabilizer of its limit set.

Lemma 4.1. *Let H be relatively quasiconvex in G and suppose H is maximal in its limit set. Then for any $g \in G \setminus H$, $g\Lambda(H) \subsetneq \Lambda(H)$.*

Proof. If $|\Lambda(H)| = 1$, then H is the maximal parabolic subgroup. In this case, the conclusion is trivial. We consider now the case $|\Lambda(H)| \geq 2$. By way of contradiction, we suppose $g\Lambda(H) \subsetneq \Lambda(H)$.

Take two distinct points x and y from $\Lambda(H)$. Since $g\Lambda(H) \subsetneq \Lambda(H)$, we have $g^n(x) \in \Lambda(H)$ and $g^n(y) \in \Lambda(H)$ for each $n \in \mathbb{N}$. Therefore we have the cosets $g^{-n}H$ belong to following set

$$\{gH \in G/H : g\Lambda(H) \cap \{x\} \neq \emptyset \text{ and } g\Lambda(H) \cap \{x\} \neq \emptyset\}.$$

By using the dynamical quasiconvexity of H, we obtain the set $\{g^{-n}H\}$ is finite. Thus we can obtain two different integers m and n such that $g^{-m}H = g^{-n}H$, and thus $g^{n-m} \in H$. Then $\Lambda(H) = g^{n-m}\Lambda(H) \subseteq g\Lambda(H) \subsetneq \Lambda(H)$. Hence $\Lambda(H) = g\Lambda(H)$, which is impossible since H is maximal in its limit set $\Lambda(H)$. □

Remark 4.2. Lemma 4.1 generalizes Lemma 2.10 in Gitik-Mitra-Rips-Sageev [14].

We now prove Theorem 1.4.

Proof of Theorem 1.4. For any $g \in G \setminus H$ such that $gHg^{-1} \subsetneq H$, by Lemma 4.1, it follows that g belongs to the stabilizer K in G of $\Lambda(H)$. By Lemma 2.6, we obtain that H is of finite index in K. So g^n belongs to H for some n. Thus we have $H = g^nHg^{-n} < gHg^{-1} < H$, which finishes the proof. □

5. Undistorted Subgroups of Groups with Nontrivial Floyd Boundary

In this section, we define the dynamical quasiconvexity in general convergence groups and establish the dynamical quasiconvexity for undistorted subgroups of finitely generated groups with nontrivial Floyd boundary. Let's first introduce the dynamical quasiconvexity in the convergence group.

Definition 5.1. A subgroup H of a convergence group G is *dynamically quasiconvex* if the following set

$$\{gH \in G/H : g\Lambda(H) \cap K \neq \emptyset \text{ and } g\Lambda(H) \cap L \neq \emptyset\}$$

is finite, whenever K and L are disjoint closed subsets of M.

In the following, we consider a finitely generated group G with a fixed finite generating set S, and denote by $Ca(G,S)$ the corresponding constructed Cayley graph with respect to S. Note that $Ca(G,S)$ is a connected graph, which naturally induces a word metric d on G by setting the length of each edge to be 1. A rectifiable path p with endpoints denoted by p_-, p_+ in $Ca(G,S)$, whose length is denoted by $l(p)$, is called *ϵ-quasigeodesic* if for any subpath q of p, we have $l(q) < \epsilon d(q_-, q_+) + \epsilon$.

Recall that a (c-)quasi-isometric map $\phi : X \to Y$ between two metric spaces X and Y is a map such that:

$$\frac{1}{c} d_X(x,y) - c \le d_Y(\phi(x), \phi(y)) \le c d_X(x,y) + c$$

where d_X, d_Y denote the metrics of X and Y respectively.

Definition 5.2. A finitely generated subgroup H of G is *undistorted* if the inclusion of H into G is a quasi-isometry with respect to the word metrics determined by finite generating sets.

Remark 5.3. It is well-known that the undistortedness of subgroups is independent of the choices of finite generating sets. For definiteness, we will fix a finite generating set T of H and without loss of generality, assume that $T \subset S$. Note that a geodesic segment of $Ca(H,T)$ is sent by the quasi-isometry to a quasigeodesic segment $Ca(G,S)$ with same endpoints.

A related notion is the quasiconvexity of subgroups which requires subgroups to be uniformly quasiconvex subsets of Caylay graph of ambient groups. Thus the quasiconvexity of subgroups depends on the choice of finite generating systems of ambient groups. Note that a quasiconvex subgroup is undistorted but the converse is not generally true. Especially in word hyperbolic groups the quasiconvexity of subgroups is equivalent to their undistortedness.

We now briefly discuss the Floyd boundary of a finitely generated group. In [9], Floyd introduced such a boundary for any finitely generated group G, which is obtained by rescaling the length of each edge e of $Ca(G,S)$ by a conformal factor $F(e)$, for example $F(e) = d(1,e)^{-2}$, and then taking the Cauchy metric completion $\overline{G}$. Denote by ρ the complete metric on $\overline{G}$. Floyd boundary $\partial(G)$ is defined as $\overline{G} \setminus G$ and depends on the choice of conformal factor F. With proper choice of conformal factor F, the Floyd boundary of a group G is a quasi-isometry invariant.

If $\partial(G)$ consists of 0, 1 or 2 points then it is said to be *trivial*. Otherwise it is uncountable and is *nontrivial*. The class of groups with nontrivial Floyd boundary includes non-elementary relatively hyperbolic groups [11], groups with infinitely many ends, and many other examples. For more details on Floyd boundary we refer to [9] and [17].

Recall that if the Floyd boundary of a finitely generated group G is nontrivial, then G acts on $\overline{G}$ and thus $\partial(G)$ as a convergence group [17]. In the sequel, we will consider this convergence action of G on its Floyd boundary and limit sets of subgroups on $\partial(G)$.

The following lemma shows that the Floyd length of a far (quasi)geodesic in $Ca(G,S)$ is small. The original version was stated in [17] for geodesics, but its proof works for general quasigeodesic of Cayley graph.

Lemma 5.4. [17] *Given $\epsilon > 0$, there is a function Θ_ϵ: $N \to R$ such that $\Theta_\epsilon(r) \to 0$ as $r \to \infty$ and having the following property. Let z, w be two points in G and let $\gamma[z,w]$ be an ϵ-quasigeodesic segment between z and w of $Ca(G,S)$. Then*

$$\rho(z,w) \le \Theta_\epsilon(d(1, \gamma[z,w]))$$

where $d(1, \gamma[z,w])$ is the distance from the identity to the ϵ-quasigeodesic segment $\gamma[z,w]$.

The following Lemma roughly says that any two limit points of a undistorted subgroup H can be connected by a quasigeodesic line lying H.

Lemma 5.5. *If H is undistorted in G, then for any two distinct points p, $q \in \Lambda(H)$, there exists an ϵ_0-quasigeodesic line γ between p and q in $Ca(G, S)$ such that all vertices of γ are elements of H. The constant ϵ_0 depends on H and G.*

Proof. Since p and q are limit points of H, there exists two sequences $\{h_n\}$ and $\{h'_n\}$ of H such that $h_n \to p$ and $h'_n \to q$. Let $\delta = d(p, q)/3$.

By continuity, we can assume for all n, $h_n \in B(p, \delta)$ and $h'_n \in B(q, \delta)$, after passage to subsequences of $\{h_n\}$ and $\{h'_n\}$ respectively. Here $B(p, \delta)$ and $B(q, \delta)$ are respectively open metric balls centered at p and q of $\overline{G}$. It then follows by the triangle inequality that $\rho(h_n, h'_n) > d(p, q)/3$ for all n.

Taking geodesic segments γ_n of Cayley graph $Ca(H, T)$ with endpoints h_n and h'_n. By the undistortedness of H, any geodesic segment of Cayley graph $Ca(H, T)$ is ϵ_0-quasigeodesic in $Ca(G, S)$ for some constant ϵ_0 depending on H. Thus γ_n are ϵ_0-quasigeodesic segments of $Ca(G, S)$. Note that the endpoints h_n, h'_n of γ_n has at least a distance δ. By Lemma 5.4, quasigeodesic segments γ_n intersect a compact ball B centered at identity in $Ca(G, S)$.

Thus using a diagonal argument based on γ_n, we can obtain an ϵ_0-quasigeodesic line which are connecting p and q, such that all vertices are elements of H. □

Remark 5.6. In contrast with hyperbolic groups, two (quasi)geodesic segments in $Ca(G, S)$ with same endpoints may not be uniformly Hausdorff distance bounded. Thus we could not guarantee that any (quasi)geodesic between p and q satisfies the statement of Lemma 5.5.

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. It suffices to establish the conclusion with the assumption $|\Lambda(H)| \geq 2$. We are going to bound the following set

$$\{gH \in G/H : g\Lambda(H) \cap L \neq \emptyset \text{ and } g\Lambda(H) \cap K \neq \emptyset\},$$

whenever K and L are disjoint closed subsets of $\partial(G)$.

Suppose we have a sequence of distinct cosets g_nH such that $g_n\Lambda(H) \cap K \neq \emptyset$ and $g_n\Lambda(H) \cap L \neq \emptyset$. Let $a_n \in g_n\Lambda(H) \cap K$ and $b_n \in g_n\Lambda(H) \cap L$. Using Lemma 5.5, we can obtain ϵ_0-quasigeodesics γ_n between a_n and b_n such that all of their vertices are elements of g_nH.

Note that $\{a_n, b_n\} \in K \times L$ and $K \times L$ is compact in $\partial(G) \times \partial(G)$. Thus we have a uniform constant μ depending on K and L, such that $\rho(a_n, b_n) \geq \mu$ for all n.

By Lemma 5.4, we have a closed ball $B(R)$ at the identity with sufficiently large radius R in $Ca(G, S)$, such that any (ϵ_0-quasi) geodesic line connecting a_n and b_n intersect nontrivially with $B(R)$. Here R can be computed by μ and the function Θ_{ϵ_0} provided by Lemma 5.4. Thus there exist points $c_n \in \gamma_n$ such that the following holds

$$c_n \in \gamma_n \cap B(R) \neq \emptyset \text{ for all } n.$$

We now have $d(1, c_n) < R$ for every n. Observe that translated ϵ_0-quasigeodesics $g_n^{-1}(\gamma_n)$ end at $g_n^{-1}(a_n), g_n^{-1}(b_n) \in \Lambda(H)$, and all of their vertices are elements of H. Therefore it follows that for each n, there exists $h_n \in H$ such that $d(g_n^{-1}(c_n), h_n) <$

1 and thus $d(g_n^{-1}, h_n) < R + 1$ for all n. Since G acts properly discontinuously on $Ca(G, S)$, we have $g_n(h_n) = g_1(h_1)$ for all n, after passage to a subsequence of $\{g_n h_n\}$.

This is a contradiction to the choice of $\{g_n H\}$, which is a sequence of different cosets of H in G. □

Observe that many results in Section 2, 3 and 4 are proved only using dynamical quasiconvexity of involved relatively quasiconvex subgroups. Thus using Theorem 1.5, we are able to restate Lemma 2.6(and its Corollaries 2.7, 2.8), Proposition 3.1, Lemma 4.1 and Theorem 4.3 in the context of finitely generated groups with nontrivial Floyd boundary. In favor of application to group theory, we only restate the following corollaries.

Corollary 5.7. *Let H be undistorted in G and $|\Lambda(H)| \geq 2$. Then H is of finite index in its commensurator. In particular, H is of finite index in its normalizer.*

Corollary 5.8. *Let H be undistorted in G and $|\Lambda(H)| \geq 2$. Then for any $g \in G \backslash H$, $gHg^{-1} \subseteq H$ implies that $gHg^{-1} = H$.*

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